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DYNAMIC BEHAVIOR OF A STOCHASTIC SIR EPIDEMIC MODEL WITH VERTICAL TRANSMISSION

XIAO-BING ZHANG, SU-QIN CHANG, HAI-FENG HUO

Abstract. This article concerns the dynamic behavior of a stochastic SIR epidemic model with vertical transmission. We present sufficient conditions which can determine the extinction and persistence in mean of the epidemic. Also we discuss the asymptotic behavior of the stochastic model around the endemic equilibrium of the corresponding model. Moreover, sufficient conditions for the existence of stationary distribution are established. The results are illustrated by numerical simulations.

1. Introduction

The SIR epidemic models is one of the most important epidemic models, which was first proposed by Kermack and Mckendrick [17], and has been extended in many ways according to the different infection characteristics and control methods (see [1, 2, 14, 26, 38, 40, 44] and the references therein). For many infectious diseases in nature, there are both horizontal and vertical transmission. These include such human diseases as rubella, herpes simplex, hepatitis B, Chagas disease and AIDS (see [21, 4] and the references therein). For human and animal diseases, horizontal transmission typically occurs through direct or indirect physical contact with infectious hosts, or through disease vectors such as mosquitos, ticks, or other biting insects. Vertical transmission is defined as the infection of newborns by their mother. For instance, vertical transmission is the main cause of HIV infection in children. Recently, the studies of epidemic models incorporating vertical transmission have become one of the important topic in the mathematical theory of epidemiology (see [3, 4, 5, 6, 9, 10, 12, 18, 25, 31, 33, 36, 39] and the references therein). For example, Busenberg and Cooke [4] constructed and analyzed various compartmental models with vertical transmission to gain insight on the role of vertical transmission in disease epidemics.

In classical SIR model, the population is divided into the susceptible S, the infective I, and the removed R. When there is vertical transmission, the newborns of the infectious may be susceptible or infectious. Ma et al. [27] introduced vertical

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transmission to SIR model and established the model

$$\begin{aligned}\frac{dS(t)}{dt} &= b(1-m)(S(t)+R(t)) - \beta S(t)I(t) + pb_I I(t) - dS(t),\\ \frac{dI(t)}{dt} &= \beta S(t)I(t) + qb_I I(t) - d_I I(t) - \gamma I(t),\\ \frac{dR(t)}{dt} &= \gamma I(t) - dR(t) + mb(S(t)+R(t)),\end{aligned} \tag{1.1}$$

where b represents the birth rate of S and R, d denotes the death rate of S and R, b_I represents the birth rate of I, d_I denotes the death rate of I, β denotes the average number of adequate contacts with susceptible for an infective individual per unit time, γ denotes the recovery rate from I to R, p stands for the probability that a child who is born from infectious mother is susceptible, q stands for the probability that a child who is born from infectious mother is infected, m denotes a fraction of vaccinated for newborns of S, R. Besides, all value are assumed to be nonnegative and $0 < m < 1$, $p + q = 1$. Obviously, when $p = 1$, that is, $q = 0$, there exists only horizontal transmission. Moreover, they assumed that the birth rate and the death rate are equal, namely $b = d$ and $b_I = d_I$. This implies that the population size $N = S + I + R$ is constant, denoted $N = 1$. Under these assumptions, system (1.1) becomes the system

$$\begin{aligned}\frac{dS(t)}{dt} &= b(1-m)(1-I) - \beta SI + pb_I I - bS,\\ \frac{dI(t)}{dt} &= \beta SI - pb_I I - \gamma I.\end{aligned} \tag{1.2}$$

For system (1.2), the basic reproduction number is $R_0 = \frac{\beta(1-m)}{pb_I+\gamma}$. It has a disease-free equilibrium $E_0 = (1-m, 0)$ and endemic equilibrium $E_* = (S^*, I^*)$, where $S^* = \frac{1-m}{R_0}$, $I^* = \frac{b(pb_I+\gamma)(R_0-1)}{\beta(b(1-m)+\gamma)}$. When $R_0 < 1$, the disease-free equilibrium E_0 is globally asymptotically stable, and therefore, the disease will die out in the end. When $R_0 > 1$, E_0 is unstable and the endemic equilibrium E_* is globally asymptotically stable, namely, the disease will prevail in population. These results of system (1.2) were investigated in [27].

In the real world, epidemic models are always affected by the environmental noise [28]. Thus, it is necessary to study how the environmental white noise affects dynamic behavior of the epidemic model. To this end, many stochastic models have been established (see [7, 8, 11, 15, 22, 23, 24, 30, 32, 34, 35, 37, 42, 43] and the references therein).

In this article, we assume that the transmission coefficient β is disturbed by environmental noise. In this case, we replace βdt with $\beta dt + \sigma dB(t)$ as [7, 11], where $B(t)$ is a standard Brownian motion with intensity $\sigma > 0$. Then, we obtain the stochastic version of system (1.2),

$$\begin{aligned}dS(t) &= [b(1-m)(1-I) - \beta SI + pb_I I - bS]dt - \sigma SI dB(t),\\ dI(t) &= [\beta SI - pb_I I - \gamma I]dt + \sigma SI dB(t).\end{aligned} \tag{1.3}$$

This article is organized as follows. In section 2, we prove that there is a unique global positive solution of system (1.3). In section 3, we establish sufficient condition for the disease to die out. The condition for the disease being persistent in mean is given in Section 4. In section 5, we discuss asymptotic behavior of system (1.3) around the endemic equilibrium (S^*, I^*) of the corresponding deterministic system

(1.2). In section 6, we show that there exists a unique stationary distribution for system (1.3). Finally, some conclusions are presented.

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets) and we let $B(t)$ be a scalar Brownian motion defined on the probability space. We denote $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0 \text{ for } 1 \leq i \leq n\}$.

In general, the d-dimensional stochastic system is

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW_t, \tag{1.4}$$

where $f(t, x)$ is an function in $\mathbb{R}^d$ defined on $[t_0, \infty] \times \mathbb{R}^d$, and $g(t, x)$ is an $d \times m$ matrix, f, g are locally Lipschitz functions in x, and W_t is an m-dimensional standard Wiener process defined on the above probability space.

We denote by $C^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$ the family of all nonnegative functions $V(x, t)$ defined on $\mathbb{R}^d \times [t_0, \infty]$ such that they are continuously twice differentiable in x and once in t. The differential operator L of (1.4) is defined [28] by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} [g^T(x, t)g(x, t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{1.5}$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$, then

$$LV(x, t) = V_t(x, t) + V_x(x, t)f(x, t) + \frac{1}{2}\operatorname{trace}[g^T(x, t)V_{xx}g(x, t)],$$

where $V_t(x, t) = \frac{\partial V}{\partial t}$, $V_x(x, t) = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d})$, $V_{xx} = (\frac{\partial^2 V}{\partial x_i x_j})_{d\times d}$. By Itô's formula, if $x(t) \in \mathbb{R}^d$, then $dV(x, t) = LV(x, t)dt + V_x(x, t)g(x, t)dW_t$.

2. Existence of uniqueness of positive solution

For a stochastic differential equation to have a unique global solution (i.e. no explosion in a finite time) for any initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition [28]. However, the coefficients of system (1.3) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of system (1.3) may explode at a finite time. It is therefore necessary to prove the solution of system (1.3) is positive and global.

