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EXISTENCE OF GLOBAL SOLUTIONS TO CAUCHY PROBLEMS FOR BIPOLAR NAVIER-STOKES-POISSON SYSTEMS

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Abstract. In this article, we consider the Cauchy problem for one-dimensional compressible bipolar Navier-Stokes-Poisson system with density-dependent viscosities. Under certain assumptions on the initial data, we prove the existence and uniqueness of a global strong solution.

1. Introduction

Bipolar Navier-Stokes-Poisson (BNSP) has been used to simulate the transport of charged particles under the influence of electrostatic force governed by the self-consistent Poisson equation. In this paper, we consider the Cauchy problem for one-dimensional isentropic compressible BNSP with density-dependent viscosities,

$$\begin{gathered}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2)_x + p(\rho)_x = \rho\Phi_x + (\mu(\rho)u_x)_x, \\
n_t + (nv)_x = 0, \\
(nv)_t + (nv^2)_x + p(n)_x = -n\Phi_x + (\mu(n)v_x)_x, \\
\Phi_{xx} = \rho - n.
\end{gathered} \tag{1.1}$$

Here $\rho(x,t) \geq 0$, $n(x,t) \geq 0$ denote the charge densities, u, v the charge velocities, Φ the electrostatic potential, $p(\rho) = \rho^\gamma$ and $p(n) = n^\gamma$, $\gamma > 1$ are the pressure of charge, such as electron and ion, and $\mu(\rho)$, $\mu(n)$ are the viscosity coefficients.

There have been extensive studies on the existence and asymptotic behavior of global solutions to the unipolar Navier-Stokes-Poisson system (NSP). The existence of global weak solutions to NSP with general initial data was proved in [4, 12, 23]. The quasi-neutral and some related asymptotic limits were studied in [3, 5, 10]. When the Poisson equation describes the self-gravitonal force for stellar gases, the existence of global weak solution and asymptotic behavior were also investigated together with the stability analysis, we refer the reader to [7] and the references therein. The results in [6, 19, 21] imply that the electric field affects the large time behavior of the solution and give rise to different asymptotic behaviors of Navier-Stokes and NSP. In addition, Hao-Li [8] proved the well-posedness of NSP in the

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Besov space. Li-Matsumura-Zhang [15] proved the existence and the optimal time convergence rates of the global classical solution. Recently, Bie-Wang-Yao in [2] proved optimal decay rate in the critical L^p framework.

For bipolar Navier-Stokes-Poisson system (1.1), there are also abundant results concerning the existence and asymptotic behavior of the global weak solution. Li-Yang-Zou [14] proved optimal L^2 time convergence rate for the global classical solution for a small initial perturbation of the constant equilibrium state. The optimal time decay rate of global strong solution is established in [9, 24]. Liu-Lian in [17] proved global existence of solution to free boundary value problem. Lin-Hao-Li [16] studied the existence and uniqueness of global strong solutions in hybrid Besov spaces with the initial data close to an equilibrium state. Wu-Wang [22] proved pointwise estimates for BNSP system. As a continuation of the study in this direction, in this paper, we will study the Cauchy problem for BNSP in one-dimension.

The rest of this paper is as follows. In section 2, we state the main results of this article. In section 3, we give some a-priori estimates for the solution. In section 4, we prove the existence and uniqueness of global strong solutions.

2. Main result

In this article, we consider the existence and uniqueness of global solutions for the Cauchy problem (1.1) in the whole space $\mathbb{R}$. Assume $\mu(\rho) = \rho^\alpha$, $\mu(n) = n^\alpha$, then (1.1) can be rewritten as

$$\begin{gathered}
\rho_t + (\rho u)_x = 0,\\
(\rho u)_t + (\rho u^2)_x + (\rho^\gamma)_x = \rho\Phi_x + (\rho^\alpha u_x)_x,\\
n_t + (nv)_x = 0,\\
(nv)_t + (nv^2)_x + (n^\gamma)_x = -n\Phi_x + (n^\alpha v_x)_x,\\
\Phi_{xx} = \rho - n,\\
\Phi_x(\pm\infty, t) = 0,\\
(\rho, u, n, v)(x, 0) = (\rho_0, u_0, n_0, v_0)(x), x \in \mathbb{R},\\
(\rho_0, u_0, n_0, v_0)(\pm\infty) = (\overline{\rho}, 0, \overline{n}, 0).
\end{gathered} \tag{2.1}$$

