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EXISTENCE AND MULTIPLICITY OF POSITIVE PERIODIC SOLUTIONS FOR FOURTH-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this article we study the existence and multiplicity of positive periodic solutions for two classes of non-autonomous fourth-order nonlinear ordinary differential equations

$$u^{iv} - pu'' - a(x)u^n + b(x)u^{n+2} = 0,$$
$$u^{iv} - pu'' + a(x)u^n - b(x)u^{n+2} = 0,$$

where n is a positive integer, $p \leq 1$, and $a(x), b(x)$ are continuous positive T-periodic functions. These equations include particular cases of the extended Fisher-Kolmogorov equations and the Swift-Hohenberg equations. By using Mawhin's continuation theorem, we obtain two multiplicity results these equations.

1. Introduction and statement of main results

In the previous years there has been an increasing interest in the study of higher order problems that arise in Biology and Physics, such as the equations

$$u^{iv} - pu'' - a(x)u + b(x)u^3 = 0, \quad x \in \mathbb{R}, \tag{1.1}$$
$$u^{iv} - pu'' + a(x)u - b(x)u^3 = 0, \quad x \in \mathbb{R}. \tag{1.2}$$

In [23], the authors prove the existence of periodic solutions to (1.1) and (1.2), when p is a positive constant, and $a(x), b(x)$ are continuous positive $2L$-periodic functions on $\mathbb{R}$.

For (1.1) and (1.2), we consider the boundary conditions

$$u(0) = u(L) = u''(0) = u''(L) = 0.$$

Existence of nontrivial solutions for (1.1) is proved using a minimization theorem and multiplicity using Clark's theorem. Existence of nontrivial solutions for (1.2) is proved using the symmetric mountain pass theorem. When $p > 0$, equations (1.1) and (1.2) are called extended Fisher-Kolmogorov (EFK) equations, which was proposed by Dee and Van Saarloos [10] in 1988 as a model for bistable systems. On the other hand, when $p < 0$, Equations (1.1) and (1.2) are called the Swift-Hohenberg (SH) equations, which was proposed by Swift and Honenberg [22] in

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1977, in the context of hydrodynamic instabilities. For more equations related to the model, see [6, 11, 20, 21].

The following two-point boundary value problem is considered In [8],

$$u^{iv} - pu'' - a(x)u + b(x)u^3 = 0, \quad 0 < x < L,$$
$$u(0) = u''(0) = u(L) = u''(L),$$

where $p \in \mathbb{R}$, and the functions $a(x)$ and $b(x)$ are positive continuous even and $2L$ periodic. This type of equations has been studied in [5, 7, 12, 17, 24, 27]. At the same time, the existence of periodic solutions of nonlinear differential equations has benn studied in [2, 3, 4, 9, 13, 14, 15, 16, 18, 25, 26].

In this paper, our purpose is to establish the existence and multiplicity of positive periodic solutions of the non-autonomous fourth-order nonlinear ordinary differential equations at resonance

$$u^{iv} - pu'' - a(x)u^n + b(x)u^{n+2} = 0, \quad x \in \mathbb{R}, \tag{1.3}$$
$$u^{iv} - pu'' + a(x)u^n - b(x)u^{n+2} = 0, \quad x \in \mathbb{R}, \tag{1.4}$$

where n is a positive integer, $p \le 1$, and $a(x), b(x)$ are continuous positive T-periodic functions on $\mathbb{R}$, where $0 < a \le a(x) \le A$, $0 < b \le b(x) \le B$.

In our work, we use coincidence degree theories to establish existence and multiplicity of positive periodic solutions for (1.3) and (1.4), under some specific assumptions on a, A, b, B, p, T to be given later. It is worth noting that when $n = 1$ Eq.uations (1.3) and (1.4) reduce to (1.1) and (1.2). Our main results are the following theorems.

Theorem 1.1. *Let*

$$1 - p \ge 0, \tag{1.5}$$

$a(x), b(x)$ be continuous positive T-periodic functions and a, A, b, B be positive constants such that

$$0 < a \le a(x) \le A, \quad 0 < b \le b(x) \le B, \quad \frac{B}{a} \le \frac{A}{b}. \tag{1.6}$$

Suppose that there exist constants $M_1, M_2, \dots, M_m$ and T such that

$$0 < M_1 < \dots < M_{r-1} < M_r < M_{r+1} < \dots < M_m, \tag{1.7}$$
$$0 < T \le \frac{1}{\beta^2(AM_m^{n-1} + BM_m^{n+1} - p + 2)}, \tag{1.8}$$

where β is the immersion constant of $H^2(0,T)$ in $C^1([0,T])$; $M_{r+1} = \sqrt{A/b} + \epsilon$ and $M_r = \sqrt{B/a} - \epsilon$ are positive constants, where $\epsilon > 0$ and small enough. Then both (1.3) and (1.4) have at least $m - 1$ positive T-periodic solutions.