Theorem 2.1. *For any given initial value $(S(0), I(0)) \in R^2_+$, system (1.3) has a unique global positive solution $(S(t), I(t)) \in R^2_+$ for all $t \geq 0$ with probability one, namely*

$$P\{(S(t), I(t)) \in R^2_+ \quad \forall t \geq 0\} = 1.$$

Proof. Obviously, the coefficients of system(1.3) are locally Lipschitz continuous. It is known that for any given initial value $(S(0), I(0)) \in R^2_+$, there is a unique local solution $(S(t), I(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [28]. Let $k_0 \geq 1$ be sufficiently large such that $S(0)$ and $I(0)$ all lie within the interval $[1/k_0, k_0]$. For each integer $k \geq 0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : \min\{S(t), I(t)\} \leq \frac{1}{k} \text{ or } \max\{S(t), I(t)\} \geq k\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Apparently, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, when $\tau_\infty \leq \tau_e$ a.s. If

we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(S(t), I(t)) \in R^2_+$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. If this statement is false, then there exist a pair of constants $T \geq 0$ and $\delta \in (0,1)$ such that

$$P\{\tau_\infty \leq T\} > \delta.$$

Hence there is an integer $k_1 \geq k_0$ such that

$$P\{\tau_k \leq T\} > \delta \quad \forall k \geq k_1. \tag{2.1}$$

Define a function $V : R^2_+ \to R_+$ by

$$V(S(t), I(t)) = (S - 1 - \ln(S)) + (I - 1 - \ln(I)).$$

The nonnegativity of this function can be seen from $u - 1 - \ln u \geq 0$, for all $u > 0$. Let $k \geq k_0$ and $T \geq 0$ be arbitrary. For $0 \leq t \leq \tau_k \wedge T$. Applying Itô's formula (see e.g. [28]), we have

$$\begin{aligned} dV &= \big(1 - \frac{1}{S}\big)dS + \frac{1}{2S^2}(dS)^2 + \big(1 - \frac{1}{I}\big)dI + \frac{1}{2I^2}(dI)^2 \\ &= LV dt + \sigma(I - S)dB(t), \end{aligned} \tag{2.2}$$

where LV is defined by

$$\begin{aligned} LV &= \big(1 - \frac{1}{S}\big)[b(1-m)(1-I) - \beta SI + pb_I I - bS] + \frac{\sigma^2 I^2}{2} \\ &\quad + \big(1 - \frac{1}{I}\big)[\beta SI - pb_I I - \gamma I] + \frac{\sigma^2 S^2}{2} \\ &= b(1-m)(1-I) - \beta SI + pb_I I - bS - \frac{b(1-m)(1-I)}{S} + \beta I \\ &\quad - \frac{pb_I I}{S} + b + \beta SI - pb_I I - \gamma I - \beta S + pb_I + \gamma + \frac{\sigma^2 I^2}{2} + \frac{\sigma^2 S^2}{2} \\ &= b(1-m)(1-I) - bS - \frac{b(1-m)(1-I)}{S} + \beta I - \frac{pb_I I}{S} + b \\ &\quad - \gamma I - \beta S + pb_I + \gamma + \frac{\sigma^2 I^2}{2} + \frac{\sigma^2 S^2}{2} \\ &\leq b(1-m)(1-I) + \beta I + b + pb_I + \gamma + \frac{\sigma^2 I^2}{2} + \frac{\sigma^2 S^2}{2} \\ &\leq b(1-m) + \beta + b + pb_I + \gamma + \sigma^2 := K. \end{aligned} \tag{2.3}$$

In view of (2.3), from (2.2) we obtain

$$dV \leq K dt + \sigma(I - S)dB(t). \tag{2.4}$$

We can now integrate both sides of (2.4) from 0 to $T \wedge \tau_k$ and then take the expectations, yields

$$E[V(S(T \wedge \tau_k), I(T \wedge \tau_k))] \leq V(S(0), I(0)) + KE(T \wedge \tau_k).$$

Hence

$$E[V(S(T \wedge \tau_k), I(T \wedge \tau_k))] \leq V(S(0), I(0)) + KT. \tag{2.5}$$

Set $\Omega_k = \{\tau_k \leq T\}$ for $k \geq k_1$ and by (2.1), we have $P(\Omega_k) \geq \delta$. Note that for every $\omega \in \Omega_k$, there is at least one of $S(\tau_k, \omega)$ and $I(\tau_k, \omega)$ equals either k or $\frac{1}{k}$. Therefore, $V(S(\tau_k, \omega), I(\tau_k, \omega))$ is no less than either

$$k - 1 - \ln k \quad \text{or} \quad \frac{1}{k} - 1 - \ln \frac{1}{k} = \frac{1}{k} - 1 + \ln k.$$

Thereby, one can see that

$$V(S(\tau_k,\omega),I(\tau_k,\omega)) \geq (k-1-\ln k)\wedge(\frac{1}{k}-1+\ln k)$$

It then follow from (2.5) that

$$\begin{aligned} V(S(0),I(0))+KT &\geq E[V(S(T\wedge\tau_k),I(T\wedge\tau_k))] \\ &\geq E[I_{\Omega_k(\omega)}V(S(\tau_k,\omega),I(\tau_k,\omega))] \\ &\geq \delta\big[(k-1-\ln k)\wedge\big(\frac{1}{k}-1+\ln k\big)\big], \end{aligned}$$

where $I_{\Omega_k(\omega)}$ is the indicator function of Ω_k. Letting $k\to\infty$ leads to a contradiction

$$\infty > V(S(0),I(0))+KT = \infty.$$

Hence, we must have $\tau_\infty=\infty$ a.s. This completes the proof. □

Remark 2.2. Theorem 2.1 and $S+I+R=1$ imply that the region $\Lambda=\{(S,I): S>0, I>0, S+I\leq 1\}$ is a positively invariant set of system (1.3). Then from now on, we assume the initial value $(S(0),I(0))\in\Lambda$.

3. Extinction

For deterministic epidemic models, we are interested in two things. One is when the disease will die out; the other is when the disease will prevail. Next, we will discuss the extinction of system in this section but leave its persistence to the next section. For convenience we introduce the following notation

$$\bar{S}=\frac{1}{t}\int_0^t S(u)du, \quad \bar{I}=\frac{1}{t}\int_0^t I(u)du,$$
$$M_t^S=\frac{1}{t}\int_0^t S(u)dB(u), \quad M_t^I=\frac{1}{t}\int_0^t I(u)dB(u).$$

Lemma 3.1 (see [28]). *Let $M=\{M_t\}_{t\geq 0}$ be a real-valued continuous local martingale vanishing at $t=0$ and $\langle M,M\rangle_t$ is the quadratic variation of $M=\{M_t\}_{t\geq 0}$. Then*

$$\lim_{t\to+\infty}\langle M,M\rangle_t=\infty \text{ a.s. } \Rightarrow \lim_{t\to+\infty}\frac{M_t}{\langle M,M\rangle_t}=0 \text{ a.s.},$$

and

$$\limsup_{t\to+\infty}\frac{\langle M,M\rangle_t}{t}<\infty \text{ a.s. } \Rightarrow \lim_{t\to+\infty}\frac{M_t}{t}=0 \text{ a.s.}$$

Theorem 3.2. *Let $(S(t),I(t))$ be the solution of system (1.3) with initial value $(S(0),I(0))\in\Lambda$. If*

$$\tilde{R}_0=R_0-\frac{\sigma^2(1-m)^2}{2(pb_I+\gamma)}=\frac{\beta(1-m)}{pb_I+\gamma}-\frac{\sigma^2(1-m)^2}{2(pb_I+\gamma)}<1 \text{ and } \sigma^2\leq\frac{\beta}{1-m}, \quad (3.1)$$

then

$$\limsup_{t\to+\infty}\frac{\ln I(t)}{t}\leq(pb_I+\gamma)(\tilde{R}_0-1)<0 \quad \text{a.s.}, \quad (3.2)$$

namely $I(t)$ tends to zero exponentially almost surely. In other words, the disease dies out with probability one. In addition

$$\lim_{t\to+\infty}I(t)=0, \quad \lim_{t\to+\infty}S(t)=(1-m) \quad \text{a.s.}$$

Proof. Note that

$$\frac{d(S+I)}{dt} = b(1-m) - bS - (b(1-m)+\gamma)I, \tag{3.3}$$

we have

$$\bar{S} = (1-m) - \frac{b(1-m)+\gamma}{b}\bar{I} - \varphi(t), \tag{3.4}$$

where

$$\varphi(t) = \frac{1}{b}\Big(\frac{S(t)+I(t)}{t} - \frac{S(0)+I(0)}{t}\Big)$$

and $\lim_{t\to+\infty}\varphi(t) = 0$ a.s.