We assume the initial data satisfy

$$\begin{gathered}
(\rho_0 - \overline{\rho}, u_0, n_0 - \overline{n}, v_0) \in H^1(\mathbb{R}),\\
0 < \rho_1 \le \rho_0(x) \le \rho_2, \quad 0 < n_1 \le n_0(x) \le n_2,\\
\Phi_{x0} = \int_{-\infty}^x (\rho_0 - n_0)(y)\,\mathrm{d}y \in L^2(\mathbb{R}),
\end{gathered} \tag{2.2}$$

where ρ_1,ρ_2,n_1,n_2,$\overline{\rho}$ and $\overline{n}$ are positive constants. We define

$$\begin{gathered}
E_{01} =: \frac{1}{2}\int_{\mathbb{R}} \rho_0 u_0^2\,\mathrm{d}x + \frac{1}{\gamma-1}\int_{\mathbb{R}} \Big((\rho_0^\gamma - \overline{\rho}^\gamma) - \gamma\overline{\rho}^{\gamma-1}(\rho_0 - \overline{\rho})\Big)\,\mathrm{d}x,\\
E_{02} =: \frac{1}{2}\int_{\mathbb{R}} n_0 v_0^2\,\mathrm{d}x + \frac{1}{\gamma-1}\int_{\mathbb{R}} \Big((n_0^\gamma - \overline{n}^\gamma) - \gamma\overline{n}^{\gamma-1}(n_0 - \overline{n})\Big)\,\mathrm{d}x,\\
E_{11} =: \frac{1}{2}\int_{\mathbb{R}} \rho_0\Big(u_0 + \frac{1}{\alpha}\rho_0^{-1}(\rho_0^\alpha)_x\Big)^2\,\mathrm{d}x + \frac{1}{\gamma-1}\int_{\mathbb{R}} \Big((\rho_0^\gamma - \overline{\rho}^\gamma) - \gamma\overline{\rho}^{\gamma-1}(\rho_0 - \overline{\rho})\Big)\,\mathrm{d}x,
\end{gathered}$$

$$E_{12} =: \frac{1}{2}\int_{\mathbb{R}} n_0\Big(v_0 + \frac{1}{\alpha}n_0^{-1}(n_0^{\alpha})_x\Big)^2\,\mathrm{d}x + \frac{1}{\gamma-1}\int_{\mathbb{R}}\Big((n_0^{\gamma}-\overline{n}^{\gamma}) - \gamma\overline{n}^{\gamma-1}(n_0-\overline{n})\Big)\,\mathrm{d}x,$$

$$E_0 =: \frac{1}{2}\int_{\mathbb{R}}\Phi_{x0}^2\,\mathrm{d}x + E_{01} + E_{02}, \quad E_1 =: \frac{1}{2}\int_{\mathbb{R}}\Phi_{x0}^2\,\mathrm{d}x + E_{11} + E_{12}.$$

Then, the main result of this paper can be stated as follows.

Theorem 2.1. *Let $\gamma > 1$, $\alpha > 0$ and $\alpha \neq 1/2$. Assume that the initial data satisfies (2.2) for $0 < \alpha < 1/2$, and (2.2) with $E_0^{1/2}(E_0+E_1)^{1/2} < \frac{1}{2\alpha-1}\bar{\rho}^{\frac{\gamma}{2}+\alpha-\frac{1}{2}}$ for $\alpha > 1/2$. Then there exist positive constants $\rho_{\pm}$ and $n_{\pm}$ with $\rho_- < \bar{\rho} < \rho_+$, $n_- < \bar{n} < n_+$ so that the unique global strong solution (ρ, n, u, v, Φ_x) to (2.1) exists and satisfies*

$$\begin{gathered} 0 < \rho_- \le \rho \le \rho^+,\ 0 < n_- \le n \le n^+, \\ u, v \in L^{\infty}([0,T]; H^1(\mathbb{R})) \cap L^2([0,T]; H^2(\mathbb{R})), \\ \rho_x, u_x, n_x, v_x \in L^{\infty}([0,T]; L^2(\mathbb{R})) \cap L^2([0,T]; L^2(\mathbb{R})), \\ \rho_t, u_t, n_t, v_t \in L^2([0,T]; H^1(\mathbb{R})), \Phi_x \in L^{\infty}([0,T]; H^2(\mathbb{R})). \end{gathered} \tag{2.3}$$

3. A-priori estimates

The proof of Theorem 2.1 consists of the basic a-priori estimates and regular analysis. Using arguments similar to those in [13], we establish the following lemmas.

