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EVENTUALLY COMPETITIVE SYSTEMS GENERATED BY PERTURBATIONS

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Abstract. A dynamical system is called eventually competitive if it preserves a partial order in backward time only after some reasonable initial transient. We show that both regular and singular perturbations of a competitive irreducible vector field are at most eventually competitive rather than competitive.

1. Introduction

A system of differential equations in $\mathbb{R}^N$ is called competitive provided that all the off-diagonal entries of its linearized Jacobian matrix are nonpositive. The flow of a competitive system preserves the vector partial order in backward time. In the biological science and population ecology, there are a lot of mathematical models of competition in which an increase of competitor's population size or density can only have a negative effect on a species per capita growth rate; see [15, 24, 30].

By time reversal, a competitive system becomes a cooperative system. And the flow of a cooperative system preserves the vector partial order in forward time. Hirsch initiated an important research branch of so called monotone dynamical systems. One may refer to the monographs and recent reviews [13, 14, 25, 27] with references therein for the theoretical developments and their enormous applications to control, biological and economic systems [3, 17, 28].

It is known that there are systems whose flows preserve the vector partial order in forward time only after some reasonable initial transient. In the terminology of linear systems, such phenomenon is called *eventual positivity* (see [8, 19, 29] and references therein) in forward time. For nonlinear systems, following Hirsch [12], the flow ϕ_t generated by such a system is called as an *eventually monotone* flow. Such property has received rapidly-increasing attention in both finite-dimensional linear systems [19] and infinite-dimensional linear systems [7, 8]. And applications to ordinary differential equations [21, 31], partial differential equations [6, 10, 11], delay differential equations [7, 8] and control theory [1, 2].

A flow ϕ_t is *eventually competitive*, if there exists a $t_* \geq 0$ such that $\phi_{-t}(x) \geq \phi_{-t}(y)$ whenever $x \geq y$ with $t \geq t_*$. In particular, ϕ_t is *competitive* if ϕ_t is eventually competitive with $t_* = 0$. In contrast, if $t_* > 0$, then there is no a priori order-preserving information for $t \in [0, t_*)$ at all. And ϕ_t is *eventually strongly*

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competitive, if ϕ_t is eventually competitive and $\phi_{-t}(x) \gg \phi_{-t}(y)$ whenever $x > y$ with $t \geq t_*$. For instance, we consider a matrix

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 0.8 & -1 & -1 \\ -1 & 0.2 & -1 \end{pmatrix},$$

and compute e^{-tA} for $t = 1, 2$ as

$$\begin{pmatrix} 0.5223 & 2.1030 & 2.6260 \\ -0.1514 & 1.8755 & 2.0262 \\ 1.2464 & 0.9743 & 3.8249 \end{pmatrix}, \begin{pmatrix} 3.2275 & 7.6010 & 15.6766 \\ 2.1625 & 5.1732 & 11.1524 \\ 5.2709 & 8.1752 & 19.8769 \end{pmatrix},$$

respectively. As a matter of fact, there exists a $t_* \in (1, 2)$ such that $e^{-tA} > 0$ for all $t \geq t_*$ (An $n \times n$ matrix $A = [a_{ij}]$ is positive, denoted by $A > 0$, if $a_{ij} > 0$ for all i and j). Thus, the linear system $\frac{dx}{dt} = Ax$ is eventually competitive.

In our previous work [20], we established the general theory of the eventually competitive system. However, for our best knowledge, it is still unclear that how to obtain an eventually competitive system. In this article, we will try to show one of the sources of eventually competitive systems. In fact, some kinds of eventually competitive systems can be obtained from the perturbations of competitive systems.

We focus on the regular perturbations of a competitive system in Section 3. We will show that any C^1-regular perturbation of a competitive irreducible vector field is at most eventually competitive rather than competitive (see Theorem 3.1). On a convex compact set $W \subset \mathbb{R}^N$, let the flow ϕ_t generated by a smooth vector field F satisfy that $D\phi_{-t}(z)[C\backslash\{0\}] \subset \operatorname{Int} C$ for any $z \in W$, $t > 0$ and $-t \in I(z)$, where $I(z) \subset \mathbb{R}$ denotes the maximal interval of existence of the solution passing though z. Let also G be a C^1 perturbed vector field of F such that W is positively invariant under the flow ψ_t generated by G. Then, we show that there exists a $t_* > 0$ such that ψ_t satisfies that $D\psi_{-t}(z)[C\backslash\{0\}] \subset \operatorname{Int} C$ for $z \in W$ and $t \in [t_*, 2t_*]$. Since W is positively invariant under the flow ψ_t, the property $D\psi_{-t}(z)[C\backslash\{0\}] \subset \operatorname{Int} C$ may not hold for $t > 2t_*$.

The fact that the flow Φ_t of a cooperative system preserves the vector partial order on a positively invariant set usually follows from the positiveness of $D\Phi_t$ and the formula $\Phi_t(x) - \Phi_t(y) = \int_0^1 D\Phi_t(sx + (1-s)y)(x - y)ds$ for $t > 0$. And a competitive system becomes a cooperative system by time reversal. Thus, the main difficulty with that formula for a competitive system is that the set of the inverse of the competitive system may not be positively invariant. Smith [26] assumed a priori property for competitive systems that the boundaries of the set are invariant. However, we will prove in this paper that the flow ψ_t is eventually competitive on the positively invariant set W by induction.