Theorem 1.2. *As in Theorem 1.1 assume that (1.5), (1.6), (1.7) hold. Also assume that*

$$0 < T \le \frac{1}{\gamma^2(AM_m^{n-1} + BM_m^{n+1} - p + 3)}, \tag{1.9}$$

where γ is the immersion constant of $H^3(0,T)$ in $C^2([0,T])$; $M_{r+1} = \sqrt{A/b} + \epsilon$ and $M_r = \sqrt{B/a} - \epsilon$ are positive constants, where $\epsilon > 0$ and small enough. Then both (1.3) and (1.4) have at least $m - 1$ positive T-periodic solutions.

2. Preliminaries

In this section, we state notation and preliminary results that will play important roles in the prove of our main results.

Definition 2.1 ([19]). Let X, Y be real Banach spaces, $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping. The mapping L is said to be a Fredholm mapping of index zero if

(a) $\operatorname{Im} L$ is closed in Y;
(b) $\dim\ker L = \operatorname{codim}\operatorname{Im} L < +\infty$.

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$\operatorname{Im} P = \ker L,$$
$$\ker Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

It follows that the restriction L_P of L to $\operatorname{Dom} L \cap \ker P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of L_P by K_P.

Definition 2.2 ([19]). Let $N : X \to Y$ be a continuous mapping. Then mapping N is said to be L-compact on Ω if Ω is an open bounded subset of X, $QN(\Omega)$ is bounded and $K_P(I - Q)N : \Omega \to X$ is compact.

Lemma 2.3 (Mawhin's Continuation Theorem [19]). *Let L be a Fredholm mapping of index zero, $\Omega \subset X$ is an open bounded set and let N is L-compact on Ω. If all the following conditions hold:*

(1) $Lx \neq \lambda Nx$ *for all* $x \in \partial\Omega \cap \operatorname{Dom} L$, *and all* $\lambda \in (0, 1]$;
(2) $QNx \neq 0$, *for all* $x \in \partial\Omega \cap \ker L$;
(3) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \ker L$ *is an isomorphism.*

Then the equation $Lx = Nx$ *has at least one solution in* $\operatorname{Dom} L \cap \Omega$.

We shall denote by $H^n_{\rm per}(0, T)$ the usual Sobolev spaces of periodic functions, that is

$$H^n_{\rm per}(0, T) = \{u \in H^n(0, T) : u^{(i)}(0) = u^{(i)}(T), i = 0, \dots, n - 1\}.$$

Then we consider $X = H^3_{\rm per}(0, T)$, $Y = L^2(0, T)$.

Define a linear operator $L : \operatorname{Dom} L \subset X \to Y$ by setting

$$Lu = u^{(iv)} - pu'', \quad u \in \operatorname{Dom} L,$$

where $\operatorname{Dom} L = H^4_{\rm per}(0, T)$. It is immediate to prove that $\ker L = \mathbb{R}$ and

$$\operatorname{Im} L = \Big\{\varphi \in L^2(0, T) : \int_0^T \varphi(t)dt = 0\Big\}.$$

It is not difficult to see that $\operatorname{Im} L$ is a closed set in Y and

$$\dim\ker L = \operatorname{codim}\operatorname{Im} L = 1.$$

Thus the operator L is a Fredholm operator with index zero.

We define the nonlinear operators $N : X \to Y$ by setting

$$Nu = a(x)u^n - b(x)u^{n+2}, \text{ or}$$
$$Nu = -a(x)u^n + b(x)u^{n+2}.$$

Now we define the projector $P : X \to \ker L$ and the projector $Q : Y \to Y$ by setting

$$Pu(t) = \bar{u} = \frac{1}{T}\int_0^T u(t)dt,$$
$$Q\varphi(t) = \bar{\varphi} = \frac{1}{T}\int_0^T \varphi(t)dt.$$

Hence, $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$. Moreover, for $\varphi \in \operatorname{Im} L$ it follows that $K_P(\varphi)$ is the unique solution $u \in H^4_{\rm per}(0,T)$ of the problem

$$u^{(iv)} - pu'' = 0,$$
$$\bar{u} = 0.$$

Lemma 2.4 ([1])**.** *There exists a constant c such that*

$$\|u\|_{H^4} \le c\|Lu\|_{L^2}$$

for every $u \in H^4_{\rm per}(0,T)$ such that $\bar{u} = 0$.

Lemma 2.5. *Let L and N be as before and assume that $a(x), b(x)$ satisfy the assumptions of Theorems 1.1 and 1.2. Then N is L-compact on Ω for any bounded set $\Omega \subset X$.*

Proof. Clearly, operators $QN : X \to Y$ by setting

$$QNu = \frac{1}{T}\int_0^T a(x)u^n - b(x)u^{n+2}, \text{ or}$$
$$QNu = \frac{1}{T}\int_0^T -a(x)u^n + b(x)u^{n+2}.$$

It is immediate that $QN(\Omega)$ is bounded. Moreover, if $\varphi = (I-Q)Nu = Nu - \overline{Nu}$, let $v = K_P(\varphi)$ the unique element of $H^4_{\rm per}(0,T)$ satisfying

$$Lv = \varphi, \quad \bar{v} = 0.$$

By Lemma 2.4, we know that there exists a constant C such that $\|v\|_{H^4} \le c\|\varphi\|_{L^2} \le C$ for any $u \in \Omega$, and compactness of $K_P(I-Q)N$ follows from the imbedding $H^4_{\rm per}(0,T) \hookrightarrow H^3_{\rm per}(0,T)$. $\square$

3. Proofs of the main results

Proof of Theorem 1.1. By Lemma 2.5, we know that N is L-compact on Ω for any open bounded set $\Omega \subset X$.

