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MULTIPLE SOLUTIONS FOR DISCONTINUOUS ELLIPTIC PROBLEMS INVOLVING THE FRACTIONAL LAPLACIAN

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Abstract. In this article, we establish the existence of three weak solutions for elliptic equations associated to the fractional Laplacian

$$(-\Delta)^s u = \lambda f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega,$$

where Ω is an open bounded subset in $\mathbb{R}^N$ with Lipschitz boundary, λ is a real parameter, $0 < s < 1$, $N > 2s$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to each variable separately. The main purpose of this paper is concretely to provide an estimate of the positive interval of the parameters λ for which the problem above with discontinuous nonlinearities admits at least three nontrivial weak solutions by applying two recent three-critical-points theorems.

1. Introduction

The celebrated mountain pass theorem by Ambrosetti and Rabinowitz [1] provided the existence of at least one critical point for a C^1-functional satisfying the Palais-Smale condition (PS) and an appropriate geometry, called mountain pass geometry. This critical point theory has become one of the forceful tools for solving ordinary and partial differentiable equations; see [3, 4, 10, 16, 18]. Pucci and Serrin [33] proved the Ambrosetti-Rabinowitz theorem with zero altitude and in particular, they investigated that a C^1-functional which has two local minimum points also admits a third critical point. Ricceri [35, 36, 37] showed the existence of at least three-critical-points for differentiable functionals by using the above Pucci-Serrin mountain pass theorem. Such results of Ricceri have been extensively studied by various researchers; see [4, 19, 20] and the references therein. For the case of nonsmooth functionals as an improvement of Ricceri's results [35, 36] for differentiable functionals, it has been extended and generalized in different directions and in different settings. Inspired by the works of Marano and Motreanu [27, 28], multiple critical points theorems for nondifferentiable functionals have been developed by Bonanno and Candito [11]; see also [14]. As considering the nondifferentiable version of Pucci-Serrin [33] with the aid of the Ekeland variational principle, Arcoya and Carmona [4] have generalized Ricceri's theorem [36] to a wide class of continuous functionals that are not necessarily differentiable. These abstract results have

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been used to study a large number of differential equations with nonsmooth potentials. As extending a smooth Ricceri's three critical-points theorem to a non-smooth case, Yuan and Huang [39] obtained the existence of at least three critical points for a $p(x)$-Laplacian differential inclusion. The existence of at least one nontrivial weak solution of elliptic equations with a variable exponent has been investigated in [5] by applying an abstract nonsmooth critical point result provided in [13] in which a recent critical point result of Bonanno (see [9]) has been extended to the nonsmooth framework.

The main goal of this article is to establish the existence of three weak solutions for elliptic equations associated to the fractional Laplacian

$$\begin{gathered} (-\Delta)^s u = \lambda f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \mathbb{R}^N \backslash \Omega. \end{gathered} \tag{1.1}$$

This equation is the counterpart of this Laplace equation

$$\begin{gathered} -\Delta u = \lambda f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \mathbb{R}^N \backslash \Omega, \end{gathered}$$

where Ω is an open bounded subset in $\mathbb{R}^N$ with Lipschitz boundary, λ is a real parameter, $0 < s < 1$, $N > 2s$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to each variable separately. Here the operator $(-\Delta)^s$ is the fractional Laplace operator, which, up to normalization factors, may be defined as

$$(-\Delta)^s u(x) := -\frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \, dy$$

for $x \in \mathbb{R}^N$.

In the previous years a great attention has been drawn to the study of fractional and nonlocal problems of elliptic type. Although such operators have been a classical topic in harmonic analysis and partial differential equations for a long time, the interest in such operators has consistently increased in view of the mathematical theory to concrete some phenomena such as social sciences, fractional quantum mechanics, materials science, continuum mechanics, phase transition phenomena, image process, game theory and Lévy processes; see [7, 8, 17, 22, 23, 25, 30, 31] and the references therein. Especially, in terms of fractional quantum mechanics, the nonlinear fractional Schrödinger equation was originally suggested by Laskin in [25, 26] as an extension of the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. Fractional operators are closely related to financial mathematics, because Lévy processes with jumps revealed as more adequate models of stock pricing in comparison with the Brownian ones used in the celebrated Black-Scholes option pricing model.

In all aspects, the main purpose of this paper is concretely to provide an estimate of the positive interval of the parameters λ for which the problem (1.1) with discontinuous nonlinearities admits at least three nontrivial weak solutions by applying two recent three-critical-points theorems. As compared with the local case, the value of $(-\Delta)^s_p u(x)$ at any point $x \in \Omega$ depends not only on the values of u on the whole Ω, but actually on the entire space $\mathbb{R}^N$. Hence more complicated analysis than the papers [5, 12] has to be carefully carried out when we investigate the accurate interval for the parameters for which the problem (1.1) possesses at least three nontrivial weak solutions. As far as we are aware, there were no such existence results for fractional Laplacian problems in this situation although our result is motivated by the paper [5].

This article is organized as follows: In Section 2, we briefly recall some properties for locally Lipschitz continuous functionals and the fractional Sobolev spaces. Also we introduce abstract three-critical-points theorems established in [11, 39]. In Section 3, we apply the main tools to investigate the accurate interval for the parameters for which problem (1.1) possesses at least three nontrivial weak solutions.