By the Itô's formula, we have

$$d\ln I(t) = [\beta S - (pb_I + \gamma + \frac{\sigma^2}{2}S^2)]dt + \sigma S dB(t).$$

Then

$$\begin{aligned}
\frac{\ln I(t)}{t} &= \frac{\ln I(0)}{t} + \beta\bar{S} - (pb_I+\gamma) - \frac{\sigma^2}{2}\frac{1}{t}\int_0^t S^2(u)du + \sigma M_t^S \\
&\le \frac{\ln I(0)}{t} + \beta\bar{S} - (pb_I+\gamma) - \frac{\sigma^2}{2}\bar{S}^2 + \sigma M_t^S \\
&= \frac{\ln I(0)}{t} + \beta\Big((1-m) - \frac{b(1-m)+\gamma}{b}\bar{I} - \varphi(t)\Big) - (pb_I+\gamma) \\
&\quad - \frac{\sigma^2}{2}\Big((1-m) - \frac{b(1-m)+\gamma}{b}\bar{I} - \varphi(t)\Big)^2 + \sigma M_t^S \\
&= \beta(1-m) - \Big(pb_I + \gamma + \frac{\sigma^2(1-m)^2}{2}\Big) \\
&\quad - \frac{(b(1-m)+\gamma)(\beta-\sigma^2(1-m))}{b}\bar{I} - \frac{\sigma^2(b(1-m)+\gamma)^2}{2b^2}\bar{I}^2 + \psi(t) \\
&= (pb_I+\gamma)(\tilde{R}_0 - 1) - \frac{(b(1-m)+\gamma)(\beta-\sigma^2(1-m))}{b}\bar{I} \\
&\quad - \frac{\sigma^2(b(1-m)+\gamma)^2}{2b^2}\bar{I}^2 + \psi(t),
\end{aligned} \tag{3.5}$$

where the first inequality is according to Schwarz inequality, and

$$\begin{aligned}
\psi(t) &= \frac{\ln I(0)}{t} - (\beta - \sigma^2(1-m))\varphi(t) - \frac{\sigma^2(b(1-m)+\gamma)}{b}\varphi(t)\bar{I} \\
&\quad - \frac{\sigma^2}{2}\varphi^2(t) + \sigma M_t^S \\
&\le \frac{\ln I(0)}{t} + \Big(\beta + \sigma^2(1-m) + \frac{\sigma^2(b(1-m)+\gamma)}{b}\Big)|\varphi(t)| - \frac{\sigma^2}{2}\varphi^2(t) \\
&\quad + \sigma M_t^S,
\end{aligned}$$

and

$$\begin{aligned}
\psi(t) &\ge \frac{\ln I(0)}{t} - \Big(\beta + \sigma^2(1-m) + \frac{\sigma^2(b(1-m)+\gamma)}{b}\Big)|\varphi(t)| - \frac{\sigma^2}{2}\varphi^2(t) \\
&\quad + \sigma M_t^S.
\end{aligned}$$

According to Lemma 3.1,

$$\lim_{t\to+\infty} M_t^S = \lim_{t\to+\infty}\frac{1}{t}\int_0^t S(u)dB(u) = 0 \quad \text{a.s.}$$

In addition $\lim_{t\to+\infty}\varphi(t)=0$ a.s. Hence, we have $\lim_{t\to+\infty}\psi(t)=0$ a.s.

If $\sigma^2\le\frac{\beta}{1-m}$, then from (3.5) it follows that

$$\frac{\ln I(t)}{t}\le(pb_I+\gamma)(\tilde{R}_0-1)+\psi(t),$$

which together with the property of $\psi(t)$ imply

$$\limsup_{t\to+\infty}\frac{\ln I(t)}{t}\le(pb_I+\gamma)(\tilde{R}_0-1).$$

We therefore obtain the desired assertion (3.2).

In view of (3.2),

$$\limsup_{t\to+\infty}\frac{\ln I(t)}{t}\le-\kappa,$$

then for a arbitrary small positive constant $\varepsilon_1<-\frac{(pb_I+\gamma)(\tilde{R}_0-1)}{2}\triangleq-\kappa$, there exists a positive constant $T_1=T_1(\omega)$ and a set Ω_{ε_1} such that $P(\Omega_{\varepsilon_1})\ge1-\varepsilon_1$ and $\ln I(t)\le-\varepsilon_1 t$ for $t\ge T_1$, $\omega\in\Omega_{\varepsilon_1}$. That is,

$$I(t)\le e^{-\varepsilon_1 t}\quad\text{for } t\ge T_1,\ \omega\in\Omega_{\varepsilon_1}.$$

Letting $t\to\infty$ and $\varepsilon_1\to0$ deduce

$$\limsup_{t\to+\infty}I(t)\le0\quad\text{a.s.}$$

which together with the positive of the solution implies $\lim_{t\to+\infty}I(t)=0$ a.s.

It follows from (3.3) that

$$\frac{d(S+I)}{dt}\le b(1-m)-b(S+I)+bme^{-\varepsilon_1 t}\quad\text{for } t\ge T_1,\ \omega\in\Omega_{\varepsilon_1}.$$

By the comparison theorem and arbitrariness of ε_1, we obtain

$$\limsup_{t\to+\infty}[S+I]\le\frac{b(1-m)}{b}\ a.s.\tag{3.6}$$

Similarly, we can obtain

$$\frac{d(S+I)}{dt}\ge b(1-m)-b(S+I)-re^{-\varepsilon_1 t}\quad\text{for } t\ge T_1,\ \omega\in\Omega_{\varepsilon_1}.$$

Using the comparison theorem and arbitrariness of ε_1,

$$\liminf_{t\to+\infty}[S+I]\ge\frac{b(1-m)}{b}\quad\text{a.s.}\tag{3.7}$$

Combining (3.6) and (3.7) leads to

$$\lim_{t\to+\infty}[S+I]=(1-m)\ a.s.,$$

which implies $\lim_{t\to+\infty}S(t)=(1-m)$ a.s. Whence the proof is complete. □

In Theorem 3.2 we require the noise intensity $\sigma^2\le\frac{\beta}{1-m}$. The following theorem covers the case when $\sigma^2>\frac{\beta}{1-m}$.