We further give some example for regular perturbations. Let P be a symmetric matrix with one positive eigenvalue and $N-1$ negative eigenvalues. A vector field $F : U(\subset \mathbb{R}^N) \to \mathbb{R}^N$ is called *strictly P-competitive*, if there is a function λ from $\mathbb{R}^N$ to $\mathbb{R}$ such that the matrix $PDF(x) + DF(x)^*P + \lambda(x)P$ is negative definite for each $x \in U$. A certain cone can be determined as $C = \{x \in \mathbb{R}^N : (Px, x) \geq 0,\ (x, v_+) \geq 0\}$ (see Ortega and Sánchez [22]). Let G be a C^1-perturbed vector field of F, then we obtain that the flow generated by G is eventually competitive. Therefore, together with our work [20], the limit sets of such an eventually competitive system are 1-codimensional.

Applications will vary widely on differing time scales, and those mathematical models can be formulated as a singularly perturbed system. In Section 4, we consider a singularly perturbed system has the form:

$$\begin{aligned} \frac{dx}{dt} &= f_0(x, y, \epsilon), \\ \epsilon \frac{dy}{dt} &= g_0(x, y, \epsilon), \end{aligned}$$

for a positive parameter ϵ near zero with $(x, y) \in U \times V$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets. And there exists a family of convex compact sets $D_\epsilon \subset U \times V$ such that D_ϵ are positively invariant for $\epsilon \in (0, \epsilon_0]$.

If there exists a critical manifold $\mathbb{C}_0$ given as

$$\{(x, h(x)) : x \in U, y = h(x), g_0(x, h(x), 0) = 0\}$$

with normally hyperbolic assumptions, by the geometric singular perturbation theory (see [9, 16, 18]), there is an invariant manifold $\mathbb{C}_\epsilon$ which behaves like a center manifold which may attract the flows in its neighborhood. And $\mathbb{C}_\epsilon$ is C^1 close to $\mathbb{C}_0$. The flow on $\mathbb{C}_\epsilon$ ($\epsilon > 0$) can be treated as a C^1 regular perturbation of the flow (for $\epsilon = 0$) on $\mathbb{C}_0$. Hence, if the flow on the manifold $\mathbb{C}_0$ is competitive, then the flow on the set $\mathbb{C}_\epsilon \cap D_\epsilon$ is eventually competitive.

Moreover, the slow manifold $\mathbb{C}_\epsilon$ possesses a stable manifold $W^s(\mathbb{C}_\epsilon)$ which is characterized in term of an exponential decay rate of solutions as $t \to \infty$, and all trajectories in D_ϵ will eventually fall into the stable manifold. As a consequence, we will show that the asymptotic phase implies that the dynamics (more specifically, the ω-limit sets) of the singularly perturbed system are equivalent to those of an eventually competitive system on $\mathbb{C}_\epsilon$ (see Theorem 4.1).

For such a singularly perturbed system, the priori property in [26] does not hold. In fact, the set D_ϵ is at most positively invariant, since all trajectories in D_ϵ will eventually fall into the stable manifold $W^s(\mathbb{C}_\epsilon)$.

Finally, we will present some example for singular perturbations. Assume that a singularly perturbed system is competitive (for $\epsilon = 0$) on the critical manifold and positive feedback acts at a comparatively fast variable, then the perturbed system is eventually competitive.

2. Definitions and notation

In this section, we introduce several useful definitions for eventually competitive systems. A nonempty closed set $C \subset \mathbb{R}^N$ is called a *convex cone* if it satisfies $C + C \subset C$, $\alpha C \subset C$ for all $\alpha \geq 0$, and $C \cap (-C) = \{0\}$. A convex cone is *solid* if $\operatorname{Int} C \neq \emptyset$. We write

$$\begin{aligned} x &\leq y \quad \text{if } y - x \in C, \\ x &< y \quad \text{if } y - x \in C \backslash \{0\}, \\ x &\ll y \quad \text{if } y - x \in \operatorname{Int} C. \end{aligned}$$

Notation such as $x \geq (>, \gg)\, y$ has the natural meanings. A subset $W \subset \mathbb{R}^N$ is *p-convex* if $x \leq y$ and $x, y \in W$ imply that W contains the line segment between x and y.

The *dual cone* of C is defined as

$$C^* = \{\lambda \in (\mathbb{R}^N)^* : \lambda(C) \geq 0\},$$

where $(\mathbb{R}^N)^*$ denotes the dual space of $\mathbb{R}^N$.

Then we have the following two properties:

$$x \in C \Longleftrightarrow \lambda(x) \geq 0, \quad \forall \lambda \in C^*, \tag{2.1}$$

$$x \in \mathrm{Int}C \Longleftrightarrow \lambda(x) > 0, \quad \forall \lambda \in C^* \backslash \{0\}. \tag{2.2}$$

See [4, Theorem 1.2.8] for (2.1), and [14, Proposition 3.1] for (2.2).

Let $\phi : \mathbb{R} \times X \to X$ be a flow on an open subset $X \subset \mathbb{R}^N$. And ϕ_t is *eventually competitive* if there exists a $t_* \geq 0$ such that $\phi_{-t}(x) \geq \phi_{-t}(y)$ whenever $x \geq y$ with $t \geq t_*$. In particular, ϕ_t is competitive if ϕ_t is eventually competitive with $t_* = 0$. And ϕ_t is *eventually strongly competitive*, if ϕ_t is eventually competitive and $\phi_{-t}(x) \gg \phi_{-t}(y)$ whenever $x > y$ with $t \geq t_*$.

3. Eventually competitive systems from regular perturbations

We consider the ODE system

$$\frac{dz}{dt} = F(z), \tag{3.1}$$

for which $F : U \to \mathbb{R}^N$ is a C^1 vector field, $U \subset \mathbb{R}^N$ is a open set. Let $I(z) \subset \mathbb{R}$ denote the maximal interval of existence of the solution of (3.1) passing though z. Assume $W \subset U$ is a convex compact subset and the flow ϕ_t of (3.1) satisfies that $D\phi_{-t}(z)[C\backslash\{0\}] \subset \mathrm{Int}\, C$ for $z \in W$, $t > 0$ and $-t \in I(z)$.