2. Basic definitions and preliminary results

In this section, we briefly introduce the following definitions and some properties for locally Lipschitz continuous functionals. For a real Banach space $(X, \|\cdot\|_X)$, we say that a functional $h : X \to \mathbb{R}$ is called locally Lipschitz when, for every $u \in X$, there correspond a neighborhood U of u and a constant $\mathcal{L} \geq 0$ such that

$$|h(v) - h(w)| \leq \mathcal{L}\|v - w\|_X \quad \text{for all } v, w \in U.$$

Let $u, v \in X$. The symbol $h^\circ(u; v)$ indicates the generalized directional derivative of h at point u along direction v, namely

$$h^\circ(u; v) := \limsup_{w\to u, t\to 0^+} \frac{h(w + tv) - h(w)}{t}.$$

The generalized gradient of the function h at u, denoted by $\partial h(u)$, is the set

$$\partial h(u) := \big\{u^* \in X^* : \langle u^*, v\rangle \leq h^\circ(u; v) \quad \text{for all} \quad v \in X\big\}.$$

A functional $h : X \to \mathbb{R}$ is called Gâteaux differentiable at $u \in X$ if there is $\varphi \in X^*$ (denoted by $h'(u)$) such that

$$\lim_{t\to 0^+} \frac{h(u + tv) - h(u)}{t} = h'(u)(v)$$

for all $v \in X$. It is called continuously Gâteaux differentiable if it is Gâteaux differentiable for any $u \in X$ and the function $u \to h'(u)$ is a continuous map from X to its dual X^*. We recall that if h is continuously Gâteaux differentiable then it is locally Lipschitz and one has $h^\circ(u; v) = h'(u)(v)$ for all $u, v \in X$. If $h : X \to \mathbb{R}$ is a locally Lipschitz functional and $x \in X$, then we say that x is a critical point of h if it satisfies the inequality

$$h^\circ(x; y) \geq 0$$

for all $y \in X$ or, equivalently, $0 \in \partial h(x)$.

We recall two results, given in the papers [11, 39], which are crucial in our further investigations. The following three-critical-points theorem is a particular case of [39, Theorem 3.1] which is a generalization of Recceri's result [37] to a wide class of nondifferentiable functionals.

Theorem 2.1. *Let $(E, \|\cdot\|_E)$ be a real reflexive Banach space. Suppose that a functional $\Phi : E \to \mathbb{R}$ is sequentially weakly lower semicontinuous and bounded on any bounded subset of X, such that Φ' is of type (S_+). Assume that $\Psi : E \to \mathbb{R}$ is a locally Lipschitz functional with compact gradient. Suppose that*

$$\text{there exists } r \in \Big(\inf_{x\in E} \Psi(x),\ \sup_{x\in E} \Psi(x)\Big) \quad \text{such that} \quad \rho_1(r) < \rho_2(r), \tag{2.1}$$

where two functions ρ_1 *and* ρ_2 *are defined by*

$$\rho_1(r) = \inf_{x\in\Psi^{-1}((-\infty,r))} \frac{\inf_{y\in\Psi^{-1}(r)}\Phi(y)-\Phi(x)}{\Psi(x)-r},$$
$$\rho_2(r) = \sup_{x\in\Psi^{-1}((r,+\infty))} \frac{\inf_{y\in\Psi^{-1}(r)}\Phi(y)-\Phi(x)}{\Psi(x)-r}$$

for every $r \in (\inf_{x\in E}\Psi(x), \sup_{x\in E}\Psi(x))$. *Moreover, assume that for each* $\lambda \in \big(\rho_1(r), \rho_2(r)\big)$*, the functional* $I_\lambda := \Phi + \lambda\Psi$ *is coercive. Then, for each compact interval* $[a,b] \subset \big(\rho_1(r),\rho_2(r)\big)$*,* I_λ *has at least three critical points in* E *for every* $\lambda \in [a,b]$.

We give the following consequence, obtained by Bonanno and Candito [11, Theorem 3.2], that we recall in a convenient form; see also [14].

Theorem 2.2. *Let* $(E, \|\cdot\|_E)$ *be a real reflexive Banach space. Let* $\Phi : E \to \mathbb{R}$ *be a sequentially weakly lower semicontinuous functional such that* Φ *is coercive on* E*. Assume that* $\Psi : E \to \mathbb{R}$ *is a locally Lipschitz functional with compact gradient such that*

$$\inf_{x\in E}\Phi(x) = \Phi(0) = \Psi(0) = 0. \tag{2.2}$$

Assume that there exist $\mu > 0$ *and* $\tilde{x} \in E$ *with* $\mu < \Phi(\tilde{x})$ *such that*

- (A1) $\frac{\sup_{x\in\Phi^{-1}((-\infty,\mu))}\Psi(x)}{\mu} < \frac{\Psi(\tilde{x})}{\Phi(\tilde{x})}$;
- (A2) *for each* $\lambda \in \Lambda_\mu := \big(\frac{\Phi(\tilde{x})}{\Psi(\tilde{x})}, \frac{\mu}{\sup_{\Phi(x)\le\mu}\Psi(x)}\big)$*, the functional* $J_\lambda := \Phi - \lambda\Psi$ *is coercive and satisfies (PS)-condition.*

Then for each $\lambda \in \Lambda_\mu$*, the functional* J_λ *has at least three distinct critical points in* E.

Proof. Since Ψ is a locally Lipschitz functional with compact gradient, it follows from [39, Lemma 2.10] that it is sequentially weakly semicontinuous. Arguing as in the proof of [11, Theorem 3.1], the conclusion holds. □

Next, we briefly recall the definitions and some elementary properties of the fractional Sobolev spaces. The Gagliardo seminorm is defined for all measurable function $u : \mathbb{R}^N \to \mathbb{R}$ by

$$[u]_{s,2}^2 := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

We define the fractional Sobolev space $W^{s,2}(\mathbb{R}^N)$ as follows

$$W^{s,2}(\mathbb{R}^N) := \big\{u \in L^2(\mathbb{R}^N) : u \text{ is measurable and } [u]_{s,2} < +\infty\big\}$$

endowed with the norm

$$\|u\|_{s,2} := \Big(\|u\|_{L^2(\mathbb{R}^N)}^2 + [u]_{s,2}^2\Big)^{1/2},$$

where

$$\|u\|_{L^2(\mathbb{R}^N)}^2 := \int_{\mathbb{R}^N} |u|^2\,dx.$$

Then $W^{s,2}(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$\langle u, \varphi\rangle_{W^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} u(x)\varphi(x)\,dx.$$

The space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,2}(\mathbb{R}^N)$, that is $W_0^{s,2}(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$ (see e.g. [32]).