Theorem 3.3. *Let $(S(t),I(t))$ be the solution of system (1.3) with initial value $(S(0),I(0))\in\Lambda$. If*

$$\sigma^2>\frac{\beta}{1-m}\vee\frac{\beta^2}{2(pb_I+\gamma)},\tag{3.8}$$

then

$$\limsup_{t\to+\infty}\frac{\ln I(t)}{t}\le -pb_I-\gamma+\frac{\beta^2}{2\sigma^2}<0\quad a.s.,$$

namely $I(t)$ tends to zero exponentially almost surely. In other words, the disease dies out with probability one. In addition

$$\lim_{t\to+\infty}I(t)=0,\quad \lim_{t\to+\infty}S(t)=(1-m),\quad a.s.$$

Proof. We use the same notation as in the proof of Theorem 3.2. From (3.5), we have

$$\frac{\ln I(t)}{t}\le (pb_I+\gamma)(\tilde{R}_0-1)+f(\bar{I})+\psi(t), \tag{3.9}$$

where

$$\begin{aligned} f(x)&\triangleq -\frac{\sigma^2(b(1-m)+\gamma)^2}{2b^2}x^2-\frac{(b(1-m)+\gamma)(\beta-\sigma^2(1-m))}{b}x\\ &=-\frac{\sigma^2(b(1-m)+\gamma)^2}{2b^2}(x+\frac{b(\beta-\sigma^2(1-m))}{(b(1-m)+\gamma)\sigma^2})^2+\frac{(\beta-\sigma^2(1-m))^2}{2\sigma^2}. \end{aligned}$$

Under condition (3.8), it is easy to confirm that $-\frac{b(\beta-\sigma^2(1-m))}{(b(1-m)+\gamma)\sigma^2}<1$ and $f(x)$ reaches its maximum value $\frac{(\beta-\sigma^2(1-m))^2}{2\sigma^2}$ at $x=-\frac{b(\beta-\sigma^2(1-m))}{(b(1-m)+\gamma)\sigma^2}$ in the interval $[0,1]$.

Consequently, from (3.9) we have

$$\begin{aligned} \frac{\ln I(t)}{t}&\le (pb_I+\gamma)(\tilde{R}_0-1)+\frac{(\beta-\sigma^2(1-m))^2}{2\sigma^2}+\psi(t)\\ &=\beta(1-m)-(pb_I+\gamma+\frac{\sigma^2(1-m)^2}{2})+\frac{(\beta-\sigma^2(1-m))^2}{2\sigma^2}+\psi(t)\\ &=\frac{\beta^2}{2\sigma^2}-(pb_I+\gamma)+\psi(t). \end{aligned}$$

Therefore,

$$\limsup_{t\to+\infty}\frac{\ln I(t)}{t}\le -pb_I-\gamma+\frac{\beta^2}{2\sigma^2}<0\ a.s.$$

The rest of the proof is the same to Theorem 3.2. □

Remark 3.4. We refer to condition (3.6), which tells us the disease will die out if $\tilde{R}_0<1$ for noise small. While if white noise is large enough such that the condition (3.6) is satisfied, then the disease will also die out even if $R_0>1$, which never happen in the corresponding deterministic system. In other words, the conditions for $I(t)$ to become extinct in the stochastic model are weaker than in the corresponding deterministic model. The following two example illustrate these results more explicitly.

Example 3.5. Choose the parameters in system (1.3) as follows:

$$b=\frac{1}{70},\quad b_I=\frac{1}{60},\quad \beta=0.15,\quad p=0.1,\quad m=0.1,\quad \gamma=0.1,\quad \sigma=0.3\,. \tag{3.10}$$

Note that

$$\tilde{R}_0=\frac{\beta(1-m)}{pb_I+\gamma}-\frac{\sigma^2(1-m)^2}{2(pb_I+\gamma)}=0.9693$$

and $\sigma^2 - \frac{\beta}{1-m} = -0.0767$, then by Theorem 3.2, the solution $(S(t), I(t))$ of system (1.3) satisfies

$$\limsup_{t\to+\infty} \frac{\ln I(t)}{t} \le (pb_I + \gamma)(\tilde{R}_0 - 1) = -0.0031 \quad \text{a.s.},$$
$$\lim_{t\to+\infty} S(t) = (1-m) = 0.9 \quad \text{a.s.},$$

with any initial value $(S(0), I(0)) \in \Lambda$. That is $I(t)$ will tend to zero exponentially with probability one. Besides, for the corresponding deterministic model (1.2)

$$R_0 = \frac{\beta(1-m)}{pb_I + \gamma} = 1.3279,$$

then the endemic equilibrium $(S^*, I^*) = (0.6778, 0.0281)$ is globally asymptotically stable in Λ. Using the method in [13], we give the simulations shown in Figure 1 to support our results.

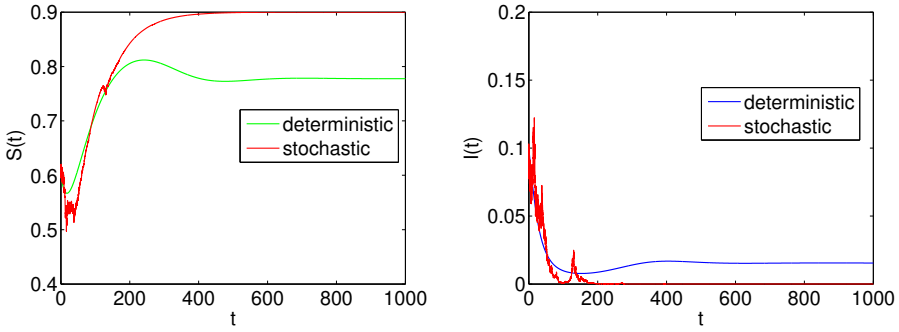

FIGURE 1. Paths $S(t)$ and $I(t)$ for models (1.2) and (1.3). The parameters are as in (3.10) with $\sigma = 0.3$.

Example 3.6. We choose the same parameters as in Example 3.5 but increase σ to 0.5. Note that $\sigma^2 > \frac{\beta}{1-m} \vee \frac{\beta^2}{2(pb_I+\gamma)} = 0.0833$, then by Theorem 3.3, the solution$(S(t), I(t))$ of system (1.3) satisfies

$$\limsup_{t\to+\infty} \frac{\ln I(t)}{t} \le -pb_I - \gamma + \frac{\beta^2}{2\sigma^2} = -0.0567 \quad \text{a.s.},$$
$$\lim_{t\to+\infty} S(t) = (1-m) = 0.9 \quad \text{a.s.}$$

That is $I(t)$ will tend to zero exponentially with probability one. For the corresponding deterministic model (1.2), since the other parameters are the same as in Example 3.5, the dynamic behavior of model (1.2) is the same as in Example 3.5. The simulations shown in Figure 2 support our results.

4. PERSISTENCE

Lemma 4.1 (see [16]). *Let $f \in C[[0,\infty) \times \Omega, (0,\infty)]$, $F \in C[[0,\infty) \times \Omega, R]$ and $\lim_{t\to+\infty} \frac{F(t)}{t} = 0$ a.s.*

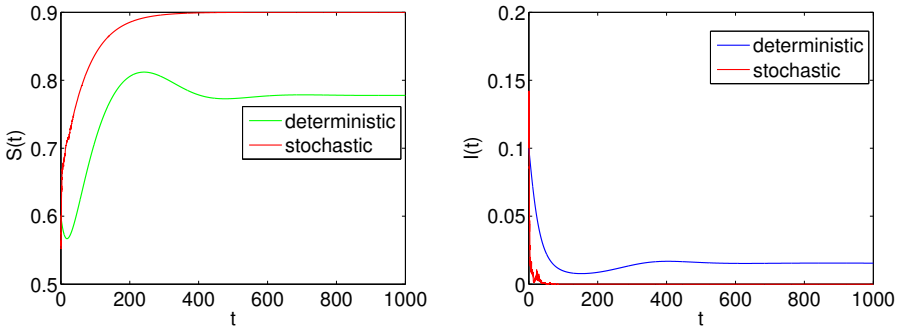

FIGURE 2. Path $S(t)$ and $I(t)$ for models (1.2) and (1.3). The parameters are as in (3.10) with $\sigma = 0.5$.

