Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 19, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

APPROXIMATE CONTROLLABILITY OF EULER-BERNOULLI VISCOELASTIC SYSTEMS

ZHIFENG YANG, ZHAOSHENG FENG

Abstract. In this article, we study an Euler-Bernoulli viscoelastic control system which is dissipative due to the presence of the viscoelastic term. The main feature which distinguishes this paper from other related works lies in the fact that we no longer impose traditional conditions such as complete monotonicity and decay property on the kernel function g. Without loss of generality, we study the system in the case of $g \equiv 1$. By means of the duality principle and the Hahn-Banach theorem, we show that the system with $g = 1$ is approximately controllable in the appropriate function space.

1. Introduction

With the development of applied mathematics and materials science, more and more research has been devoted to the study of the mathematical models of viscoelastic materials which have both instantaneous elastic response and sustained internal friction effects under the action of a load. The mechanical response of these materials is to be influenced by the previous behavior of the materials themselves. This memory property is usually described by an integro-differential operator in mathematics. So, the so-called viscoelastic model is usually an integro-differential equation with various initial-boundary conditions. A number of theoretical issues concerning mathematical theory of viscoelasticity have received considerable attention, for example, see [5, 6, 11, 12, 13, 18] etc. In particular, the Hilbert uniqueness method (HUM), proposed by Lions in [13, Chapter 4] has been widely used in the study of the exact controllability of distributed parametric systems.

Let Ω be a bounded domain with a smooth boundary Γ, and $T > 0$ be the time variable. Lions [13] considered the exact controllability of the Euler-Bernoulli

2010 *Mathematics Subject Classification.* 93B05, 93C20, 35Q93.
Key words and phrases. Euler-Bernoulli viscoelastic system; approximate controllability; duality principle; Hahn-Banach theorem.

Submitted February 28, 2018. Published January 30, 2019.

system

$$
\begin{gathered}
u_{tt} + \Delta^2 u = 0, \quad (x,t) \in \Omega \times (0,T), \\
u(x,0) = u^{(0)}(x), \quad u_t(x,0) = u_t^{(0)}(x), \quad x \in \Omega, \\
u = \begin{cases} 0, & (x,t) \in \Gamma \backslash \Gamma_0 \times (0,T), \\ v_0, & (x,t) \in \Gamma_0 \times (0,T), \end{cases} \\
\Delta u = \begin{cases} 0, & (x,t) \in \Gamma \backslash \Gamma_0 \times (0,T), \\ v_1, & (x,t) \in \Gamma_0 \times (0,T), \end{cases}
\end{gathered} \tag{1.1}
$$

by using the HUM framework, where Γ_0 is a part of the boundary Γ, and the two control functions v_0 and v_1 act on the boundary. Here, v_0 and v_1 are dependent each other. Up to now, how to use a single control function ($v_0 = 0$ or $v_1 = 0$) to achieve the exact controllability of system (1.1) is still an interesting problem. Exponential decay rates for the solutions of Euler-Bernoulli equations with boundary dissipation occurring in the moments only was investigated by Lasiecka [11], and the exact controllability of the Euler-Bernoulli equation with boundary controls for displacement and moment was established by Lasiecka and Triggiani [12].

For the study on the control problem of the viscoelastic heat equation

$$
u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = f(u), \tag{1.2}
$$

we refer the reader to [17, 9, 22, 21, 7, 23, 1, 4] and the references therein. For example, the controllability and identification problem for heat equations with memory were studied by Pandolfi [17]. Based on the theory of interpolation, Ivanov et al [9] showed that the one-dimensional heat equation with memory cannot be controlled to rest for large classes of memory kernels and controls. The approximate controllability of a parabolic equation with memory was studied by using the duality method [22]. As we know, the null controllability property of the heat equation with a memory term fails for a special set of initial data [7]. The null controllability of the heat equations with memory was also discussed by developing a new weighted Carleman inequality [21, 4]. Moreover, a characterization of the set of nontrivial initial data which can be driven to zero with a boundary control was described in [23].

For the hyperbolic equation with memory

$$
u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = f(u), \tag{1.3}
$$

the reachability, observability and controllability of a viscoelastic string were presented in [15, 14]. The exact controllability and the boundedness of the control function was shown in [19]. Moreover, the memory-type null controllability property of vicoelastic wave equations with exponential decay kernel function was considered by the duality principle and an observability inequality [10]. The approximate controllability of semilinear beam equations with impulses, memory and delay was studied in [2].

It is notable that the results on equation (1.3) are derived usually through imposing some restrictions on the kernel function g, such as completely monotonicity or decay properties. If $g \equiv 1$, that is, the kernel function has no support and it does not satisfy the conditions like those in the above references, the methods used in the previous works become invalid for equation (1.2) or (1.3).

In this article, we consider an Euler-Bernoulli viscoelastic control system

$$
\begin{gathered}
u_{tt}+u_{xxxx}-\int_0^t u_{xxxx}(s)ds=0, \quad (x,t)\in(0,\pi)\times(0,T),\\
u(x,0)=u^{(0)}(x),\ u_t(x,0)=u_t^{(0)}(x), \quad x\in(0,\pi),\\
u(0,t)=u_{xx}(0,t)=u_{xx}(\pi,t)=0, \quad t\in(0,T),\\
u(\pi,t)=v(t), \quad t\in(0,T),
\end{gathered}
\tag{1.4}
$$

where $u^{(0)}, u_t^{(0)}$ are the given initial data, and v is the control function acting on the boundary. Compared with system (1.1), this viscoelastic system contains an integro-differential term (i.e. the viscoelastic term with the kernel function $g\equiv 1$) and only one control function v. Because of the role of the viscoelastic term, the energy of system (1.4) is not conserved, but decayed. As we know, the so-called observability inequality is the key to prove the exact controllability in the HUM framework. But, the conservation of energy provides a great convenience to establish the observability inequality. So, from the perspective of system control, it is difficult for us to make effective control to the system behavior if we can not catch the energy which is decayed. Moreover, in the process of estimating the norm of the solution for system (1.4), the viscoelastic term is very difficult to be absorbed by other global integral term. It always stays in the side of the local integral term. Thus, the classical Carleman estimate can not be attained. As a result, one can not use the local term to control the global term. So, the problem becomes challenging while we study the exact controllability of system (1.4).