Lemma 3.1. *Let $T > 0$, and (ρ, n, u, v, Φ_x) with $\rho > 0$, $n > 0$ be a solution to (2.1) for $t \in [0,T]$ under the conditions in Theorem 2.1. Then*

$$\begin{aligned} &\int_{\mathbb{R}}\frac{1}{2}\Big(\rho u^2 + nv^2 + \Phi_x^2\Big)\,\mathrm{d}x + \frac{1}{\gamma-1}\int_{\mathbb{R}}\Big((\rho^{\gamma}-\overline{\rho}^{\gamma}) - \gamma\overline{\rho}^{\gamma-1}(\rho-\overline{\rho})\Big)\,\mathrm{d}x \\ &+ \frac{1}{\gamma-1}\int_{\mathbb{R}}\Big((n^{\gamma}-\overline{n}^{\gamma}) - \gamma\overline{n}^{\gamma-1}(n-\overline{n})\Big)\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}}\Big(\rho^{\alpha}u_x^2 + n^{\alpha}v_x^2\Big)\,\mathrm{d}x\,\mathrm{d}s \\ &= E_0. \end{aligned} \tag{3.1}$$

Proof. Taking the product of $(2.1)_2$ with u and integrating on $\mathbb{R}$, and using $(2.1)_1$ and integrating by parts, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\rho u^2\,\mathrm{d}x + \int_{\mathbb{R}}(\rho^{\gamma})_x u\,\mathrm{d}x + \int_{\mathbb{R}}\rho^{\alpha}u_x^2\,\mathrm{d}x = \int_{\mathbb{R}}\rho_t\Phi\,\mathrm{d}x, \tag{3.2}$$

where

$$\int_{\mathbb{R}}(\rho^{\gamma})_x u\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\Big(\frac{1}{\gamma-1}(\rho^{\gamma}-\overline{\rho}^{\gamma}) - \frac{\gamma}{\gamma-1}\overline{\rho}^{\gamma-1}(\rho-\overline{\rho})\Big)\,\mathrm{d}x, \tag{3.3}$$

integrating with respect to $t \in [0,T]$, we have

$$\begin{aligned} &\int_{\mathbb{R}}\Big(\frac{1}{2}\rho u^2 + \frac{1}{\gamma-1}(\rho^{\gamma}-\overline{\rho}^{\gamma}) - \frac{\gamma}{\gamma-1}\overline{\rho}^{\gamma-1}(\rho-\overline{\rho})\Big)\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}}\rho^{\alpha}u_x^2\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^t\int_{\mathbb{R}}\rho_s\Phi\,\mathrm{d}x\,\mathrm{d}s + E_{01}. \end{aligned} \tag{3.4}$$

Meanwhile, we have

$$\begin{aligned}&\int_{\mathbb{R}}\Big(\frac{1}{2}nv^2+\frac{1}{\gamma-1}(n^\gamma-\overline{n}^\gamma)-\frac{\gamma}{\gamma-1}\overline{n}^{\gamma-1}(n-\overline{n})\Big)\,\mathrm{d}x+\int_0^t\int_{\mathbb{R}}n^\alpha v_x^2\,\mathrm{d}x\,\mathrm{d}s\\&=-\int_0^t\int_{\mathbb{R}}n_s\Phi\,\mathrm{d}x\,\mathrm{d}s+E_{02}.\end{aligned}\tag{3.5}$$

Adding (3.4) to (3.5), we obtain

$$\int_{\mathbb{R}}(\rho_t-n_t)\Phi\,\mathrm{d}x=\int_{\mathbb{R}}\Phi_{xxt}\Phi\,\mathrm{d}x=-\int_{\mathbb{R}}\Phi_x\Phi_{xt}\,\mathrm{d}x=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\Phi_x^2\,\mathrm{d}x.\tag{3.6}$$

The combination of (3.4), (3.5) and (3.6) gives rise to (3.1). □

Lemma 3.2. *Under the assumptions in Lemma 3.1, it holds*

$$\begin{aligned}&\int_{\mathbb{R}}\Big(\frac{1}{2}\rho(u+\frac{1}{\alpha}\rho^{-1}(\rho^\alpha)_x)^2+\frac{1}{\gamma-1}(\rho^\gamma-\overline{\rho}^\gamma)-\frac{\gamma}{\gamma-1}\overline{\rho}^{\gamma-1}(\rho-\overline{\rho})\Big)\,\mathrm{d}x\\&+\int_{\mathbb{R}}\Big(\frac{1}{2}n(v+\frac{1}{\alpha}n^{-1}(n^\alpha)_x)^2+\frac{1}{\gamma-1}(n^\gamma-\overline{n}^\gamma)-\frac{\gamma}{\gamma-1}\overline{n}^{\gamma-1}(n-\overline{n})\Big)\,\mathrm{d}x\\&+\frac{1}{2}\int_{\mathbb{R}}\Phi_x^2\,\mathrm{d}x+\gamma\int_0^t\int_{\mathbb{R}}\Big(\rho^{\gamma+\alpha-3}\rho_x^2+n^{\gamma+\alpha-3}n_x^2\Big)\,\mathrm{d}x\,\mathrm{d}s\\&+\int_0^t\int_{\mathbb{R}}(\rho^\alpha-n^\alpha)(\rho-n)\,\mathrm{d}x\,\mathrm{d}s=E_1.\end{aligned}\tag{3.7}$$

