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# EXISTENCE, REGULARITY AND POSITIVITY OF GROUND STATES FOR NONLOCAL NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We study ground states of a nonlinear Schrödinger equation driven by the infinitesimal generator of a rotationally invariant Lévy process. The equation includes many special cases such as classical Schrödinger equations, fractional Schrödinger equations and relativistic Schrödinger equations, etc. It is proved that the equation possesses ground states in a suitable space of functions, then the regularity of solutions to the equation is examined, in particular, any solution is Hölder continuous, and, if the process involves diffusion terms, any solution is twice differentiable further. Finally, we show that any ground state is either positive or negative.

## 1. Introduction

The well known nonlinear Schrödinger equation

$$-\Delta u + u = |u|^{p-2}u, \tag{1.1}$$

which is driven by the infinitesimal generator of a Brownian motion, has been studied by many authors. There are many references to equation (1.1), see for example [13, 11, 9].

Noting that the Brownian motion is a special rotationally invariant stable Lévy process (i.e., its index is 2), one would like to consider the equation

$$(-\Delta)^{\alpha/2}u + u = |u|^{p-2}u, \tag{1.2}$$

where $0 < \alpha \le 2$, since $-(-\Delta)^{\alpha/2}$ is the infinitesimal generator of a rotationally invariant *stable* Lévy process with index $\alpha$. Laskin obtained the fractional Schrödinger equation through the path integral approach [15, 16]. Many authors investigated Schrödinger equations involving fractional Laplacians.

Naturally, removing *stable*, we are interested in the (nonlocal) Schrödinger equation

$$-2Au + u = |u|^{p-2}u, \tag{1.3}$$

where $A$ is the infinitesimal generator of a rotationally invariant Lévy process. Zhang and Zhu [22] and Zhang and Zhou [23] explored this equation for the rotationally invariant Lévy process with a non-degenerate diffusion term and a finite Lévy measure.

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Equation (1.3) also stems from looking for the standing wave $\psi(t,x)=e^{\mathrm{i}t}u(x)$ of the equation

$$\mathrm{i}\frac{\partial\psi}{\partial t}=-2A\psi-|\psi|^{p-2}\psi.$$

After clarifying the required assumptions on the infinitesimal generator $A$, we provide some examples satisfying the assumptions. Let $\sigma_A$ be the symbol of $A$. We assume the following:

(H1) There are positive constants $s$, $c$ and $K$ such that

$$-\sigma_A(\xi)\geq c|\xi|^{2s}\quad\text{for all }\xi\in\mathbb{R}^N\text{ with }|\xi|\geq K.$$

(H2) $(1+|\xi|^{2s})/(1-2\sigma_A(\xi))$ is an $L^q$-Fourier multiplier for all $q\in[2,+\infty)$.

**Remark 1.1.** (1) Since $A$ is the infinitesimal generator of a rotationally invariant Lévy process, we have, by [3, p. 128, Exercise 2.4.23],

$$\sigma_A(\xi)=-\frac{a}{2}|\xi|^2+\int_{\mathbb{R}^N\setminus\{0\}}(\cos(\xi\cdot x)-1)\,\nu(\mathrm{d}x),\tag{1.4}$$

where $a\geq 0$ and $\nu$ is an $\mathbf{O}(N)$-invariant Lévy measure. Thus $s\leq 1$, $\sigma_A(\xi)\leq 0$ for all $\xi\in\mathbb{R}^N$, and

$$\begin{aligned}&\sup\big\{s:\text{there are constants }c\text{ and }K\text{ such that}\\&-\sigma_A(\xi)\geq c|\xi|^{2s}\text{ for all }\xi\in\mathbb{R}^N\text{ with }|\xi|\geq K\big\}\\&\geq 0.\end{aligned}$$

(2) By (1.4) and

$$\lim_{|\xi|\to\infty}|\xi|^{-2}\int_{\mathbb{R}^N\setminus\{0\}}(\cos(\xi\cdot x)-1)\,\nu(\mathrm{d}x)=0$$

(cf. [6, p. 17]), we have that $s=1$ if and only if $a>0$.

**Example 1.2.** The infinitesimal generators $-(-\Delta)^s/2$ of some rotationally invariant stable Lévy processes with index $2s$ fulfill (H1) and (H2), where $0<s\leq 1$.

We go a step further. Let $\phi:[0,+\infty)\to\mathbb{R}$ be a Borel measurable function such that $\phi(r)\geq\varepsilon>0$. Define the symbol in (1.4) by $a:=0$ and $\nu(\mathrm{d}x):=\phi(|x|)/|x|^{N+2s}\mathrm{d}x$, where $s\in(0,1)$. Then the symbol $\sigma_A$ fulfills (H1) and, by [4, Theorem 1], (H2). In particular, if $\phi(\cdot)\equiv 1$, the associated operator is $-(-\Delta)^s/2$ up to some constant coefficient.