We consider problem (1.1) in the closed linear subspace

$$X_s(\Omega) = \{u \in W^{s,2}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\},$$

which can be equivalently renormed by setting $\|\cdot\|_{X_s(\Omega)} = [\cdot]_{s,2}$ (see [32, Theorem 7.1]). It is readily seen that $(X_s(\Omega), \|\cdot\|_{X_s(\Omega)})$ is a Hilbert spaces and its dual space is denoted by $(X_s(\Omega)^*, \|\cdot\|_*)$.

Lemma 2.3 ([32]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Then we have the following continuous embeddings:*

$$\begin{gathered} X_s(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for all } q \in [1, 2_s^*], \quad \text{if } 2s \le N, \\ X_s(\Omega) \hookrightarrow C_b^{0,\alpha}(\Omega) \quad \text{for all } \alpha < s - N/2 \quad \text{if } 2s > N, \end{gathered}$$

where 2_s^* is the fractional critical Sobolev exponent, that is

$$2_s^* := \begin{cases} \frac{2N}{N-2s} & \text{if } 2s < N, \\ +\infty & \text{if } 2s \ge N. \end{cases}$$

In particular, the space $X_s(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, 2_s^*)$ and $2s \le N$.

Let $2s < N$. From the Sobolev embedding theorem there exists a positive constant $C(N,s)$ such that for all $u \in W^{s,2}(\mathbb{R}^N)$,

$$\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)} \le C(N,s)\, [u]^2_{s,2}, \tag{2.3}$$

where

$$C(N,s) = \frac{16(N+4)^6 2^{(N+1)(N+2)} s(1-s)}{N^{2/2_s^*} \omega_{N-1}^{\frac{2s}{N}+1}(N-2s)};$$

see [29] and [21, Theorem 1.1].

Remark 2.4. It is possible to obtain an estimate of the embedding's constants C_1 and C_q. Fixed $q \in [1, 2_s^*]$, for each $u \in X_s(\Omega)$ we have $u \in L^q(\Omega)$. Put $l = \frac{2_s^*}{q}$ and $l' = \frac{2_s^*}{2_s^*-q}$. Since $|u|^q \in L^{\frac{2_s^*}{q}}(\Omega)$, the Hölder inequality ensures that $|u|^q \in L^1(\Omega)$ and

$$\begin{aligned} \int_\Omega |u(x)|^q\,dx &= \|u^q\|_{L^1(\Omega)} \le \|u^q\|_{L^l(\Omega)} \|1\|_{L^{l'}(\Omega)} \\ &\le \Big(\int_\Omega |u(x)|^{2_s^*}\,dx\Big)^{q/2_s^*} |\Omega|^{1/l'} = \|u\|^q_{L^{2_s^*}(\Omega)} |\Omega|^{1/l'} \end{aligned}$$

and hence

$$\|u\|_{L^q(\Omega)} \le \|u\|_{L^{2_s^*}(\Omega)} |\Omega|^{\frac{1}{l'q}} \le \|u\|_{L^{2_s^*}(\mathbb{R}^N)} |\Omega|^{\frac{1}{l'q}}.$$

Combining this with (2.3), we obtain

$$\|u\|_{L^q(\Omega)} \le C(N,s)^{1/2} |\Omega|^{\frac{1}{l'q}} [u]_{s,2} \le C(N,s)^{1/2} |\Omega|^{\frac{2_s^*-q}{2_s^* q}} \|u\|_{X_s(\Omega)}.$$

Therefore, if C_q is the embedding's constant of $X_s(\Omega) \hookrightarrow L^q(\Omega)$, we have

$$C_q \le C(N,s)^{1/2} |\Omega|^{\frac{2_s^*-q}{2_s^* q}}.$$

Also, when $q = 1$ and C_1 is the embedding's constant of $X_s(\Omega) \hookrightarrow L^1(\Omega)$, we obtain that

$$\|u\|_{L^1(\Omega)} \le C(N,s)^{1/2}|\Omega|^{\frac{2_s^*-1}{2_s^*}}\|u\|_{X_s(\Omega)} \quad \text{and} \quad C_1 \le C(N,s)^{1/2}|\Omega|^{\frac{2_s^*-1}{2_s^*}}.$$

Let us define the functional $\Phi_s : X_s(\Omega) \to \mathbb{R}$ by

$$\Phi_s(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

It follows that the functional Φ_s is well defined on $X_s(\Omega)$, $\Phi_s \in C^1(X_s(\Omega),\mathbb{R})$, and its Fréchet derivative is given by

$$\langle \Phi_s'(u), v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy$$

for any $v \in X_s(\Omega)$.

Lemma 2.5 ([2, 34])**.** *The functional $\Phi_s : X_s(\Omega) \to \mathbb{R}$ is convex, sequentially weakly lower semicontinuous, and coercive. Moreover, the functional $\Phi_s' : X_s(\Omega) \to X_s(\Omega)^*$ is of type (S_+), i.e., if $u_n \rightharpoonup u$ in $X_s(\Omega)$ as $n \to \infty$ and*

$$\limsup_{n\to\infty}\langle \Phi_s'(u_n) - \Phi_s'(u), u_n - u\rangle \le 0,$$

then $u_n \to u$ in $X_s(\Omega)$ as $n \to \infty$.

We suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(A3) f is measurable with respect to each variable separately;

(A4) there exist q with $1 < q < 2^*$ and a positive constant a such that

$$|f(x,t)| \le a(1+|t|^{q-1})$$

for each $(x,t) \in \Omega \times \mathbb{R}$;

(A5) for almost every $x \in \Omega$ and each $z \in D_f$ such that $\underline{f}(x,z) \le 0 \le \overline{f}(x,z)$ one has $f(x,z) = 0$, where

$$\underline{f}(x,z) := \lim_{\delta\to 0^+} \operatorname{ess\,inf}_{|\xi-z|<\delta} f(x,\xi) \quad \text{and} \quad \overline{f}(x,z) := \lim_{\delta\to 0^+} \operatorname{ess\,sup}_{|\xi-z|<\delta} f(x,\xi).$$

We denote by $\mathcal{G}$ the family of all locally bounded functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(A6) $f(\cdot, z)$ is measurable for every $z \in \mathbb{R}$;

(A7) there exists a set $\Omega_0 \subseteq \Omega$ with $m(\Omega_0) = 0$ such that the set

$$D_f := \cup_{x\in\Omega\setminus\Omega_0}\{z \in \mathbb{R} : f(x,\cdot) \text{ is discontinuous at } z\}$$

has measure zero;

(A8) $\underline{f}$ and $\overline{f}$ are superpositionally measurable, that is, $\underline{f}(\cdot, u(\cdot))$ and $\overline{f}(\cdot, u(\cdot))$ are measurable on Ω for any measurable function $u : \Omega \to \mathbb{R}$.

Clearly, if $f \in \mathcal{G}$ then f satisfies (A3). In the sequel, with F, we denote the function

$$F(x,\xi) := \int_0^\xi f(x,t)dt \quad \text{for } (x,\xi) \in \Omega \times \mathbb{R}.$$

Define the functional $\Upsilon : X_s(\Omega) \to \mathbb{R}$ by

$$\Upsilon(u) = \int_\Omega F(x,u(x))\,dx.$$

Next we define the integral functional $I_\lambda : X_s(\Omega) \to \mathbb{R}$ related to the problem (1.1) by

$$I_\lambda(u) = \Phi_s(u) - \lambda\Upsilon(u).$$

Definition 2.6. We say that $u \in X_s(\Omega)$ is a weak solution of problem (1.1) if

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy = \lambda\int_\Omega f(x,u)v\,dx$$

for all $v \in X_s(\Omega)$.

We recall the following consequences for locally Lipschitz functionals, given in [24], which will be used in the next section.

Lemma 2.7. *If $f \in \mathcal{G}$ satisfies* (A4)*, then $\Upsilon : X_s(\Omega) \to \mathbb{R}$ is a locally Lipschitz functional with compact gradient.*

Lemma 2.8. *If $f \in \mathcal{G}$ satisfies* (A4) *and* (A5)*, then for each $\lambda > 0$, the critical points of the functional I_λ are weak solutions for the problem* (1.1).

3. Main results

In this section, we investigate the accurate interval for the parameters for which problem (1.1) possesses at least three nontrivial weak solutions by applying Theorems 2.1 and 2.2.

3.1. **Application of Theorem 2.1.** In this subsection we localize precisely the intervals of λ's for which the problem (1.1) has either only the trivial solution or at least two nontrivial solutions, by applying the three-critical-points theorem 2.1.

The positivity of the infimum of all eigenvalues for problem (3.1) below is important to assert our main result in this subsection.

Lemma 3.1 ([38])**.** *Let us consider the eigenvalue problem*

$$\begin{gathered}(-\Delta)^s u = \lambda u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \mathbb{R}^N\backslash\Omega.\end{gathered} \tag{3.1}$$

Denote the quantity

$$\lambda_1 = \inf_{u\in X_0(\Omega)\backslash\{0\}} \frac{\|u\|^2_{X_s(\Omega)}}{\|u\|^2_{L^2(\Omega)}}. \tag{3.2}$$

Then there is $u_1 \in X$ with $\int_\Omega |u_1|^2\,dx = 1$ such that the infimum λ_1 in (3.2) *will be attained and u_1 represents an eigenfunction for the problem* (3.1) *corresponding to λ_1, that is, λ_1 is a positive eigenvalue of problem* (3.1)*. In particular,*

$$\lambda_1\int_\Omega |u|^2\,dx \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy$$

for every $u \in X_s(\Omega)$.

We assume that

- (A9) there exist a positive constant a_0 and γ_0 with $1 < \gamma_0 < 2$ such that
$$|f(x,t)| \le a_0(1+|t|^{\gamma_0-1})$$
for each $(x,t) \in \Omega\times\mathbb{R}^+$;
- (A10) $\limsup_{s\to 0}\frac{|f(x,s)|}{|s|^{\xi_1-1}} < +\infty$ uniformly for almost all $x \in \Omega$, where $2 < \xi_1 < 2^*_s$.