Inspired by this fact and the results described in [4, 11, 12, 13, 21], in this study we first attempt to work on the expression of the solutions to the associated dual system of the viscoelastic system, then explore the observability inequality by making appropriate estimates to the solutions, and finally prove the approximate controllability. Before processing our discussions, we have to figure out two issues: (i) Which functional space is the dual system represented in? and (ii) can we return to some classical functional spaces in which we can deal with the approximate controllability of the original system? Fortunately, there have been helpful attempts to such a problem. For example, the duality method was applied to consider the approximate controllability of a perturbed wave system [20, 22]:

$$
\begin{gathered}
y_t-y_{xx}-\varepsilon u_{txx}=0, \quad (x,t)\in(0,1)\times(0,T),\\
y(x,0)=y^{(0)}(x),\ y_t(x,0)=y_t^{(0)}(x), \quad x\in(0,1),\\
y(0,t)=0,\ y(1,t)=h(t), \quad t\in(0,T),
\end{gathered}
\tag{1.5}
$$

and a partial integro-differential system

$$
\begin{gathered}
y_t-y_{xx}+\int_0^t y(x,s)ds=0, \quad (x,t)\in(0,1)\times(0,T),\\
y(x,0)=y^{(0)}(x), \quad x\in(0,1),\\
y(0,t)=0,\ y(1,t)=h(t), \quad t\in(0,T),
\end{gathered}
\tag{1.6}
$$

respectively. As we know, the eigenvalues of classical heat equations are less than zero and have the negative infinity as the limit. This property guarantees that the solutions of the heat equation will naturally decay. In other words, after a sufficiently long time, the solutions of the heat equation will naturally decay to zero without any control to the system. For the viscoelastic parabolic system, like (1.6), the eigenvalues of its principal operator are also less than zero. But there is a class of eigenvalues which tend to zero, while others tend to $-\infty$. Thus, this fact motivates us to think of adding an appropriate control to the system, then we might able to obtain the approximate controllability of the viscoelastic parabolic system. It is worth mentioning that the method used in [20, 22] works in the case where the system possesses negative eigenvalues. It may not be applicable for the case of positive eigenvalues or complex eigenvalues which arise from some systems like (1.4) as we had attempted. Nevertheless, it provides us some useful insight which encourages us to analyze system (1.4) by appropriately expanding the function space.

The rest of this article is organized as follows. In Section 2, we introduce some preliminary definitions and state our main results. In Section 3, by defining a Hilbert space $H_{\theta,k}$ for all $\theta \in R$ and $k \geq 0$, we derive the expression of solutions of the corresponding dual system and present the properties of solutions in the space $H_{\theta,k}$. Section 4 is dedicated to the approximate controllability of system (1.4) by means of the duality method and the Hahn-Banach theorem in the product space $H_{\theta,k} \times H_{\theta,k}$.

2. Preliminaries and statement of main results

Throughout this article, we use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H^s(\Omega)$ with the usual norms $\|\cdot\|_p$ and $\|\cdot\|_{H^s(\Omega)}$, respectively. We denote $H_0^s(\Omega)$ by the complete space of $C_c^\infty(\Omega)$ according to the norm $\|\cdot\|_{H^s(\Omega)}$, and $(\cdot,\cdot)_{L^2(\Omega)}$ by the inner product in $L^2(\Omega)$. In addition, X and V are the state space and the control space, respectively. $O(x;d)$ denotes a neighbourhood with the center x and the radius d.

To make the paper sufficiently self-contained and present our discussions in a straightforward manner, let us briefly recall the definitions of exact controllability and approximate controllability of system (1.4).

Definition 2.1 (Exact controllability)**.** The control system (1.4) is said to be exactly controllable if, for the given target state $\big(u^{(1)}(x), u_t^{(1)}(x)\big) \in X$, there exist $t^* \in (0,T)$ and a control function $v(t) \in V$ which drives the solution $\big(u(x,t;v), u_t(x,t;v)\big)$ from the initial state $\big(u^{(0)}, u_t^{(0)}\big)$ to the prescribed target, that is,

$$(u(x,t^*;v), u_t(x,t^*;v)) = \big(u^{(1)}(x), u_t^{(1)}(x)\big).$$

Definition 2.2 (Approximate controllability)**.** The control system (1.4) is said to be approximately controllable if, for the given target state $\big(u^{(1)}(x), u_t^{(1)}(x)\big) \in X$, there exist $t^* \in (0,T)$, $\varepsilon > 0$ and a control function $v(t) \in V$ which drives the solution $\big(u(x,t;v), u_t(x,t;v)\big)$ from the initial state $\big(u^{(0)}, u_t^{(0)}\big)$ to the ε-neighbourhood of the prescribed target; that is,

$$(u(x,t^*;v), u_t(x,t^*;v)) \in O\big(\big(u^{(1)}(x), u_t^{(1)}(x)\big); \varepsilon\big).$$

Denote by Φ the input mapping of the control system (1.4). We know that the well-posedness of system (1.4) can be established by using the Faedo-Galerkin method [3, 16], and Φ is unique under the given initial data $\big(u^{(0)}, u_t^{(0)}\big)$ and the control v. The range of Φ is the so-called reachable set:

$$R(T) := \big\{(u(T,x), u_t(T,x)) : u(T) = u\big(T; u^{(0)}, u_t^{(0)}, v\big)\big\},$$

where T is a given positive constant, and u is the solution of system (1.4). The controllability can also be described from the perspective of the input mapping [8].

Definition 2.3. The control system (1.4) is said to be approximately controllable if the reachable set $R(t)$ is dense in the state space X. Moreover, system (1.4) is said to be exactly controllable if $R(T) \equiv X$.

Remark 2.4. Let

$$\widetilde{R}(T) := \{(u_t(T,x), -u(T,x)) : u(T) = u\big(T; u^{(0)}, u_t^{(0)}, v\big)\}.$$

Note that the mapping $\Gamma : R(T) \to \widetilde{R}(T)$ given by

$$\big(u(T,x), u_t(T,x)\big) \mapsto \big(u_t(T,x), -u(T,x)\big)$$

is an isomorphism, and the two sets $R(T)$ and $\widetilde{R}(T)$ are equivalent in the sense of algebraic structure.

For any integrable function $u : (0,\pi) \to \mathbb{R}$, the n-th Fourier coefficient (with respect to the orthonormal basis $\{\sin(nx)\}_{n\geq 1}$ of $L^2(0,\pi)$) of u is defined by

$$\hat{u}_n = \int_0^\pi u(x)\sin(nx)dx,$$

from which it is easy to deduce that

$$u(x) = \sum_{n=1}^\infty \hat{u}_n \sin(nx).$$

For all $\theta \in \mathbb{R}$ and $k \geq 0$, let

$$H_{\theta,k} := \big\{u : (0,\pi) \to \mathbb{R} : \sum_{n=1}^\infty n^{2\theta}|\hat{u}_n|^2 e^{-kt} < \infty\big\},$$

endowed with the inner product

$$(u,v)_{\theta,k} = \sum_{n=1}^\infty n^{2\theta}\hat{u}_n\hat{v}_n e^{-kt}.$$

Then $H_{\theta,k}$ becomes a Hilbert space. Furthermore, when $k_2 > k_1 > 0$, we have

