Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 124, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND STABILITY OF STEADY STATES FOR HIERARCHICAL AGE-STRUCTURED POPULATION MODELS

ZE-RONG HE, DONGDONG NI, SHUPING WANG

Abstract. This article concerns the stability of equilibria of a hierarchical age-structured population system. We establish the existence of positive steady states via a fixed point result. Also we derive some criteria from the model parameters for asymptotical stability or instability, by means of spectrum and semigroups of linear operators. In addition, we present some numerical experiments.

1. Introduction

Because of ecological reality and mathematical challenges, researchers have proposed and analyzed a number of hierarchical models (see [1]-[8], [10]-[20] and references therein). Problems investigated include animal behavior [8], allocation of resources [18], ecological stability [12], system dynamics [1, 2, 3, 4, 5, 6, 7, 10, 11, 14, 15, 16, 17, 21] and state approximation [13, 20].

In this article, we continue the research originated in [13]. Our focus is the steady states of the model system; specifically, we consider the existence of nontrivial equilibria and their stability, since they are both special solutions and of significance in describing the asymptotical behaviors of the population system. According to general theory of linear stability for nonlinear infinite-dimensional system, the zero solution is asymptotically stable if the growth bound of the semigroup generated by the linear operator in the linearized system is negative, and unstable if the point spectrum of the generator contains some eigenvalues with positive real part. Unfortunately, estimation of the growth bound is often quite difficult. On the other hand, providing conditions that guaranteeing the existence of characteristic values with positive real part is by no means trivial if one cannot derive the characteristic equation. That is the case in our situation. We have to proceed along another way. Among the existing works in this respect are [11] and [10], in which Farkas, Hinow and Hagen analyzed two classes of size-structured models, and established (or assumed) the existence and conditions for linear stability of positive equilibria. Since the models used by them have different features, their results are not applicable to our system. Besides the theoretical results, we present some numerical experiments to show the behaviors of the population system when theoretical methods fail.

2010 *Mathematics Subject Classification.* 92D05, 47D06, 35B35.
Key words and phrases. Hierarchy of age; population system; steady states; stability; semigroup of operators.

Submitted October 27, 2018. Published November 21, 2019.

The present article is organized as follows. In the next section, we describe the model, the basic assumptions and the well-posedness result. Section 3 is devoted to the existence of positive steady states treated by a special fixed point theorem, and then we investigate the linear stability of equilibria in section 4 by means of spectrum and semigroups of operators. The final two sections consist of some numerical examples and remarks.

2. Description of the model

Because the main aim of this paper is steady states, we consider the following autonomous system in stead of the time-varying one in [13]:

$$\begin{gathered}\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}=-\mu(a,E(p)(a,t))p(a,t),\quad (a,t)\in D,\\ p(0,t)=\int_0^A\beta(a,E(p)(a,t))p(a,t)da,\quad 0<t<\infty,\\ p(a,0)=p_0(a),\quad 0\le a\le A,\end{gathered}\tag{2.1}$$

where $D=(0,A)\times(0,\infty)$, and $A>0$ is the maximum age of individuals. The "environment" $E(p)$ is given by

$$E(p)(a,t)=\alpha\int_0^a p(r,t)dr+\int_a^A p(r,t)dr,\ 0\le\alpha<1.\tag{2.2}$$

Here the constant α shows the rank weight of individuals with age smaller than a. Functions μ and β are mortality and fertility, respectively. The form of μ and β indicates that, apart from age, the vital rates of an individual of age a depend more on the number of individuals with age equal to or larger than a. Finally, $p_0(a)$ gives the initial age distribution of the individuals.

Throughout this article, we assume that the following conditions:

- (A1) $\mu(a,x)>0$, $\beta(a,x)\ge 0$ for all $(a,x)\in[0,A]\times R_+$. For any $x\in R_+$, $\beta(\cdot,x)\in L^\infty[0,A]$, and $\mu(\cdot,x)\in C[0,A)\cap L^1_{loc}[0,A)$; $\int_0^A\mu(a,x)da=+\infty$.
- (A2) β and μ are locally Lipschitz functions with respect to the second variable, that is, for each $M>0$, there exists $L(M)>0$, such that

$$\begin{gathered}|\beta(a,x_1)-\beta(a,x_2)|\le L(M)|x_1-x_2|,\\ |\mu(a,x_1)-\mu(a,x_2)|\le L(M)|x_1-x_2|\end{gathered}$$

  for all x_1, x_2 with $|x_i|\le M$, $i=1,2$.
- (A3) For given $a\in[0,A]$, $\beta(a,\cdot)$ is non-increasing and $\mu(a,\cdot)$ is nondecreasing.
- (A4) There is a positive constant $\overline{p_0}$, such that $0\le p_0(a)\le\overline{p_0}$ for all $a\in[0,A]$.

In [13], we proved that, for any given $T>0$, the system (2.1)-(2.2) (actually a more general system) admits a unique bounded solution on $[0,T]$. Therefore, we have the following result.

Theorem 2.1. *System* (2.1)-(2.2) *has a unique solution* $p\in C([0,+\infty),L^\infty[0,A])$ *with* $p(a,t)\ge 0$ *and* $p(A,t)=0$ *for all* $t>0$.

3. Existence of positive equilibria

In this section, we show that the system (2.1)-(2.2) has at least one positive equilibrium, by means of the following fixed point result [19, Theorem A].

Lemma 3.1 ([19]). *Let X be a Banach space, $K \subset X$ a closed convex cone, $K_r = K \cap B_r(0)$, where $B_r(0)$ denotes the ball with radius r and center 0. The map $F : K_r \to K$ is continuous and $F(K_r)$ is relatively compact. Suppose that*

(i) *$Fx \neq \lambda x$ for all $\|x\| = r$, $\lambda > 1$.*
(ii) *There are $\rho \in (0, r)$ and $e \in K/\{0\}$ such that $x - Fx \neq \lambda e$ for all $\|x\| = \rho$, $\lambda > 0$.*

Then F has at least one fixed point $x^* \in \{x \in K : \rho \leq \|x\| \leq r\}$.