There exists an $\epsilon > 0$ small enough such that

$$0 < \epsilon + \sqrt{\frac{A}{b}} = M_{r+1} < M_{r+2}.$$

Let

$$\Omega_{r+1} = \{u \in X : M_{r+1} < u(x) < M_{r+2}\}, \tag{3.1}$$

which is an open set in X. Suppose that there exist $0 < \lambda \le 1$ and u be such that

$$u^{iv} - pu'' - \lambda a(x)u^n + \lambda b(x)u^{n+2} = 0.$$

Multiplying by u and the integrating from 0 to T, we have

$$\int_0^T u^{iv}u - pu''u - \lambda a(x)u^{n+1} + \lambda b(x)u^{n+3}dx = 0.$$

By (3.1), if $u \in \partial\Omega_{r+1}$, then $M_{r+1} \leq |u|_\infty \leq M_{r+2}$, where $|u|_\infty = \max_{t\in[0,T]}|u(x)|$. By (1.5), (1.6), (1.7) and (1.8), we have

$$\begin{aligned}
0 &= \int_0^T u^{iv}u - pu''u - \lambda a(x)u^{n+1} + \lambda b(x)u^{n+3}dx \\
&= \int_0^T (u'')^2 + p(u')^2 dx - \int_0^T \lambda a(x)u^{n+1} - \lambda b(x)u^{n+3}dx \\
&> \int_0^T (u'')^2 + p(u')^2 dx - \int_0^T \lambda a(x)u^{n+1} + \lambda b(x)u^{n+3}dx \\
&\geq \int_0^T (u'')^2 + p(u')^2 dx - \int_0^T a(x)u^{n+1} + b(x)u^{n+3}dx \\
&= \int_0^T (u'')^2 + (u')^2 + u^2 dx - \int_0^T -p(u')^2 + (u')^2 + u^2 dx \\
&\quad - \int_0^T a(x)u^{n+1} + b(x)u^{n+3}dx \\
&\geq \|u\|^2_{H^2(0,T)} - \int_0^T (1-p)(u')^2 + u^2 dx - \int_0^T u^2(A|u|_\infty^{n-1} + B|u|_\infty^{n+1})dx \\
&\geq \|u\|^2_{H^2(0,T)} - \int_0^T (1-p)\|u\|^2_{C^1([0,T])} + \|u\|^2_{C^1([0,T])}dx \\
&\quad - \int_0^T \|u\|^2_{C^1([0,T])}(AM_{r+2}^{n-1} + BM_{r+2}^{n+1})dx \\
&\geq \frac{\|u\|^2_{C^1([0,T])}}{\beta^2} - T\|u\|^2_{C^1([0,T])}(AM_{r+2}^{n-1} + BM_{r+2}^{n+1} - p + 2) \\
&> \frac{\|u\|^2_{C^1([0,T])}}{\beta^2} - T\|u\|^2_{C^1([0,T])}(AM_m^{n-1} + BM_m^{n+1} - p + 2) \\
&= \left[\frac{1}{\beta^2} - T(AM_m^{n-1} + BM_m^{n+1} - p + 2)\right]\|u\|^2_{C^1([0,T])} \geq 0,
\end{aligned}$$

where β is the immersion constant of $H^2(0,T)$ in $C^1([0,T])$. But this is contradiction. Therefore condition (1) of Lemma 2.3 holds for Ω_{r+1}.

It is easy to see that

$$a(x) - b(x)M_{r+2}^2 < 0, \tag{3.2}$$

$$a(x) - b(x)M_{r+1}^2 < 0\,. \tag{3.3}$$

Taking $u \in \partial\Omega_{r+1} \cap \ker L$, we have $u = M_{r+1}$ or $u = M_{r+2}$. By (3.2) and (3.3), we know that for all $u \in \partial\Omega_{r+1} \cap \ker L$, we obtain that

$$QNu = \frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx \neq 0. \tag{3.4}$$

Therefore condition (2) of Lemma 2.3 holds for Ω_{r+1}.