Proof. Multiplying $(2.1)_1$ by $\rho^{\alpha-1}$, and then differentiting with respect to x, then using $(2.1)_2$ and direct computations, we obtain

$$\rho(u+\frac{1}{\alpha}\rho^{-1}(\rho^\alpha)_x)_t+\rho\,u(u+\frac{1}{\alpha}\rho^{-1}(\rho^\alpha)_x)_x+(\rho^\gamma)_x=\rho\Phi_x.\tag{3.8}$$

Then, multiplying (3.8) by $u+\frac{1}{\alpha}\rho^{-1}(\rho^\alpha)_x$, and integrating over $\mathbb{R}$ (by parts), using $(2.1)_1$ and the boundary conditions, after direct computations, we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\Big(\frac{1}{2}\rho(u+\frac{1}{\alpha}\rho^{-1}(\rho^\alpha)_x)^2+\frac{1}{\gamma-1}(\rho^\gamma-\overline{\rho}^\gamma)-\frac{\gamma}{\gamma-1}\overline{\rho}^{\gamma-1}(\rho-\overline{\rho})\Big)\,\mathrm{d}x\\&+\gamma\int_0^t\int_{\mathbb{R}}\rho^{\gamma+\alpha-3}\rho_x^2\,\mathrm{d}x\,\mathrm{d}s\\&=\int_0^t\int_{\mathbb{R}}\rho_s\Phi\,\mathrm{d}x\,\mathrm{d}s+\frac{1}{\alpha}\int_0^t\int_{\mathbb{R}}(\rho^\alpha)_x\Phi_x\,\mathrm{d}x\,\mathrm{d}s+E_{11}.\end{aligned}\tag{3.9}$$

Similarly, we have

$$\begin{aligned}&\int_{\mathbb{R}}\Big(\frac{1}{2}n(v+\frac{1}{\alpha}n^{-1}(n^\alpha)_x)^2+\frac{1}{\gamma-1}(n^\gamma-\overline{n}^\gamma)-\frac{\gamma}{\gamma-1}\overline{n}^{\gamma-1}(n-\overline{n})\Big)\,\mathrm{d}x\\&+\gamma\int_0^t\int_{\mathbb{R}}n^{\gamma+\alpha-3}n_x^2\,\mathrm{d}x\,\mathrm{d}s\\&=-\int_0^t\int_{\mathbb{R}}n_s\Phi\,\mathrm{d}x\,\mathrm{d}s-\frac{1}{\alpha}\int_0^t\int_{\mathbb{R}}(n^\alpha)_x\Phi_x\,\mathrm{d}x\,\mathrm{d}s+E_{12}.\end{aligned}\tag{3.10}$$

Adding (3.9) to (3.10), we obtain

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}(\rho^\alpha-n^\alpha)_x\Phi_x\,\mathrm{d}x\,\mathrm{d}s&=-\int_0^t\int_{\mathbb{R}}(\rho^\alpha-n^\alpha)\Phi_{xx}\,\mathrm{d}x\,\mathrm{d}s\\&=-\int_0^t\int_{\mathbb{R}}(\rho^\alpha-n^\alpha)(\rho-n)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}\tag{3.11}$$

Combining (3.9), (3.10), (3.11) and (3.6) gives rise to (3.7). □

Lemma 3.3. *Under the assumptions in Lemma 3.1, we have*

$$0<\rho_-\le\rho\le\rho^+,\quad 0<n_-\le n\le n^+.\tag{3.12}$$

Proof. Denote

$$\varphi(\rho):=\frac{1}{\gamma-1}\Big(\rho^\gamma-\bar\rho^\gamma-\gamma\bar\rho^{\gamma-1}(\rho-\bar\rho)\Big),\tag{3.13}$$

$$\psi(\rho):=\int_{\bar\rho}^\rho\varphi(\eta)^{\frac12}\eta^{\alpha-\frac32}\,\mathrm{d}\eta.\tag{3.14}$$