$$0 < e^{-k_2\varphi_n t} < e^{-k_1\varphi_n t} < 1.$$

So, we obtain

$$\sum_{n=1}^\infty n^{2\theta}|\hat{u}_n|^2 e^{-k_2 t} < \sum_{n=1}^\infty n^{2\theta}|\hat{u}_n|^2 e^{-k_1 t} < \sum_{n=1}^\infty n^{2\theta}|\hat{u}_n|^2,$$

which implies

$$H_{\theta,0} \subset H_{\theta,k_1} \subset H_{\theta,k_2}.$$

In addition, for any $\theta \geq 0$, one can verify that $H_{-\theta,k}$ is the dual space of $H_{\theta,k}$ with respect to the central space $H_{0,k}$. Hence, we can define the dual product of the product spaces $H^2_{\theta,k} := H_{\theta,k} \times H_{\theta,k}$ and $H^2_{-\theta,k} := H_{-\theta,k} \times H_{-\theta,k}$ by

$$\langle (u_1,u_2),(w_1,w_2)\rangle_{H^2_{\theta,k},H^2_{-\theta,k}} := \int_0^\pi (u_1w_1 + u_2w_2)dx.$$

Remark 2.5. From the equivalence of norms, one can verify that

$$H_{0,0} = L^2(0,\pi), \quad H_{1,0} = H_0^1(0,\pi),$$
$$H_{-1,0} = H^{-1}(0,\pi), \quad H_{2,0} = H^2(0,\pi) \cap H_0^1(0,\pi).$$

To prove our main result, we need the following technical lemma.

Lemma 2.6 ([22, 20])**.** *Let $\{\beta_n\}$ and $\{\lambda_n\}$ be two sequences of complex numbers such that*

$$\sum_{n=1}^{\infty} |\beta_n| < \infty, \quad \mathrm{Re}\lambda_n < \Theta,$$

for each $n \geq 1$ and some number $\Theta \in \mathbb{R}$. Assume that the λ_n's are pairwise distinct, and that

$$\sum_{n=1}^{\infty} \beta_n e^{\lambda_n t} = 0$$

for a.e. $t \in (0,T)$. Then $\beta_n = 0$ for all $n \geq 1$.

Denote

$$V := \big\{\varphi \in L^2(0,T) : \int_0^T \varphi(t) e^t dt = 0\big\}. \tag{2.1}$$

Now, we are ready to summarize our main result.

Theorem 2.7 (Approximate controllability)**.** *There exists a boundary control function $v(t) \in V$ such that system (1.4) is approximately controllable in $H^2_{\theta,k}$, where $\theta < -\frac{7}{2}$ and $k > 0$.*

3. Spectral properties

In this section, we are concerned with an explicit solution of the following homogeneous initial boundary value problem

$$\begin{gathered} u_{tt} + u_{xxxx} - \int_0^t u_{xxxx}(s) ds = 0, \quad (x,t) \in (0,\pi) \times (0,T), \\ u(x,0) = u^{(0)}(x),\ u_t(x,0) = u_t^{(0)}(x), \quad x \in (0,\pi), \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \quad t \in (0,T), \end{gathered} \tag{3.1}$$

by the method of separation of variables. Then we discuss its properties in the space $H_{\theta,k}$.

3.1. Explicit solutions in the homogeneous case.

Let $u(x,t) = T(t)X(x) \neq 0$. Substituting it into the first equation of system (3.1), we have

$$\frac{T''(t)}{\int_0^t T(s) ds - T(t)} = \frac{X^{(4)}(x)}{X(x)}.$$

Obviously, this identity is true if and only if both sides are equal to the same nonzero constant μ. That is,

$$\begin{gathered} X^{(4)}(x) = \mu X(x), \quad x \in (0,\pi), \\ T''(t) + \mu T(t) - \mu \int_0^t T(s) ds = 0, \quad t > 0. \end{gathered} \tag{3.2}$$

By the boundary value conditions in (3.1), it induces an eigenvalue problem,

$$\begin{gathered} X^{(4)}(x) = \mu X(x), \quad x \in (0,\pi), \\ X(0) = X(\pi) = X''(0) = X''(\pi) = 0. \end{gathered} \tag{3.3}$$

A direct calculation yields

$$\begin{gathered} \mu = \mu_n = n^4, \quad n = 1,2,\dots; \\ X_n(x) = B_0 \sin(nx), \quad n = 1,2,\dots, \end{gathered}$$

where B_0 is an arbitrary constant.

Consider the resulting integro-differential equation

$$T_n''(t) + \mu_n T_n(t) - \mu_n \int_0^t T_n(s)ds = 0, \quad t > 0. \tag{3.4}$$

Taking differentiation on both sides of equation (3.4) with respect to the variable t, we obtain a 3rd order linear differential equation

$$T_n'''(t) + \mu_n T_n'(t) - \mu_n T_n(t) = 0 \tag{3.5}$$

with the characteristic equation

$$\lambda^3 + \mu_n \lambda - \mu_n = 0, \quad \mu_n > 0. \tag{3.6}$$

In view of the fact that $\sigma = y + z$ is a solution to the equation

$$\sigma^3 - 3yz\sigma - (y^3 + z^3) = 0, \tag{3.7}$$

for equation (3.6), we can try to find the solution in the form $\lambda = y + z$. So, the coefficient μ_n must satisfy

$$\mu_n = -3yz = (y^3 + z^3).$$

To find y and z satisfying the above equation, we note that $y^3 z^3 = -\mu_n^3/27$ and $y^3 + z^3 = \mu_n$, so y^3 and z^3 must be the roots of the quadratic equation

$$r^2 - \mu_n r - \frac{\mu_n^3}{27} = 0. \tag{3.8}$$

Let

$$\Delta_n := \frac{\Delta}{4},$$

where $\Delta = \mu_n^2 + \frac{4}{27}\mu_n^3$ is the discriminant of equation (3.8). Since $\Delta > 0$, two solutions of equation (3.8) can be expressed as

$$r_{1,2} = \frac{\mu_n}{2} \pm \sqrt{\Delta_n}.$$

By making the transformations:

$$y_n = \Big(\frac{\mu_n}{2} + \sqrt{\Delta_n}\Big)^{1/3}, \quad z_n = \Big(\frac{\mu_n}{2} - \sqrt{\Delta_n}\Big)^{1/3},$$

three sets of solutions of equation (3.6) are

$$\lambda_{1,n} = y_n + z_n, \tag{3.9}$$

$$\lambda_{2,n} = y_n e^{2\pi i/3} + z_n e^{-2\pi i/3} = -\frac{1}{2}(y_n + z_n) + i\frac{\sqrt{3}}{2}(y_n - z_n), \tag{3.10}$$

$$\lambda_{3,n} = y_n e^{-\frac{2\pi i}{3}} + z_n e^{2\pi i/3} = -\frac{1}{2}(y_n + z_n) - i\frac{\sqrt{3}}{2}(y_n - z_n). \tag{3.11}$$