Let $p^*(a)$ be a positive equilibrium of (2.1)-(2.2). Then the following holds

$$\begin{gathered} \frac{dp^*}{da} = -\mu(a, E(p^*)(a))p^*(a), \\ p^*(0) = \int_0^A \beta(a, E(p^*)(a))p^*(a)da, \\ E(p^*)(a) = \alpha \int_0^a p^*(r)dr + \int_a^A p^*(r)dr. \end{gathered} \tag{3.1}$$

From the first equation in (3.1) it follows that

$$p^*(a) = p^*(0) \exp\Big\{ -\int_0^a \mu(r, E(p^*)(r))dr \Big\}, \quad p^*(0) > 0. \tag{3.2}$$

Substituting (3.2) into the second equation in (3.1), we obtain

$$p^*(0) = p^*(0) \int_0^A \beta(a, E(p^*)(a)) \exp\Big\{ -\int_0^a \mu(r, E(p^*)(r))dr \Big\} da. \tag{3.3}$$

Obviously, a necessary condition for the existence of positive equilibrium is

$$R(0) := \int_0^A \beta(a, 0)) \exp\Big\{ -\int_0^a \mu(r, 0)dr \Big\} da > 1.$$

Otherwise, the assumptions on β and μ give

$$\int_0^A \beta(a, E(p^*)(a)) \exp\Big\{ -\int_0^a \mu(r, E(p^*)(r))dr \Big\} da < R(0) \leq 1,$$

which contradicts the equation (3.3).

On the other hand, if there exists a function $p^* \in L^1[0, A]$, then $E(p^*) \in C[0, A]$. Consequently, the function given by (3.2) is differentiable and satisfies (3.1); that is, p^* is a steady state of the system (2.1)-(2.2).

Let $\mathcal{X} = \mathcal{R} \times L^1(0, A)$, $\mathcal{K} = [0, +\infty) \times L^1_+(0, A)$; For given $r > 0$, $B_r(0) = \{(h, q) \in \mathcal{X} : |h| + \|q\| \leq r\}$, where $\|q\| = \int_0^A |q(a)|da$; and $\mathcal{K}_r = \mathcal{K} \times B_r(0)$. Let $\phi : \mathcal{K}_r \to \mathcal{K}$, be defined by

$$[\phi(h, q)](a) = \begin{pmatrix} h \int_0^A \beta(a, E(q)(a)) \exp\{-\int_0^a \mu(r, E(q)(r))dr\} da \\ h \exp\{-\int_0^a \mu(r, E(q)(r))dr\} \end{pmatrix}$$

for $(h, q) \in \mathcal{K}_r$.

Lemma 3.2. *The mapping ϕ is continuous.*

Proof. Let $(h, q) \to (h_0, q_0)$, that is, $|h - h_0| \to 0, \|q - q_0\| \to 0$. Since

$$[E(q) - E(q_0)](a) = \alpha \int_0^a [q(r) - q_0(r)]dr + \int_a^A [q(r) - q_0(r)]dr, \tag{3.4}$$

it follows that

$$\begin{aligned}\|E(q)-E(q_0)\| &= \int_0^A \Big|\alpha\int_0^a [q(r)-q_0(r)]dr + \int_a^A [q(r)-q_0(r)]dr\Big| da \\ &\le \int_0^A \Big\{\alpha\int_0^a |q(r)-q_0(r)|dr + \int_a^A |q(r)-q_0(r)|dr\Big\} da \\ &\le \int_0^A\int_0^A |q(r)-q_0(r)|\,dr\,da \to 0.\end{aligned} \tag{3.5}$$

Thus, $E(q)(a)\to E(q_0)(a)$ a.e. $a\in(0,A)$. As a result, we have

$$\begin{aligned}&h\int_0^A \beta(a,E(q)(a))\exp\Big\{-\int_0^a \mu(r,E(q)(r))dr\Big\}da \\ &\to h_0\int_0^A \beta(a,E(q_0)(a))\exp\Big\{-\int_0^a \mu(r,E(q_0)(r))dr\Big\}da,\end{aligned}$$

for a.e. $a\in(0,A)$ when $|h-h_0|\to 0$, $\|q-q_0\|\to 0$. In addition,

$$\begin{aligned}&\|\exp\big\{-\int_0^\cdot \mu(r,E(q)(r))dr\big\} - \exp\big\{-\int_0^\cdot \mu(r,E(q_0)(r))dr\big\}\| \\ &= \int_0^A \Big|\exp\big\{-\int_0^a \mu(r,E(q)(r))dr\big\} - \exp\big\{-\int_0^a \mu(r,E(q_0)(r))dr\big\}\Big| da \\ &\le \int_0^A\int_0^a |\mu(r,E(q)(r))-\mu(r,E(q_0)(r))|\,dr\,da \\ &\le C\int_0^A |E(q)(a)-E(q_0)(a)|da \to 0,\end{aligned}$$

where C is a constant. The proof is complete. □

Next, we show that $\phi(\mathcal{K}_r)$ is relatively compact by means of Fréchet-Kolmogorov theorem [22, p. 275]. The verification is trivial so it is omitted.

Now we examine condition (i) in Lemma 3.1. For $\lambda>1$, $\|(h,q)\|=r$, we prove that $\phi(h,q)\ne\lambda(h,q)$. Otherwise, we see that

$$1<\lambda=\int_0^A \beta(a,E(q)(a))\exp\big\{-\int_0^a \mu(r,E(q)(r))dr\big\}da$$

is fixed, and $h\exp\big\{-\int_0^a \mu(r,E(q)(r))dr\big\}=\lambda q(a)$. Then

$$\|q\|\le\lambda\|q\| = h\big\|\exp\big\{-\int_0^a \mu(r,E(q)(r))dr\big\}\big\|\le hA.$$

Therefore, $h+\|q\|\le(1+A)h$. $\|(h,q)\|$ will be very small when h is small enough. It is impossible for (h,q) to meet $\|(h,q)\|=r$. The following result verifies the condition (ii) in Lemma 3.1.