It is easy to verify that $\varphi(\rho)\ge0$ and $\psi'(\rho)\ge0$. In addition, as $\rho\to+\infty$ it holds

$$\begin{aligned}\lim_{\rho\to+\infty}\psi(\rho)&\to(\gamma-1)^{-1/2}\lim_{\rho\to+\infty}\int_{\bar\rho}^\rho\eta^{\frac{\gamma+2\alpha-3}{2}}\,\mathrm{d}\eta\\&=\lim_{\rho\to+\infty}\frac{2}{(\gamma+2\alpha-1)\sqrt{\gamma-1}}(\rho^{\frac{\gamma+2\alpha-1}{2}}-\bar\rho^{\frac{\gamma+2\alpha-1}{2}})\to+\infty,\end{aligned}\tag{3.15}$$

and as $\rho\to0$,

$$\begin{aligned}\lim_{\rho\to0}\psi(\rho)&\to\lim_{\rho\to0}\int_{\bar\rho}^\rho\bar\rho^{\frac\gamma2}\eta^{\alpha-\frac32}\,\mathrm{d}\eta\\&=\lim_{\rho\to0}\frac{2}{2\alpha-1}\bar\rho^{\frac\gamma2}(\rho^{\alpha-\frac12}-\bar\rho^{\alpha-\frac12}),\\&\to\begin{cases}-\infty,&\text{if }0<\alpha<\frac12,\\-\frac{2}{2\alpha-1}\bar\rho^{\frac\gamma2+\alpha-\frac12}&\text{if }\alpha>\frac12.\end{cases}\end{aligned}\tag{3.16}$$

We can choose two constants $\rho_\pm>0$ with $\rho_-<\bar\rho<\rho_+$ and $\rho_+=\psi^{-1}(-\psi(\rho_-))$ so that

$$\begin{gathered}2E_0^{1/2}(E_0+E_1)^{1/2}<-\psi(\rho_-),\quad\alpha\in(0,\frac12),\\2E_0^{1/2}(E_0+E_1)^{1/2}<-\psi(\rho_-)<\frac{2}{2\alpha-1}\bar\rho^{\frac\gamma2+\alpha-\frac12},\quad\alpha>\frac12,\end{gathered}\tag{3.17}$$

which obviously satisfies

$$\psi(\rho_-)<-2E_0^{1/2}(E_0+E_1)^{1/2}<2E_0^{1/2}(E_0+E_1)^{1/2}<\psi(\rho_+).\tag{3.18}$$

From (3.1) and (3.7) it follows that

$$\begin{aligned}|\psi(\rho(x))|&\le|\int_{\mathbb{R}}\partial_x\psi(\rho)\,\mathrm{d}x|\le\Big|\int_{\mathbb{R}}\varphi(\rho)^{1/2}\rho_x\rho^{\alpha-\frac32}\,\mathrm{d}x\Big|\\&\le\Big(\int_{\mathbb{R}}\varphi(\rho)\,\mathrm{d}x\Big)^{1/2}\Big(\int_{\mathbb{R}}\big(\tfrac{2}{2\alpha-1}(\rho^{\alpha-\frac12})_x\big)^2\,\mathrm{d}x\Big)^{1/2}\\&\le2E_0^{1/2}(E_0+E_1)^{1/2},\end{aligned}\tag{3.19}$$

from which we obtain the half of (3.12) with ρ_- and ρ^+ determined as above.

Similarly, we have the another half of (3.12). The proof is complete. □

Lemma 3.4. *Under the same assumptions as in Lemma 3.1, it holds that*

$$
\begin{aligned}
&\int_{\mathbb{R}} u_x^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} u_s^2 \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_{\mathbb{R}} u_{xx}^2 \,\mathrm{d}x \,\mathrm{d}s \\
&+ \int_{\mathbb{R}} v_x^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} v_s^2 \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_{\mathbb{R}} v_{xx}^2 \,\mathrm{d}x \,\mathrm{d}s
\end{aligned}
\le C(T). \tag{3.20}
$$

Proof. First we estimate for u. Multiplying $(2.1)_2$ by $\rho^{-\alpha} u_t$, and integrate over $\mathbb{R}$. With the help of $(2.1)_1$ and the boundary conditions, after direct computations, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \Big(\frac{1}{2} u_x^2 - \rho^{\gamma-\alpha} u_x\Big) \,\mathrm{d}x + \int_{\mathbb{R}} \rho^{1-\alpha} u_t^2 \,\mathrm{d}x \\
&= -\int_{\mathbb{R}} \rho^{1-\alpha} u u_x u_t \,\mathrm{d}x - \alpha \int_{\mathbb{R}} \rho^{\gamma-\alpha-1} \rho_x u_t \,\mathrm{d}x + (\gamma-\alpha) \int_{\mathbb{R}} \rho^{\gamma-\alpha-1} \rho_x u u_x \,\mathrm{d}x \\
&\quad + (\gamma-\alpha) \int_{\mathbb{R}} \rho^{\gamma-\alpha} u_x^2 \,\mathrm{d}x + \alpha \int_{\mathbb{R}} \rho^{-1} \rho_x u_x u_t \,\mathrm{d}x + \int_{\mathbb{R}} \rho^{1-\alpha} u_t \Phi_x \,\mathrm{d}x.
\end{aligned} \tag{3.21}
$$