Hence, the general solution of equation (3.5) reads

$$\begin{aligned} T_n(t) &= B_1 e^{\lambda_{1,n} t} + B_2 e^{-\frac{t}{2}(y_n+z_n)} \sin\Big(\frac{\sqrt{3}}{2}(y_n - z_n)t\Big) \\ &\quad + B_3 e^{-\frac{t}{2}(y_n+z_n)} \cos\Big(\frac{\sqrt{3}}{2}(y_n - z_n)t\Big) \\ &= B_1 e^{\varphi_n t} + B_2 e^{-\frac{\varphi_n}{2}t} \sin\Big(\frac{\sqrt{3}\phi_n}{2}t\Big) + B_3 e^{-\frac{\varphi_n}{2}t} \cos\Big(\frac{\sqrt{3}\phi_n}{2}t\Big), \end{aligned} \tag{3.12}$$

where $\varphi_n = y_n + z_n$, $\phi_n = y_n - z_n$, and B_i $(i = 1, 2, 3)$ are arbitrary constants. So, direct calculations give

$$\begin{aligned} T_n''(t) &= B_1 \varphi_n^2 e^{\varphi_n t} + \Big(B_2 \frac{\varphi_n^2 - 3\phi_n^2}{4} + B_3 \frac{\sqrt{3}\varphi_n\phi_n}{2}\Big) e^{-\frac{\varphi_n}{2}t} \sin\Big(\frac{\sqrt{3}\phi_n}{2}t\Big) \\ &\quad + \Big(B_3 \frac{\varphi_n^2 - 3\phi_n^2}{4} - B_2 \frac{\sqrt{3}\varphi_n\phi_n}{2}\Big) e^{-\frac{\varphi_n}{2}t} \cos\Big(\frac{\sqrt{3}\phi_n}{2}t\Big). \end{aligned} \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.4) yields

$$A_1 e^{\varphi_n t} + A_2 e^{-\frac{\varphi_n t}{2}} \sin\Big(\frac{\sqrt{3}\phi_n}{2}t\Big) + A_3 e^{-\frac{\varphi_n t}{2}} \cos\Big(\frac{\sqrt{3}\phi_n}{2}t\Big) + A_4 = 0,$$

where

$$\begin{gathered} A_1 = \Big(\varphi_n^2 + \mu_n - \frac{\mu_n}{\varphi_n}\Big) B_1, \\ A_2 = \Big(\frac{\varphi_n^2 - 3\phi_n^2}{4} + \mu_n + \frac{2\varphi_n\mu_n}{\varphi_n^2 + 3\phi_n^2}\Big) B_2 + \Big(\frac{\sqrt{3}\varphi_n\varphi_n}{2} - \frac{2\sqrt{3}\phi_n\mu_n}{\varphi_n^2 + 3\phi_n^2}\Big) B_3, \\ A_3 = \Big(\frac{\varphi_n^2 - 3\phi_n^2}{4} + \mu_n + \frac{2\varphi_n\mu_n}{\varphi_n^2 + 3\phi_n^2}\Big) B_3 + \Big(\frac{2\sqrt{3}\phi_n\mu_n}{\varphi_n^2 + 3\phi_n^2} - \frac{\sqrt{3}\varphi_n\varphi_n}{2}\Big) B_2, \\ A_4 = \Big(\frac{B_1}{\varphi_n} - \frac{2\sqrt{3}\phi_n B_2}{\varphi_n^2 + 3\phi_n^2} - \frac{2\varphi_n B_3}{\varphi_n^2 + 3\phi_n^2}\Big) \mu_n\,. \end{gathered}$$

Note that $\lambda_{i,n}$ $(i = 1, 2, 3)$ are the eigenvalues of equation (3.6), then we can derive that $A_i = 0$ $(i = 1, 2, 3)$. This indicates that $A_4 = 0$. Since $\mu_n = n^4 > 0$, there holds

$$B_1 = \frac{2\sqrt{3}\varphi_n\phi_n}{\varphi_n^2 + 3\phi_n^2} B_2 + \frac{2\varphi_n^2}{\varphi_n^2 + 3\phi_n^2} B_3.$$

Thus, the solution of the second equation of (3.2) reads

$$\begin{aligned} T_n(t) &= \Big(\frac{2\sqrt{3}\varphi_n\phi_n}{\varphi_n^2 + 3\phi_n^2} B_2 + \frac{2\varphi_n^2}{\varphi_n^2 + 3\phi_n^2} B_3\Big) e^{\varphi_n t} \\ &\quad + B_2 e^{-\varphi_n t/2} \sin\Big(\frac{\sqrt{3}\phi_n t}{2}\Big) + B_3 e^{-\varphi_n t/2} \cos\Big(\frac{\sqrt{3}\phi_n t}{2}\Big). \end{aligned}$$

Taking differentiation gives

$$\begin{aligned} T_n'(t) &= \Big(\frac{2\sqrt{3}\varphi_n^2\phi_n}{\varphi_n^2 + 3\phi_n^2} B_2 + \frac{2\varphi_n^3}{\varphi_n^2 + 3\phi_n^2} B_3\Big) e^{\varphi_n t} \\ &\quad + \Big(-\frac{\varphi_n}{2} B_2 - \frac{\sqrt{3}\phi_n}{2} B_3\Big) e^{-\varphi_n t/2} \sin\Big(\frac{\sqrt{3}\phi_n t}{2}\Big) \\ &\quad + \Big(\frac{\sqrt{3}\phi_n}{2} B_2 - \frac{\varphi_n}{2} B_3\Big) e^{-\varphi_n t/2} \cos\Big(\frac{\sqrt{3}\phi_n t}{2}\Big). \end{aligned}$$

Thus, we can deduce the following lemma.

Lemma 3.1 (Representation of solution)**.** *If the initial data $u^{(0)}$ and $u_t^{(0)}$ can be expanded to the following sine series*

$$u^{(0)}(x) = \sum_{n=1}^{\infty} c_n \sin(nx), \quad u_t^{(0)}(x) = \sum_{n=1}^{\infty} d_n \sin(nx), \tag{3.14}$$

where $\{c_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$ are two sequences of complex numbers, then the solution of system (3.1) *can be expressed as*

$$u(x,t) = \sum_{n=1}^{\infty} f(c_n, d_n, \varphi_n, \phi_n, t) \sin(nx), \tag{3.15}$$

where

$$\begin{aligned} f(c_n, d_n, \varphi_n, \phi_n, t) &= (D_1 c_n + D_2 d_n) e^{\varphi_n t} + (D_3 c_n + D_4 d_n) e^{-\varphi_n t/2} \sin\Big(\frac{\sqrt{3}\phi_n t}{2}\Big) \\ &\quad + (D_5 c_n + D_6 d_n) e^{-\varphi_n t/2} \cos\Big(\frac{\sqrt{3}\phi_n t}{2}\Big) \end{aligned}$$