Theorem 3.1. *There exists a $\delta > 0$ with the following property. Let G denote a C^1 vector field such that $\|F(z) - G(z)\| + \|DF(z) - DG(z)\| < \delta$ for all $z \in U$ and W is positively invariant under the flow ψ_t generated by G. Then there exists a $t_* > 0$ such that the flow ψ_t is eventually competitive in W.*

Proof. Since W is a compact set, there exists such a fixed $t_* > 0$ that $[-2t_*, 0] \subset I(z)$ for $z \in W$. When $t \in [t_*, 2t_*]$, the property of $D\phi_{-t}(z)[C\backslash\{0\}] \subset \mathrm{Int}\, C$ implies that there exists a $\delta_1 > 0$, which is independent to t and z, such that $B(D\phi_{-t}(z)v, \delta_1) \subset \mathrm{Int}\, C$ for all $v \in C\backslash\{0\}$, $|v| = 1$. Next, a positive δ can be found such that if $\|F(z)-G(z)\|+\|DF(z)-DG(z)\| < \delta$ then $\|D\phi_{-t}(z)v - D\psi_{-t}(z)v\| < \frac{\delta_1}{2}$ hold for $t \in [t_*, 2t_*]$ and $z \in W$.

In fact, by the definition of flow, we have

$$\begin{aligned} \frac{d}{dt}\phi_{-t}(z) &= F(\phi_{-t}(z)), \\ \frac{d}{dt}\psi_{-t}(z) &= G(\psi_{-t}(z)); \end{aligned}$$

and the variational equations

$$\begin{aligned} \frac{d}{dt}D\phi_{-t}(z) &= DF(\phi_{-t}(z))D\phi_{-t}(z), \\ \frac{d}{dt}D\psi_{-t}(z) &= DG(\psi_{-t}(z))D\psi_{-t}(z). \end{aligned}$$

Then we obtain the estimate

$$\begin{aligned} \|D\phi_{-t}(z) - D\psi_{-t}(z)\| \leq &\int_0^t \|DF(\phi_{-s}(z))\|\|D\phi_{-s}(z) - D\psi_{-s}(z)\| ds \\ &+ \int_0^t \|DF(\phi_{-s}(z)) - DG(\psi_{-s}(z))\|\|D\psi_{-s}(z)\| ds. \end{aligned}$$

By Gronwall's inequality (see [5]),

$$\begin{aligned}\|D\phi_{-t}(z) - D\psi_{-t}(z)\| \le e^{\int_0^t \|DF(\phi_{-s}(z))\|ds} \int_0^t \|DF(\phi_{-s}(z)) \\ - DG(\psi_{-s}(z))\|\|D\psi_{-s}(z)\|ds.\end{aligned}$$

Since F, G are C^1 and $t_* \le t \le 2t_*$, there exist some $M > 0$ and $N > 0$ such that $\|DF(\phi_{-s}(z))\| \le M$, $\|D\psi_{-s}(z)\| \le N$ for all $s \in [0, t]$.

We also obtain that, for all $s \in [0, t]$,

$$\begin{aligned}\|DF(\phi_{-s}(z)) - DG(\psi_{-s}(z))\| &\le \|DF(\phi_{-s}(z)) - DG(\phi_{-s}(z))\| \\ &\quad + \|DG(\phi_{-s}(z)) - DG(\psi_{-s}(z))\| \\ &\le \delta + \|DG(\phi_{-s}(z)) - DG(\psi_{-s}(z))\|,\end{aligned}$$

and

$$\begin{aligned}\|\phi_{-s}(z) - \psi_{-s}(z)\| &\le \int_0^s \|F(\phi_{-k}(z)) - G(\psi_{-k}(z))\|dk \\ &\le \int_0^s \|F(\phi_{-k}(z)) - G(\phi_{-k}(z))\|dk \\ &\quad + \int_0^s \|G(\phi_{-k}(z)) - G(\psi_{-k}(z))\|dk \\ &\le \delta s + \int_0^s K\|\phi_{-k}(z) - \psi_{-k}(z)\|dk;\end{aligned}$$

then

$$\|\phi_{-s}(z) - \psi_{-s}(z)\| \le \delta 2t_* e^{K2t_*}.$$

The last estimate is obtained by Gronwall's inequality and $||DG|| \le K$. If we choose a $\delta < \frac{\delta_1}{8Nt_* e^{2t_* M}}$ small enough such that $\|DG(\phi_{-s}(z)) - DG(\psi_{-s}(z))\| < \frac{\delta_1}{8Nt_* e^{2t_* M}}$. Then

$$\|D\phi_{-t}(z) - D\psi_{-t}(z)\| < \frac{\delta_1}{2}.$$

Thus, we obtain that $D\psi_{-t}(z)v \in \operatorname{Int} C$ for $t \in [t_*, 2t_*]$ and $z \in W$.

Next, we assert that $x \gg y$ whenever $\psi_t(x) > \psi_t(y)$ with $x, y \in W$ and $t \in [t_*, 2t_*]$. In fact, for any $\lambda \in C^* \backslash \{0\}$, one has $\lambda(x - y) = \lambda(\psi_{-t}(\psi_t x) - \psi_{-t}(\psi_t y)) = \int_0^1 \lambda(D\psi_{-t}(k\psi_t x + (1-k)\psi_t y)(\psi_t x - \psi_t y))dk$. Since W is a positively invariant convex set, we have $k\psi_t x + (1-k)\psi_t y \in W$ for all $k \in [0, 1]$. If $\psi_t x > \psi_t y$, then $\int_0^1 \lambda(D\psi_{-t}(k\psi_t x + (1-k)\psi_t y)(\psi_t x - \psi_t y))dk > 0$. Thus, $\lambda(x - y) > 0$ and $x \gg y$ by the property (2.2).