Integrating (3.21) over $t \in [0, T]$ and direct computations yield

$$
\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}} u_x^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} \rho^{1-\alpha} u_s^2 \,\mathrm{d}x \,\mathrm{d}s \\
&= \int_{\mathbb{R}} \rho^{\gamma-\alpha} u_x \,\mathrm{d}x - \int_0^t \int_{\mathbb{R}} \rho^{1-\alpha} u u_x u_s \,\mathrm{d}x \,\mathrm{d}s - \alpha \int_0^t \int_{\mathbb{R}} \rho^{\gamma-\alpha-1} \rho_x u_s \,\mathrm{d}x \,\mathrm{d}s \\
&\quad + (\gamma-\alpha) \int_0^t \int_{\mathbb{R}} \rho^{\gamma-\alpha-1} u \rho_x u_x \,\mathrm{d}x \,\mathrm{d}s + \alpha \int_0^t \int_{\mathbb{R}} \rho^{-1} \rho_x u_x u_s \,\mathrm{d}x \,\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{R}} \rho^{1-\alpha} u_s \Phi_x \,\mathrm{d}x \,\mathrm{d}s + (\gamma-\alpha) \int_0^t \int_{\mathbb{R}} \rho^{\gamma-\alpha} u_x^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\quad + \int_{\mathbb{R}} \Big(\frac{1}{2} u_{x0}^2 - \rho_0^{\gamma-\alpha} u_{x0}\Big) \,\mathrm{d}x.
\end{aligned}
$$

With the help of (2.2), Lemmas 3.1, 3.2 and 3.3, and Young's inequality, direct computation yield

$$
\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}} u_x^2 \,\mathrm{d}x + \frac{3}{5} \int_0^t \int_{\mathbb{R}} \rho^{1-\alpha} u_s^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\le C(T) + C \int_0^t \int_{\mathbb{R}} u^2 u_x^2 \,\mathrm{d}x \,\mathrm{d}s + C \int_0^t \int_{\mathbb{R}} \rho_x^2 u_x^2 \,\mathrm{d}x \,\mathrm{d}s.
\end{aligned} \tag{3.22}
$$

Next, we estimate $\int_0^t \int_{\mathbb{R}} u_{xx}^2 \,\mathrm{d}x \,\mathrm{d}s$. From $(2.1)_1$ and $(2.1)_2$, we have

$$
u_{xx} = \rho^{1-\alpha} u_t + \rho^{1-\alpha} u u_x + \gamma \rho^{-\alpha-\gamma-1} \rho_x - \rho^{1-\alpha} \Phi_x - \alpha \rho^{-1} \rho_x u_x. \tag{3.23}
$$

Combination Lemma 3.2 and Young's inequality, we obtain

$$
\int_{\mathbb{R}} u_{xx}^2 \,\mathrm{d}x \le \frac{1}{10} \int_{\mathbb{R}} \rho^{1-\alpha} u_t^2 \,\mathrm{d}x + C \int_{\mathbb{R}} u^2 u_x^2 \,\mathrm{d}x + C \int_{\mathbb{R}} \rho_x^2 u_x^2 \,\mathrm{d}x + C. \tag{3.24}
$$

Integrating (3.24) over $t \in [0,T]$, combining (3.22), Lemmas 3.1 and 3.2, and using Gagliardo-Nirenberg-Sobolev inequality, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} u_x^2 \,\mathrm{d}x + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \rho^{1-\alpha} u_s^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}} u_{xx}^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\le C\int_0^t \int_{\mathbb{R}} (u^2 + \rho_x^2) u_x^2 \,\mathrm{d}x\,\mathrm{d}s + C(T) \\
&\le C\int_0^t \|u_x\|_{L^\infty}^2 \,\mathrm{d}s + C(T) \\
&\le \frac{1}{4}\int_0^t \int_{\mathbb{R}} u_{xx}^2 \,\mathrm{d}x\,\mathrm{d}s + C(T)\,.
\end{aligned} \tag{3.25}$$

Then

$$\int_{\mathbb{R}} u_x^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} u_s^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}} u_{xx}^2 \,\mathrm{d}x\,\mathrm{d}s \le C(T). \tag{3.26}$$