Now we consider $(M_{r+1} + M_{r+2})/2$, the arithmetic mean of M_{r+1} and M_{r+2}. We define a continuous function

$$H(u,\mu) = -(1-\mu)\Big(u + \frac{M_{r+1} + M_{r+2}}{2}\Big) + \mu\frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx,$$

for $\mu \in [0,1]$. By (3.4), we obtain

$$H(u,\mu) \neq 0, \quad \forall u \in \partial\Omega_{r+1} \cap \ker L.$$

By using the homotopy invariance theorem, we find that

$$\begin{aligned}
&\deg(QN, \Omega_{r+1} \cap \ker L, 0) \\
&= \deg\Big(\frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx, \Omega_{r+1} \cap \ker L, 0\Big) \\
&= \deg\Big(-\Big(u + \frac{M_{r+1} + M_{r+2}}{2}\Big), \Omega_{r+1} \cap \ker L, 0\Big) \neq 0.
\end{aligned}$$

Therefore condition (3) of Lemma 2.3 holds for Ω_{r+1}. So we conclude from Lemma 2.3 that (1.3) has a solution in Ω_{r+1}. By the method above, we can prove that (1.3) has a solution in $\Omega_{r+l} = \{u \in X : M_{r+l} < u(x) < M_{r+l+1}\}$, $l = 2, 3, \dots, m-r-1$.

There exists an $\epsilon > 0$ small enough such that

$$0 < M_r = \sqrt{\frac{B}{a}} - \epsilon < \sqrt{\frac{A}{b}} + \epsilon = M_{r+1}.$$

Let

$$\Omega_r = \{u \in X : M_r < u(x) < M_{r+1}\}, \tag{3.5}$$

an open set in X. Suppose that there exist $0 < \lambda \leq 1$ and u be such that

$$u^{iv} - pu'' - \lambda a(x)u^n + \lambda b(x)u^{n+2} = 0.$$

Multiplying by u and the integrating from 0 to T,

$$\int_0^T u^{iv}u - pu''u - \lambda a(x)u^{n+1} + \lambda b(x)u^{n+3}dx = 0.$$

By (3.5), if $u \in \partial\Omega_r$, we have $M_r \leq |u|_\infty \leq M_{r+1}$, where $|u|_\infty = \max_{t\in[0,T]}|u(x)|$. By (1.5), (1.6), (1.7) and (1.8), we have

$$\begin{aligned}
0 &= \int_0^T u^{iv}u - pu''u - \lambda a(x)u^{n+1} + \lambda b(x)u^{n+3}dx \\
&> \int_0^T (u'')^2 + p(u')^2 dx - \int_0^T a(x)u^{n+1} + b(x)u^{n+3}dx \\
&\geq \|u\|^2_{H^2(0,T)} - \int_0^T (1-p)(u')^2 + u^2 dx - \int_0^T u^2(AM_{r+1}^{n-1} + BM_{r+1}^{n+1})dx \\
&\geq \|u\|^2_{H^2(0,T)} - \int_0^T (1-p)\|u\|^2_{C^1([0,T])} + \|u\|^2_{C^1([0,T])}dx \\
&\quad - \int_0^T \|u\|^2_{C^1([0,T])}(AM_{r+1}^{n-1} + BM_{r+1}^{n+1})dx \\
&\geq \frac{\|u\|^2_{C^1[0,T]}}{\beta^2} - T\|u\|^2_{C^1([0,T])}(AM_{r+1}^{n-1} + BM_{r+1}^{n+1} - p + 2) \\
&> \Big[\frac{1}{\beta^2} - T(AM_m^{n-1} + BM_m^{n+1} - p + 2)\Big]\|u\|^2_{C^1([0,T])} \geq 0,
\end{aligned}$$

where β is the immersion constant of $H^2(0,T)$ in $C^1([0,T])$. But this is contradiction. Therefore condition (1) of Lemma 2.3 holds for Ω_r.

It is easy to show that

$$a(x) - b(x)M_{r+1}^2 < 0. \tag{3.6}$$

$$a(x) - b(x)M_r^2 > 0. \tag{3.7}$$

Taking $u \in \partial\Omega_r \cap \ker L$, we have $u = M_{r+1}$ or $u = M_r$. By (3.6) and (3.7), we know that for all $u \in \partial\Omega_r \cap \ker L$. Then

$$QNu = \frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx \neq 0. \tag{3.8}$$

Therefore condition (2) of Lemma 2.3 holds for Ω_r.

Now we consider $(M_{r+1} + M_r)/2$, the arithmetic mean of M_{r+1} and M_r. We define a continuous function

$$H(u,\mu) = -(1-\mu)\Big(u - \frac{M_{r+1}+M_r}{2}\Big) + \mu\frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx,$$

for $\mu \in [0,1]$. By (3.8), we obtain

$$H(u,\mu) \neq 0, \quad \forall u \in \partial\Omega_r \cap \ker L.$$

By using the homotopy invariance theorem, we find that

$$\begin{aligned}
&\deg(QN, \Omega_r \cap \ker L, 0)\\
&= \deg\Big(\frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx, \Omega_r \cap \ker L, 0\Big)\\
&= \deg\left(-\left(u - \frac{M_{r+1}+M_r}{2}\right), \Omega_r \cap \ker L, 0\right) \neq 0.
\end{aligned}$$

Therefore condition (3) of Lemma 2.3 holds for Ω_r. So we conclude from Lemma 2.3 that (1.3) has a solution in Ω_r.