**Example 1.3.** Assume (H1) and

(H2') There constants $B$ and $R$ such that $|\xi^\alpha\partial^\alpha\sigma_A(\xi)|\leq B|\sigma_A(\xi)|$ for $\alpha\in\{0,1\}^N$ and $|\xi|>R$.

Then from [18, p. 117, Theorem 2.8.2] it follows that $(1+|\xi|^{2s})/(1-2\sigma_A(\xi))$ is an $L^q$-Fourier multiplier for all $q\in[2,+\infty)$. Also refer to [19, p. 54, Theorem 1.5.4] or [7, p. 87, Lemma 4.1].

**Example 1.4.** (Relativistic Schrödinger operators [3, pp. 166–167, Example 3.3.9]) Fix $m,c>0$. The (minus) relativistic Schrödinger operator $A$ is defined through

$$A:=-\left(\sqrt{m^2c^4-c^2\Delta}-mc^2\right).$$

Then the symbol of $A$ satisfies (H1) and (H2) with $s=1/2$ by Example 1.3.

More generally, the operator

$$A := -\left((m^2c^4 - c^2\Delta)^s - m^{2s}c^{4s}\right), \text{ where } 0 < s < 1,$$

fulfills (H1) and (H2) by Example 1.3.

**Remark 1.5.** Equation (1.3) covers equations (1.1) and (1.2). Equation (1.3) also covers relativistic Schrödinger equations (cf. [8, 10]) such as

$$\left(\sqrt{m^2c^4 - c^2\Delta} - mc^2\right)u + u = |u|^{p-2}u,$$

and more generally (see, for example, [2])

$$\left((m^2c^4 - c^2\Delta)^s - m^{2s}c^{4s}\right)u + u = |u|^{p-2}u, \text{ where } 0 < s < 1.$$

In what follows, we assume that (H1) and (H2) hold and $2 < p < 2_s^*$ with $2_s^* := +\infty$ if $N \le 2s$, and $2_s^* := 2N/(N-2s)$ if $N > 2s$.

To see that (1.3) has a variational structure, we introduce the Hilbert space $H_A^1(\mathbb{R}^N) := \{u : u \in \mathcal{S}'(\mathbb{R}^N) \text{ and } (1 - 2\sigma_A(\cdot))^{1/2}\widehat{u}(\cdot) \in L^2(\mathbb{R}^N)\}$ with the inner product

$$(u, v) := (2\pi)^{-N}\left((1 - 2\sigma_A(\cdot))^{1/2}\widehat{u}(\cdot), (1 - 2\sigma_A(\cdot))^{1/2}\widehat{v}(\cdot)\right)_{L^2},$$

and the induced norm denoted by $\|\cdot\|$.

Define a functional $E : H_A^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$E(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x. \tag{1.5}$$

It follows from Lemma 2.2 and [21, p.11, Corollary 1.13] that $E \in C^2(H_A^1(\mathbb{R}^N), \mathbb{R})$.

Equation (1.3) has a variational structure: $u \in H_A^1(\mathbb{R}^N)$ solves equation (1.3) if and only if $u$ is a critical point of the functional $E$.

Our main results are summarized in the following theorem.

**Theorem 1.6.**
- (i) *(existence) There is a nonzero function $v \in H_A^1(\mathbb{R}^N)$ such that*

$$-2Av + v = |v|^{p-2}v$$

*in the distribution sense. Moreover, $E(v) > 0$ and*

$$E(v) = \inf\{E(u) : u \in H_A^1(\mathbb{R}^N) \setminus \{0\} \text{ and } \|u\|^2 = \|u\|_{L^p}^p\}. \tag{1.6}$$

- (ii) *(regularity) Any weak solution $u$ to equation (1.3) in $H_A^1(\mathbb{R}^N)$ belongs to $H^{2s,q}(\mathbb{R}^N)$ for all $q \ge \max\{2, 2_s^*/(p-1)\}$. Moreover, if $s' \le s$ and $0 \le \mu \le 2s' - N/q < 1$, then $u \in C^{0,\mu}(\mathbb{R}^N)$ and, if $s = 1$, $u \in C_{loc}^{2,\mu}(\mathbb{R}^N)$. Consequently, $u(x) \to 0$ as $|x| \to \infty$.*
- (iii) *(positivity) Any ground state of equation (1.3) is either positive or negative. (A solution to equation (1.3) such that (1.6) holds is called by definition a* ground state*, see [21, p. 71]).*

The rest of this article is organized as follows. In Section 2, we prove that equation (1.3) has ground states. In Section 3, we establish the regularity of solutions to equation (1.3). In Section 4, we show that any ground state of equation (1.3) is either positive or negative.