with

$$\begin{aligned} D_1 &= \frac{\big(6 - 10\sqrt{3}\big)\varphi_n^4 \phi_n + \big(6 - 6\sqrt{3}\big)\varphi_n^2 \phi_n^3}{-9\sqrt{3}\varphi_n^4 \phi_n - 30\sqrt{3}\varphi_n^2 \phi_n^3 - 9\sqrt{3}\varphi_n^5}, \\ D_2 &= \frac{-4\sqrt{3}\varphi_n^3 \phi_n - 12\sqrt{3}\varphi_n \phi_n^3}{-9\sqrt{3}\varphi_n^4 \phi_n - 30\sqrt{3}\varphi_n^2 \phi_n^3 - 9\sqrt{3}\varphi_n^5}, \\ D_3 &= \frac{\big(3\varphi_n^3 + 3\varphi_n \phi_n^2\big)\big(\varphi_n^2 + 3\phi_n^2\big)}{-9\sqrt{3}\varphi_n^4 \phi_n - 30\sqrt{3}\varphi_n^2 \phi_n^3 - 9\sqrt{3}\varphi_n^5}, \\ D_4 &= \frac{-6\big(\varphi_n^2 + \phi_n^2\big)\big(\varphi_n^2 + 3\phi_n^2\big)}{-9\sqrt{3}\varphi_n^4 \phi_n - 30\sqrt{3}\varphi_n^2 \phi_n^3 - 9\sqrt{3}\varphi_n^5}, \\ D_5 &= \frac{\big(-5\sqrt{3}\varphi_n^2 \phi_n - 3\sqrt{3}\phi_n^3\big)\big(\varphi_n^2 + 3\phi_n^2\big)}{-9\sqrt{3}\varphi_n^4 \phi_n - 30\sqrt{3}\varphi_n^2 \phi_n^3 - 9\sqrt{3}\varphi_n^5}, \\ D_6 &= \frac{4\sqrt{3}\varphi_n \phi_n \big(\varphi_n^2 + 3\phi_n^2\big)}{-9\sqrt{3}\varphi_n^4 \phi_n - 30\sqrt{3}\varphi_n^2 \phi_n^3 - 9\sqrt{3}\varphi_n^5}. \end{aligned}$$

3.2. **Properties of solutions in $H_{\theta,k}$.** In this subsection, we will deduce some properties of the solutions in the Hilbert space $H_{\theta,k}$.

Proposition 3.2. *Assume that $\theta \in \mathbb{R}$. If the initial data $u^{(0)}, u_t^{(0)} \in H_{\theta,0}$ and the given condition* (3.14) *in Lemma 3.1 holds, then we have*

$$u \in C(\mathbb{R}^+; H_{\theta,2}), \quad u_t \in C(\mathbb{R}^+; H_{\theta,2}).$$

Furthermore, if $\theta > 7/2$, then we have

$$\sum_{n=1}^{\infty} n^3 \left(|D_1 c_n + D_2 d_n| + |D_3 c_n + D_4 d_n| + |D_5 c_n + D_6 d_n|\right) < \infty$$

and $u_{xxx}(0,\cdot) \in C(\mathbb{R}^+, H_{\theta,1})$.

Proof. Since $u^{(0)}, u_t^{(0)} \in H_{\theta,0}$, we have

$$\sum_{n=1}^{\infty} n^{2\theta}|c_n|^2 e^{-2\varphi_n t} < \sum_{n=1}^{\infty} n^{2\theta}|c_n|^2 < \infty,$$
$$\sum_{n=1}^{\infty} n^{2\theta}|d_n|^2 e^{-2\varphi_n t} < \sum_{n=1}^{\infty} n^{2\theta}|d_n|^2 < \infty.$$

From $0 < e^{-2\varphi_n t} < 1$ and the boundedness of sine (cosine) functions, it is easy to see that

$$\begin{aligned}
& n^{2\theta}\left|f(c_n,d_n,\varphi_n,\phi_n,t)\right|^2 e^{-2\varphi_n t} \\
&\leq 2n^{2\theta}\left|(D_1c_n + D_2d_n)e^{\varphi_n t}\right|^2 e^{-2\varphi_n t} \\
&\quad + 2n^{2\theta}\left|(D_3c_n + D_4d_n)e^{-\varphi_n t/2}\sin\Big(\frac{\sqrt{3}\phi_n t}{2}\Big)\right|^2 e^{-2\varphi_n t} \\
&\quad + 2n^{2\theta}\Big|(D_5c_n + D_6d_n)e^{-\varphi_n t/2}\cos\Big(\frac{\sqrt{3}\phi_n t}{2}\Big)\Big|^2 e^{-2\varphi_n t} \\
&\leq 2n^{2\theta}\Big(|D_1c_n + D_2d_n|^2 + |D_3c_n + D_4d_n|^2 + |D_5c_n + D_6d_n|^2\Big) \\
&\leq Mn^{2\theta}\left(|c_n|^2 + |d_n|^2\right),
\end{aligned}$$

where M is a positive constant.

Similarly, there exists another positive number M_1 such that

$$n^{2\theta}\left|f'(c_n,d_n,\varphi_n,\phi_n,t)\right|^2 e^{-2\varphi_n t} \leq M_1 n^{2\theta}\left(|c_n|^2 + |d_n|^2\right).$$

Hence, we have $u \in C(\mathbb{R}^+; H_{\theta,2})$, $u_t \in C(\mathbb{R}^+; H_{\theta,2})$. Moreover, by (3.15), we can obtain

$$u_{xxx}(0,t) = \sum_{n=1}^{\infty}(-n)^3 f(c_n,d_n,\varphi_n,\phi_n,t).$$

If $\theta > 7/2$, owing to

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^3\left|(D_1c_n + D_2d_n)e^{\varphi_n t}\right| e^{-\varphi_n t} \\
&= \sum_{n=1}^{\infty} n^3\left|D_1c_n + D_2d_n\right| \\
&\leq \Big(\sum_{n=1}^{\infty} n^{-2(\theta-3)}\Big)^{1/2}\Big(\sum_{n=1}^{\infty} n^{2\theta}\left|(D_1c_n + D_2d_n)\right|^2\Big)^{1/2}, \\
& \sum_{n=1}^{\infty} n^3\Big|(D_3c_n + D_4d_n)e^{-\varphi_n t/2}\sin\Big(\frac{\sqrt{3}\phi_n t}{2}\Big)\Big| \cdot e^{-\varphi_n t} \\
&\leq \sum_{n=1}^{\infty} n^3|D_3c_n + D_4d_n| \\
&\leq \Big(\sum_{n=1}^{\infty} n^{-2(\theta-3)}\Big)^{1/2}\Big(\sum_{n=1}^{\infty} n^{2\theta}|(D_3c_n + D_4d_n)|^2\Big)^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^3 \Big|(D_5c_n + D_6d_n)e^{-\varphi_n t/2}\cos\Big(\frac{\sqrt{3}\phi_n t}{2}\Big)\Big|e^{-\varphi_n t}\\
&\le \sum_{n=1}^{\infty} n^3\,|D_5c_n + D_6d_n|\\
&\le \Big(\sum_{n=1}^{\infty} n^{-2(\theta-3)}\Big)^{1/2}\Big(\sum_{n=1}^{\infty} n^{2\theta}|(D_5c_n + D_6d_n)|^2\Big)^{1/2},
\end{aligned}$$