(i) *If there exist positive constants* λ_0, λ *such that*

$$\ln f(t) \geq \lambda t - \lambda_0 \int_0^t f(u)du + F(t) \quad a.s.$$

for all $t > 0$, *then*

$$\liminf_{t\to+\infty} \frac{1}{t} \int_0^t f(u)du \geq \frac{\lambda}{\lambda_0} \quad a.s.$$

(ii) *If there exist positive constants* λ_0, λ *such that*

$$\ln f(t) \leq \lambda t - \lambda_0 \int_0^t f(u)du + F(t) \quad a.s.$$

for all $t > 0$, *then*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t f(u)du \leq \frac{\lambda}{\lambda_0} \quad a.s.$$

Theorem 4.2. *If*

$$\tilde{R}_{0*} = R_0 - \frac{\sigma^2}{2(pb_I + \gamma)} > 1, \quad \sigma^2 \leq \frac{\beta}{1-m},$$

then for any initial value $(S(0), I(0)) \in R^2_+$, *the solution of system* (1.3) *satisfies*

$$\limsup_{t\to+\infty} \bar{I} \leq \frac{b(pb_I + \gamma)(\tilde{R}_0 - 1)}{(b(1-m) + \gamma)[\beta - \sigma^2(1-m)]} \quad a.s.,$$

$$\liminf_{t\to+\infty} \bar{S} \geq (1-m) - \frac{(pb_I + \gamma)(\tilde{R}_0 - 1)}{\beta - \sigma^2(1-m)} \quad a.s.,$$

and

$$\liminf_{t\to+\infty} \bar{I} \geq \frac{b(pb_I + \gamma)(\tilde{R}_{0*} - 1)}{\beta(b(1-m) + \gamma)} \quad a.s.,$$

$$\limsup_{t\to+\infty} \bar{S} \leq (1-m) - \frac{(pb_I + \gamma)(\tilde{R}_{0*} - 1)}{\beta} \quad a.s.,$$

where

$$\tilde{R}_{0*} = \frac{\beta(1-m)}{pb_I+\gamma} - \frac{\sigma^2}{2(pb_I+\gamma)} < \tilde{R}_0 = \frac{\beta(1-m)}{pb_I+\gamma} - \frac{\sigma^2(1-m)^2}{2(pb_I+\gamma)}.$$

Proof. If $\tilde{R}_0 > 1$ and $\sigma^2 \le \frac{\beta}{1-m}$, then from (3.5) we have

$$\frac{\ln I(t)}{t} \le (pb_I+\gamma)(\tilde{R}_0-1) - \frac{(b(1-m)+\gamma)(\beta-\sigma^2(1-m))}{b}\bar{I} + \psi(t).$$

By the Lemma 4.1, we have

$$\limsup_{t\to+\infty} \bar{I} \le \frac{b(pb_I+\gamma)(\tilde{R}_0-1)}{(b(1-m)+\gamma)[\beta-\sigma^2(1-m)]} \quad \text{a.s.}$$

This means that for any $\varepsilon_2 > 0$ ($\varepsilon_2 < \frac{b(pb_I+\gamma)(\tilde{R}_0-1)}{(b(1-m)+\gamma)[\beta-\sigma^2(1-m)]}$), there is a $T_2(\omega)$ such that for $t > T_2(\omega)$,

$$\bar{I} \le \frac{b(pb_I+\gamma)(\tilde{R}_0-1)}{(b(1-m)+\gamma)(\beta-\sigma^2(1-m))} + \varepsilon_2.$$

Then from (3.6), we obtain

$$\bar{S} = (1-m) - \frac{b(1-m)+\gamma}{b}\bar{I} - \varphi(t), \tag{4.1}$$

From this and (4.1),

$$\bar{S} \ge (1-m) - \frac{b(1-m)+\gamma}{b}\Big(\frac{b(pb_I+\gamma)(\tilde{R}_0-1)}{(b(1-m)+\gamma)(\beta-\sigma^2(1-m))} + \varepsilon_2\Big) - \varphi(t).$$

Letting $t\to\infty$ with ε_2 arbitrary, we obtain

$$\liminf_{t\to+\infty} \bar{S} \ge (1-m) - \frac{(pb_I+\gamma)(\tilde{R}_0-1)}{\beta-\sigma^2(1-m)} \quad \text{a.s.}$$

On the other hand,

$$\begin{aligned}
&\frac{\ln I(t)}{t}\\
&= \frac{\ln I(0)}{t} + \beta\bar{S} - (pb_I+\gamma) - \frac{\sigma^2}{2}\frac{1}{t}\int_0^t S^2(u)du + \sigma M_t^S\\
&\ge \frac{\ln I(0)}{t} + \beta\bar{S} - (pb_I+\gamma) - \frac{\sigma^2}{2} + \sigma M_t^S\\
&\ge \frac{\ln I(0)}{t} + \beta((1-m) - \frac{b(1-m)+\gamma}{b}\bar{I} - \varphi(t)) - (pb_I+\gamma) - \frac{\sigma^2}{2} + \sigma M_t^S\\
&= \beta(1-m) - \Big(pb_I+\gamma+\frac{\sigma^2}{2}\Big) - \beta\frac{b(1-m)+\gamma}{b}\bar{I} + \Theta(t)\\
&= (pb_I+\gamma)[\frac{\beta(1-m)}{pb_I+\gamma} - 1 - \frac{\sigma^2}{2(pb_I+\gamma)}] - \beta\frac{b(1-m)+\gamma}{b}\bar{I} + \Theta(t)\\
&= (pb_I+\gamma)(\tilde{R}_{0*}-1) - \beta\frac{b(1-m)+\gamma}{b}\bar{I} + \Theta(t),
\end{aligned}$$

where $\Theta(t) = \frac{\ln I(0)}{t} - \beta\varphi(t) + \sigma M_t^S$ and $\limsup_{t\to+\infty}\Theta(t) = 0$ a.s. Then by the Lemma 4.1, we have

$$\liminf_{t\to+\infty} \bar{I} \ge \frac{b(pb_I+\gamma)(\tilde{R}_{0*}-1)}{\beta(b(1-m)+\gamma)} \ a.s.$$

For any $\forall \varepsilon_3 > 0$ ($\varepsilon_3 < \frac{b(pb_I+\gamma)(\tilde{R}_{0*}-1)}{\beta(b(1-m)+\gamma)}$), there is a $T_3(\omega)$, such that

$$\bar{I} \geq \frac{b(pb_I+\gamma)(\tilde{R}_{0*}-1)}{\beta(b(1-m)+\gamma)} - \varepsilon_3. \tag{4.2}$$

Combining (4.1) and (4.2) leads to

$$\bar{S} \leq (1-m) - \frac{b(1-m)+\gamma}{b}\Big(\frac{b(pb_I+\gamma)(\tilde{R}_{0*}-1)}{\beta(b(1-m)+\gamma)} - \varepsilon_3\Big) - \varphi(t).$$

Letting $t \to \infty$ and ε_3 arbitrary, we obtain

$$\limsup_{t\to+\infty} \bar{S} \leq (1-m) - \frac{(pb_I+\gamma)(\tilde{R}_{0*}-1)}{\beta} \ a.s.$$