Lemma 3.3. *There are $\rho\in(0,r)$ and $(\overline{h},\overline{q})\in\mathcal{K}-\{0\}$ such that*

$$(h,q)-\phi(h,q)\ne\lambda(\overline{h},\overline{q}) \quad \textit{for all } (h,q) \textit{ with } \|(h,q)\|=\rho \textit{ and } \lambda>0.$$

Proof. We prove the conclusion in the following cases.

Case 1. If there is $(\overline{h},\overline{q})\in\mathcal{K}-\{0\}$, such that for all $\rho\in(0,r)$,

$$(h,q)-\phi(h,q)=\lambda(\overline{h},\overline{q}) \quad \text{for some } \|(h,q)\|=\rho, \forall\ \lambda>0.$$

Then using the definition of ϕ, we have

$$
\begin{gathered}
h-h\int_0^A \beta(a,E(q)(a))\exp\big\{-\int_0^a \mu(r,E(q)(r))dr\big\}da=\lambda\bar{h},\\
q(a)-h\exp\big\{-\int_0^a \mu(r,E(q)(r))dr\big\}=\lambda\bar{q}(a).
\end{gathered}
\tag{3.6}
$$

Let $\lambda=1$, $\rho\to 0$, from the above equations, we obtain $(\bar{h},\bar{q})=0$, which is a contradiction.

Case 2. If there is $\rho\in(0,r)$, such that for all $(\bar{h},\bar{q})\in\mathcal{K}-\{0\}$,

$$
(h,q)-\phi(h,q)=\lambda(\bar{h},\bar{q})\quad\text{for some } \|(h,q)\|=\rho,\ \forall\lambda>0.
$$

Let $\lambda\to 0$, we have $(h,q)=\phi(h,q)$, which implies that (h,q) is a positive equilibrium.

Case 3. If there is $(\bar{h},\bar{q})\in\mathcal{K}-\{0\}$, such that for all $\rho\in(0,r)$,

$$
(h,q)-\phi(h,q)=\lambda_0(\bar{h},\bar{q})\quad\text{for some } \lambda_0>0,\ \forall\|(h,q)\|=\rho,
$$

then, let $\rho\to 0$ we have $\lambda_0(\bar{h},\bar{q})=0$, which is impossible.

Case 4. If there is $\rho\in(0,r)$, such that for all $(\bar{h},\bar{q})\in\mathcal{K}-\{0\}$,

$$
(h,q)-\phi(h,q)=\lambda_0(\bar{h},\bar{q})\quad\text{for some } \lambda_0>0,\ \forall\|(h,q)\|=\rho.
$$

Then choosing $(\bar{h},\bar{q})=\frac{2}{\lambda_0}(h,q)$, we derive $\phi(h,q)=-(h,q)$, which is absurd, because $\phi(h,q)\in\mathcal{K}$ implies that $\phi(h,q))$ is a nonnegative vector. $\square$

Summarizing the above results and using Lemma 3.1, we obtain the following result.

Theorem 3.4. *If $R(0)\le 1$, then system (2.1)-(2.2) has no positive steady state. If $R(0)>1$, then the system has at least one positive steady state.*

4. Stability of equilibria

In this section, we investigate the linear stability of steady states. Firstly, a general result is derived by means of linearization and theory of semigroup of operators. Then, some conditions for stability of the trivial equilibrium are given as a particular situation.

Let $p^*(a)$ be a nonnegative equilibrium and $p(a,t)=p^*(a)+u(a,t)$, where $u(a,t)$ is a perturbation. Noticing that $\frac{dp^*(a)}{da}=-\mu(a,E(p^*)(a))p^*(a)$, we obtain the linearization of the system (2.1)-(2.2) about $p^*(a)$ as follows:

$$
\begin{gathered}
\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}=-\mu(a,E(p^*)(a))u(a,t)-p^*(a)\mu_E(a,E(p^*)(a))E(u)(a,t),\\
u(0,t)=\int_0^A[\beta(a,E(p^*)(a))u(a,t)+p^*(a)\beta_E(a,E(p^*)(a))E(u)(a,t)]da,\\
u(a,0)=p_0(a)-p^*(a),
\end{gathered}
\tag{4.1}
$$

where μ_E and β_E denote the partial derivatives of μ and β with respect to the second variable.

Let $X=L^1([0,A])$, define three linear operators as follows.

$$
\mathcal{A}:\ D(\mathcal{A})\subset W^{1,1}[0,A]\subset X\to X,\quad \mathcal{A}u=-\frac{\partial u}{\partial a},
$$

where

$$\begin{gathered}D(\mathcal{A})=\big\{u\in W^{1,1}[0,A]:u(0)=\int_0^A[\beta(\cdot,E(p^*)(\cdot))u\\ +p^*\beta_E(\cdot,E(p^*)(\cdot))E(u)(\cdot)](a)da\big\};\\ \mathcal{B}:X\to X,\quad \mathcal{B}u=-\mu(\cdot,E(p^*)(\cdot))u;\\ \mathcal{C}:\ X\to X,\quad \mathcal{C}u=-p^*(\cdot)\mu_E(\cdot,E(p^*)(\cdot))E(u)(\cdot).\end{gathered}$$

Then system (4.1) can be rewritten as the abstract Cauchy problem in X:

$$\begin{gathered}\frac{du}{dt}=(\mathcal{A}+\mathcal{B}+\mathcal{C})u,\\ u(0)=u_0:=p_0-p^*.\end{gathered}\tag{4.2}$$

Theorem 4.1 (Asymptotic stability)**.** *If, for almost all $a\in(0,A)$,*

$$\begin{aligned}\mu(a,E(p^*)(a))-\beta(a,E(p^*)(a))>\alpha\int_a^A p^*(r)[\mu_E-\beta_E](r,E(p^*)(r))dr\\ +\int_0^a p^*(r)[\mu_E-\beta_E](r,E(p^*)(r))dr,\end{aligned}\tag{4.3}$$

then the steady state $p^*(a)$ is asymptotically stable.