For $x, y \in W$, we will show that

$$x \gg y, \quad \text{whenever } \psi_t(x) > \psi_t(y) \text{ with } t \in [nt_*, (n+1)t_*), \text{ for all } n \ge 1. \tag{3.2}$$

We will prove (3.2) by induction. It is clear that (3.2) hold for $n = 1$. Let (3.2) hold for $k = n - 1$. For $t \in [nt_*, (n+1)t_*)$, let $t = t_* + s$ with $s \in [(n-1)t_*, nt_*)$. So, $\psi_t(x) = \psi_s(\psi_{t_*} x)$ and $\psi_t(y) = \psi_s(\psi_{t_*} y)$. If $\psi_t(x) > \psi_t(y)$, then $\psi_{t_*}(x) \gg \psi_{t_*}(y)$ for $\psi_{t_*}(x), \psi_{t_*}(y) \in W$ with $k = n - 1$. Consequently, $x \gg y$ by $n = 1$. Thus, we have proved (3.2) and the flow ψ_t is eventually competitive. $\square$

Remark 3.2. Let F be a C^1 vector field, if the off-diagonal entries of $DF(z)$ are nonpositive for all $z \in U$ and $Df(z)$ is also irreducible for all $z \in U$. Then, by the Kamke Condition, the flow ϕ_t generated by F satisfies $D\phi_{-t}(z)[C \backslash \{0\}] \subset \operatorname{Int} C$ for

$z \in U$, $t > 0$ and $-t \in I(z)$. See e.g. [12, Theorem 1.1], and [25, Theorem 4.1.1]. The results of Theorem 3.1 mean that C^1 regular perturbation of a competitive irreducible vector field is at most eventually competitive rather than competitive.

Next, we give an example of P-competitive systems with perturbation. Assume that there exists a symmetric matrix P of order N having 1 positive eigenvalue and $N-1$ negative eigenvalues, where every eigenvalue is counted according to its multiplicity. System (3.1) is called *strictly P-competitive* if there is a function λ from $\mathbb{R}^N$ to $\mathbb{R}$ such that the matrix

$$PDF(x) + DF(x)^* P + \lambda(x) P$$

is negative definite for each $x \in U$(see [22]). Here $DF(x)^*$ stands for the transpose of $DF(x)$.

With the symmetric matrix P, we can obtain a cone C with nonempty interior:

$$C = \{x \in \mathbb{R}^N : (Px, x) \geq 0, \quad (x, v_+) \geq 0\},$$

where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^N$ and v_+ is an eigenvector with respect to the positive eigenvalue λ_+ of P. The following Property can be found in [22, Theorem 2].

Proposition 3.3. *If system* (3.1) *is strictly P-competitive, then the flow ϕ_t of* (3.1) *satisfies $D\phi_{-t}(z)[C\backslash\{0\}] \subset \operatorname{Int} C$ for $z \in U$, $t > 0$ and $-t \in I(z)$.*

Now we consider the C^1-perturbed system of (3.1),

$$\frac{dz}{dt} = G(z), \tag{3.3}$$

where $G : U \to \mathbb{R}^N$ is a C^1 vector field. And G satisfies that there exists a $\delta > 0$ such that $\|F(z) - G(z)\| + \|DF(z) - DG(z)\| < \delta$ for all $z \in U$. Let ψ_t denote the flow of (3.3). Assume that $W \subset U$ is a convex compact subset and W is positively invariant under ψ_t.

Theorem 3.4. *Assume that* (3.1) *is strictly P-competitive. For system* (3.3)*, if $p \in W$ has complete orbit in W, then the flow on the omega limit set $\omega(p)$ is topologically equivalent to a flow on a compact invariant set of a Lipschitz system of differential equations in $\mathbb{R}^{N-1}$.*

Proof. Since W is positively invariant and G is a perturbation of F in the space of C^1 vector fields, one has that the flow ψ_t is eventually competitive in W by Theorem 3.1. If $p \in W$ has complete orbit in W, one can use the *Non-oscillation Principle* in [20] to obtain the non-ordering of the ω-limit set $\omega(p)$, which follows from [20, Theorem 3.1]. Then, one can repeat the argument in the proof of [20, Theorem 3.3] to obtain that the flow on $\omega(p)$ is topologically equivalent to a flow on a compact invariant set of a Lipschitz system of differential equations in $\mathbb{R}^{n-1}$. □

4. Eventually competitive systems rom singular perturbations

In this section, we study singular perturbed differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= f_0(x, y, \epsilon), \\ \epsilon \frac{dy}{dt} &= g_0(x, y, \epsilon). \end{aligned} \tag{4.1}$$

This system can be reformulated with a change of time scale as

$$\begin{aligned} \frac{dx}{d\tau} &= \epsilon f_0(x, y, \epsilon), \\ \frac{dy}{d\tau} &= g_0(x, y, \epsilon), \end{aligned} \tag{4.2}$$

where $\tau = t/\epsilon$. The time scale given by τ is said to be fast whereas that for t is slow. When ϵ is small enough, we call (4.1) the slow system and (4.2) the fast system. And the two systems are equivalent as long as $\epsilon \neq 0$.