we can deduce that

$$\sum_{n=1}^{\infty} n^3\left(|D_1c_n + D_2d_n| + |D_3c_n + D_4d_n| + |D_5c_n + D_6d_n|\right) < \infty.$$

and $u_{xxx}(0,\cdot) \in C(\mathbb{R}^+, H_{\theta,1})$. □

4. Approximate Controllability

In this section, we study the approximate controllability of system (1.4). To this end, we first consider the dual system of (1.4) as follows:

$$\begin{gathered}
w_{tt} - w_{xxxx} + \int_t^T w_{xxxx}(s)ds = 0, \quad (x,t) \in (0,\pi)\times(0,T),\\
w(x,T) = w^{(T)}(x),\ w_t(x,T) = w_t^{(T)}(x), \quad x \in (0,\pi),\\
w(0,t) = w(\pi,t) = w_{xx}(0,t) = w_{xx}(\pi,t) = 0, \quad t \in (0,T).
\end{gathered} \tag{4.1}$$

Assume that $w^{(T)}$ and $w_t^{(T)}$ can be expanded as

$$w^{(T)}(x) = \sum_{n=1}^{\infty} \tilde{c}_n \sin(nx), \tag{4.2}$$

$$w_t^{(T)}(x) = \sum_{n=1}^{\infty} \tilde{d}_n \sin(nx), \tag{4.3}$$

respectively, where $\{\tilde{c}_n\}_{n\ge1}$ and $\{\tilde{d}_n\}_{n\ge1}$ belong to $\mathbb{C}$. Similar to Lemma 3.1, the solution of system (4.1) can be expressed as

$$w(x,t) = \sum_{n=1}^{\infty} \tilde{f}(\tilde{c}_n, \tilde{d}_n, \varphi_n, \phi_n, t)\sin(nx), \tag{4.4}$$

where

$$\begin{aligned}
\tilde{f}(\tilde{c}_n, \tilde{d}_n, \varphi_n, \phi_n, t) &= (D_1\tilde{c}_n + D_2\tilde{d}_n)e^{\varphi_n(T-t)}\\
&\quad + (D_3\tilde{c}_n + D_4\tilde{d}_n)e^{-\frac{\varphi_n(T-t)}{2}}\sin\Big(\frac{\sqrt{3}\phi_n(T-t)}{2}\Big)\\
&\quad + (D_5\tilde{c}_n + D_6\tilde{d}_n)e^{-\frac{\varphi_n(T-t)}{2}}\cos\Big(\frac{\sqrt{3}\phi_n(T-t)}{2}\Big),
\end{aligned}$$

and D_i $(i = 1,2,3,4,5,6)$ is the same as given in Lemma 3.1. So, by using an analogous argument as shown in Proposition 3.2, we obtain the following result.

Proposition 4.1. *Assume that* $\theta \in \mathbb{R}$. *If* $(w^{(T)}, w_t^{(T)}) \in H_{-\theta,0} \times H_{-\theta,0}$, *then*

$$w \in C(\mathbb{R}^+; H_{-\theta,2}), \quad w_t \in C(\mathbb{R}^+; H_{-\theta,2}).$$

Furthermore, if $\theta < -7/2$, *then we have*

$$\sum_{n=1}^{\infty} n^3 \Big(|D_1\tilde{c}_n + D_2\tilde{d}_n| + |D_3\tilde{c}_n + D_4\tilde{d}_n| + |D_5\tilde{c}_n + D_6\tilde{d}_n| \Big) < \infty,$$

and $w_{xxx}(\pi, \cdot) \in C(\mathbb{R}^+, H_{-\theta,1})$.

Without loss of generality, we assume that the initial data $u^{(0)} = u_t^{(0)} = 0$ in system (1.4). We can obtain the following lemma regarding the approximate controllability of system (1.4).

Lemma 4.2. *Assume that for all* $v = v(t)$,

$$\int_0^T v(t)\Big(w_{xxx}(0,t) - \int_t^T w_{xxx}(0,s)ds\Big)dt = 0 \tag{4.5}$$

holds if and only if $u^{(T)} = u_t^{(T)} = 0$. *Then system* (1.4) *is approximately controllable in the product space* $H_{\theta,k} \times H_{\theta,k}$ *(*$k \ge 0$*).*

Remark 4.3. The significance of this lemma is somehow similar to the uniqueness theorem in the HUM framework. It will play a critical role in the proof of the approximate controllability of system (1.4).

Remark 4.4. From the physical point of view, the term

$$w_{xxx}(0,t) - \int_t^T w_{xxx}(0,s)ds$$

represents the traction acting on the boundary, and its impact on the system is equivalent to $w_{xxx}(0,t)$, see [14].

Proof of Lemma 4.2. Let w be the solution of the dual system (4.1). Multiplying both sides of the first equation of system (1.4) by w and then integrating it on $(0,\pi) \times (0,T)$ leads to

$$\int_0^T \int_0^\pi u_{tt} w \, dx \, dt - \int_0^T \int_0^\pi u_{xxxx} w \, dx \, dt + \int_0^T \int_0^\pi \Big(\int_0^t u_{xxxx}(x,s)ds\Big) w \, dx \, dt = 0.$$

Using the initial value, terminal value and boundary value, by integration by parts, we have

$$\begin{aligned}
\int_0^T \int_0^\pi u_{tt} w \, dx \, dt &= \int_0^\pi \int_0^T u(x,t) w_{tt}(x,t) \, dt \, dx + \int_0^\pi (u_t w - u w_t)\,|_0^T dx \\
&= \int_0^\pi \int_0^T u(x,t) w_{tt}(x,t) \, dt \, dx \\
&\quad + \int_0^\pi \Big(u_t(T,x) w^{(T)}(x) - u(T,x) w_t^{(T)}(x) \Big) dx,
\end{aligned}$$

$$\begin{aligned}
\int_0^T \int_0^\pi u_{xxxx} w \, dx \, dt &= \int_0^T \int_0^\pi u w_{xxxx} \, dx \, dt + \int_0^T (u_{xxx} w - u w_{xxx})|_0^\pi dt \\
&\quad + \int_0^T \int_0^\pi (u_x w_{xxx} - u_{xxx} w_x) \, dx \, dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_0^\pi u w_{xxxx}\, dx\, dt + \int_0^T (u_{xxx} w - u w_{xxx})|_0^\pi dt \\
&\quad + \int_0^T (u_x w_{xx} - u_{xx} w_x)|_0^\pi dt \\
&= \int_0^T \int_0^\pi u w_{xxxx}\, dx\, dt + \int_0^T v(t) w_{xxx}(0,t) dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \int_0^\pi \Big(\int_0^t u_{xxxx}(x,s) ds \Big) w\, dx\, dt \\
&= \int_0^T \int_0^\pi \Big(\int_t^T w_{xxxx}(x,s) ds \Big) u\, dx\, dt + \int_0^T v(t) \int_t^T w_{xxx}(0,s) ds dt.
\end{aligned}
$$