Proof. Let $\mathcal{T}(t)$ be the semigroup generated by $\mathcal{A}+\mathcal{B}+\mathcal{C}$, we will prove that $\lim_{t\to+\infty}\|\mathcal{T}(t)\|=0$.

Firstly we show that the operator $\mathcal{A}+\mathcal{B}+\mathcal{C}+kI$ is dissipative [9, p. 82, Definition 3.13], where k is a positive constant (will be determined later) and I the identity operator. Let

$$u-\lambda(\mathcal{A}+\mathcal{B}+\mathcal{C}+kI)u=h,\quad \lambda>0.\tag{4.4}$$

Then

$$\begin{aligned}\|u\|&=\int_0^A u(a)\operatorname{sgn}u(a))da\\ &=\int_0^A h(a)\operatorname{sgn}u(a))da-\lambda\int_0^A u'(a)\operatorname{sgn}u(a))da\\ &\quad+\lambda\int_0^A[k-\mu(a,E(p^*)(a))]u(a)\operatorname{sgn}u(a))da\\ &\quad-\lambda\int_0^A p^*(a)\mu_E(a,E(p^*)(a))E(u)(a)\operatorname{sgn}u(a))da.\end{aligned}\tag{4.5}$$

Firstly,

$$\begin{aligned}&-\lambda\int_0^A p^*(a)\mu_E(a,E(p^*)(a))E(u)(a)\operatorname{sgn}u(a))da\\ &=-\lambda\int_0^A p^*(a)\mu_E(a,E(p^*)(a))\operatorname{sgn}u(a))\Big[\alpha\int_0^a u(r)dr+\int_a^A u(r)dr\Big]da\\ &=-\lambda\alpha\int_0^A u(r)\int_r^A p^*(a)\mu_E(a,E(p^*)(a))\operatorname{sgn}u(a))\,da\,dr\\ &\quad-\lambda\int_0^A u(r)\int_0^r p^*(a)\mu_E(a,E(p^*)(a))\operatorname{sgn}u(a))\,da\,dr\end{aligned}$$

$$
\begin{aligned}
&= -\lambda\alpha \int_0^A u(a) \int_a^A p^*(r)\mu_E(r, E(p^*)(r)) \operatorname{sgn} u(r))\, dr\, da \\
&\quad - \lambda \int_0^A u(a) \int_0^a p^*(r)\mu_E(r, E(p^*)(r)) \operatorname{sgn} u(r))\, dr\, da \\
&\le \lambda \int_0^A |u(a)|[\alpha \int_a^A p^*(r)\mu_E(r, E(p^*)(r))dr \\
&\quad + \int_0^a p^*(r)\mu_E(r, E(p^*)(r))dr]da.
\end{aligned} \tag{4.6}
$$

Let $U = \{a \in (0, A) : u(a) \neq 0\}$. Then $U = \sum_{i=1}^{\infty}(a_i, b_i)$, $u(a) \neq 0$, $a \in (a_i, b_i)$; $(a_i, b_i) \cap (a_j, b_j) = \emptyset$, $i \neq j$. If $u(a) > 0$, $a \in (a_i, b_i)$, then

$$
\begin{aligned}
&\int_{a_i}^{b_i} h(a) \operatorname{sgn} u(a))da - \lambda \int_{a_i}^{b_i} u'(a) \operatorname{sgn} u(a))da \\
&+ \lambda \int_{a_i}^{b_i} [k - \mu(a, E(p^*)(a))]u(a) \operatorname{sgn} u(a))da \\
&\le \int_{a_i}^{b_i} |h(a)|da + \lambda u(a_i) + \lambda \int_{a_i}^{b_i} [k - \mu(a, E(p^*)(a))]|u(a)|da.
\end{aligned} \tag{4.7}
$$

If $u(a) < 0$, $a \in (a_i, b_i)$, then we proceed similarly,

$$
\begin{aligned}
&\int_{a_i}^{b_i} h(a) \operatorname{sgn} u(a))da - \lambda \int_{a_i}^{b_i} u'(a) \operatorname{sgn} u(a))da \\
&+ \lambda \int_{a_i}^{b_i} [k - \mu(a, E(p^*)(a))]u(a) \operatorname{sgn} u(a))da \\
&\le \int_{a_i}^{b_i} |h(a)|da - \lambda u(a_i) + \lambda \int_{a_i}^{b_i} [k - \mu(a, E(p^*)(a))]|u(a)|da.
\end{aligned} \tag{4.8}
$$

Substituting inequalities (4.6)–(4.8) into (4.5), we arrive at

$$
\begin{aligned}
\|u\| &= \sum_i \int_{a_i}^{b_i} u(a) \operatorname{sgn} u(a))da \\
&\le \|h\| + \lambda u(0) \ + \lambda \int_0^A |u(a)|[k - \mu(a, E(p^*)(a)) \\
&\quad + \alpha \int_a^A p^*(r)\mu_E(r, E(p^*)(r))dr + \int_0^a p^*(r)\mu_E(r, E(p^*)(r))dr]da.
\end{aligned} \tag{4.9}
$$