Motivated by Wang and Sontag [32], we list a series of basic assumptions, and the following definitions may make it easier to state. Let C_b^r denote a class of functions such that a function f is in C_b^r if it is in C^r and its derivatives up to order r as well as f are bounded. Throughout this paper, let $Df(x_0)$ denote the derivatives of f evaluated at x_0 with respect to variables x other than t or τ. Moreover, $D_x f(x_0, y_0)$ and $D_y f(x_0, y_0)$ denote the partial derivatives of f with respect to x and y evaluated at (x_0, y_0), respectively. Then, we have the following assumptions, where the integer $r > 1$ and the positive number ϵ_0 are fixed from now on.

- (A1) Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open and bounded sets. The functions $f_0 : U \times V \times [0, \epsilon_0] \to \mathbb{R}^n$ and $g_0 : U \times V \times [0, \epsilon_0] \to \mathbb{R}^m$ are both of class C_b^r.
- (A2) There is a function $h_0 : U \to V$ in C_b^r such that $g_0(x, h_0(x), 0) = 0$ for all x in U.
- (A3) All eigenvalues of the matrix $D_y g_0(x, h_0(x), 0)$ have negative real parts for every $x \in U$.
- (A4) There exists a family of convex compact sets $D_\epsilon \subset U \times V$, which depend continuously on $\epsilon \in [0, \epsilon_0]$, such that (4.1) is positively invariant on D_ϵ for $\epsilon \in (0, \epsilon_0]$.
- (A5) For each $x \in U$, the system

$$\frac{dz}{d\tau} = g_1(x, z, 0) \triangleq g_0(x, z + h_0(x), 0) \tag{4.3}$$

  is given on $\{z \in \mathbb{R}^m : z + h_0(x_0) \in V\}$. And the steady state $z = 0$ of (4.3) is globally asymptotically stable on $\{z : z + h_0(x_0) \in V\}$.
- (A6) ψ_t^0 is the flow of the system

$$\frac{dx}{dt} = f_0(x, h_0(x), 0), \tag{4.4}$$

  on U; and K_0 is the projection of $D_0 \cap \{(x, y) : y = h_0(x), x \in U\}$ onto the x-axis. For any $x \in K_0$, $D\psi_{-t}^0(x)[C \backslash \{0\}] \subset \operatorname{Int} C$ for $t > 0$ and $-t \in I(x)$.

Then we have the following theorem.

Theorem 4.1. *Assume that* (A1)–(A6) *hold. Then there exists a positive constant $\epsilon^* < \epsilon_0$ such that for each $\epsilon \in (0, \epsilon^*)$, the ω-limit set of* (4.1) *is equivalent to a ω-limit set of an eventually competitive system.*

Before proving Theorem 4.1, we state a result about the singular perturbation theory which is a restatement of the [23, Theorems 2.1 and 3.1] for the vector fields on $\mathbb{R}^n \times \mathbb{R}^m \times [0, \epsilon_0]$.

Lemma 4.2. *The system*

$$\begin{aligned} \frac{dx}{d\tau} &= \epsilon f(x,y,\epsilon), \\ \frac{dy}{d\tau} &= g(x,y,\epsilon), \end{aligned} \tag{4.5}$$

satisfies $f:\mathbb{R}^n\times\mathbb{R}^m\times[0,\epsilon_0]\to\mathbb{R}^n$ *is* C_b^r, $g:\mathbb{R}^n\times\mathbb{R}^m\times[0,\epsilon_0]\to\mathbb{R}^m$ *is* C_b^r. *For* $x\in\mathbb{R}^n$, *there is a* C_b^r *function* $\overline{h}:\mathbb{R}^n\to\mathbb{R}^m$ *such that* $g(x,\overline{h}(x),0)=0$. *And all eigenvalues of the matrix* $D_y g(x,\overline{h}(x),0)$ *have negative real parts less than* $-\mu$ *for every* $x\in\mathbb{R}^n$.

Then, there exists a $\epsilon_1<\epsilon_0$ *such that for every* $\epsilon\in(0,\epsilon_1]$:

(1) *There exists a* C_b^{r-1} *function* $h:\mathbb{R}^n\times[0,\epsilon_1]\to\mathbb{R}^m$, *such that the set* $\mathbb{C}_\epsilon$ *defined by*

$$\mathbb{C}_\epsilon=\{(x,h(x,\epsilon)):x\in\mathbb{R}^n\},\ \epsilon\in(0,\epsilon_1]$$

is invariant under the flow generated by (4.5), *and*

$$\sup_{x\in\mathbb{R}^n}\{|h(x,\epsilon)-\overline{h}(x)|:x\in\mathbb{R}^n\}=O(\epsilon),\quad \epsilon\to 0.$$

In particular, we have $h(x,0)=\overline{h}(x)$ *for all* $x\in\mathbb{R}^n$.

(2) *There is a* C^{r-1}*-immersed submanifold* $W^s(\mathbb{C}_\epsilon)$ *in* $\mathbb{R}^n\times\mathbb{R}^m$ *of dimension* $n+m$. *It is characterized by*

$$W^s(\mathbb{C}_\epsilon)=\{(x_0,y_0):\sup_{\tau\geq 0}:y(\tau;x_0,y_0)-h_\epsilon(x(\tau;x_0,y_0))|e^{\frac{\mu\tau}{4}}<\infty\},$$

where $(x(\tau;x_0,y_0),y(\tau;x_0,y_0))$ *is the solution of* (4.5) *passing through the point* (x_0,y_0), *and* $h_\epsilon(x)=h(x,\epsilon)$ *is the function defining* $\mathbb{C}_\epsilon$.