Let us introduce the crucial value

$$\mathcal{C}_f = \operatorname{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|f(x,s)|}{|s|}.$$

Hence, under (A9) and (A10), the same arguments in [20] imply that $\mathcal{C}_f$ is well defined and a positive constant, and furthermore the following relation holds,

$$\operatorname{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|F(x,s)|}{|s|^2} = \frac{\mathcal{C}_f}{2}. \tag{3.3}$$

Lemma 3.2. *Assume that $f\in\mathcal{G}$ satisfies* (A5), (A9), (A10)*. Then the functional $I_\lambda : X_s(\Omega)\to\mathbb{R}$ is coercive.*

Proof. It follows from condition (A10), Lemma 2.3 and the Hölder inequality that

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\|u\|^2_{X_s(\Omega)} - \lambda\int_\Omega F(x,u)\,dx \\
&\geq \frac{1}{2}\|u\|^2_{X_s(\Omega)} - \lambda a_0\int_\Omega 1+|u|^{\gamma_0}\,dx \\
&\geq \frac{1}{2}\|u\|^2_{X_s(\Omega)} - \lambda a_0\int_\Omega |u|^{\gamma_0}\,dx - \lambda d_1 \\
&= \frac{1}{2}\|u\|^2_{X_s(\Omega)} - \lambda a_0\|u\|^{\gamma_0}_{L_{\gamma_0}(\Omega)} - \lambda d_1 \\
&\geq \frac{1}{2}\|u\|^2_{X_s(\Omega)} - \lambda d_2\|u\|^{\gamma_0}_{X_s(\Omega)} - \lambda d_1
\end{aligned}$$

where d_1 and d_2 are positive constants. Since $\gamma_0 < 2$, we deduce that $I_\lambda(u)\to\infty$ as $\|u\|_{X_s(\Omega)}\to\infty$. □

Theorem 3.3. *Assume that $f\in\mathcal{G}$ satisfies* (A5), (A9) *and* (A10)*. Then we have:*

(i) for every $\theta\in\mathbb{R}$, there exists $\ell_* = \lambda_1/\mathcal{C}_f$ such that the problem (1.1) *has only the trivial solution for all $\lambda\in[0,\ell_*)$, where λ_1 is the positive real number in* (3.2) *in Lemma 3.1.*

(ii) if furthermore f satisfies the assumption

(A11) *$\int_\Omega F(x,u_1(x))dx > 1/2$ holds, where u_1 is the eigenfunction corresponding to the principle eigenvalue of* (3.1) *satisfying $\int_\Omega |u_1|^2\,dx = 1$,*

then the problem (1.1) *has at least two distinct nontrivial solutions for each compact interval $[a_0,b_0]\subset(\ell^*,\lambda_1)$, where $\ell^* = \rho_1(0) < \lambda_1$ with $\ell^*\geq\ell_*$ and for every $\lambda\in[a_0,b_0]$.*

Proof. Our aim is to apply Theorem 2.1 to the space $X = X_s(\Omega)$ with the usual norm and to the functionals $\Phi := \Phi_s$ and $\Psi := -\Upsilon$, where

$$\Phi_s(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy,$$
$$\Upsilon(u) = \int_\Omega F(x,u(x))\,dx$$

for all $u\in X_s(\Omega)$. Taking into account Lemma 2.5, the functional Φ_s is convex, sequentially weakly lower semicontinuous, coercive, and the functional $\Phi_s' : X_s(\Omega)\to X_s(\Omega)^*$ is of type (S_+). Moreover, according to Lemma 2.7, the functional Υ is a locally Lipschitz functional with compact gradient. Thus all of the assumptions in Theorem 2.1 except the condition (2.1) are satisfied.

Now we prove the assertion (i). Let $u \in X_s(\Omega)$ be a nontrivial weak solution of the problem (1.1). Then it is clear that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy=\lambda\int_\Omega f(x,u)v\,dx$$

for any $v\in X_s(\Omega)$. If we put $v=u$, then it follows from the relation (3.2) and the definition of $\mathcal{C}_f$ that

$$\begin{aligned}
\lambda_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy&=\lambda_1\Big(\lambda\int_\Omega f(x,u)u\,dx\Big)\\
&\le\lambda_1\Big(\lambda\int_\Omega\frac{f(x,u)}{|u|}|u|^2\,dx\Big)\\
&\le\lambda\mathcal{C}_f\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)\\
&\le\lambda\mathcal{C}_f\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big).
\end{aligned}$$

Thus if u is a nontrivial weak solution of the problem (1.1), then necessarily $\lambda\ge\ell_*=\lambda_1/\mathcal{C}_f$, as claimed.

Next, we show that the assertion (ii) holds. From (A11), it is clear that the crucial positive number

$$\ell^*=\rho_1(0)=\inf_{u\in(-\Upsilon)^{-1}((-\infty,0))}\Big(\frac{\Phi_s(u)}{\Upsilon(u)}\Big)$$

is well defined. Hence by the definition of u_1 and the assumption (A10), we have that

$$\begin{aligned}
\ell^*=\rho_1(0)&=\inf_{u\in(-\Upsilon)^{-1}((-\infty,0))}\Big(\frac{\Phi_s(u)}{\Upsilon(u)}\Big)\le\frac{\Phi_s(u_1)}{\Upsilon(u_1)}\\
&=\frac{\frac12\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_1(x)-u_1(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\int_\Omega F(x,u_1)\,dx}\\
&<\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_1(x)-u_1(y)|^2}{|x-y|^{N+2s}}\,dx\,dy=\lambda_1.
\end{aligned}$$

In addition, to assert $\ell^*\ge\ell_*$, let u be in $X_s(\Omega)$ with $u\not\equiv0$. From (3.3), we obtain

$$\begin{aligned}
\frac{\Phi_s(u)}{|\Upsilon(u)|}&=\frac{\frac12\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\big|\int_\Omega F(x,u)\,dx\big|}\\
&\ge\frac{\frac12\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\int_\Omega\frac{|F(x,u)|}{|u|^2}|u|^2\,dx}\\
&\ge\frac{\frac12\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\frac{\mathcal{C}_f}{2}\int_\Omega|u|^2\,dx}\\
&\ge\frac{\lambda_1}{2\mathcal{C}_f}=\ell_*.
\end{aligned}$$