Noticing that

$$
\begin{aligned}
|u(0)| &\le \int_0^A \beta(a, E(p^*)(a))|u(a)|da \\
&\quad + |\int_0^A p^*(a)\beta_E(a, E(p^*)(a))E(u)(a)da| \\
&\le \int_0^A \beta(a, E(p^*)(a))|u(a)|da \\
&\quad + \int_0^A |u(a)|[-\alpha \int_a^A p^*(r)\beta_E(r, E(p^*)(r))dr
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^a p^*(r)\beta_E(r,E(p^*)(r))dr]da \\
&= \int_0^A |u(a)|[\beta(a,E(p^*)(a)) - \alpha \int_a^A p^*(r)\beta_E(r,E(p^*)(r))dr \\
&- \int_0^a p^*(r)\beta_E(r,E(p^*)(r))dr]da,
\end{aligned}
$$

and choosing

$$
\begin{aligned}
k \le\ & \mu(a,E(p^*)(a)) - \beta(a,E(p^*)(a)) \\
& - \alpha \int_a^A p^*(r)[\mu_E - \beta_E](r,E(p^*)(r))dr \\
& - \int_0^a p^*(r)[\mu_E - \beta_E](r,E(p^*)(r))dr,
\end{aligned}
$$

we obtain from (4.9) that $\|u\| \le \|h\|$, which implies the dissipativity desired.

Next, we prove that the range $R(\lambda I - \mathcal{A} - \mathcal{B} - \mathcal{C} - kI) = X$. Consider the equation

$$
(\lambda I - \mathcal{A})u = h, \quad \forall h \in X,\ \lambda > 0; \tag{4.10}
$$

that is, $u'(a) + \lambda u(a) = h(a)$, which has a unique (formal) solution

$$
u(a) = u(0)e^{-\lambda a} + \int_0^a h(r)e^{-\lambda(a-r)}dr.
$$

To determine the value $u(0)$, noting that

$$
\begin{aligned}
u(0) =& \int_0^A \beta(a,E(p^*)(a))u(a)da + \int_0^A p^*(a)\beta_E(a,E(p^*)(a))E(u)(a)da \\
=& \int_0^A \beta(a,E(p^*)(a)[u(0)e^{-\lambda a} + \int_0^a h(r)e^{-\lambda(a-r)}dr]da \\
&+ \int_0^A p^*(a)\beta_E(a,E(p^*)(a))[\alpha \int_0^a (u(0)e^{-\lambda r} + \int_0^r h(\theta)e^{-\lambda(r-\theta)}d\theta)dr \\
&+ \int_a^A (u(0)e^{-\lambda r} + \int_0^r h(\theta)e^{-\lambda(r-\theta)}d\theta)dr]da \\
=& u(0)\{\int_0^A e^{-\lambda a}\beta(a,E(p^*)(a)da \\
&+ \int_0^A p^*(a)\beta_E(a,E(p^*)(a))[\alpha \int_0^a e^{-\lambda r}dr + \int_a^A e^{-\lambda r}dr]da\} \\
&+ \int_0^A \beta(a,E(p^*)(a) \int_0^a h(r)e^{-\lambda(a-r)}\,dr\,da \\
&+ \int_0^A p^*(a)\beta_E(a,E(p^*)(a))[\alpha \int_0^a \int_0^r h(\theta)e^{-\lambda(r-\theta)}d\theta dr \\
&+ \int_a^A \int_0^r h(\theta)e^{-\lambda(r-\theta)}d\theta dr]da,
\end{aligned}
$$

we derive

$$
\begin{aligned}
u(0) = \Big(1 - \int_0^A \beta(a, E(p^*)(a)) e^{-\lambda a} da \\
- \int_0^A p^*(a)\beta_E(a, E(p^*)(a)) \frac{\alpha - e^{-\lambda A} + (1-\alpha)e^{-\lambda a}}{\lambda} da\Big)^{-1} \\
\times \int_0^A p^*(a)\beta_E(a, E(p^*)(a)) \Big[\alpha \int_0^a \int_0^r h(\theta) e^{-\lambda(r-\theta)} d\theta dr \\
+ \int_a^A \int_0^r h(\theta) e^{-\lambda(r-\theta)} d\theta dr\Big] da.
\end{aligned}
\tag{4.11}
$$

The above relation makes sense for $\lambda > 0$ large enough.

According to Lumer-Phillips theorem [9, p. 83], the operator $\mathcal{A} + \mathcal{B} + \mathcal{C} + kI$ generates a contraction semigroup. Therefore, if $\mathcal{T}(t)$ denotes the semigroup generated by $\mathcal{A} + \mathcal{B} + \mathcal{C}$, then

$$
\|\mathcal{T}(t)\| \le e^{-kt}, \ t \ge 0.
$$

Under the condition of the theorem, $k > 0$. The proof is complete. $\square$

Let $p^*(a) \equiv 0$.

Corollary 4.2. *If, for almost every $a \in (0, A)$, $\mu(a, 0) > \beta(a, 0)$, then the trivial steady state is asymptotically stable.*

In what follows, we derive some conditions for instability of the steady state.

Theorem 4.3 (Instability)**.** *If*

$$
\begin{aligned}
&\int_0^A \beta(a, E(p^*)(a)) \exp\Big\{ - \int_0^a [\lambda_0 + \mu(r, E(p^*)(r))] dr\Big\} da \\
&+ \int_0^A p^*(a)\beta_E(a, E(p^*)(a)) \Big(\alpha \int_0^a \exp\Big\{ - \int_0^r [\lambda_0 + \mu(\theta, E(p^*)(\theta))] d\theta\Big\} dr \\
&+ \int_a^A \exp\Big\{ - \int_0^r [\lambda_0 + \mu(\theta, E(p^*)(\theta))] d\theta]\Big\} dr\Big) da > 1,
\end{aligned}
\tag{4.12}
$$

where $\lambda_0 = \int_0^A p^*(a)\mu_E(a, E(p^*)(a)) da$, then the steady state $p^*(a)$ is unstable.

