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L^p-ESTIMATES FOR A SCHRÖDINGER EQUATION ASSOCIATED WITH THE HARMONIC OSCILLATOR

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Abstract. In this article we obtain Strichartz estimates for a Schrödinger equation associated with the harmonic oscillator and the Laplacian. Our main tools are embeddings between Lebesgue and Triebel-Lizorkin spaces.

1. Introduction

In this article we consider the quantum harmonic oscillator $H := -\Delta + |x|^2$ on $\mathbb{R}^n$ where Δ is the standard Laplacian. We obtain regularity for the Schrödinger equation (associated with H)

$$iu_t(t,x) - Hu(t,x) = 0, \tag{1.1}$$

with initial data $u(0,\cdot) = f$. It is well known that this is an important model in quantum mechanics, see for example Feynman and Hibbs [6]. As a consequence of the regularity we have estimates for the classical Schrödinger equation

$$iu_t(t,x) + \Delta u(t,x) = 0. \tag{1.2}$$

Regularity for (1.1) has been extensively studied; see for example Thangavelu [17, Section 5], Bongioanni and Torrea [2], Bongioanni and Rogers [3], Yajima [19], and the references therein. On the other hand, regularity properties for (1.2) can be found in the seminal work by Ginibre and Velo [8], also in Moyua and Vega [9], in Keel and Tao [11], and in their references. The works by Carleson [4] and Dahlberg and Kenig [5] include pointwise convergence theorems for the solution $u(x,t) = e^{it\Delta}f$.

The following sharp result was proved in [9]: when $\frac{2(n+2)}{n} \leq p \leq \infty$ and $2 \leq q < \infty$ with $\frac{1}{q} \leq \frac{n}{2}(\frac{1}{2} - \frac{1}{p})$, the estimate

$$\|u(t,x)\|_{L^p_x(\mathbb{R}^n, L^q_t[0,2\pi])} \leq C_s\|f\|_{\mathcal{H}^s(\mathbb{R}^n)} \tag{1.3}$$

holds for all $s \geq s_{n,p,q} := n(\frac{1}{2} - \frac{1}{p}) - \frac{2}{q}$. Also if $s < s_{n,p,q}$, then (1.3) is false. In the result above $\mathcal{H}^s$ is the Sobolev space associated with H and with the norm $\|f\|_{\mathcal{H}^s} := \|H^{s/2}f\|_{L^2}$. The proof of (1.3) involves Strichartz estimates by Keel and

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Tao [11], and Wainger's Sobolev embedding theorem. It is important to mention that the machinery for the work by Keel and Tao [11] implies the estimate

$$\|u(t,x)\|_{L^q_t([0,2\pi],L^p_x(\mathbb{R}^n))} \leq C_p\|f\|_{L^2(\mathbb{R}^n)}, \tag{1.4}$$

for $2 \leq q < \infty$ and $\frac{1}{q} = \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, excluding the case $(p,q,n) = (\infty,2,2)$. On the other hand, Koch and Tataru proved estimate (1.4) for Schrödinger type operators in more general contexts, including the operator H. They also proved that estimates of this type cannot be obtained for $2 \leq p < \frac{2n}{n-2}$.

The following is a remarkable formula that links the solution of (1.1) to that of the classical Schrödinger equation (see Sjögren and Torrea [16]),

$$\|e^{-it((-\Delta+|x|^2))}f\|_{L^q[(0,\frac{\pi}{4}),L^p_x(\mathbb{R}^d)]} = \|e^{it\Delta}f\|_{L^q[(0,\infty),L^p_x(\mathbb{R}^d)]} \tag{1.5}$$

for $1 \leq p,q \leq \infty$ and $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{p})$. As it was pointed out in [16], the interval of integration in the t variable is now bounded, (1.4) remains true if the equality in (1.5) is replaced by the inequality $n(\frac{1}{2} - \frac{1}{p}) \leq \frac{2}{q}$, and the interval $(0,\pi/4)$ can be replaced by $(0,\pi/2)$. In such case the two norms are equivalent, for real functions f. In particular, (1.5) shows that (1.4) is equivalent to the following Strichartz estimate (see [12])

$$\|e^{it\Delta}f\|_{L^q[(0,\infty),L^p_x(\mathbb{R}^d)]} \leq C\|f\|_{L^2(\mathbb{R}^n)} \tag{1.6}$$

which holds if and only if $2 \leq p \leq \infty$ for $n = 1$, $2 \leq p < \infty$ for $n = 2$, and $2 \leq p < \frac{2n}{n-2}$ for $n = 2$ for $n \geq 3$.

The novelty of this article is that we provide regularity results for the Schödinger equation associated with H, involving L^p-Sobolev norms for the initial data instead of the L^2 and L^2-Sobolev bounds mentioned above. Our main result in this article is the following theorem.