(3) *The manifold* $W^s(\mathbb{C}_\epsilon)$ *is a disjoint union of* C^{r-1}*-immersed manifold* $W^s((\xi,h_\epsilon(\xi)))$ *of dimension* m:

$$W^s(\mathbb{C}_\epsilon)=\bigcup_{\xi\in\mathbb{R}^n}W^s((\xi,h_\epsilon(\xi))).$$

Moreover, this manifold is characterized as

$$W^s((\xi,h_\epsilon(\xi)))=\{(x_0,y_0):\sup_{\tau\geq 0}|\widetilde{x}(\tau)|e^{\frac{\mu\tau}{4}}<\infty,\ \sup_{\tau\geq 0}|\widetilde{y}(\tau)|e^{\frac{\mu\tau}{4}}<\infty\},$$

where $\widetilde{x}(\tau)=x(\tau;x_0,y_0)-H_\epsilon(\xi)(\tau)$, $\widetilde{y}(\tau)=y(\tau;x_0,y_0)-h_\epsilon(H_\epsilon(\xi)(\tau))$, *and* $H_\epsilon(\xi)(\tau)$ *stands for a unique solution of* $\frac{dx}{d\tau}=\epsilon f(x,h_\epsilon(x),\epsilon)$, $x(0)=\xi\in\mathbb{R}^n$.

(4) *There is a constant* $\delta_0>0$ *such that if a solution* $(x(\tau),y(\tau))$ *of* (4.5) *satisfies*

$$\sup_{\tau\geq 0}|y(\tau)-h_\epsilon(x(\tau))|<\delta_0,$$

then $(x(0),y(0))\in W^s(\mathbb{C}_\epsilon)$.

(5) *The fibers are positively invariant in the sense that*

$$\begin{aligned} &W^s((H_\epsilon(\xi)(\tau),h_\epsilon(H_\epsilon(\xi)(\tau)))) \\ &=\{(x(\tau;,x_0,y_0),y(\tau;x_0,y_0)):(x_0,y_0)\in W^s((\xi,h_\epsilon(\xi)))\}, \end{aligned}$$

for each $\tau\geq 0$.

Remark 4.3. (1) The δ_0 in property (4) of Lemma 4.2 can be chosen uniformly for $\epsilon\in(0,\epsilon_0]$.

(2) Property (3) of Lemma 4.2 is often referred to as the asymptotic phase property in the way

$$|x(\tau; x_0, y_0) - H_\epsilon(\xi)(\tau)| \to 0,$$
$$|y(\tau; x_0, y_0) - h_\epsilon(H_\epsilon(\xi)(\tau)| \to 0,$$

as $\tau \to \infty$.

To use Sakamoto's results in Lemma 4.2, we firstly extend the vector fields from $U \times V$ to $\mathbb{R}^n \times \mathbb{R}^m$ for $\epsilon \in [0, \epsilon_0]$. The technique is standard by [18], which can also be found in [32], such that the extended system:

$$\begin{aligned} \frac{dx}{d\tau} &= \epsilon f(x, y, \epsilon), \\ \frac{dy}{d\tau} &= g(x, y, \epsilon), \end{aligned} \tag{4.6}$$

satisfies the assumptions (A1)–(A6) and the assumptions for the geometric singular perturbation in Lemma 4.2. Moreover, $\overline{h}(x)$ coincides with $h_0(x)$ on K, f and g coincide with f_0, g_0 on Ω_{d_1}, respectively. Where K is a compact set with $K_0 \subset K \subset U$, $\Omega_{d_1} \triangleq \{(x, y) : x \in K, y \in V, |y - h_0(x)| \le d_1\}$ and $d_1 > 0$ is fixed such that δ_0 in Lemma 4.2 is less than d_1.

Proof of Theorem 4.1. First, we focus on the solutions on the invariant manifold $\mathbb{C}_\epsilon$ satisfying

$$\begin{aligned} \frac{dx}{dt} &= f(x, h_\epsilon(x), \epsilon), \\ y(t) &= h_\epsilon(x(t)). \end{aligned} \tag{4.7}$$

For brevity, we just mention the x directions on the invariant manifold $\mathbb{C}_\epsilon$ since $y = h_\epsilon(x)$. It is clear that the limiting of (4.7) is (4.4) when ϵ approaches zero. And a flow ψ_t^0 of the limiting system satisfies $D\psi_{-t}^0(x)[C\backslash\{0\}] \subset \operatorname{Int} C$ for $x \in K_0$, $t > 0$ and $-t \in I(x)$. By the continuity of D_ϵ and $h_\epsilon(x)$ at $\epsilon = 0$, we can pick an $\epsilon_2 < \epsilon_1$ small enough such that $D\psi_{-t}^0(x)[C\backslash\{0\}] \subset \operatorname{Int} C$ for $x \in K_\epsilon$ and $\epsilon \in (0, \epsilon_2)$, where K_ϵ is the projection of $\mathbb{C}_\epsilon \cap D_\epsilon$ to the x-axis. We also have that D_ϵ is positively invariant under (4.7) and $\mathbb{C}_\epsilon$ is an invariant manifold, thus K_ϵ is positively invariant under the flow ψ_t^ϵ of (4.7). Applying Theorem 3.1, we obtain that there exist an $\epsilon_3 \le \epsilon_2$ and a $t_* > 0$ such that for each $\epsilon \in (0, \epsilon_3)$, ψ_t^ϵ is eventually strongly competitive.