## 2. Existence

In this section, we prove that equation (1.3) possesses a ground state. To this end, we first introduce the definition of the Banach space $H_A^{s,q}(\mathbb{R}^N)$, and then we present some embedding results and a concentration compactness principle. After these preparations, we show that functional (1.5) has a nontrivial critical point in Theorem 2.4, and this critical point is a ground state in Theorem 2.5, respectively.

**Definition 2.1** ([14, Chapter 3])**.** For $s \in \mathbb{R}$ and $q \in (1, +\infty)$, we define $H_A^{s,q}(\mathbb{R}^N)$ to be the set of all tempered distributions $u$ for which $\mathscr{F}^{-1}((1-2\sigma_A(\cdot))^{s/2}\mathscr{F}(u)(\cdot))$ is a function in $L^q(\mathbb{R}^N)$, i.e.,

$$H_A^{s,q}(\mathbb{R}^N) := \{u : u \in \mathcal{S}'(\mathbb{R}^N), \quad \mathscr{F}^{-1}((1-2\sigma_A(\cdot))^{\frac{s}{2}}\mathscr{F}(u)(\cdot)) \in L^q(\mathbb{R}^N)\}.$$

If $q = 2$, we define $H_A^s(\mathbb{R}^N)$, as usual, to be $H_A^{s,2}(\mathbb{R}^N)$.

**Lemma 2.2.** (i) *The following embeddings are continuous:*

$$H_A^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad N \le 2s \text{ and } q \ge 2,$$
$$H_A^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad N > 2s \text{ and } 2 \le q \le 2_s^*,$$
$$H_A^{2,q}(\mathbb{R}^N) \hookrightarrow H^{2s,q}(\mathbb{R}^N), \quad q > 1.$$

(ii) *Let $\Omega$ be a bounded domain of $\mathbb{R}^N$. If $2 \le q < 2_s^*$, then every bounded sequence in $H_A^1(\mathbb{R}^N)$ has a convergent subsequence in $L^q(\Omega)$.*

*Proof.* (i) It follows from (H1) that the embedding $H_A^1(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$ is continuous. Then, thanks to [1, p. 221, Theorem 7.63], we obtain the first and the second continuous embeddings in (i). The last embedding follows from (H2) and [14, p. 289, Theorem 3.3.28].

(ii) The conclusion is a consequence of (i) and [12, Lemma 2.1]. □

**Lemma 2.3** (concentration compactness principle)**.** *Let $r > 0$ and $2 \le q < 2_s^*$. If $\{u_n\}_{n=1}^\infty$ is bounded in $H_A^1(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |u_n(x)|^q \,dx = 0,$$

*then $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2_s^*$.*

*Proof.* It follows from (H1) that the embedding $H_A^1(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$ is continuous; consequently, $\{u_n\}_{n=1}^\infty$ is bounded in $H^s(\mathbb{R}^N)$. The remains of the proof are similar to that of [21, p. 16, Lemma 1.21]. Also refer to [12, Lemma 2.2]. □

**Theorem 2.4.** *The functional $E$ defined by (1.5) has a nontrivial critical point.*

*Proof.* Step 1. Let

$$\Gamma := \{\gamma : \gamma \in C([0,1], H_A^1(\mathbb{R}^N)) \text{ such that } \gamma(0) = 0 \text{ and } E(\gamma(1)) < 0\}. \tag{2.1}$$

Since $p > 2$, for $T$ large enough we have

$$E(T\exp(-|\cdot|^2)) = \frac{T^2}{2}\|\exp(-|\cdot|^2)\|^2 - \frac{T^p}{p}\int_{\mathbb{R}^N} \exp(-p|x|^2)\mathrm{d}x < 0.$$

Thus $\Gamma \ne \emptyset$.

Step 2. Define

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} E(\gamma(t)). \tag{2.2}$$

By Lemma 2.2, there is a positive constant $C$ such that

$$\|u\|_{L^p} \le C\|u\| \quad \text{for all } u \in H^1_A(\mathbb{R}^N). \tag{2.3}$$

Then it follows from the definition of the functional $E$ that

$$E(u) \ge \frac{1}{2}\|u\|^2 - \frac{C^p}{p}\|u\|^p.$$

Setting $r := (p/(4C^p))^{1/(p-2)}$, we have

$$\min_{\|u\|\le r} E(u) = 0 \quad \text{and} \quad \min_{\|u\|=r} E(u) \ge \frac{1}{4}\Big(\frac{p}{4C^p}\Big)^{\frac{2}{p-2}} > 0. \tag{2.4}$$

It follows from the above fact that $c \ge (p/(4C^p))^{2/(p-2)}/4 > 0$ (see Figure 1). Therefore, by [21, p. 41, Theorem 2.9], there exists a sequence $\{u_n\}_{n=1}^\infty \subset H^1_A(\mathbb{R}^N)$ satisfying

$$E(u_n) \to c \quad \text{and} \quad E'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.5}$$

(See Step 5 of the proof of Theorem 2.5).