There exists an $\epsilon > 0$ small enough such that

$$0 < M_{r-1} < \sqrt{\frac{B}{a}} - \epsilon = M_r.$$

Let

$$\Omega_{r-1} = \{u \in X : M_{r-1} < u(x) < M_r\}, \tag{3.9}$$

which is an open set in X. Suppose that there exist $0 < \lambda \leq 1$ and u be such that

$$u^{iv} - pu'' - \lambda a(x)u^n + \lambda b(x)u^{n+2} = 0.$$

Multiplying by u and the integrating from 0 to T, it is immediate that

$$\int_0^T u^{iv}u - pu''u - \lambda a(x)u^{n+1} + \lambda b(x)u^{n+3}dx = 0.$$

By (3.9), if $u \in \partial\Omega_{r-1}$, we have $M_{r-1} \leq |u|_\infty \leq M_r$, where $|u|_\infty = \max_{t\in[0,T]}|u(x)|$. By (1.5), (1.6), (1.7) and (1.8), we have

$$\begin{aligned}
0 &= \int_0^T u^{iv}u - pu''u - \lambda a(x)u^{n+1} + \lambda b(x)u^{n+3}dx\\
&> \int_0^T (u'')^2 + p(u')^2dx - \int_0^T a(x)u^{n+1} + b(x)u^{n+3}dx\\
&\geq \|u\|_{H^2(0,T)}^2 - \int_0^T (1-p)(u')^2 + u^2dx - \int_0^T u^2(AM_r^{n-1} + BM_r^{n+1})dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \|u\|^2_{H^2(0,T)} - \int_0^T (1-p)\|u\|^2_{C^1([0,T])} + \|u\|^2_{C^1([0,T])}dx \\
&\quad - \int_0^T \|u\|^2_{C^1([0,T])}(AM_r^{n-1} + BM_r^{n+1})dx \\
&\geq \frac{\|u\|^2_{C^1([0,T])}}{\beta^2} - T\|u\|^2_{C^1([0,T])}(AM_r^{n-1} + BM_r^{n+1} - p + 2) \\
&> \Big[\frac{1}{\beta^2} - T(AM_m^{n-1} + BM_m^{n+1} - p + 2)\Big]\|u\|^2_{C^1([0,T])} \geq 0,
\end{aligned}
$$

where β is the immersion constant of $H^2(0,T)$ in $C^1([0,T])$. But this is contradiction. Therefore condition (1) of Lemma 2.3 holds for Ω_{r-1}.

It is easy to show that

$$a(x) - b(x)M_{r-1}^2 > 0, \tag{3.10}$$

$$a(x) - b(x)M_r^2 > 0. \tag{3.11}$$

Taking $u \in \partial\Omega_{r-1} \cap \ker L$, we have $u = M_{r-1}$ or $u = M_r$. By (3.10) and (3.11). We know that for all $u \in \partial\Omega_{r-1} \cap \ker L$, we obtain

$$QNu = \frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx \neq 0. \tag{3.12}$$

Therefore condition (2) of Lemma 2.3 holds for Ω_{r-1}.

Now we consider $(M_r + M_{r-1}/2$, the arithmetic mean of M_{r-1} and M_r. We define a continuous function

$$H(u,\mu) = (1-\mu)\Big(u + \frac{M_r + M_{r-1}}{2}\Big) + \mu\frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx,$$

for $\mu \in [0,1]$. It follows from (3.12) that

$$H(u,\mu) \neq 0, \quad \forall u \in \partial\Omega_{r-1} \cap \ker L.$$

By using the homotopy invariance theorem, we find that

$$
\begin{aligned}
&\deg(QN, \Omega_{r-1} \cap \ker L, 0) \\
&= \deg\left(\frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx, \Omega_{r-1} \cap \ker L, 0\right) \\
&= \deg\left(u + \frac{M_r + M_{r-1}}{2}, \Omega_{r-1} \cap \ker L, 0\right) \neq 0.
\end{aligned}
$$

Therefore condition (3) of Lemma 2.3 holds for Ω_{r-1}. So we conclude from Lemma 2.3 that (1.3) has a solution in Ω_{r-1}. By the method above, we can prove that (1.3) has a solution in $\Omega_k = \{u \in X : M_k < u(x) < M_{k+1}\}$, $k = 1, 2, \dots, r-2$.

By (1.7), we know that $\Omega_i \cap \Omega_j = \emptyset$, $i = 1, 2, 3 \dots m$, $j = 1, 2, 3, \dots m$, $i \neq j$.