Applying similar arguments we obtain

$$\int_{\mathbb{R}} v_x^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} v_s^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}} v_{xx}^2 \,\mathrm{d}x\,\mathrm{d}s \le C(T); \tag{3.27}$$

thus (3.20) follows. □

Lemma 3.5. *Under the assumptions in Lemma 3.1, the solution* (ρ, n, u, v, Φ_x) *satisfies*

$$\int_{\mathbb{R}} u_t^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} u_{xs}^2 \,\mathrm{d}x\,\mathrm{d}s + \int_{\mathbb{R}} v_t^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} v_{xs}^2 \,\mathrm{d}x\,\mathrm{d}s \le C(T). \tag{3.28}$$

Proof. Differentiating $(2.1)_2$ with respect to t, then multiplying by u_t and integrating over $\mathbb{R}$ (by parts), with $(2.1)_1$ and Young's equality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \rho u_t^2 \,\mathrm{d}x + \int_{\mathbb{R}} \rho^\alpha u_{xt}^2 \,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}} \rho_t u_t^2 \,\mathrm{d}x - \int_{\mathbb{R}} (\rho u u_x)_t u_t \,\mathrm{d}x + \int_{\mathbb{R}} (\rho^\gamma)_t u_{tx} \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} (\rho^\alpha)_t u_x u_{xt} \,\mathrm{d}x + \int_{\mathbb{R}} (\rho \Phi_x)_t u_t \,\mathrm{d}x \\
&\le \frac{1}{2}\int_{\mathbb{R}} \rho^\alpha u_{xt}^2 \,\mathrm{d}x + C\int_{\mathbb{R}} (u_x^2 + u_t^2 + \rho_x^2) \,\mathrm{d}x + \int_{\mathbb{R}} (\rho\Phi_x)_t u_t \,\mathrm{d}x.
\end{aligned} \tag{3.29}$$

Integrating (3.29) over $[0,t]$, we have

$$\frac{1}{2}\int_{\mathbb{R}} \rho u_t^2 \,\mathrm{d}x + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \rho^\alpha u_{xs}^2 \,\mathrm{d}x\,\mathrm{d}s \le C(T) + \int_0^t \int_{\mathbb{R}} (\rho\Phi_x)_s u_s \,\mathrm{d}x\,\mathrm{d}s. \tag{3.30}$$

Next we estimate $\int_0^t \int_{\mathbb{R}} (\rho\Phi_x)_s u_s \,\mathrm{d}x\,\mathrm{d}s$. From $(2.1)_1$ and $(2.1)_5$, it follows that

$$(\rho\Phi_x)_s u_s = -\rho_x u u_s \Phi_x - \rho u_x u_s \Phi_x - \rho u_s(\rho u - nv). \tag{3.31}$$

Using $(2.1)_5$, $(2.1)_6$, we have

$$\int_{\mathbb{R}} \Phi_{xxx}\Phi_x \,\mathrm{d}x = -\int_{\mathbb{R}} \Phi_{xx}^2 \,\mathrm{d}x = \int_{\mathbb{R}} \Phi_x (\rho - n)_x \,\mathrm{d}x,$$

which implies

$$\int_{\mathbb{R}} \Phi_{xx}^2 \,\mathrm{d}x \le \frac{1}{2}\int_{\mathbb{R}} (2\Phi_x^2 + \rho_x^2 + n_x^2)\,\mathrm{d}x.$$

Combining Lemmas 3.1 and 3.2, and Sobolev embedding theorem yields

$$\|\Phi_x\|_{L^\infty} \le C. \tag{3.32}$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}} (\rho\Phi_x)_s u_s \,\mathrm{d}x \\
&= -\int_{\mathbb{R}} \Big(\rho_x u u_s \Phi_x + \rho u_x u_s \Phi_x + \rho u_s(\rho u - nv)\Big)\,\mathrm{d}x \\
&\le |\int_{\mathbb{R}} \rho_x u u_s \Phi_x \,\mathrm{d}x| + |\int_{\mathbb{R}} \rho u_x u_s \Phi_x \,\mathrm{d}x| + |\int_{\mathbb{R}} \rho u_s(\rho u - nv)\,\mathrm{d}x| \\
&\le C(\|\rho_x\|_{L^2}^2 + \|u_t\|_{L^2}^2) + C|\int_{\mathbb{R}} (u u_{tx}\Phi_x + u u_t \Phi_{xx})\,\mathrm{d}x| \\
&\quad + C\big(\|u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|v\|_{L^2}^2\big) \\
&\le \frac{1}{4}\|\rho^{\frac{\alpha}{2}} u_{tx}\|_{L^2}^2 + C\Big(\|\rho_x\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\Phi_x\|_{L^2}^2\Big) + C \\
&\le \frac{1}{4}\|\rho^{\frac{\alpha}{2}} u_{tx}\|_{L^2}^2 + C\|u_t\|_{L^2}^2 + C\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} (\rho\Phi_x)_s u_s \,\mathrm{d}x\,\mathrm{d}s &\le C(T) + \frac{1}{4}\int_0^t \int_{\mathbb{R}} \rho^\alpha u_{sx}^2 \,\mathrm{d}x\,\mathrm{d}s + C\int_0^t \int_{\mathbb{R}} u_s^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\le C(T) + \frac{1}{4}\int_0^t \int_{\mathbb{R}} \rho^\alpha u_{sx}^2 \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.33}$$