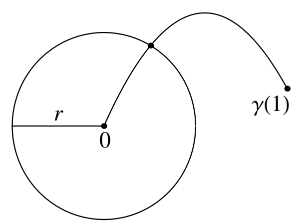


FIGURE 1. $c > 0$ and $\tau > 0$.

Step 3. By (2.5), for $n$ large enough, we have

$$c + 1 + \|u_n\| \ge E(u_n) - \frac{1}{p}\langle E'(u_n), u_n\rangle = \big(\frac{1}{2} - \frac{1}{p}\big)\|u_n\|^2.$$

It follows that $\{u_n\}_{n=1}^\infty$ is bounded in $H^1_A(\mathbb{R}^N)$. Thus $\{u_n\}_{n=1}^\infty$ possesses a subsequence, again denoted by $\{u_n\}_{n=1}^\infty$, such that

$$u_n \rightharpoonup u \quad \text{in } H^1_A(\mathbb{R}^N) \tag{2.6}$$

for some $u \in H^1_A(\mathbb{R}^N)$, and, by Lemma 2.2,

$$u_n \to u \quad \text{in } L^p_{\rm loc}(\mathbb{R}^N). \tag{2.7}$$

Therefore, by (2.5)–(2.7), we have

$$E'(u)\varphi = \lim_{n\to\infty} E'(u_n)\varphi = 0 \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^N),$$

i.e., $u$ is a critical point of $E$.

Step 4. We prove

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n(x)|^2 \mathrm{d}x > 0 \tag{2.8}$$

by contradiction. If (2.8) fails, it follows from Lemma 2.3 that

$$u_n \to 0 \quad \text{in } L^p(\mathbb{R}^N).$$

For $n$ large enough, by (2.5), we have

$$\frac{1}{2}c \leq E(u_n) - \frac{1}{2}E'(u_n)u_n = \big(\frac{1}{2} - \frac{1}{p}\big)\int_{\mathbb{R}^N} |u_n(x)|^p \mathrm{d}x.$$

Thus, $c \leq 0$, which is contradictory to $c > 0$ (see Step 2).

Step 5. By (2.8), there is subsequence of $\{u_n\}_{n=1}^{\infty}$, again denoted by $\{u_n\}_{n=1}^{\infty}$, such that

$$\int_{B(y_n,1)} u_n(x)^2 \mathrm{d}x > \varepsilon$$

for some positive number $\varepsilon$ and sequence $\{y_n\}_{n=1}^{\infty}$ with $y_n \in \mathbb{R}^N$. Define $v_n(\cdot) := u_n(\cdot + y_n)$, $n = 1, 2, \dots$. Then

$$\int_{B(0,1)} v_n(x)^2 \mathrm{d}x > \varepsilon, \tag{2.9}$$

and

$$E(v_n) \to c \quad \text{and} \quad E'(v_n) \to 0 \quad \text{as } n \to \infty. \tag{2.10}$$

By repeating Step 3, $\{v_n\}_{n=1}^{\infty}$ possesses a subsequence, again denoted by $\{v_n\}_{n=1}^{\infty}$, such that

$$v_n \rightharpoonup v \quad \text{in } H_A^1(\mathbb{R}^N) \tag{2.11}$$

for some $v \in H_A^1(\mathbb{R}^N)$,

$$v_n \to v \quad \text{in } L_{\rm loc}^p(\mathbb{R}^N), \tag{2.12}$$

and $v$ is a critical point of $E$. Moreover, by (2.9) and (2.12), $v$ is nontrivial. $\square$

In the next theorem we prove that the function $v$ in (2.12) is a ground state of equation (1.3).