Theorem 1.1. *Let $n > 2$, $2 \leq q < \infty$ and $1 \leq p \leq 2$ satisfy $|\frac{1}{2} - \frac{1}{p}| < \frac{1}{2n}$. Then the estimate*

$$\|u(t,x)\|_{L^{p'}_x[\mathbb{R}^n,L^q_t[0,2\pi]]} \leq C\|f\|_{W^{2s,p,H}(\mathbb{R}^n)} \tag{1.7}$$

holds for every $s \geq s_q := \frac{1}{2} - \frac{1}{q}$. In particular, if $q = 2$ we have

$$\|u(t,x)\|_{L^{p'}_x[\mathbb{R}^n,L^2_t[0,2\pi]]} \leq C\|f\|_{L^p(\mathbb{R}^n)}. \tag{1.8}$$

Moreover, for $n > 2$, $1 \leq p \leq 2$, and $1 \leq q \leq p'$, we have

$$\|u(t,x)\|_{L^{p'}_x[\mathbb{R}^n,L^q_t[0,2\pi]]} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \tag{1.9}$$

provided that $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{nq}$.

In the following remarks, we briefly discuss some consequences of our main result.

The main contributions of Theorem 1.1 are the estimates (1.7) and (1.9). This theorem laso provides an analogue to the Littlewood-Paley theorem (see (2.13) below). Littlewood-Paley type results can be understood as substitutes of the Plancherel identity on L^p-spaces.

An important consequence of Theorem 1.1 are the estimates:

$$\|e^{it\Delta}f\|_{L^q[(0,\infty),L^p_x(\mathbb{R}^d)]} \asymp \|u(t,x)\|_{L^q_t([0,2\pi],L^p_x(\mathbb{R}^n))} \leq C\|f\|_{F^s_{p,2}(\mathbb{R}^n)}, \tag{1.10}$$

for $s \geq s_q$, $2 \leq p \leq q < \infty$, $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{p})$, (see Theorem 3.6). The inequality

$$\|e^{it\Delta}f\|_{L^q[(0,\infty),L^{p'}_x(\mathbb{R}^d)]} \asymp \|u(t,x)\|_{L^q_t([0,2\pi],L^{p'}_x(\mathbb{R}^n))} \leq C\|f\|_{W^{2s,p,H}(\mathbb{R}^n)}, \tag{1.11}$$

holds for $s \geq s_q$, $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n}$, $1 < p < 2$, $n > 2$ and $\frac{2}{q} = n(\frac{1}{p} - \frac{1}{2})$, (compare (1.11) and (1.4)). The estimate

$$\|f\|_{F^0_{p,2}(\mathbb{R}^n)} \leq C\|e^{it\Delta} f\|_{L^q[(0,\infty), L^p_x(\mathbb{R}^n)]} \asymp C\|u(t,x)\|_{L^q_t([0,2\pi], L^p_x(\mathbb{R}^n))} \tag{1.12}$$

holds when $2 \leq q \leq p < \infty$ provided that $n(\frac{1}{2} - \frac{1}{p}) = \frac{2}{q}$. In the results above, the spaces $F^s_{p,2}$ are Triebel-Lizorkin spaces associated with H, to be introduced in the next section.

Estimate (1.10) links our results to those in [11, 16]. For $\frac{1}{q} = \frac{n}{2}(\frac{1}{2} - \frac{1}{p})$, Corollary 3.7 shows that

$$\|u(t,x)\|_{L^p_x(\mathbb{R}^n\,, L^q_t[0,\pi/4])} \leq C_s\|f\|_{L^2(\mathbb{R}^n)} \tag{1.13}$$

holds provided that $2 \leq p \leq \infty$ for $n = 1$, $2 \leq p < \infty$ for $n = 2$, and $2 \leq p < \frac{2n}{n-2}$ for $n \geq 3$. As a consequence of the embedding $\mathcal{H}^s \hookrightarrow L^2$ for $s \geq 0$, estimate (1.13) improves (1.3) in the case above.

This article is organized as follows. In section 2 we present some basics on the spectral decomposition of the harmonic oscillator and we discuss our analogue of the Littlewood-Paley theorem. Finally, in the last section we provide our regularity results.