Proof. For the eigenvalues of the operator $\mathcal{A} + \mathcal{B}$, we consider

$$
(\lambda - \mathcal{A} - \mathcal{B})u = 0,
$$

whose vector solutions have the form

$$
u(a) = u(0) \exp\Big\{ - \int_0^a [\lambda + \mu(r, E(p^*)(r))] dr\Big\}, \quad u(0) \neq 0.
$$

Substituting the above relation into the condition satisfied by $u(0)$ in the domain $D(\mathcal{A})$, we obtain

$$
\begin{aligned}
1 = &\int_0^A \beta(a, E(p^*)(a)) \exp\Big\{ - \int_0^a [\lambda + \mu(r, E(p^*)(r))] dr\Big\} da \\
&+ \int_0^A p^*(a)\beta_E(a, E(p^*)(a)) \Big(\alpha \int_0^a \exp\Big\{ - \int_0^r [\lambda + \mu(\theta, E(p^*)(\theta))] d\theta\Big\} dr
\end{aligned}
$$

$$+\int_a^A \exp\Big\{-\int_0^r [\lambda+\mu(\theta,E(p^*)(\theta))]d\theta\Big\}dr\Big)da =: F(\lambda).$$

One can see that $F'(\lambda)<0$ for $\lambda>0$, and $F(\lambda)\to 0$ as $\lambda\to+\infty$. The condition in the theorem implies that $F(\lambda_0)>1$. Consequently, the operator $\mathcal{A}+\mathcal{B}$ has a unique positive eigenvalue λ^*, with $\lambda^*>\lambda_0>0$. Since $\mathcal{C}$ is bounded, λ^* is also an eigenvalue of the operator $\mathcal{A}+\mathcal{B}+\mathcal{C}$ [9, p. 159]. The proof is complete. $\square$

Let $p^*(a)\equiv 0$.

Corollary 4.4. *If $R(0) := \int_0^A \beta(a,0)\exp\big\{-\int_0^a \mu(r,0)dr\big\}da > 1$, then the trivial steady state is unstable.*

As far as the stability of the trivial equilibrium is concerned, we can prove the following result.

Theorem 4.5. (i) *If $\mu(a,0)>\beta(a,0)$ a.e. $a\in(0,A)$, then the trivial equilibrium is globally asymptotically stable;*
(ii) *If $\mu(a,0)<\beta(a,0)$ a.e. $a\in(0,A)$, then the trivial equilibrium is unstable.*

Proof. (i) Consider the Lyapunov function

$$V(t)=\int_0^A p(a,t)da.$$

From the equations of the system and the assumptions, it follows that

$$\begin{aligned}\frac{dV}{dt} &= \int_0^A \frac{\partial p}{\partial t}da\\ &= \int_0^A [-\frac{\partial p}{\partial a}-\mu(a,E(p)(a,t))p(a,t)]da\\ &= p(0,t)-\int_0^A \mu(a,E(p)(a,t))p(a,t)da\\ &= \int_0^A [\beta(a,E(p)(a,t))-\mu(a,E(p)(a,t))]p(a,t)da\\ &\le \int_0^A [\beta(a,0)-\mu(a,0)]p(a,t)da<0,\end{aligned}\tag{4.13}$$

which implies the global stability.

(ii) From the conditions in the theorem, we derive

$$\begin{aligned}&\int_0^A \beta(a,0)\exp\{-\int_0^a \mu(r,0)dr\}da\\ &> \int_0^A \mu(a,0)\exp\{-\int_0^a \mu(r,0)dr\}da = 1.\end{aligned}\tag{4.14}$$

Consequently, Corollary 4.4 provides the conclusion. $\square$

5. Numerical results

This section we present some numerical experiments which show the stability of zero solution intuitively. We apply the algorithm proposed in [13] and MATLAB to carry out the computations and draw the population surfaces. Let $A=10$, and the step size of age and time be $h=0.1$; so that the number of iterates is 100.

Example 5.1 (Instability)**.** Let the parameters be as follows: $\alpha = 0.5$, $p_0(a) = 0.3(10-a)(1+\sin a)$;

$$\mu(a, E(p)) = \begin{cases} 0.2\cos(a) + 0.01E(p) + \frac{(2-a)^2}{100}, & a \in [0,2), \\ 0.2\cos(a) + 0.01E(p), & a \in [2,8), \\ 0.2\cos(a) + 0.01E(p) + \frac{a-8}{10-a}, & a \in [8,10), \end{cases} \tag{5.1}$$

$$\beta(a, E(p)) = \begin{cases} 0, & a \in [0,1) \cup [9,10), \\ 0.5(1+\sin a)(a-1)^2, & a \in [1,2), \\ 0.5(1+\sin a), & a \in [2,8), \\ 0.5(1+\sin a)(a-9)^2, & a \in [8,9). \end{cases} \tag{5.2}$$

In this case, $R(0) = 4.3628$. We can see that the population surface is gradually away from the horizontal plane (zero equilibrium) over time, which is consistent with the conclusion of Corollary 4.4. See Figure 1 .

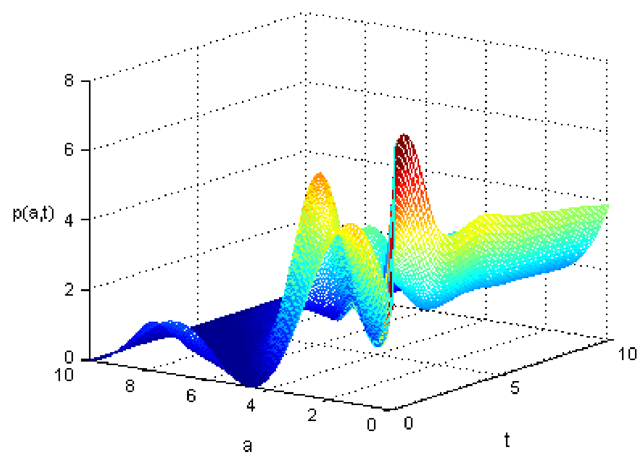


FIGURE 1. The zero solution is unstable.