Hence we have $\ell^* \ge \ell_*$. Now we claim that there exists a real number r satisfying the condition (2.1). For any $u \in (-\Upsilon)^{-1}((-\infty, 0))$, we deduce that

$$\rho_1(r) \le \frac{\Phi_s(u)}{r + \Upsilon(u)}$$

for all $r \in (-\Upsilon(u), 0)$. This implies

$$\limsup_{r\to 0-} \rho_1(r) \le \frac{\Phi_s(u)}{\Upsilon(u)}$$

for all $u \in (-\Upsilon)^{-1}((-\infty, 0))$. Hence we have

$$\limsup_{r\to 0-} \rho_1(r) \le \rho_1(0) = \ell^*.$$

From (A9) and (A10), it is obvious that there exists a positive real number $\mathcal{C}_*$ such that

$$|F(x,s)| \le \mathcal{C}_* |s|^{\xi_1}$$

for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$. Thus it follows that

$$|\Upsilon(u)| \le \int_\Omega \mathcal{C}_* |u|^{\xi_1}\,dx \le \mathcal{C}_* \|u\|_{X_s(\Omega)}^{\xi_1} + \frac{1}{2\lambda_1}\|u\|_{X_s(\Omega)}^2$$

for all $u \in X$. If $r < 0$ and $v \in (-\Upsilon)^{-1}(r)$, then it follows that

$$\begin{aligned} 2r &= -2\Upsilon(v) \\ &\ge -2\mathcal{C}_* \|v\|_{X_s(\Omega)}^{\xi_1} - \frac{1}{\lambda_1}\|v\|_{X_s(\Omega)}^2 \\ &\ge -2\mathcal{C}_*(2)^{\frac{\xi_1}{2}+1}\Phi_s(v)^{\xi_1/2} - \frac{2}{\lambda_1}\Phi_s(v). \end{aligned} \tag{3.4}$$

Since $u = 0 \in (-\Upsilon)^{-1}((r, +\infty))$, by the definition of ρ_2, we have

$$\rho_2(r) \ge \frac{1}{|r|} \inf_{v\in(-\Upsilon)^{-1}((r,+\infty))} \Phi_s(v),$$

and hence there exists an element $u_r \in (-\Upsilon)^{-1}((r, +\infty))$ such that $\Phi_s(u_r) = \inf_{v\in(-\Upsilon)^{-1}((r,+\infty))} \Phi_s(v)$; see [6, Theorem 6.1.1]. According to (3.4), we obtain

$$\begin{aligned} 2 &\le \hat{C}|r|^{\frac{\xi_1}{2}-1}\Big(\frac{\Phi_s(u_0)}{|r|}\Big)^{\xi_1/2} + \frac{2}{\lambda_1}\frac{\Phi_s(u_0)}{|r|} \\ &\le \hat{C}|r|^{\frac{\xi_1}{2}-1}\rho_2(r)^{\xi_1/2} + \frac{2}{\lambda_1}\rho_2(r), \end{aligned} \tag{3.5}$$

where

$$\hat{C} = \mathcal{C}_* 2^{\frac{\xi_1}{2}+1}.$$

Then there are two possibilities to be considered: either ρ_2 is locally bounded at $0-$ so that relation (3.5) shows $\liminf_{r\to 0-} \rho_2(r) \ge \lambda_1$ because $\xi_1 > 2$ or $\limsup_{r\to 0-} \rho_2(r) = \infty$.

Since the functional $\Phi_s - \lambda\Upsilon$ is coercive for all $\lambda \in \mathbb{R}$ by Theorem 3.2. For all integers $n \ge n^* := 1 + 2/[\lambda_1 - \ell^*]$, there exists a negative sequence $\{r_n\}$ converging to 0 as $n \to \infty$ such that $\rho_1(r_n) < \ell^* + 1/n < \lambda_1 - 1/n < \rho_2(r_n)$. Bearing in mind Lemma 2.1, we conclude that $u \equiv 0$ is a critical point of the functional $\Phi_s - \lambda\Upsilon$

and, in view of Lemma 2.8, the problem (1.1) admits at least two distinct weak solutions for each compact interval

$$[a_0, b_0] \subset \big(\ell^*, \lambda_1\big) = \cup_{n=n^*}^{\infty} \big[\ell^* + \frac{1}{n}, \lambda_1 - \frac{1}{n}\big] \subset \cup_{n=n^*}^{\infty} (\rho_1(r_n), \rho_2(r_n))$$

and for every $\lambda \in [a_0, b_0]$. This completes the proof. $\square$

3.2. **Application of Theorem 2.2.** In this subsection, we prove the existence of nontrivial weak solutions for the problem (1.1) under suitable assumptions. Putting

$$\delta(x) = \sup\{\delta > 0 : B(x, \delta) \subseteq \Omega\}$$

for all $x \in \Omega$, we can show that there exists $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$, where

$$D = \sup_{x\in\Omega} \delta(x). \tag{3.6}$$

We introduce the following conditions:

(A12) there exist $c \in (0, +\infty)$ and γ with $1 < \gamma < 2$ such that

$$F(x, t) \le c(1 + |t|^{\gamma})$$

for each $(x, t) \in \Omega \times \mathbb{R}$;

(A13) $F(x, |t|) \ge 0$ for each $(x, t) \in \Omega \times \mathbb{R}$;

(A14) There exist $\mu > 0$ and $\delta > 0$ with $|\delta|^2\omega_N^2 D^{N-2s}\mathcal{M} < 1$ such that

$$aC_1\sqrt{2\mu} + \frac{aC_q^q(2\mu)^{q/2}}{q} < \frac{D^{2s}\inf_{x\in\Omega} F(x, \delta)}{2^N\delta^2\omega_N\mathcal{M}},$$

where D is given in (3.6), ω_N is the volume of $B(x_0, D) := \{x \in \mathbb{R}^N : |x - x_0| < D\}$ in $\mathbb{R}^N$, $\mathcal{M} = \frac{2^{2+N-2s}}{(1-s)(N-2s+2)} + \frac{1}{2^{N-2s}s(N-2s+2)} + \frac{1}{2s(N-2s)}$.