**Theorem 2.5.** *Define the* Nehari manifold $\mathcal{N}$ *through*

$$\mathcal{N} := \{u : u \in H_A^1(\mathbb{R}^N) \setminus \{0\} \text{ and } E'(u)u = 0\}.$$

*Then the number $c$ defined in* (2.2) *satisfies $c = \inf_{u\in\mathcal{N}} E(u)$. Moreover, the function $v$ in* (2.12) *is a critical point of the critical value $c$.*

*Proof.* Step 1. For $u \in \mathcal{N}$, we have $E'(u)u = 0$, i.e.,

$$\|u\|^2 - \int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x = 0. \tag{2.13}$$

Noting that $u \neq 0$ as $u \in \mathcal{N}$, we obtain

$$\int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x > 0.$$

Thus, for $n \in \mathbb{N}$ large enough, as $p > 2$, we have

$$E(nu) = \frac{n^2}{2}\|u\|^2 - \frac{n^p}{p}\int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x < 0. \tag{2.14}$$

Define the path $\widetilde{\gamma}(t) := tnu$, where $t \in [0, 1]$. Then, thanks to (2.14), $\widetilde{\gamma} \in \Gamma$ (for the definition of $\Gamma$, see (2.1)). Consequently, we obtain

$$c \leq \sup_{t\in[0,1]} E(\widetilde{\gamma}(t)). \tag{2.15}$$

Step 2. From (2.4) it follows that

$$\min_{\|u\|=r} E(u) \geq \frac{1}{4}\big(\frac{p}{4C^p}\big)^{\frac{2}{p-2}} > 0. \tag{2.16}$$

Take $n > 1$ large enough such that $n\|u\| > r$. Then, by (2.14) and (2.16),
$$E(\widetilde{\gamma}(\cdot)) : [0,1] \to \mathbb{R} \text{ reaches its maximum at a point } t \in (0,1). \tag{2.17}$$

Step 3. Note that
$$\frac{\mathrm{d}}{\mathrm{d}t}E(\widetilde{\gamma}(t)) = tn^2\|u\|^2 - t^{p-1}n^p \int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x.$$
This, (2.13) and (2.17) yield that the function $E(\widetilde{\gamma}(\cdot)) : [0,1] \to \mathbb{R}$ reaches its maximum at the point $t = n^{-1}$. Thus, by noting (2.15), we obtain
$$c \le E(\widetilde{\gamma}(n^{-1})) = E(u) \text{ for any } u \in \mathcal{N}. \tag{2.18}$$

Step 4. Let $\gamma \in \Gamma$. Then $E(\gamma(1)) < 0$, i.e.,
$$\frac{1}{2}\|\gamma(1)\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} |\gamma(1)(x)|^p \mathrm{d}x < 0.$$
As $p > 2$, we obtain
$$\|\gamma(1)\|^2 - \int_{\mathbb{R}^N} |\gamma(1)(x)|^p \mathrm{d}x < 0. \tag{2.19}$$

Step 5. Set $\tau := \sup\{t : E'(\gamma(t))\gamma(t) \ge 0,\ t \in [0,1]\}$. Note (2.3) and
$$E'(u)u = \|u\|^2 - \int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x.$$
By taking $r := (1/(4C^p))^{1/(p-2)}$, we have
$$\min_{\|u\| \le r} E(u) = 0 \quad \text{and} \quad \min_{\|u\| = r} E'(u)u \ge \frac{3}{4}\Big(\frac{1}{4C^p}\Big)^{\frac{2}{p-2}}.$$
The above fact, $\gamma \in \Gamma$ and $E(\gamma(\cdot)) \in C([0,1],\mathbb{R})$ yield $\tau > 0$ (cf. Figure 1). Furthermore, $E'(\gamma(\cdot))\gamma(\cdot) \in C([0,1],\mathbb{R})$, $E'(\gamma(\tau))\gamma(\tau) \ge 0$ and (2.19) imply there is a point $t_0 \in [\tau, 1)$ such that $E'(\gamma(t_0))\gamma(t_0) = 0$.

We prove $\gamma(t_0) \ne 0$ by contradiction. If $\gamma(t_0) = 0$, then, by the same argument as above, there a number $\tau'$ such that $\tau < \tau' < 1$ and $E'(\gamma(\tau'))\gamma(\tau') \ge 0$ which is contradictory to the definition of $\tau$. In summary, $\gamma(t_0) \in \mathcal{N}$, i.e., $\gamma([0,1]) \cap \mathcal{N} \ne \emptyset$.

Step 6. It follows from $\gamma([0,1]) \cap \mathcal{N} \ne \emptyset$ that $c \ge \inf_{\mathcal{N}} E(u)$. This and (2.18) show us that $c = \inf_{u \in \mathcal{N}} E(u)$.