Next, we consider the flows on $W^s(\mathbb{C}_\epsilon)$. By the positive invariance and the asymptotic phase property mentioned in property (5) of Lemma 4.2 and Remark 4.3(2), the dynamics of the flows on $W^s(\mathbb{C}_\epsilon)$ totally depends on those for $\mathbb{C}_\epsilon$, or rather, $\omega(q_0)$ is equivalent to $\omega(p_0)$, whenever $p_0 = (\xi_0, h_\epsilon(\xi_0)) \in \mathbb{C}_\epsilon$ and $q_0 \in W^s(p_0)$. In fact, let q_0 be a point on the fiber $W^s(p_0)$, where $(\xi_0, h_\epsilon(\xi_0)) = p_0 \in \mathbb{C}_\epsilon$, and the solution of (4.6) starting from $p_0 \in \mathbb{C}_\epsilon$ tends to $p_1 = (\xi_1, h_\epsilon(\xi_1)) \in \mathbb{C}_\epsilon$ at time τ_1, then the solution of (4.6) starting from $q_0 \in W^s(p_0)$ will tend to $q_1 \in W^s(p_1)$ at time τ_1, which means that two solutions are always on the same fiber. Moreover, $|x(\tau; q_0) - H_\epsilon(\xi_0)(\tau)| \to 0$, $|y(\tau; q_0) - h_\epsilon(H_\epsilon(\xi_0)(\tau))| \to 0$, as $\tau \to \infty$, that is, the solution $(x(\tau; q_0), y(\tau; q_0))$ converges to the solution $(H_\epsilon(\xi_0)(\tau), h_\epsilon(H_\epsilon(\xi_0)(\tau)))$, as $\tau \to \infty$.

At last, we will show that all trajectories in D_ϵ eventually stay in $W^s(\mathbb{C}_\epsilon)$. We firstly claim that there exist an $\epsilon_4 > 0$ and $\delta_0 > d > 0$ such that $(x_0, y_0) \in W^s(\mathbb{C}_\epsilon)$ if $(x_0, y_0) \in D_\epsilon$ satisfies $|y_0 - h_\epsilon(x_0)| < d$ for each $\epsilon \in (0, \epsilon_4)$. This claim follows from [32, Lemma 7]. Moreover, for any $(x_0, y_0) \in D_\epsilon$, there exist some uniformly positive τ_0 and an $\epsilon_5 < \epsilon_4$ such that $|y(\tau_0) - h_\epsilon(x(\tau_0))| < d$ for all $\epsilon \in (0, \epsilon_5)$, then $(x(\tau_0), y(\tau_0)) \in W^s(\mathbb{C}_\epsilon)$, where $(x(\tau), y(\tau))$ is the solution to (4.2) with $(x(0), y(0)) = (x_0, y_0)$. The corresponding proof is similar to [32, Lemma 8] with the positive invariance of D_ϵ. Hence, the limit set $\omega((x_0, y_0))$ is equivalent to an ω-limit set $\omega(\xi)$ on $\mathbb{C}_\epsilon$, where $(x_0, y_0) \in D_\epsilon$ and $\xi \in K_\epsilon$.

We complete the proof of Theorem 4.1 by taking $\epsilon^* = \min\{\epsilon_3, \epsilon_5\}$. □

Remark 4.4. For any point $x \in K_\epsilon$, if the orbits of (4.7) is complete in K_ϵ, then the non-ordering of the ω-limit set $\omega(x)$ directly follows from the [20, Theorem 3.1]. Similarly as Theorem 3.4, the flow on $\omega(x)$ is topologically equivalent to a flow on a compact invariant set of a Lipschitz ODE system in $\mathbb{R}^{n-1}$.

Next we give an example a singularly perturbed system. We fix $k \neq l \in \{1, \dots, n\}$ and consider the system

$$\begin{aligned} \frac{dx}{dt} &= f(x) + \Gamma(y), \quad x \in U \subset \mathbb{R}^n, \\ \epsilon \frac{dy}{dt} &= -py + g(x_l),\ p > 0, \quad y \in V \subset \mathbb{R}^1,\ 0 < \epsilon \ll 1, \end{aligned} \tag{4.8}$$

with a positive feedback $x_l \longrightarrow y$ (hence $g'(x_l) > 0$), and a negative feedback $y \longrightarrow x_k$ (hence $\Gamma(y) = (0, \dots, 0, \overset{k}{\gamma(y)}, 0, \dots, 0)$ with $\gamma'(y) < 0$). Moreover, the system

$$\frac{dx}{dt} = f(x), \quad x \in U \subset \mathbb{R}^n \tag{4.9}$$

satisfies $\frac{\partial f_i}{\partial x_j} < 0$, $i \neq j$, $i, j \in \{1, \dots, n\}$.

We also assume that the functions f, Γ, g are of class C_b^r for sufficiently large bounded sets, $|g(x_l)| \le M$ for any $x_l \in \mathbb{R}$, and

$$f_i(x_1, \dots, -x_i, \dots, x_n) = -f_i(x_1, \dots, x_i, \dots, x_n),$$

for $i \in \{1, \dots, n\}$. Then we take

$$D_\epsilon = \{(x, y) : |x_i| \le a_i, |y| \le b, i = 1, 2, \dots, n\},$$

where $b > \frac{M}{p}$, $N = \max\limits_{|y| \le b} |\gamma(y)|$ and $a_i > 0$ such that $-f_i(\dots, \overset{i}{a_i}, \dots) > N$, $i = 1, 2, \dots, n$. So, the vector fields point transversely inside on the boundary of D_ϵ, which enables us to assume that D_ϵ is positively invariant under (4.8).

Let U and V be bounded open sets such that $D_\epsilon \subset U \times V$. Then (A1)–(A5) follow naturally. Then the limiting system can be obtained as

$$\frac{dx}{dt} = f(x) + \Gamma(\frac{1}{p} g(x_l)) = h(x). \tag{4.10}$$

Since system (4.9) is a competitive system with $\frac{\partial f_i}{\partial x_j} < 0$, $i \neq j$, and $g'(x_l) > 0$, $\gamma'(y) < 0$, we obtain that $\frac{\partial h_i}{\partial x_j} < 0$, $i \neq j$. Hence, (A6) holds, by Kamke's theorem, which means theorem 4.1 holds for system (4.8), i.e., the ω-limit set of (4.8) is equivalent to a ω-limit set of an eventually competitive system.