Theorem 3.4. *Let $f \in \mathcal{G}$ satisfy* (A4) *and* (A5). *Assume also that conditions* (A12)–(A14) *are satisfied. Then, for every*

$$\lambda \in \tilde{\Lambda} := \Big(\frac{2^N\delta^2\omega_N\mathcal{M}}{D^{2s}\inf_{x\in\Omega} F(x, \delta)}, \frac{q}{qaC_1\sqrt{2\mu} + aC_q^q(2\mu)^{q/2}}\Big),$$

the problem (1.1) *admits at least three weak solutions.*

Proof. Without loss of generality, we can assume $f(x, t) = 0$ for all $x \in \Omega$ and for all $t \le 0$. Apply Theorem 2.2 to the functionals $\Phi := \Phi_s$ and $\Psi := \Upsilon$ as in Theorem 3.3.

Now, let

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus B(x_0, D), \\ \delta & \text{if } x \in B(x_0, \frac{D}{2}), \\ \frac{2\delta}{D}(D - |x - x_0|) & \text{if } x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}), \end{cases} \tag{3.7}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. Then it is clear that $\tilde{u} \in X_s(\Omega)$ and $0 \le \tilde{u}(x) \le \delta$ for all $x \in \Omega$, and so $\tilde{u} \in X_s(\Omega)$. Denote $B_D := B(x_0, D)$. Then, it follows that

$$\begin{aligned} \Phi_s(\tilde{u}) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^{N+2s}}\,dx\,dy \\ &= \frac{1}{2}\int_{B_D\setminus B_{\frac{D}{2}}}\int_{B_D\setminus B_{\frac{D}{2}}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^{N+2s}}\,dx\,dy \end{aligned}$$

$$
\begin{aligned}
&+\int_{B_D\setminus B_{\frac{D}{2}}}\int_{\mathbb{R}^N\setminus B_D}\frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&+\int_{B_{\frac{D}{2}}}\int_{B_D\setminus B_{\frac{D}{2}}}\frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&+\int_{\mathbb{R}^N\setminus B_D}\int_{B_{\frac{D}{2}}}\frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
=:&\ \frac{1}{2}I_1+I_2+I_3+I_4.
\end{aligned}
$$

Next we estimate I_1–I_4, by direct calculations.

• Estimate of I_1: For any positive constant ε small enough,

$$
\begin{aligned}
I_1&=\int_{B_D\setminus B_{\frac{D}{2}}}\int_{B_D\setminus B_{\frac{D}{2}}}\frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\le\frac{2^2|\delta|^2}{D^2}\int_{B_D\setminus B_{\frac{D}{2}}}\int_{B_D\setminus B_{\frac{D}{2}}}\frac{|x-y|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\le\frac{2^2|\delta|^2\omega_N}{D^2}\int_{B_D\setminus B_{\frac{D}{2}}}\int_{\varepsilon}^{D+|y|}r^{2-2s-1}\,drdy\\
&\le\frac{2^2|\delta|^2\omega_N}{D^2}\int_{B_D\setminus B_{\frac{D}{2}}}\frac{(D+|y|)^{2-2s}}{2-2s}\,dy\\
&=\frac{2^2|\delta|^2\omega_N^2}{(2-2s)D^2}\int_{\frac{3}{2}D}^{2D}r^{2+N-2s-1}\,dr\\
&=\frac{2|\delta|^2\omega_N^2D^{N-2s}}{(1-s)(2+N-2s)}\Big(2^{2+N-2s}-(\frac{3}{2})^{2+N-2s}\Big).
\end{aligned}
$$

• Estimate of I_2:

$$
\begin{aligned}
I_2&=\int_{B_D\setminus B_{\frac{D}{2}}}\int_{\mathbb{R}^N\setminus B_D}\frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\le\frac{2^2|\delta|^2}{D^2}\int_{B_D\setminus B_{\frac{D}{2}}}\int_{\mathbb{R}^N\setminus B_D}\frac{|D-|y-x_0||^2}{|x-y|^{N+2s}}\,dx\,dy\\
&=\frac{2^2|\delta|^2\omega_N}{D^2}\int_{B_D\setminus B_{\frac{D}{2}}}\int_{D-|y-x_0|}^{\infty}\frac{|D-|y-x_0||^2}{r^{2s+1}}\,drdy\\
&=\frac{2^2|\delta|^2\omega_N}{D^22s}\int_{B_D\setminus B_{\frac{D}{2}}}|D-|y-x_0||^{2-2s}\,dy\\
&=\frac{2|\delta|^2\omega_N^2}{D^2s}\int_0^{\frac{D}{2}}r^{N+2-2s-1}\,dr\\
&=\frac{|\delta|^2\omega_N^2D^{N-2s}}{2^{N-2s+1}s(N-2s+2)}.
\end{aligned}
$$