The proof is complete. □

Example 4.3. We keep all the system (1.3) parameters the same as in Example 3.5 except that σ is reduced to 0.05. Note that $\tilde{R}_{0*} = \frac{\beta(1-m)}{pb_I+\gamma} - \frac{\sigma^2}{2(pb_I+\gamma)} = 1.3156$, and $\sigma^2 - \frac{\beta}{1-m} = -0.1642$. Then by Theorem 4.2, for any initial value $(S(0), I(0)) \in \Lambda$ the solution $(S(t), I(t))$ of system (1.3) satisfies

$$\begin{aligned}
\limsup_{t\to+\infty} \frac{1}{t}\int_0^t I(u)du &\leq \frac{b(pb_I+\gamma)(\tilde{R}_0-1)}{(b(1-m)+\gamma)[\beta-\sigma^2(1-m)]} = 0.0277 \quad \text{a.s.},\\
\liminf_{t\to+\infty} \frac{1}{t}\int_0^t S(u)du &\geq (1-m) - \frac{(pb_I+\gamma)(\tilde{R}_0-1)}{\beta-\sigma^2(1-m)} = 0.6812 \quad \text{a.s.},\\
\liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(u)du &\geq \frac{b(pb_I+\gamma)(\tilde{R}_{0*}-1)}{\beta(b(1-m)+\gamma)} = 0.0271 \quad \text{a.s.},\\
\limsup_{t\to+\infty} \frac{1}{t}\int_0^t S(u)du &\leq (1-m) - \frac{(pb_I+\gamma)(\tilde{R}_{0*}-1)}{\beta} = 0.6861 \quad \text{a.s.}
\end{aligned}$$

That is to say, the disease will prevail. The simulations shown in Figure 3 support our results.

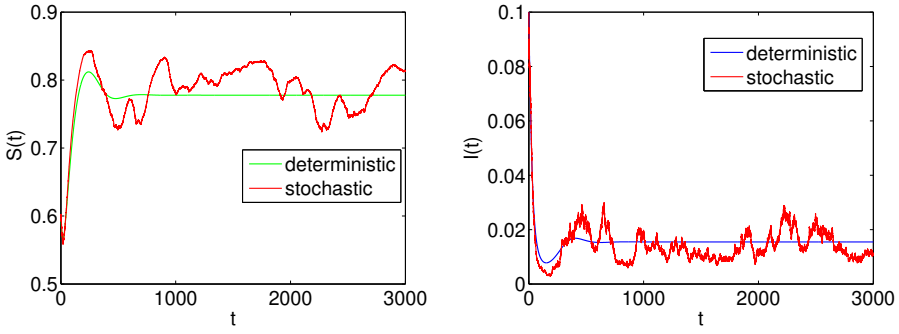

FIGURE 3. Paths $S(t)$ and $I(t)$ for model (1.2) and (1.3). The parameters are as in 3.10 with $\sigma = 0.05$.

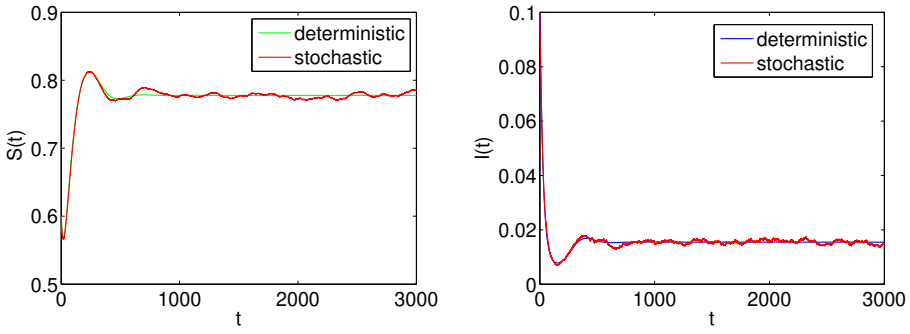

FIGURE 4. Paths $S(t)$ and $I(t)$ for model (1.2) and (1.3). The parameters are as in (3.10) with $\sigma = 0.01$.

To further illustrate the effect of the noise intensity σ on model (1.3), we keep all the parameters of (1.3) unchanged but reduced σ to 0.01. In this case,

$$\tilde{R}_{0*} = \frac{\beta(1-m)}{pb_I+\gamma} - \frac{\sigma^2}{2(pb_I+\gamma)} = 1.3274, \sigma^2 - \frac{\beta}{1-m} = -0.1666$$

which satisfy the assumption in Theorem 4.2. We give the simulations shown in Figure 4. Comparing the Figure 3, with the noise getting smaller, the fluctuation of the solution of system (1.3) is getting weaker.

5. Asymptotic behavior around the endemic equilibrium

For the deterministic system (1.2), the endemic equilibrium exists and is globally asymptotically stable. However, for the stochastic system (1.3), there exists no endemic equilibrium. In this section, we discuss how stochastic fluctuations affect the endemic equilibrium (S^*, I^*) of the deterministic system (1.2).

Theorem 5.1. *If $R_0 > 1$, then for any given initial value $(S(0), I(0)) \in \Lambda = \{(x,y) : x > 0, y > 0, x+y \le 1\}$ the solution of model (1.3) satisfies*

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t}\int_0^t [b(S(u)-S^*)^2 + [b(1-m)+\gamma](I(u)-I^*)^2]du\\ &\le \frac{1}{2}\sigma^2 I^* \frac{[b(1-m)+\gamma+b]}{\beta}.\end{aligned}$$

Proof. Define a C^2-function $V : (0,1)\times(0,1) \to \mathbb{R}_+$ by

$$V(S,I) = \frac{[b(1-m)+\gamma+b]}{\beta} V_1 + V_2,$$

where $V_1(I) = I - I^* - I^* \ln \frac{I}{I^*}$, $V_2(S,I) = \frac{1}{2}(S - S^* + I - I^*)^2$.

An application of the differential operator L to V_1 yields

$$\begin{aligned}LV_1 &= (1-\frac{I^*}{I})(\beta SI - pb_I I - \gamma I) + \frac{1}{2}\sigma^2 I^* S^2\\ &= (I-I^*)\beta(S-S^*) + \frac{1}{2}\sigma^2 I^* S^2 \qquad (5.1)\\ &\le \beta(I-I^*)(S-S^*) + \frac{1}{2}\sigma^2 I^*.\end{aligned}$$

and

$$\begin{aligned} LV_2 &= (S-S^*+I-I^*)[-[b(1-m)+\gamma](I-I^*)-b(S-S^*)] \\ &= -b(S-S^*)^2-[b(1-m)+\gamma](I-I^*)^2 \\ &\quad -[b(1-m)+\gamma+b](I-I^*)(S-S^*). \end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), we obtain

$$LV \le -b(S-S^*)^2-[b(1-m)+\gamma](I-I^*)^2+\frac{1}{2}\sigma^2 I^*\frac{[b(1-m)+\gamma+b]}{\beta}. \tag{5.3}$$

Then

$$\begin{aligned} dV &= LV\,dt-\sigma(I-I^*)S\,dB(t) \\ &\le -b(S-S^*)^2-[b(1-m)+\gamma](I-I^*)^2 \\ &\quad +\frac{1}{2}\sigma^2 I^*\frac{[b(1-m)+\gamma+b]}{\beta}-\sigma S(I-I^*)dB(t). \end{aligned} \tag{5.4}$$

Integrating both sides of (5.4) from 0 to t yields

$$\begin{aligned} V(t)-V(0) \le \int_0^t \Big[&-b(S(u)-S^*)^2-[b(1-m)+\gamma](I(u)-I^*)^2 \\ &+\frac{1}{2}\sigma^2 I^*\frac{[b(1-m)+\gamma+b]}{\beta}\Big]du-\int_0^t \sigma S(u)(I(u)-I^*)dB(u). \end{aligned}$$