Step 7. We have proved in Theorem 2.4 that $v$ is a nontrivial critical point of $E$. Particularly, it follows that $v \in \mathcal{N}$. In this step, we prove $E(v) = c$. First we have $E(v) \ge c$ since $v \in \mathcal{N}$ and $c = \inf_{u \in \mathcal{N}} E(u)$. In the following, we show that $E(v) \le c$. Note that $p > 2$ and
$$E(v_n) - \frac{1}{2}E'(v_n)v_n = \big(\frac{1}{2} - \frac{1}{p}\big)\int_{\mathbb{R}^N} |v_n(x)|^p \mathrm{d}x.$$
Then, for any positive number $R$, we have
$$E(v_n) - \frac{1}{2}E'(v_n)v_n \ge \big(\frac{1}{2} - \frac{1}{p}\big)\int_{B(0,R)} |v_n(x)|^p \mathrm{d}x.$$
Thanks to (2.10)–(2.12), taking limits in the above inequality, we obtain
$$c \ge \big(\frac{1}{2} - \frac{1}{p}\big)\int_{B(0,R)} |v(x)|^p \mathrm{d}x,$$
i.e., as $R$ is arbitrary,
$$c \ge \big(\frac{1}{2} - \frac{1}{p}\big)\int_{\mathbb{R}^N} |v(x)|^p \mathrm{d}x.$$

Therefore,

$$\begin{aligned} c &\geq \Big(\frac{1}{2}-\frac{1}{p}\Big)\int_{\mathbb{R}^N}|v(x)|^p\mathrm{d}x+\frac{1}{2}\|v\|^2-\frac{1}{2}\|v\|^2 \\ &= E(v)-\frac{1}{2}E'(v)v=E(v), \end{aligned}$$

where we have used that $v$ is a critical point of $E$ in the last identity. □

## 3. Regularity

In this section, we investigate the regularity of weak solutions to equation (1.3).

**Theorem 3.1.** *If $u$ is a weak solution to the equation*

$$-2Au+u=|u|^{p-2}u$$

*in $H_A^1(\mathbb{R}^N)$, then $u\in H^{2s,q}(\mathbb{R}^N)$ for any $q\in[\max\{2,2_s^*/(p-1)\},+\infty)$. Moreover, if $s'\leq s$ and $0\leq\mu\leq 2s'-N/q<1$, then $u\in C^{0,\mu}(\mathbb{R}^N)$ and, if $s=1$, $u\in C_{loc}^{2,\mu}(\mathbb{R}^N)$.*

*Proof.* The proof is similar to that of [23, Theorem 2].

Step 1. If $N\leq 2s$, then, by (i) of Lemma 2.2, $u\in L^q(\mathbb{R}^N)$ for any $q\in[2,+\infty)$. Consequently, by Definition 2.1, we have $u\in H_A^{2,q}(\mathbb{R}^N)$. Furthermore, it follows from (i) of Lemma 2.2 that $u\in H^{2s,q}(\mathbb{R}^N)$ for any $q\in[2,+\infty)$.

Step 2. Assume that $N>2s$. We conclude the result for this case by following the *bootstrapping procedure* (cf. [5, pp. 50–51]). Recall $2_s^*=2N/(N-2s)$. By Lemma 2.2, we have $u\in L^{2_s^*}(\mathbb{R}^N)$, and then $|u|^{p-1}\in L^{q_1}(\mathbb{R}^N)$, where $q_1:=2_s^*/(p-1)$. Furthermore, by Definition 2.1, we obtain $u\in H_A^{2,q_1}(\mathbb{R}^N)$. This and (i) of Lemma 2.2 show us that $u\in H^{2s,q_1}(\mathbb{R}^N)$.

Step 3. If $N\leq 2sq_1$, it follows from $u\in H^{2s,q_1}(\mathbb{R}^N)$ and [1, p. 221, Theorem 7.63] that $u\in L^q(\mathbb{R}^N)$ for any $q\in[2_s^*/(p-1),+\infty)$. Then by an argument similar to Step 1, we have $u\in H^{2s,q}(\mathbb{R}^N)$ for any $q\in[2_s^*/(p-1),+\infty)$. If $N>2sq_1$, it follows from $u\in H^{2s,q_1}(\mathbb{R}^N)$ and [1, p. 221, Theorem 7.63] that $u\in L^{q_1N/(N-2sq_1)}(\mathbb{R}^N)$, and then $|u|^{p-1}\in L^{q_2}(\mathbb{R}^N)$, where $q_2:=q_1N/((N-2sq_1)(p-1))$. By the same reason as Step 2, we find $u\in H^{2s,q_2}(\mathbb{R}^N)$.

For $n=3,4,\dots$, we define $q_n$ by induction as follows, $q_n:=q_{n-1}N/((N-2sq_{n-1})(p-1))$ until $\max\{n:N>2sq_{n-1}\}$. Then we have $u\in H^{2s,q_n}(\mathbb{R}^N)$ for $n=1,2\dots$.