In view of the discussion above, we know that (1.3) has at least $m-1$ positive T-periodic solutions. Similarly, we can prove (1.4) has at least $m-1$ positive T-periodic solutions. □

Proof of Theorem 1.2. By Lemma 2.5, we know that N is L-compact on Ω for any open bounded set $\Omega \subset X$. There exists an $\epsilon > 0$ small enough such that

$$0 < M_{r+1} = \sqrt{\frac{A}{b}} + \epsilon < M_{r+2}.$$

Let

$$\Omega_{r+1} = \{u \in X : M_{r+1} < u(x) < M_{r+2}\}, \tag{3.13}$$

an open set in X. Suppose that there exist $0 < \lambda \le 1$ and u be such that

$$u^{iv} - pu'' - \lambda a(x)u^n + \lambda b(x)u^{n+2} = 0.$$

Multiplying by u'' and the integrating from 0 to T, we have

$$\int_0^T u^{iv}u'' - pu''u'' - \lambda a(x)u^n u'' + \lambda b(x)u^{n+2}u''dx = 0.$$

By (3.13), if $u \in \partial\Omega_{r+1}$, then $M_{r+1} \le |u|_\infty \le M_{r+2}$, where $|u|_\infty = \max_{t\in[0,T]}|u(x)|$. By (1.5), (1.6), (1.7) and (1.9), we have

$$\begin{aligned}
0 &= \int_0^T u^{iv}u'' - pu''u'' - \lambda a(x)u^n u'' + \lambda b(x)u^{n+2}u''dx \\
&= \int_0^T (u''')^2 + p(u'')^2 + \lambda a(x)u^n u'' - \lambda b(x)u^{n+2}u''dx \\
&> \int_0^T (u''')^2 + p(u'')^2 dx - \int_0^T \lambda a(x)u^n|u''| + \lambda b(x)u^{n+2}|u''|dx \\
&\ge \int_0^T (u''')^2 + p(u'')^2 dx - \int_0^T a(x)u^n|u''| + b(x)u^{n+2}|u''|dx \\
&\ge \|u\|^2_{H^3(0,T)} - \int_0^T (1-p)(u'')^2 + (u')^2 + u^2 dx \\
&\quad - \int_0^T u|u''|(A|u|_\infty^{n-1} + B|u|_\infty^{n+1})dx \\
&\ge \|u\|^2_{H^3(0,T)} - \int_0^T (3-p)\|u\|^2_{C^2([0,T])}dx \\
&\quad - \int_0^T \|u\|^2_{C^2([0,T])}(AM_{r+2}^{n-1} + BM_{r+2}^{n+1})dx \\
&\ge \frac{\|u\|^2_{C^2([0,T])}}{\gamma^2} - T\|u\|^2_{C^2([0,T])}(AM_{r+2}^{n-1} + BM_{r+2}^{n+1} - p + 3) \\
&> \frac{\|u\|^2_{C^2([0,T])}}{\gamma^2} - T\|u\|^2_{C^2([0,T])}(AM_m^{n-1} + BM_m^{n+1} - p + 3) \\
&= \left[\frac{1}{\gamma^2} - T(AM_m^{n-1} + BM_m^{n+1} - p + 3)\right]\|u\|^2_{C^2([0,T])} \ge 0,
\end{aligned}$$

where γ is the immersion constant of $H^3(0,T)$ in $C^2([0,T])$. But this is contradiction. Therefore condition (1) of Lemma 2.3 holds for Ω_{r+1}.

Obviously, for all $u \in \partial\Omega_{r+1} \cap \ker L$, we obtain

$$QNu = \frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx \ne 0. \tag{3.14}$$

Therefore condition (2) of Lemma 2.3 holds for Ω_{r+1}.

Now we consider $(M_{r+1} + M_{r+2})/2$, the arithmetic mean of M_{r+1} and M_{r+2}. We define a continuous function

$$H(u, \mu) = -(1-\mu)\left(u + \frac{M_{r+1} + M_{r+2}}{2}\right) + \mu \frac{1}{T}\int_0^T u^n (a(x) - b(x)u^2) dx,$$

for $\mu \in [0, 1]$. By (3.14), we obtain

$$H(u, \mu) \neq 0, \quad \forall u \in \partial\Omega_{r+1} \cap \ker L.$$

By using the homotopy invariance theorem, we find that

$$\begin{aligned}
&\deg(QN, \Omega_{r+1} \cap \ker L, 0) \\
&= \deg\Big(\frac{1}{T}\int_0^T u^n (a(x) - b(x)u^2) dx, \Omega_{r+1} \cap \ker L, 0\Big) \\
&= \deg\Big(-\Big(u + \frac{M_{r+1} + M_{r+2}}{2}\Big), \Omega_{r+1} \cap \ker L, 0\Big) \neq 0.
\end{aligned}$$

Therefore condition (3) of Lemma 2.3 holds for Ω_{r+1}. So we conclude from Lemma 2.3 that (1.3) has a solution in Ω_{r+1}. By the method above, we can prove that (1.3) has a solution in $\Omega_{r+l} = \{u \in X : M_{r+l} < u(x) < M_{r+l+1}\}$, $l = 2, 3, \dots, m-r-1$.