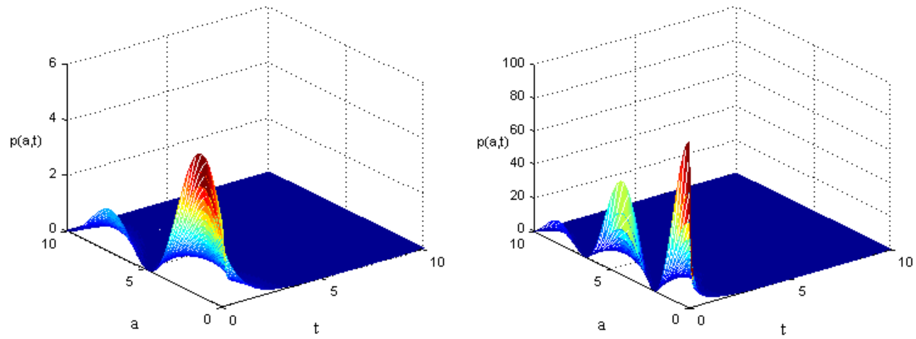
Example 5.2 (Asymptotical stability)**.** Let $\alpha = 0.5$, and

$$\mu(a, E(p)) = \begin{cases} 0.4 + 0.01E(p) + \frac{(2-a)^2}{100}, & a \in [0,2), \\ 0.4 + 0.01E(p), & a \in [2,8), \\ 0.4 + 0.01E(p) + \frac{a-8}{10-a}, & a \in [8,10); \end{cases} \tag{5.3}$$

$$\beta(a, E(p)) = \begin{cases} 0, & a \in [0,1) \cup [9,10), \\ 0.2(a-1)^2, & a \in [1,2), \\ 0.2, & a \in [2,9). \end{cases} \tag{5.4}$$

Note that $\mu(a, 0) > \beta(a)$, a.e. $a \in (0, 10)$, the result agrees with Corollary 4.2. Figure 2 shows that the zero solution is asymptotically stable.

Note that the the conditions in the Theorems 4.3 and 4.5 are not necessary.



$p_0(a) = 0.3(10 - a)(1 + \sin a)$ $\quad p_0(a) = 5(10 - a)[1 + \cos(1.4a)$

FIGURE 2. The zero solution is asymptotically stable

Example 5.3. For other cases we have:

(i) Let $\alpha = 0.5$, and

$$\mu(a, E(p)) = \begin{cases} 0.04 + 0.02E(p) + \frac{(2-a)^2}{100}, & a \in [0, 2), \\ 0.04 + 0.02E(p), & a \in [2, 8), \\ 0.04 + 0.02E(p) + \frac{a-8}{10-a}, & a \in [8, 10); \end{cases} \tag{5.5}$$

$$\beta(a, E(p)) = \begin{cases} 0, & a \in [0, 1) \cup [9, 10), \\ 0.2(1 + \sin a)(a - 1)^2, & a \in [1, 2), \\ 0.2(1 + \sin a), & a \in [2, 9). \end{cases} \tag{5.6}$$

In this case, $\beta(a, 0) > \mu(a, 0)$ for $a \in (2\pi, \frac{5\pi}{2}) \subset (2, 8)$, which is against the conditions (i) and (ii) in Theorem 4.5. Figure 3 shows that zero solution is still asymptotically stable.

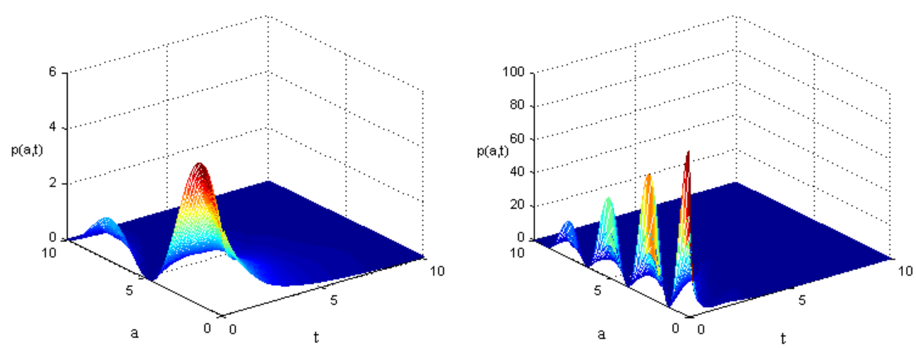


FIGURE 3. The zero solution is asymptotically stable

(ii) Let $\alpha = 0.6$, $p_0(a) = 0.5(10 - a)(1 + \cos a)$, and

$$\mu(a, E(p)) = \frac{0.006a + 0.03E(p)}{10 - a}, \tag{5.7}$$

$$\beta(a, E(p)) = \begin{cases} 0, & a \in [0, 0.039) \cup [9.911, 10), \\ 0.45(1+\cos a)(a-0.0039)^2, & a \in [0.0039, 0.0042), \\ 0.15(1+\cos a), & a \in [0.0042, 9.911). \end{cases} \tag{5.8}$$

In this case, $R(0) = 0.8987$ and $\beta(a,0) < \mu(a,0)$ for $a \in [0, 0.039) \cup [9.911, 10)$. Consequently the conditions in Theorems 4.3 and 4.5 do not hold. Figure 4 shows that zero solution is still unstable.

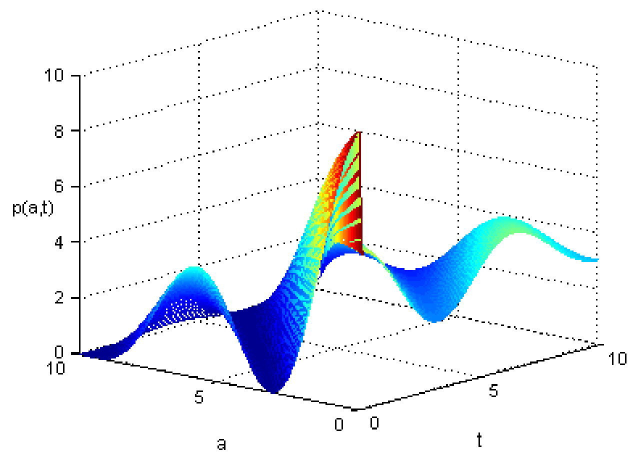


FIGURE 4. The zero solution is unstable.