Step 4. We prove $q_n/q_{n-1}>1+\varepsilon$, for some positive number $\varepsilon$ independent of $n$ and $n=1,2\dots$, by induction. Note that $p<2_s^*$. There is a positive number $\varepsilon$ such that $q_1=2N(1+\varepsilon)/(N+2s)$. Therefore, after some calculations, we have

$$\frac{q_2}{q_1}=\frac{(1+\varepsilon)(N-2s)}{N-2s-4\varepsilon}>1+\varepsilon.$$

Suppose that $q_n/q_{n-1}>1+\varepsilon$. Then

$$\frac{q_{n+1}}{q_n}=\frac{q_n}{q_{n-1}}\cdot\frac{N-2sq_{n-1}}{N-2sq_n}>\frac{q_n}{q_{n-1}}>1+\varepsilon.$$

Step 5. After finite steps, we must have $u\in H^{2s,q}(\mathbb{R}^N)$ for any $q\geq q_{n_0}$ and some $n_0\in\mathbb{N}$. Thus, by Hölder inequality, we have $u\in H^{2s,q}(\mathbb{R}^N)$ for all $q\geq 2_s^*/(p-1)$. Furthermore, it follows from [1, p. 221, Theorem 7.63] that $u\in H^{2s',q}(\mathbb{R}^N)$ for all $s'\leq s$, and so $u\in C^{0,\mu}(\mathbb{R}^N)$ for any $\mu$ such that $0\leq\mu\leq 2s'-N/q<1$. In addition, by Schauder estimate, we obtain $u\in C_{\rm loc}^{2,\mu}(\mathbb{R}^N)$ if $s=1$. □

**Corollary 3.2.** *If $u$ is a weak solution to the equation*

$$-2Au + u = |u|^{p-2}u$$

*in $H^1_A(\mathbb{R}^N)$, then $u(x) \to 0$ as $|x| \to \infty$.*

*Proof.* Step 1. If $u(x) \nrightarrow 0$ as $|x| \to \infty$, there is a positive number $\varepsilon$ and a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ such that $|x_n| > |x_{n-1}| + 2$ and $|u(x_n)| > 2\varepsilon$.

Step 2. By Theorem 3.1, the function $u$ is uniformly continuous. Thus there is a positive number $\delta < 1$ such that $|u(x) - u(y)| < \varepsilon$ if $|x - y| < \delta$.

Step 3. By Steps 1 and 2, we have $|u(x)| > \varepsilon$ for $x \in B(x_n, \delta)$ and $n = 1, 2, \dots$. Consequently,

$$\int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x \geq \sum_{n=1}^{\infty} \int_{B(x_n,\delta)} |u(x)|^2 \mathrm{d}x = \infty,$$

which contradicts $u \in H^1_A(\mathbb{R}^N)$. □

**Corollary 3.3.** *Assume that $u \in H^1_A(\mathbb{R}^N)$ is a positive solution to the equation $-2Au + u = |u|^{p-2}u$. Let $x_0 \in \mathbb{R}^N$ is a maximizer of the function $u$. Then $u(x_0) \geq 1$.*

*Proof.* Following the proofs of [22, Corollary 2.4] and [23, Corollary 6], we have, by the positive maximum principle (see, for example, [20, p. 283, Proposition 1.5] or [3, p. 181, Theorem 3.5.2]), $Au(x_0) \leq 0$. Therefore,

$$u(x_0)^{p-1} - u(x_0) = -2Au(x_0) \geq 0.$$

So the inequality $u(x_0) \geq 1$ holds. □

## 4. Positivity

In this section, we examine the positivity of ground states of (1.3).

**Lemma 4.1.** *Let $f, g$ be real functions in $H^1_A(\mathbb{R}^N)$. Then*

$$\|\sqrt{f^2 + g^2}\|^2 \leq \|f\|^2 + \|g\|^2.$$

*Proof.* It follows from (1.4) and the definition of $H^1_A(\mathbb{R}^N)$ that for $\psi \in H^1_A(\mathbb{R}^N)$,

$$\|\psi\|^2 = \|\psi\|^2_{L^2} + a\|\nabla\psi\|^2_{L^2} + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus\{0\}} \big(\psi(x) - \psi(y)\big)^2 \nu(-x + \mathrm{d}y)\mathrm{d}x.$$

Then with the help of [17, p. 177, Theorem 7.8] and following the proof of [17, p. 185, Theorem 7.13], we complete this proof. □

**Theorem 4.2.** *If $w \in H^1_A(\mathbb{R}^N)$ is a ground state of the equation*

$$-2Au + u = |u|^{p-2}u,$$

*then $w > 0$ or $w < 0$.*

*Proof.* Step 1. We prove that $|w| \in \mathcal{N}$ and $E(|w|) = E(w)$. First we have $\||w|\| \leq \|w\|$ by Lemma 4.1. Recall the Nehari manifold $\mathcal{N}$,

$$\mathcal{N} = \{u : u \in H^1_A(\mathbb{R}^N) \setminus \{0\} \text{ and } \|u\|^2 = \|u\|^p_{L^p}\}.$$