There exists an $\epsilon > 0$ small enough such that

$$0 < M_r = \sqrt{\frac{B}{a}} - \epsilon < \sqrt{\frac{A}{b}} + \epsilon = M_{r+1}.$$

Let

$$\Omega_r = \{u \in X : M_r < u(x) < M_{r+1}\}, \tag{3.15}$$

an open set in X. Suppose that there exist $0 < \lambda \leq 1$ and u be such that

$$u^{iv} - pu'' - \lambda a(x)u^n + \lambda b(x)u^{n+2} = 0.$$

Multiplying by u'' and the integrating from 0 to T,

$$\int_0^T u^{iv}u'' - pu''u'' - \lambda a(x)u^n u'' + \lambda b(x)u^{n+2}u'' dx = 0.$$

By (3.15), if $u \in \partial\Omega_{r+1}$, we have $M_r \leq |u|_\infty \leq M_{r+1}$, where $|u|_\infty = \max_{t\in[0,T]}|u(x)|$. By (1.5), (1.6), (1.7) and (1.9). We have

$$\begin{aligned}
0 &= \int_0^T u^{iv}u'' - pu''u'' - \lambda a(x)u^n u'' + \lambda b(x)u^{n+2}u'' dx \\
&> \int_0^T (u''')^2 + p(u'')^2 dx - \int_0^T \lambda a(x)u^n |u''| + \lambda b(x)u^{n+2}|u''| dx \\
&\geq \|u\|^2_{H^3(0,T)} - \int_0^T (1-p)(u'')^2 + (u')^2 + u^2 dx \\
&\quad - \int_0^T u|u''|(A|u|^{n-1}_\infty + B|u|^{n+1}_\infty) dx \\
&\geq \frac{\|u\|^2_{C^2([0,T])}}{\gamma^2} - T\|u\|^2_{C^2([0,T])}(AM^{n-1}_{r+1} + BM^{n+1}_{r+1} - p + 3) \\
&> \frac{\|u\|^2_{C^2([0,T])}}{\gamma^2} - T\|u\|^2_{C^2([0,T])}(AM^{n-1}_m + BM^{n+1}_m - p + 3) \geq 0,
\end{aligned}$$

where γ is the immersion constant of $H^3(0,T)$ in $C^2([0,T])$. But this is contradiction. Therefore condition (1) of Lemma 2.3 holds for Ω_r.

Obviously, for all $u \in \partial\Omega_r \cap \ker L$, we obtain that

$$QNu = \frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx \neq 0. \tag{3.16}$$

Therefore condition (2) of Lemma 2.3 holds for Ω_r.

Now we consider $(M_{r+1} + M_r)/2$, the arithmetic mean of M_{r+1} and M_r. We define a continuous function

$$H(u,\mu) = -(1-\mu)\Big(u - \frac{M_{r+1} + M_r}{2}\Big) + \mu\frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx,$$

for $\mu \in [0,1]$. By (3.16), we obtain

$$H(u,\mu) \neq 0, \quad \forall u \in \partial\Omega_r \cap \ker L.$$

By using the homotopy invariance theorem, we find that

$$\begin{aligned}
&\deg(QN, \Omega_r \cap \ker L, 0)\\
&= \deg\Big(\frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx, \Omega_r \cap \ker L, 0\Big)\\
&= \deg\Big(-\Big(u - \frac{M_{r+1} + M_r}{2}\Big), \Omega_r \cap \ker L,\ 0\Big) \neq 0.
\end{aligned}$$

Therefore condition (3) of Lemma 2.3 holds for Ω_r. So we conclude from Lemma 2.3 that (1.3) has a solution in Ω_r.

There exists an $\epsilon > 0$ small enough such that

$$0 < M_{r-1} < \sqrt{\frac{B}{a}} - \epsilon = M_r.$$

Let

$$\Omega_{r-1} = \{u \in X : M_{r-1} < u(x) < M_r\}, \tag{3.17}$$

which is an open set in X. Suppose that there exist $0 < \lambda \leq 1$ and u be such that

$$u^{iv} - pu'' - \lambda a(x)u^n + \lambda b(x)u^{n+2} = 0.$$

Multiplying by u'' and the integrating from 0 to T,

$$\int_0^T u^{iv}u'' - pu''u'' - \lambda a(x)u^n u'' + \lambda b(x)u^{n+2}u''dx = 0.$$

By (3.17), if $u \in \partial\Omega_{r-1}$, we have $M_{r-1} \leq |u|_\infty \leq M_r$, where $|u|_\infty = \max_{t\in[0,T]}|u(x)|$. By (1.5), (1.6), (1.7) and (1.9). We have