Using (3.33) and (3.30), we obtain

$$\int_{\mathbb{R}} u_t^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} u_{sx}^2 \,\mathrm{d}x\,\mathrm{d}s \le C(T). \tag{3.34}$$

Applying similar arguments we obtain

$$\int_{\mathbb{R}} v_t^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} v_{sx}^2 \,\mathrm{d}x\,\mathrm{d}s \le C(T). \tag{3.35}$$

Then (3.34) and (3.35) give rise to (3.28). □

4. Proof of main results

Proof of Theorem 2.1. We prove only the existence of the solution (ρ, u); existence of (n, v) can be proved by the same method.

Let (ρ_0, u_0) be the initial data as described in the theorem, and let $\rho_0^\delta := j_\delta * \rho_0$, $u_0^\delta := j_\delta * u_0$, where $j_\delta = \delta^{-1} j(x/\delta)$ is the standard mollifier. Then, for any $0 < \beta < 1$ we have $\rho_0^\delta \in C^{1+\beta}(\mathbb{R})$ and $u_0^\delta \in C^{2+\beta}(\mathbb{R})$. This implies $\rho_0^\delta \to \rho_0$ in $W^{1,2}(\mathbb{R})$, and $u_0^\delta \to u_0$ in $L^2(\mathbb{R})$, as $\delta \to 0$.

Next, we consider the Cauchy problem $(2.1)_1$ and $(2.1)_2$ with the initial data (ρ_0, u_0) replaced by $(\rho_0^\delta, u_0^\delta)$, Φ_x be regarded as external force. For this problem we can apply the standard argument (the energy estimates and the contraction mapping theorem) to obtain the existence of a unique local solution (ρ^δ, u^δ) with ρ^δ, ρ_x^δ, ρ_t^δ, ρ_{tx}^δ, u^δ, u_x^δ, u_t^δ, $u_{xx}^\delta \in C^{\beta,\beta/2}(\mathbb{R}\times[0,T^*])$ for some $T^* > 0$. Furthermore, from Lemmas 3.1-3.5, we see that ρ^δ is pointwise bounded from below and above, $u^\delta,\ \rho^\delta_x \in L^\infty([0,T];L^2(\mathbb{R}))$, $u^\delta_x \in L^2([0,T];L^2(\mathbb{R}))$, $\rho^\delta,\ \rho^\delta_x,\ \rho^\delta_t,\ \rho^\delta_{tx},\ u^\delta,\ u^\delta_x,\ u^\delta_t,\ u^\delta_{xx} \in C^{\beta,\beta/2}(\mathbb{R}\times[0,T])$ for any $T>0$. Therefore, we can continue the local solution globally in time and deduce that there exists a unique global solution (ρ^δ,u^δ) of the Cauchy problem $(2.1)_1$ and $(2.1)_2$ with (ρ_0,u_0) replaced by $(\rho^\delta_0,u^\delta_0)$, which is carried out as in [1].

Thus, we extract a subsequence of (ρ^δ,u^δ), still denoted by (ρ^δ,u^δ), such that as $\delta\to 0$,

$$u^\delta \rightharpoonup u \quad \text{weak}{*} \text{ in } L^\infty([0,T];L^2(\mathbb{R})), \tag{4.1}$$

$$\rho^\delta \rightharpoonup \rho \quad \text{weak}{*} \text{ in } L^\infty([0,T];L^2(\mathbb{R})), \tag{4.2}$$

$$(\rho^\delta_t,u^\delta_x) \to (\rho_t,u_x) \quad \text{weak in } L^2([0,T];L^2(\mathbb{R})). \tag{4.3}$$

Moreover, from (3.1), (3.7) and (3.12), the existence of a global weak solution to the Cauchy problem $(2.1)_1$ and $(2.1)_2$ can be proved directly as in [11]. As a matter of fact, because of (3.20) and (3.28), (ρ,u) is also a global strong solution. Uniqueness of this strong solution can be proved as in [11]. We omit the details here. □

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