That is,

$$\begin{aligned} &\frac{1}{t}\int_0^t [b(S(u)-S^*)^2+[b(1-m)+\gamma](I(u)-I^*)^2]du \\ &\le \frac{1}{2}\sigma^2 I^*\frac{[b(1-m)+\gamma+b]}{\beta}-\frac{V(t)-V(0)}{t}-\frac{1}{t}\int_0^t \sigma S(u)(I(u)-I^*)dB(u). \end{aligned}$$

From the Lemma 3.1 it follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma(I(u)-I^*)S(u)dB(u)=0 \quad \text{a.s.}$$

which implies

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\int_0^t [b(S(u)-S^*)^2+[b(1-m)+\gamma](I(u)-I^*)^2]du \\ &\le \frac{1}{2}\sigma^2 I^*\frac{[b(1-m)+\gamma+b]}{\beta}. \end{aligned}$$

□

Remark 5.2. From Theorem 5.1, we have $\frac{1}{2}\sigma^2 I^*\frac{[b(1-m)+\gamma+b]}{\beta}\to 0$ as $\sigma^2\to 0$. This means that the solution of model (1.3) fluctuates around the endemic equilibrium (S^*,I^*) of model (1.2) and with the values of σ^2 decreasing, the difference between them also decreases.

6. Stationary distribution

First, we give a definition about stationary distribution and some lemmas.

Definition 6.1 (see [29, 19]). Let $P(t, X_0, \cdot)$ denote the probability measure induced by $X(t) = (S(t), I(t))$ with initial value $X_0 = (S(0), I(0))$; that is,

$$P_{X_0}(X \in B) = \mathbf{P}\{X(t) \in B : X(0) = X_0\} \text{ for any Borel set } B \subset \mathbb{R}^2_+.$$

If there exists a probability measure $\pi(\cdot)$ on the measurable space $(\mathbb{R}^2_+, \mathbb{B}(\mathbb{R}^2_+))$ such that

$$\lim_{t\to\infty} P_{X_0}(X \in B) = \pi(B) \quad \text{for any } X_0 \in \mathbb{R}^2_+,$$

we then say that model has a stationary distribution $\pi(\cdot)$.

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^n_+$ described by

$$dX(t) = b(X)dt + \sum_{r=1}^{k} \sigma_r(X)dB_r(t).$$

The diffusion matrix is defined as

$$A(X) = (a_{ij}(x)),\ a_{ij}(x) = \sum_{r=1}^{k} \sigma_r^i(x)\sigma_r^j(x).$$

To show the existence of a stationary distribution, we cite a known result from Zhu and Yin [45, Remark 3.2, Theorems 3.13, 4.2 ,4.4]; see also [20].

Lemma 6.2. *The Markov process $X(t)$ has a unique stationary distribution $\pi(\cdot)$ if there exists a bounded domain $U \in \mathbb{R}^d$ with regular boundary such that its closure $\bar{U} \subset \mathbb{R}^d$, having the following properties:*

(i) *There exist some $i = 1, 2, \ldots, n$, and a positive constant η such that $a_{ii}(x) \geq \eta$ for any $x \in U$.*

(ii) *There is a nonnegative C^2-function $V(x)$, and a neighborhood U such that for some constants $\kappa > 0$, $LV(x) < -\kappa$, $x \in \Lambda \setminus U$.*

Moreover, if $f(\cdot)$ is a function integrable with respect to the measure $\pi(\cdot)$, then

$$P\Big\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X^x(t)) = \int_{\mathbb{R}^d} f(x)\pi(dx)\Big\} = 1,$$

for all $x \in \mathbb{R}^d$.

Theorem 6.3. *Let the assumptions in Theorem 5.1 hold and*

$$0 < \Psi < \min(bS^{*2}, [b(1-m)+\gamma]I^{*2}). \tag{6.1}$$

Then for any given initial value $(S(0), I(0)) \in \Lambda$, there exists a unique stationary distribution $\pi(\cdot)$, and the solution of (1.3) is ergodic, where $\Psi = \frac{1}{2}\sigma^2 I^* \frac{[b(1-m)+\gamma+b]}{\beta}$ and (S^*, I^*) is the unique endemic equilibrium of (1.2).

Proof. To validate condition (ii), we use the nonnegative C^2-function $V(S, I)$ as Theorem 5.1. From (5.3) it follows that

$$\begin{aligned} LV &\leq -b(S-S^*)^2 - [b(1-m)+\gamma](I-I^*)^2 + \frac{1}{2}\sigma^2 I^* \frac{[b(1-m)+\gamma+b]}{\beta} \\ &= -b(S-S^*)^2 - [b(1-m)+\gamma](I-I^*)^2 + \Psi. \end{aligned}$$

Since

$$0 < \Psi < \min(bS^{*2}, [b(1-m)+\gamma]I^{*2}),$$

the ellipsoid

$$b(S-S^*)^2 + [b(1-m)+\gamma](I-I^*)^2 = \Psi$$

lies entirely in $\mathbb{R}^2_+$. One can then take U as any neighborhood of the ellipsoid such that $\bar{U} \subset \mathbb{R}^2_+$, where $\bar{U}$ is the closure of U. Thus, we have $LV(S,I) < 0$ for $(S,I) \in \mathbb{R}^2_+ \setminus \bar{U}$, which implies that condition (ii) in Lemma 6.2 is satisfied.

On the other hand, for system (1.3), the diffusion matrix is

$$A(S,I) = \sigma^2 S^2 I^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Since $\bar{U} \subset \mathbb{R}^2_+$, it follows that $a_{11}(S,I) = \sigma^2 S^2 I^2 \geq \min_{(S,I)\in\bar{U}} \sigma^2 S^2 I^2 > 0$. We have therefore verified condition (i) in Lemma 6.2. As a consequence, system (1.3) has a stationary distribution $\pi(\cdot)$ and is ergodic. □

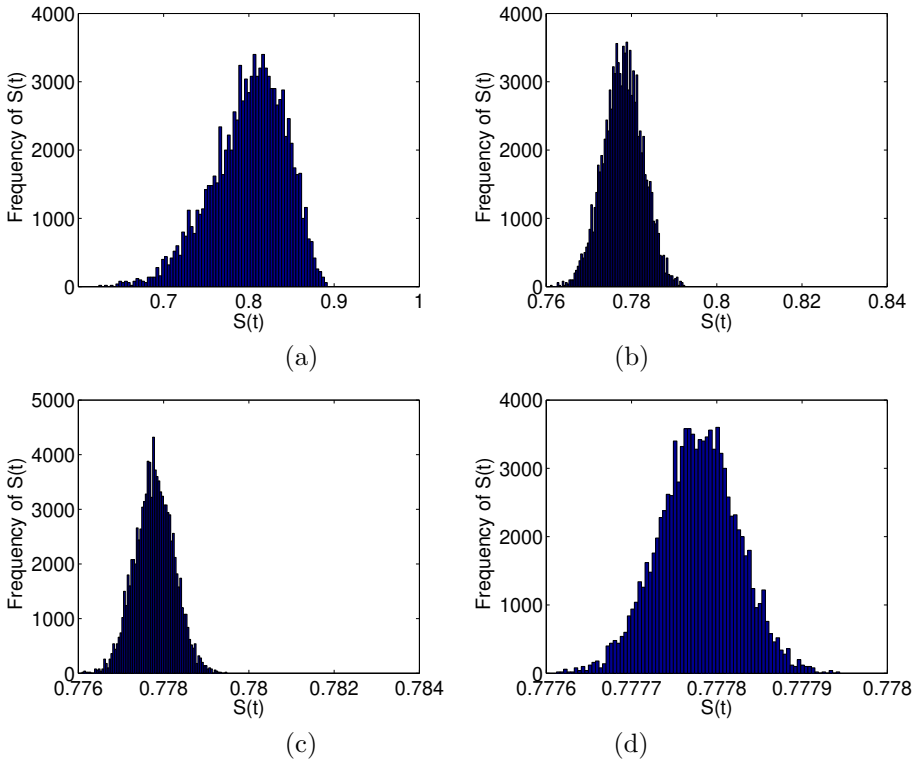

FIGURE 5. Frequency histograms of $S(t)$ at $t = 3000$ obtained from 100000 simulations with: (a) $\sigma = 0.1$, (b) $\sigma = 0.01$, (c) $\sigma = 0.001$, (d) $\sigma = 0.0001$.