$$\begin{aligned}
0 &= \int_0^T u^{iv}u'' - pu''u'' - \lambda a(x)u^n u'' + \lambda b(x)u^{n+2}u''dx\\
&> \int_0^T (u''')^2 + p(u'')^2dx - \int_0^T \lambda a(x)u^n|u''| + \lambda b(x)u^{n+2}|u''|dx\\
&\geq \|u\|^2_{H^3(0,T)} - \int_0^T (1-p)(u'')^2 + (u')^2 + u^2dx\\
&\quad - \int_0^T u|u''|(A|u|_\infty^{n-1} + B|u|_\infty^{n+1})dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\|u\|^2_{C^2([0,T])}}{\gamma^2} - T\|u\|^2_{C^2([0,T])}(AM_r^{n-1} + BM_r^{n+1} - p + 3) \\
&> \frac{\|u\|^2_{C^2([0,T])}}{\gamma^2} - T\|u\|^2_{C^2([0,T])}(AM_m^{n-1} + BM_m^{n+1} - p + 3) \geq 0,
\end{aligned}
$$

where γ is the immersion constant of $H^3(0,T)$ in $C^2([0,T])$. But this is contradiction. Therefore condition (1) of Lemma 2.3 holds for Ω_{r-1}.

Obviously, for all $u \in \partial\Omega_{r-1} \cap \ker L$, we obtain that

$$
QNu = \frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx \neq 0. \tag{3.18}
$$

Therefore condition (2) of Lemma 2.3 holds for Ω_{r-1}.

Now we consider $(M_r + M_{r-1})/2$, the arithmetic mean of M_{r-1} and M_r. We define a continuous function

$$
H(u,\mu) = (1-\mu)\Big(u + \frac{M_r + M_{r-1}}{2}\Big) + \mu\frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx,
$$

for $\mu \in [0,1]$. By (3.18), we obtain

$$
H(u,\mu) \neq 0, \quad \forall u \in \partial\Omega_{r-1} \cap \ker L.
$$

By using the homotopy invariance theorem, we find that

$$
\begin{aligned}
&\deg(QN, \Omega_{r-1} \cap \ker L, 0) \\
&= \deg\Big(\frac{1}{T}\int_0^T u^n(a(x) - b(x)u^2)dx, \Omega_{r-1} \cap \ker L, 0\Big) \\
&= \deg\Big(u + \frac{M_r + M_{r-1}}{2}, \Omega_{r-1} \cap \ker L, 0\Big) \neq 0.
\end{aligned}
$$

Therefore condition (3) of Lemma 2.3 holds for Ω_{r-1}. so we conclude from Lemma 2.3 that (1.3) has a solution in Ω_{r-1}. by the above method, we prove that (1.3) has a solution in $\Omega_k = \{u \in X : M_k < u(x) < M_{k+1}\}$, $k = 1, 2, \dots, r-2$.

By (1.7), we know that $\Omega_i \cap \Omega_j = \emptyset$, $i = 1, 2, 3 \dots m$, $j = 1, 2, 3, \dots m$, $i \neq j$. In view of the discussion above, we know that (1.3) has at least $m-1$ positive T-periodic solutions. Similarly, we can prove (1.4) have at least $m-1$ positive T-periodic solutions. □

4. Example

Consider(1.3) and (1.4) with $a(x) = \cos(\frac{2\pi x}{T}) + 7$, $b(x) = \sin(\frac{2\pi x}{T}) + 5$. Define $a = 6$, $A = 8$, $b = 4$, $B = 6$, and $n = 1$. We have that

$$
\frac{B}{a} = 1 \leq 2 = \frac{A}{b}.
$$

Let $\epsilon = 0.01$, $M_r = 1 - 0.01 = 0.99 > 0$, $M_{r+1} = \sqrt{2} + 0.01 > 0$, $M_1 = 0.1$, $M_m = 100$.

When $p = 1$, we have

$$
1 - p = 0 \geq 0,
$$

$$
0 < T \leq \frac{1}{\beta^2(AM_m^{n-1} + BM_m^{n+1} - p + 2)} = \frac{1}{60009\beta^2}.
$$

Theorem 1.1 guarantees that both equations

$$u^{iv} - u'' - \Big(\cos\big(\frac{2\pi x}{T}\big) + 7\Big)u + \Big(\sin\big(\frac{2\pi x}{T}\big) + 5\Big)u^3 = 0,$$
$$u^{iv} - u'' + \Big(\cos\big(\frac{2\pi x}{T}\big) + 7\Big)u - \Big(\sin\big(\frac{2\pi x}{T}\big) + 5\Big)u^3 = 0,$$

have at least $m-1$ positive T-periodic solutions.

When $p=-1$, we have that

$$1-p=2\geq 0,$$
$$0<T\leq \frac{1}{\beta^2(AM_m^{n-1}+BM_m^{n+1}-p+3)}=\frac{1}{60012\gamma^2}.$$

Theorem 1.2 guarantees that both equations

$$u^{iv} + u'' - \Big(\cos\big(\frac{2\pi x}{T}\big) + 7\Big)u + \Big(\sin\big(\frac{2\pi x}{T}\big) + 5\Big)u^3 = 0,$$
$$u^{iv} + u'' + \Big(\cos\big(\frac{2\pi x}{T}\big) + 7\Big)u - \Big(\sin\big(\frac{2\pi x}{T}\big) + 5\Big)u^3 = 0,$$

have at least $m-1$ positive T-periodic solutions.

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