• Estimate of I_3:

$$\begin{aligned}
I_3 &= \int_{B_{\frac{D}{2}}}\int_{B_D\setminus B_{\frac{D}{2}}} \frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&= \frac{2^2|\delta|^2}{D^2}\int_{B_{\frac{D}{2}}}\int_{B_D\setminus B_{\frac{D}{2}}} \frac{|-\frac{D}{2}+|x-x_0||^2}{|x-y|^{N+2s}}\,dx\,dy\\
&= \frac{2^2|\delta|^2}{D^2}\int_{B_D\setminus B_{\frac{D}{2}}}\int_{B_{\frac{D}{2}}} \frac{|-\frac{D}{2}+|x-x_0||^2}{|x-y|^{N+2s}}\,dydx\\
&= \frac{2^2|\delta|^2\omega_N}{D^2}\int_{B_D\setminus B_{\frac{D}{2}}} \big|-\frac{D}{2}+|x-x_0|\big|^2\int_{|x-x_0|-\frac{D}{2}}^{|x-x_0|+\frac{D}{2}}\frac{1}{r^{2s+1}}\,drdx\\
&\le \frac{2|\delta|^2\omega_N}{D^2 s}\int_{B_D\setminus B_{\frac{D}{2}}} \big|-\frac{D}{2}+|x-x_0|\big|^{2-2s}\,dx\\
&= \frac{2|\delta|^2\omega_N^2}{D^2 s}\int_0^{\frac{D}{2}} t^{N-2s+1}\,dt\\
&= \frac{|\delta|^2\omega_N^2 D^{N-2s}}{2^{N-2s+1}s(N-2s+2)}
\end{aligned}$$

• Estimate of I_4:

$$\begin{aligned}
I_4 &= \int_{B_{\frac{D}{2}}}\int_{\mathbb{R}^N\setminus B_D} \frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&= |\delta|^2\int_{B_{\frac{D}{2}}}\int_{\mathbb{R}^N\setminus B_D} \frac{1}{|x-y|^{N+2s}}\,dx\,dy\\
&= |\delta|^2\omega_N\int_{B_{\frac{D}{2}}}\int_{D-|y-x_0|}^{\infty} r^{-2s-1}\,drdy\\
&= |\delta|^2\omega_N\int_{B_{\frac{D}{2}}}\frac{1}{2s(D-|y-x_0|)^{2s}}\,dy\\
&= \frac{|\delta|^2\omega_N^2}{2s}\int_{\frac{D}{2}}^{D} t^{N-2s-1}\,dt\\
&= \frac{|\delta|^2\omega_N^2 D^{N-2s}}{2s(N-2s)}\Big(1-\frac{1}{2^{N-2s}}\Big)\\
&= \frac{|\delta|^2\omega_N^2 D^{N-2s}}{2s(N-2s)}.
\end{aligned}$$

Hence, it follows from (A14) that

$$\Phi_s(\tilde{u}) \le |\delta|^2\omega_N^2\, D^{N-2s}\mathcal{M} < 1,$$

where

$$\mathcal{M} = \frac{2^{2+N-2s}}{(1-s)(N-2s+2)} + \frac{1}{2^{N-2s}s(N-2s+2)} + \frac{1}{2s(N-2s)}.$$

Owing to the assumption (A13) and the definition (3.7), we deduce that

$$\Upsilon(\tilde{u}) \geq \int_{B_{\frac{D}{2}}} F(x,\tilde{u})\,dx \geq \inf_{x\in\Omega} F(x,\delta)\Big(\frac{\omega_N D^N}{2^N}\Big)$$

and thus

$$\frac{\Upsilon(\tilde{u})}{\Phi_s(\tilde{u})} \geq \frac{D^{2s}\inf_{x\in\Omega}F(x,\delta)}{2^N\delta^2\omega_N\mathcal{M}}. \tag{3.8}$$

Also by (A4), Lemma 2.3 and the best constants C_1, C_q, we have

$$\begin{aligned}
\Upsilon(u) &= \int_\Omega F(x,u)\,dx \\
&\leq a\int_\Omega \Big\{|u(x)| + \frac{1}{q}|u(x)|^q\Big\}\,dx \\
&= a\|u\|_{L^1(\Omega)} + \frac{a}{q}\|u\|^q_{L^q(\Omega)} \\
&\leq aC_1\|u\|_{X_s(\Omega)} + \frac{a}{q}C_q^q\|u\|^q_{X_s(\Omega)}.
\end{aligned}$$

For each $u\in\Phi_s^{-1}((-\infty,\mu])$, it follows that

$$\Upsilon(u) \leq aC_1\sqrt{2\mu} + \frac{aC_q^q(2\mu)^{q/2}}{q}$$

and hence

$$\sup_{u\in\Phi_s^{-1}((-\infty,\mu])}\Upsilon(u) \leq aC_1\sqrt{2\mu} + \frac{aC_q^q(2\mu)^{q/2}}{q}.$$

From inequality (3.8) and the assumption (A14), we have

$$\sup_{u\in\Phi_s^{-1}((-\infty,1])}\Upsilon(u) < \frac{\Upsilon(\tilde{u})}{\Phi_s(\tilde{u})}.$$

Therefore,

$$\tilde{\Lambda} \subseteq \Big(\frac{\Phi_s(\tilde{u})}{\Upsilon(\tilde{u})}, \frac{1}{\sup_{\Phi_s(u)\leq 1}\Upsilon(u)}\Big).$$

Since condition (2.2) is easily verified and $J_\lambda = \Phi_s - \lambda\Upsilon$ is coercive by (A12), all conditions of Theorem 2.2 are satisfied for every $\lambda\in\tilde{\Lambda}$. Hence, by applying Theorem 2.2 and Lemma 2.8, we conclude that for each $\lambda\in\tilde{\Lambda}$, the functional $J_\lambda = \Phi_s - \lambda\Upsilon$ admits three critical points which are weak solutions for the problem (1.1). This completes the proof. □

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