Example 6.4. To verify the conditions mentioned in Theorem 6.3, we choose the parameter values as follows:

$$b = 0.1, \quad b_I = 0.1, \quad \beta = 0.15, \quad p = 0.1, \quad m = 0.1, \quad \gamma = 0.1.$$

Furthermore, to display the effect of the noise intensity on the stationary distribution, let σ change from 0.1, 0.01, 0.001 to 0.0001. We find $R_0 = 1.2273$, $\min(bS^{*2}, [b(1-m)+\gamma]I^{*2}) = 0.0015$. For convenience, $\Psi(\sigma) = \frac{1}{2}\sigma^2 I^* \frac{[b(1-m)+\gamma+b]}{\beta}$,

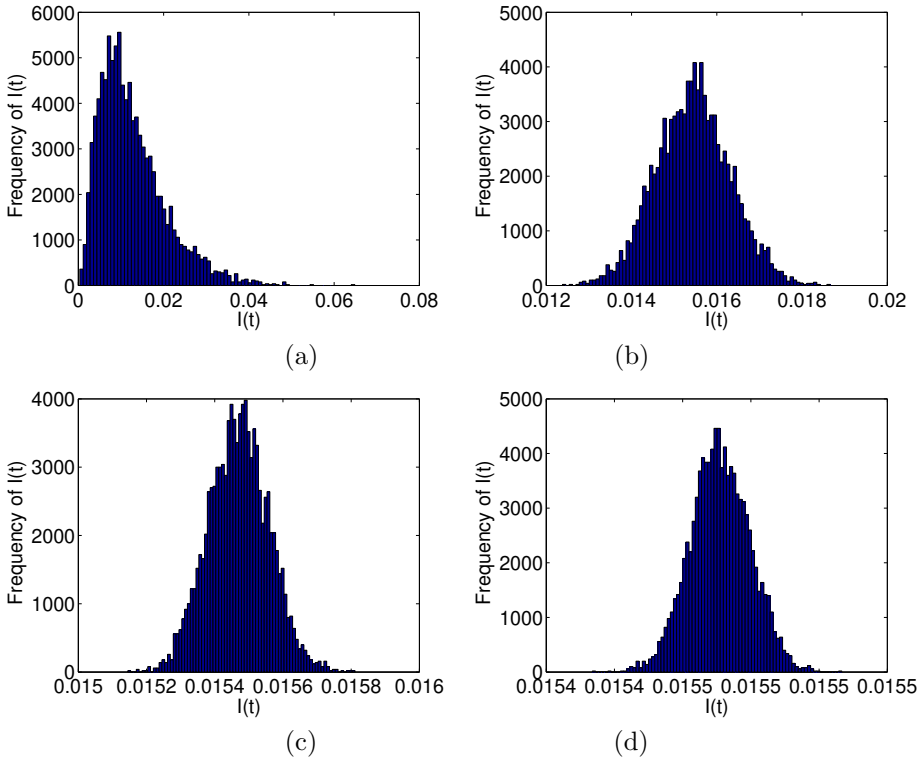

FIGURE 6. Frequency histograms of $I(t)$ at $t = 3000$ obtained from 100000 simulations with: (a) $\sigma = 0.1$, (b) $\sigma = 0.01$, (c) $\sigma = 0.001$, (d) $\sigma = 0.0001$.

we obtain $\Psi(0.1) = 8.4795 \times 10^{-4}$, $\Psi(0.01) = 8.4795 \times 10^{-6}$, $\Psi(0.001) = 8.4795 \times 10^{-8}$, $\Psi(0.0001) = 8.4795 \times 10^{-10}$. Hence the desired conditions for the existence of stationary distribution are satisfied. We have run the numerical simulation 100000 times and collected the values of $S(t)$ and $I(t)$ at $t = 3000$, and their distributions are exhibited in Fig.5 and Fig.6. The distributions presented at Fig.5 and Fig.6 do not change with time, hence they are stationary in nature. From Fig.5 and Fig.6, we can see that the profile of the stationary distribution becomes steeper with the noise intensity increasing.

7. Conclusions

In this paper, a stochastic SIR epidemic model with vertical infection is presented. When the noise is small, Theorem 3.2 shows that if $\tilde{R}_0 = R_0 - \frac{\sigma^2(1-m)^2}{2(pb_I+\gamma)} < 1$ the disease always dies out in the end, where $R_0 = \frac{\beta(1-m)}{pb_I+\gamma}$ is the basic reproduction number of the corresponding deterministic model (1.2)). However, when the noise is large, Theorem 3.3 shows that the disease decays even if $\tilde{R}_0 > 1$ (or $R_0 > 1$). These results imply that the noise can suppress the spread of the disease. On the other hand, when the noise is small, Theorem 4.2 shows that if $\tilde{R}_{0*} = R_0 - \frac{\sigma^2}{2(pb_I+\gamma)} > 1$ the disease is persistent in mean. In addition, we discuss asymptotic behavior of

the stochastic model (1.3) around the endemic equilibrium (S^*, I^*) of the deterministic model (1.2), Theorem 5.1 reveals that the solution of model (1.3) fluctuates around the endemic equilibrium (S^*, I^*) of model (1.2) and with the values of σ^2 decreasing, the difference between them also decreases. Moreover, we show that when Theorem 5.1 holds, there exists a unique stationary distribution for system (1.3) and the solution of system (1.3) is ergodic (see Theorem 6.3).

Finally we point out that some issues deserve further investigation. For instance, we have established sufficient conditions of the extinction and persistence in mean of the disease, as well as the existence of stationary distribution. However, obtaining necessary and sufficient conditions of these problems remain open. Another interesting continuation of this work might be to introduce independent random perturbations into the model (1.3) as in [41] and have the model

$$\begin{aligned} dS(t) &= [b(1-m)(S(t)+R(t)) - \beta S(t)I(t) + pb_I I(t) - dS(t)]dt + \sigma_1 S(t)dB_1(t), \\ dI(t) &= [\beta S(t)I(t) + qb_I I(t) - d_I I(t) - \gamma I(t)]dt + \sigma_2 I(t)dB_2(t), \\ dR(t) &= [\gamma I(t) - dR(t) + mb(S(t)+R(t))]dt + \sigma_3 R(t)dB_3(t). \end{aligned}$$

We leave these topics for a future work.

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Xiao-Bing Zhang
College of Electrical and Information Engineering, Lanzhou University of Technology, Lanzhou, Gansu 730050, China.
Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China
Email address: 1037823915@qq.com

Su-Qin Chang
Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China
Email address: 11234678911@qq.com

Hai-Feng Huo (corresponding author)
College of Electrical and Information Engineering, Lanzhou University of Technology, Lanzhou, Gansu 730050, China.
Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China
Email address: hfhuo@lut.cn