For any $u \in H^1_A(\mathbb{R}^N) \setminus \{0\}$, we have

$$\widetilde{u} := \Big(\frac{\|u\|^2}{\|u\|^p_{L^p}}\Big)^{\frac{1}{p-2}} u \in \mathcal{N}.$$

Thus $E(\widetilde{u}) \geq E(w)$, i.e.,

$$\Big(\frac{\|u\|}{\|u\|_{L^p}}\Big)^{\frac{2p}{p-2}} \geq \|w\|^2.$$

In particular, taking $u := |w|$, we obtain $\||w|\| \geq \|w\|$ and then $\||w|\| = \|w\|$. Consequently, we obtain that $|w| \in \mathcal{N}$ and $E(|w|) = E(w)$.

Step 2. Define a functional $G$ from $H^1_A(\mathbb{R}^N)$ to $\mathbb{R}$ by $G(\psi) := \|\psi\|^2 - \|\psi\|^p_{L^p}$. Then $G'(|w|) : H^1_A(\mathbb{R}^N) \to \mathbb{R}$ is surjective. Thanks to the Lagrange multiplier rule, there exists a number $\lambda$ such that

$$(|w|, \psi) - (|w|^{p-1}, \psi)_{L^2} = \lambda(2(|w|, \psi) - p(|w|^{p-1}, \psi)_{L^2}) \quad \text{for any } \psi \in H^1_A(\mathbb{R}^N).$$

Choosing $\psi := |w|$, we find $\lambda = 0$ since $\||w|\|^2 = \||w|\|^p_{L^p}$ by Step 1.

Step 3. By Steps 1 and 2, $|w|$ is also a ground state of the equation

$$-2Au + u = |u|^{p-2}u.$$

Thus, without loss of generality, we assume $w \geq 0$.

Step 4. If $\nu = 0$, then the strong maximum principle shows us that $w > 0$.

Step 5. Assume that $\nu \neq 0$. We prove that $w > 0$ by contradiction. Let $x_0$ be a global minimizer of $w$ on $\mathbb{R}^N$ and $w(x_0) = 0$. Noting that

$$-2Aw = -a\Delta w + \int_{\mathbb{R}^N\setminus\{0\}} (2w(\cdot) - w(\cdot - y) - w(\cdot + y))\nu(\mathrm{d}y)$$

by (1.4), we find

$$0 > -2Aw(x_0) + w(x_0) = w(x_0)^{p-1} = 0,$$

which is a contradiction. $\square$

**Corollary 4.3.** *The minimization problem*

$$\textit{minimize } \frac{\|u\|}{\|u\|_{L^p}} \textit{ over } u \in H^1_A(\mathbb{R}^N) \setminus \{0\} \tag{4.1}$$

*has a solution which is a (positive) ground state of the equation* $-2Au+u = |u|^{p-2}u$.

*Proof.* By Theorem 4.2, let $w$ be a (positive) ground state of the equation $-2Au + u = |u|^{p-2}u$. Then it follows from Step 1 of the proof of Theorem 4.2 that

$$\frac{\|u\|}{\|u\|_{L^p}} \geq \|w\|^{\frac{p-2}{p}}.$$

The equality of the above inequality holds for $u = w$. $\square$

**Remark 4.4.** (1) Problem (4.1) can be solved by a solution $\varphi$ with $\varphi > 0$ and $\|\varphi\|_{L^p} = 1$, and is equivalent to

$$\inf_{u\in\mathcal{M}} \|u\|, \quad \text{where } \mathcal{M} := \{u : u \in H^1_A(\mathbb{R}^N) \text{ and } \|u\|_{L^p} = 1\}. \tag{4.2}$$

(2) Define $S := \inf_{u\in\mathcal{M}} \|u\|$. If $\varphi$ is a solution to (4.2), then $w := S^{\frac{2}{p-2}}\varphi$ is a ground state of the equation $-2Au + u = |u|^{p-2}u$. To see this, we note the facts
   - $w \in \mathcal{N}$;
   - $\inf_{u\in\mathcal{N}} E(u) = \inf_{u\in\mathcal{N}} \big(\frac{1}{2} - \frac{1}{p}\big)\|u\|^2$;
   - it follows from the Lagrange multiplier rule that $-2A\varphi+\varphi = S^2|\varphi|^{p-2}\varphi$.

(3) To find ground states of the equation $-2Au + u = |u|^{p-2}u$, one may solve (4.2), and vice versa.

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