Discussion

The main results obtained in previous sections are Theorems 3.4, 4.1, 4.3, 4.5. We can see that the zero equilibrium is unstable and there exists at least one positive steady state if $R(0) > 1$. It should be noted that the conditions in Theorems 4.1 and 4.3 involve positive equilibrium $p^*(a)$, solutions to the nonlinear system (3.1) with a global-feedback boundary condition. This system may be solved with given accuracy by existing numerical methods, and consequently the conditions in theorems 4.1 and 4.3 can be checked.

When the conditions in Theorems 4.1 and 4.3 do not hold, the situation is quite complicated and extensive theoretical analysis seems unlikely, even if for the stability of zero equilibrium. Example 5.3 shows the possibilities.

Acknowledgments. This work was supported by National Natural Science Foundation of China (11871185), and by Zhejiang Provincial Natural Science Foundation of China (LY18A010010).

References

[1] Ackleh, A. S.; Deng, K.; *Monotone approximation for a hierarchical age-structured population model.* Dynamics of Continuous, Discrete and Impulsive Systems, 2005, 2 (2): 203-214.
[2] Ackleh, A. S.; Deng, K.; Hu, S.; *A quasilinear hierarchical size-structured model: well-posedness and approximation.* Applied Mathematics and Optimization, 2005, 51 (1): 35-59.

[3] Aniţa, S.; *Analysis and Control of Age-Dependent Population Dynamics.* Kluwer Academic Publishers, Dordrecht/Boston/London, 2000.
[4] Calsina, A.; Saldana, J.; *Asymptotic behaviour of a model of hierarchically structured population dynamics.* Journal of Mathematical Biology, 1997, 35 (8): 967-987.
[5] Calsina, A.; Saldana, J.; *Basic theory for a class of models of hierarchically structured population dynamics with distributed states in the recruitment.* Mathematical Models and Methods in Applied Sciences, 2006, 16 (10): 1695-1722.
[6] Cushing, J. M.; *The dynamics of hierarchical age-structured populations.* Journal of Mathematical Biology, 1994, 32 (7): 705-729.
[7] Cushing, J. M.; Li J.; *Oscillations caused by cannibalism in a size-structured population model.* Canadian Applied Mathematics Quarterly, 1995, 3 (2):155-172.
[8] Dewsbery, D. A.; *Dominance rank, copulatory behavior, and differential reproduction.* The Quarterly Review of Biology, 1982, 57 (2): 135-159.
[9] Engel, K.-J.; Nagel, R.; *One-Parameter Semigroups for Linear Evolution Equations.* Springer-Verlag, New York, 2000.
[10] Farkas, J. Z.; Hagen, T.; *Asymptotic analysis of a size-structured cannibalism model with infinite dimensional environment feedback.* Communications on Pure and Applied Analysis, 2009, 8(6), 1825-1839.
[11] Farkas, J. Z.; Hinow, P.; *Steady states in hierarchical structured populations with distributed states at birth.* Discrete and Continuous Dynamical Systems, Ser. B, 2012, 17 (8), 2671-2690.
[12] Gurney W. S. C.; Nisbet, R. M.; *Ecological stability and social hierarchy.* Theoretical Population Biology, 1979, 16(1): 48-80.
[13] He, Z.; Ni, D.; Liu, Y.; *Theory and Approximation of Solutions to a Controlled Hierarchical Age-Structured Population Model*, Journal of Applied Analysis and Computation, 2018, 8 (5): 1326-1341.
[14] Henson, S. M.; Cushing, J. M.; *Hierarchical models of intra-specific competition: scramble versus contest.* Journal of Mathematical Biology, 1996, 34 (7): 755-772.
[15] Jang, R. J.; Cushing, J. M.; *A discrete hierarchical model of intra-specific competition.* Journal of Mathematical Analysis and Applications, 2003, 280 (1): 102-122.
[16] Kraev, E. A.; *Existence and uniqueness for height structured hierarchical population models.* Natural Resource Modeling, 2001, 14 (1): 45-70.
[17] Liu, Y.; He, Z.; *On the well-posedness of a nonlinear hierarchical size-structured population model.* ANZIAM Journal, 2017, 58 (3-4): 482-490.
[18] Lominicki, A; *Individual differences between animals and the natural regulation of their numbers.* Journal of Animal Ecology, 1978, 47 (2): 461-475.
[19] Prüss, J.; *On the qualitative behavior of populations with age-specific interaction.* Comp. & Maths. with Appls., 1983, 9 (3), 327-339.
[20] Shen, J.; Shu, C. W.; Zhang, M.; *A High Order WENO Scheme for a Hierarchical Size-Structured Population Model.* Journal of Scientific Computing, 2007, 33 (3): 279-291.
[21] Sounvoravong, B.; Guo, S.; Bai, Y. Z.; *Bifurcation and stability of a diffusive SIRS epidemic model with time delay.* Electronic Journal of Differential Equations, 2019, 45, 1-16.
[22] Yosida, K.; *Functional Analysis* (Sixth Edition). Springer-Verlag, Berlin, 1980.

Ze-Rong He (corresponding author)
Department of Mathematics, Institute of Operational Research and Cybernetics, Hangzhou Dianzi University, Hangzhou 310018, China
Email address: zrhe@hdu.edu.cn

Dongdong Ni
Department of Mathematics, Institute of Operational Research and Cybernetics, Hangzhou Dianzi University, Hangzhou 310018, China
Email address: 2290811277@qq.com

Shuping Wang
Department of Mathematics, Institute of Operational Research and Cybernetics, Hangzhou Dianzi University, Hangzhou 310018, China
Email address: wsp_2011@hdu.edu.cn