So, we further deduce that

$$
\begin{aligned}
&\int_0^\pi \int_0^T u(x,t) \Big(w_{tt} - w_{xxxx} + \int_t^T w_{xxxx}(x,s) ds \Big)\, dt\, dx \\
&+ \int_0^\pi \Big(u_t(T,x) w^{(T)}(x) - u(T,x) w_t^{(T)}(x) \Big)\, dx - \int_0^T v(t) w_{xxx}(0,t) dt \\
&+ \int_0^T v(t) \int_t^T w_{xxx}(0,s) ds dt \\
&= 0.
\end{aligned}
$$

Note that w is the solution of the dual system (4.1). Then we have

$$
\begin{aligned}
&\int_0^\pi \Big(u_t(T,x) w^{(T)}(x) - u(T,x) w_t^{(T)}(x) \Big) dx \\
&= \int_0^T v(t) \Big(w_{xxx}(0,t) - \int_t^T w_{xxx}(0,s) ds \Big) dt,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
&\langle \big(u_t(T,x), -u(T,x) \big), \big(w^{(T)}(x), w_t^{(T)}(x) \big) \rangle_{H^2_{\theta,k}, H^2_{-\theta,k}} \\
&= \int_0^T v(t) \Big(w_{xxx}(0,t) - \int_t^T w_{xxx}(0,s) ds \Big) dt.
\end{aligned} \tag{4.6}
$$

In view of Definition 2.3, to prove the approximate controllability of system (1.4) in $H^2_{\theta,k}$ ($k \geq 0$), we just need to show that the reachable set $R(T)$ is dense in $H^2_{\theta,k}$ in the sense of isomorphism.

By way of contradiction, suppose that $R(T)$ is not dense in $H^2_{\theta,k}$. By the Hahn-Banach theorem, there exists

$$
(0,0) \neq (w^{(T)}, w_t^{(T)}) \in H^2_{-\theta,0} \tag{4.7}
$$

such that

$$
\langle (u_t(T,x), -u(T,x)), (w^{(T)}(x), w_t^{(T)}(x)) \rangle_{H^2_{\theta,k}, H^2_{-\theta,k}} = 0,
$$

for all $(u(T,x), u_t(T,x)) \in R(T)$. By (4.6) we have

$$
\int_0^T v(t) \Big(w_{xxx}(0,t) - \int_t^T w_{xxx}(0,s) ds \Big) dt = 0.
$$

However, according to condition (4.5), it is equivalent to

$$(w^{(T)}, w_t^{(T)}) = (0, 0).$$

This is a contradiction to (4.7). So, $R(T)$ is dense in $H^2_{\theta,k}$. This implies that system (4.1) is approximately controllable in $H^2_{\theta,k}$. □

Proof of Theorem 2.7. We first claim that, if $\theta < -\frac{7}{2}$ and $\left(w^{(T)}, w_t^{(T)}\right) \in H^2_{-\theta,0}$, then there exists a control function $v(t)$ such that $R(T)$ is dense in $H^2_{\theta,k}$. In fact, by Lemma 4.2, if

$$\int_0^T v(t)\Big(w_{xxx}(0,t) - \int_t^T w_{xxx}(0,s)ds\Big)dt = 0$$

for all $v(t) \in V$, we only need to prove that $(w^{(T)}, w_t^{(T)}) = (0, 0)$. Furthermore, by (4.2) and (4.3), it is equivalent to prove that $\tilde{c}_n = \tilde{d}_n = 0$ for all n. However, for all $v(t) \in \operatorname{span}\{e^t\}^\perp$, we have

$$\Big(v(t), \Big(w_{xxx}(0,t) - \int_t^T w_{xxx}(0,s)ds\Big)\Big)_{L^2(0,T)} = 0,$$

which implies that

$$w_{xxx}(0,t) \in \operatorname{span}\{e^t\}^{\perp\perp} = \operatorname{span}\{e^t\}.$$

Hence, by (4.4), there exists a real constant C such that

$$\sum_{n=1}^{\infty}(-n^3)\tilde{f}(\tilde{c}_n, \tilde{d}_n, \varphi_n, \phi_n, t) = Ce^t \tag{4.8}$$

for a.e. $t \in (0,T)$. Let

$$\begin{aligned} b_{1,n} &= (-n^3)(D_1\tilde{c}_n + D_2\tilde{d}_n),\\ b_{2,n} &= (-n^3)(D_3\tilde{c}_n + D_4\tilde{d}_n),\\ b_{3,n} &= (-n^3)(D_5\tilde{c}_n + D_6\tilde{d}_n). \end{aligned}$$

By Proposition 4.1, equation (4.8) becomes

$$\begin{aligned} &\sum_{n=1}^{\infty}\Big[b_{1,n}e^{\varphi_n(T-t)} + b_{2,n}\operatorname{Im}\Big(e^{-\frac{\varphi_n(T-t)}{2} + i(\frac{\sqrt{3}\phi_n(T-t)}{2})}\Big)\Big]\\ &+ \sum_{n=1}^{\infty} b_{3,n}\operatorname{Re}\Big(e^{-\frac{\varphi_n(T-t)}{2} + i(\frac{\sqrt{3}\phi_n(T-t)}{2})}\Big) - Ce^t = 0. \end{aligned}$$