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References

[1] C. Altafini; *Representing externally positive systems through minimal eventually positive realizations*, Proc IEEE Conf Decision Control, 2015, 6385–6390.

[2] C. Altafini, G. Lini; *Predictable dynamics of opinion forming for networks with antagonistic interactions*, IEEE Trans. Autom. Control, **60** (2015), 342–357.

[3] D. Angeli and E. Sontag; *Monotone control systems*, IEEE Trans. Autom. Control, **48** (2003), 1684–1698.

[4] A. Berman, M. Neumann, R. Stern; *Nonnegative matrices in dynamic systems*, John Wiley & Sons, New York (1989).

[5] C. Chicone; *Ordinary differential equations with applications*, Springer, New York (2006).

[6] D. Daners; *Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator*, Positivity, **18** (2014), 235–256.

[7] D. Daners, J. Glück, J. B. Kennedy; *Eventually positive semigroups of linear operators*, J. Math. Anal. Appl., **433** (2016), 1561–1593.

[8] D. Daners, J. Glück, J. B. Kennedy; *Eventually and asymptotically positive semigroups on Banach lattices*, J. Diff. Eqns., **261** (2016), 2607–2649.

[9] N. Fenichel; *Geometric singular perturbation theory for ordinary differential equations*, J. Diff. Equ., **31** (1979), 53–98.

[10] A. Ferrero, F. Gazzola, H.-C. Grunau; *Decay and eventual local positivity for biharmonic parabolic equations*, Discrete Contin. Dyn. Syst., **21** (2008), 1129–1157.

[11] F. Gazzola, H.-C. Grunau; *Eventual local positivity for a biharmonic heat equation in* $\mathbb{R}^n$, Discrete Contin. Dyn. Syst.-S, **1** (2008), 83–87.

[12] M. W. Hirsch; *Systems of differential equations that are competitive or cooperative. II. Convergence almost everywhere*, SIAM J. Math. Anal., **16**(1985), 423–439.

[13] M. W. Hirsch, H. L. Smith; *Monotone Systems, A Mini-review, Proceedings of the First Multidisciplinary Symposium on Positive Systems (POSTA 2003)*, Luca Benvenuti, Alberto De Santis and Lorenzo Farina (Eds.) Lecture Notes on Control and Information Sciences vol. 294, Springer-Verlag, Heidelberg, 2003.

[14] M. W. Hirsch, H. L. Smith; *Monotone dynamical systems*, Handbook of Differential Equations: Ordinary Differential Equations, vol. 2, Elsevier, Amsterdam 2005.

[15] J. Hofbauer, K. Sigmund; *Evolutionary Games and Population Dynamics*, Cambridge University Press, 1998.

[16] C. K. R. T. Jones; *Geometric singular perturbation theory*, Dynamical Systems (Montecatini Terme,1994). Lect. Notes in Math., 1609, Springer, Berlin, 1995.

[17] W. W. Leontief; *Input-output Economics*, Oxford University Press on Demand, 1986.

[18] K. Nipp; *Smooth attractive invariant manifolds of singularly perturbed ODE's*, Research Report, (1992), 92–13.

[19] D. Noutsos, M. J. Tsatsomeros; *Reachability and holdability of nonnegative states*, SIAM J. Matrix Anal. Appl., **30** (2008), 700–712.

[20] L. Niu, Y. Wang; *Non-oscillation principle for eventually competitive and cooperative systems*, Discrete Contin. Dyn. Syst.-B, in press, Preprint Available at https: //arxiv.org/abs/1809.10068.

[21] D. D. Olesky, M. J. Tsatsomeros, P. van den Driessche; *M_v-matrices: A generalization of M-matrices based on eventually nonnegative matrices*, Electron. J. Linear Algebra, **18** (2009), 339–351.

[22] R. Ortega, L. Sánchez; *Abstract competitive systems and orbital stability in* $\mathbb{R}^3$, Proc. Amer. Math. Soc., **128** (2000), 2911–2919.

[23] K. Sakamoto; *Invariant manifolds in singular perturbation problems for ordinary differential equations*, Proc. R. Soc. Edinb. A, **116** (1990), 45–78.

[24] K. Sigmund; *The Population Dynamics of Conflict and Cooperation*, Proceedings of ICM 98, Documenta Mathematica, **1** (1998), 487–506.

[25] H. L. Smith; *Monotone Dynamical Systems, an introduction to the theory of competitive and cooperative systems*, Math. Surveys and Monographs, 41, Amer. Math. Soc., Providence, Rhode Island 1995.

[26] H. L. Smith; *Periodic solutions of periodic competitive and cooperative systems*, SIAM J. Math. Anal., **17** (1986), 1289–1318.

[27] H. L. Smith; *Monotone dynamical systems: Reflections on new advances and applications*, Discrete Contin. Dyn. Syst., **37** (2017), 485–504.
[28] E. D. Sontag; *Monotone and near-monotone biochemical networks*, Syst. Synthetic Biol., **1** (2007), 59–87.
[29] A. Sootla, A. Mauroy; *Operator-Theoretic Characterization of Eventually Monotone Systems*, arXiv:1510.01149.
[30] H. L. Smith, H. Thieme; *Dynamical Systems and Population Persistence*, American Mathematical Society, **118**, (2011), Graduate Studies in Mathematics.
[31] E. R. Stern, H. Wolkowicz; *Exponential nonnegativity on the ice cream cone*, SIAM J. Matrix Anal. Appl., **12** (1991), 160–165.
[32] L. Wang, E. D. Sontag; *Singularly perturbed monotone systems and an application to double phosphorylation cycles*, J. Nonlinear Sci., **18**(2008), 527-550.

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