Take $\tau = T - t$ and $b_0 = -Ce^T$. Then we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\Big[b_{1,n}e^{\varphi_n\tau} + b_{2,n}\operatorname{Im}\Big(e^{-\frac{\varphi_n\tau}{2} + i(\frac{\sqrt{3}\phi_n\tau}{2})}\Big)\Big]\\ &+ \sum_{n=1}^{\infty} b_{3,n}\operatorname{Re}\Big(e^{-\frac{\varphi_n\tau}{2} + i(\frac{\sqrt{3}\phi_n\tau}{2})}\Big) + b_0e^{-\tau} = 0. \end{aligned}$$

for a.e. $\tau \in (0,T)$. Since $\theta < -7/2$ and $\left(w^{(T)}, w_t^{(T)}\right) \in H^2_{-\theta,0}$, by Proposition 4.1, we deduce that

$$\sum_{n=1}^{\infty}\left(|b_{1,n}| + |b_{2,n}| + |b_{3,n}|\right) < \infty.$$

According to Lemma 2.6, we have $b_{i,n} = 0$ $(i = 1, 2, 3)$. Moreover, it is easy to verify that the system

$$(-n^3)(D_1\tilde{c}_n + D_2\tilde{d}_n) = 0,$$
$$(-n^3)(D_3\tilde{c}_n + D_4\tilde{d}_n) = 0,$$
$$(-n^3)(D_5\tilde{c}_n + D_6\tilde{d}_n) = 0.$$

has only the zero solution. Thus, we obtain $\tilde{c}_n = \tilde{d}_n = 0$ and $\big(w^{(T)}, w_t^{(T)}\big) = (0, 0)$. So far, we have found a control function $v \in V$ such that the reachable set $R(T)$ is dense in $H^2_{\theta,k}$, where $\theta < -\frac{7}{2}$ and $k > 0$. Consequently, system (1.4) is approximately controllable in the Hilbert space $H^2_{\theta,k}$ with $\theta < -\frac{7}{2}$ and $k > 0$. □

Remark 4.5. It is notable that our approach can also be extended to the distributed parameter systems with positive eigenvalues of the principal operators. For the case of parabolic control systems with negative eigenvalues of the principal operators, we only need to consider the Hilbert space $H_{\theta,0}$, which is equivalent to the space H_α in [22].

Acknowledgments. This work was supported by the National Science Foundation of China under 11671128, by the Science and Technology Plan Project of Hunan Province under 2016TP1020, by the Science Research Project of Hengyang Normal University under 16D01, by the Application-oriented Special Disciplines, Double First-Class University Project of Hunan Province under Xiangjiaotong [2018] 469, and by the Science Research Project of Education Department of Hunan Province under 17A029.

The first author would like to thank the School of Mathematical and Statistical Sciences of University of Texas-Rio Grande Valley for its hospitality and generous support during his visiting from September 2017 to February 2018. He would also like to thank Professor Qiuyi Dai for his useful suggestions.

References

[1] Barbu, V.; Iannelli, M.; *Controllability of the heat equation with memory*, Differential Integral Equations, 13 (2000), 1393-1412.

[2] Carrasco, A.; Guevara, G.; Leiva, H.; *Controllability of the impulsive semilinear beam equation with memory and delay*, IMA J. Math. Control Inform., 34 (2017), https://doi.org/10.1093/imamci/dnx042.

[3] Cavalcanti, M. M.; Oquendo ,H. P.; *Frictional versus viscoelastic damping in a semilinear wave equation*, SIAM J. Control Optim., 42 (4) (2003), 1310-1324.

[4] Chaves-Silva, Felipe W.; Zhang, X.; Zuazua, E.; *Controllability of Evolution Equations with Memory*, SIAM J. Control Optim., 55 (4) (2017), 2437-2459.

[5] Dafermos, C. M.; *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal., 37(4) (1970), 297-308.

[6] Dafermos, C. M.; *An abstract Volterra equation with applications to linear viscoelasticity*, J. Differential Equations, 7 (3) (1970), 554-569.

[7] Guerrero, S.; Imanuvilov, O. Y.; *Remarks on non-controllability of the heat equation with memory*, ESAIM Control Optim. Calc. Var., 19 (1) (2013), 288-300.

[8] Guo, B. Z.; Chai, S. G.; Control Theory of Linear Infinite-Dimensional Systems (in Chinese), Science Press, Beijing, 2012.

[9] Ivanov, S.; Pandolfi, L.; *Heat equation with memory: Lack of controllability to rest*, J. Math. Anal. Appl. 355(1) (2009) 1-11.

[10] Lu, Q.; Zhang, X.; Zuazua, E.; *Null controllability for wave equations with memory*, J. Math. Pure Appl., 108 (4) (2017), 500-531.

[11] Lasiecka, I.; *Exponential decay rates for the solutions of the Euler-Bernoulli equations with boundary dissipation occuring in the moments only*, J. Differential Equations, 95 (1) (1992), 169-182.
[12] Lasiecka, I.; Triggiani, R.; *Exact controllability of the Euler-Bernoulli equation with boundary controls for displacement and moment*, J. Math. Anal. Appl., 146(1) (1990), 1-33.
[13] Lions, J.L.; *Controlabilite Exacte, Perturbations et Stabilisation de Systems Distribues*, Masson, Paris, 1988.
[14] Loreti, P.; Pandolfi, L.; Sforza, D.; *Boundary controllability and observability of a viscoelastic string*, SIAM J. Control Optim. 50(2) (2012) 820-844.
[15] Loreti, P.; Sforza, D.; *Reachability problems for a class of integro-differential equations*, J. Differential Equations, 248 (7) (2010), 1711-1755.
[16] Munoz Rivera, J. E.; Quispe Gomez, F.P.; *Existence and decay in non linear viscoelasticity*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., 6(8) (2003), 1-37.
[17] Pandolfi, L.; *Riesz systems and an identification problem for heat equations with memory*, Discrete Contin. Dyn. Syst. Ser. S, 4 (3) (2011), 745-759.
[18] Renardy, M.; Hrusa, W. J.; Nohel J. A.; *Mathematical Problems in Viscoelasticity*, Longman Scientific and Technical, New York, 1987.
[19] Romanov, I.; Shamaev, A.; *Exact controllability of the distributed system, governed by string equation with memory*, J. Dyn. Control Syst., 19(4) (2013), 611-623.
[20] Rosier, L.; Rouchon, P.; *On the controllability of a wave equation with structural damping*, Int. J. Tomogr. Stat., 5 (2007), 79-84.
[21] Tao, Q.; Gao, H.; *On the null controllability of heat equation with memory*, J. Math. Anal. Appl., 440 (1) (2016), 1-13.
[22] Tao, Q.; Gao, H.; Zhang, B.; *Approximate controllability of a parabolic equation with memory*, Nonlinear Anal. Hybrid Syst. 6(2) (2012) 839-845.
[23] Zhou, X.; Gao, H.; *Controllability of a class of heat equations with memory in one dimension*, Math. Methods Appl. Sci., 40 (8) (2017), 3066-3078.

Zhifeng Yang
College of Mathematics and Statistics, Hengyang Normal University, Hengyang, Hunan 421002, China.
Hunan Provincial Key Lab of Intelligent Information Processing and Applications, Hunan 421002, China
E-mail address: yzfmath@hynu.edu.cn

Zhaosheng Feng
Department of Mathematics, University of Texas Rio Grande Valley, Edinburg, TX 78539, USA
E-mail address: zhaosheng.feng@utrgv.edu