2. Spectral decomposition of the harmonic oscillator and a Littlewood-Paley type result

Let $H = -\Delta + |x|^2$ be the Hermite operator or (quantum) *harmonic oscillator*. This operator extends to an unbounded self-adjoint operator on $L^2(\mathbb{R}^n)$, and its spectrum consists of the discrete set $\lambda_\nu := 2|\nu| + n$, $\nu \in \mathbb{N}^n_0$, with a set of *real eigenfunctions* ϕ_ν, $\nu \in \mathbb{N}^n_0$, (called Hermite functions) which provide an orthonormal basis of $L^2(\mathbb{R}^n)$. Every Hermite function ϕ_ν on $\mathbb{R}^n$ has the form

$$\phi_\nu = \Pi^n_{j=1}\phi_{\nu_j}, \quad \phi_{\nu_j}(x_j) = (2^{\nu_j}\nu_j!\sqrt{\pi})^{-1/2} H_{\nu_j}(x_j) e^{-x_j^2/2}, \tag{2.1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n_0$, and

$$H_{\nu_j}(x_j) := (-1)^{\nu_j} e^{x_j^2} \frac{d^k}{dx_j^k}(e^{-x_j^2})$$

denotes the Hermite polynomial of order ν_j. By the spectral theorem, for every $f \in \mathscr{D}(\mathbb{R}^n)$ we have

$$Hf(x) = \sum_{\nu \in \mathbb{N}^n_0} \lambda_\nu \widehat{f}(\phi_\nu)\phi_\nu(x), \tag{2.2}$$

where $\widehat{f}(\phi_v)$ is the Hermite-Fourier transform of f at ν defined by

$$\widehat{f}(\phi_\nu) := \langle f, \phi_\nu \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x)\phi_\nu(x)\,dx. \tag{2.3}$$

The main tool in the harmonic analysis of the harmonic oscillator is the Hermite semigroup, which we introduce as follows. If P_ℓ, $\ell \in 2\mathbb{N}_0 + n$, is the projection on $L^2(\mathbb{R}^n)$ given by

$$P_\ell f(x) := \sum_{2|\nu|+n=\ell} \widehat{f}(\phi_\nu)\phi_\nu(x), \tag{2.4}$$

then the Hermite semigroup (semigroup associated with the harmonic oscillator) $T_t := e^{-tH}$, $t > 0$ is given by

$$e^{-tH} f(x) = \sum_{\ell} e^{-t\ell} P_\ell f(x). \tag{2.5}$$

For each $t > 0$, the operator e^{-tH} has Schwartz kernel

$$K_t(x, y) = \sum_{\nu \in \mathbb{N}_0^n} e^{-t(2|\nu|+n)} \phi_\nu(x)\phi_\nu(y). \tag{2.6}$$

In view of Mehler's formula (see Thangavelu [18]) the above series can be summed and we obtain

$$K_t(x, y) = (2\pi)^{-n/2} \sinh(2t)^{-n/2} e^{-(\frac{1}{2}|x|^2+|y|^2)\coth(2t)+xy\,\mathrm{csch}(2t))}. \tag{2.7}$$

In this article we estimate the mixed norms $L^p_x(L^q_t)$ of solutions to Schrődinger equations by using the following version of Triebel-Lizorkin space associated with H.

Definition 2.1. Let $0 < p \leq \infty$, $r \in \mathbb{R}$ and $0 < q \leq \infty$. The Triebel-Lizorkin space associated with H, the family of projections P_ℓ, $\ell \in 2\mathbb{N} + n$, and the parameters p, q and r is defined by the complex functions f satisfying

$$\|f\|_{F^r_{p,q}(\mathbb{R}^n)} := \Big\| \Big(\sum_{\ell} \ell^{rq} |P_\ell f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty. \tag{2.8}$$

The above definition differs from those arising with dyadic decompositions [1, 13]. The following are natural embedding properties of such spaces. Let $\mathcal{H}^s$ denote the Sobolev space associated with H and defined by the norm $\|f\|_{\mathcal{H}^s} := \|H^{s/2} f\|_{L^2}$. Sobolev spaces $W^{2s,p,H}$ in L^p-spaces and associated with H, can be defined by the norm $\|f\|_{W^{2s,p,H}} := \|H^s f\|_{L^p}$. Then we have

(1) $F^{r+\varepsilon}_{p,q_1} \hookrightarrow F^r_{p,q_1} \hookrightarrow F^r_{p,q_2} \hookrightarrow F^r_{p,\infty}$, $\varepsilon > 0$, $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$.
(2) $F^{r+\varepsilon}_{p,q_1} \hookrightarrow F^r_{p,q_2}$, $\varepsilon > 0$, $0 < p \leq \infty$, $1 \leq q_2 < q_1 < \infty$.
(3) $F^0_{2,2} = L^2$ and consequently, for every $s \in \mathbb{R}$, $\mathcal{H}^{2s} = F^s_{2,2}$.

Some other properties associated with Sobolev spaces of the harmonic oscillator can be found in [1, 2, 13].

Now we discuss a close relation between $F^0_{p,2}$ and Lebesgue spaces. If ψ is a smooth function supported in $[1/4, 2]$, such that $\psi = 1$ on $[1/2, 1]$,

$$\sum_{k=0}^{\infty} \psi_k(t) = 1, \quad \psi_k(t) := \psi(2^{-k}t), \tag{2.9}$$

and A is an elliptic pseudo-differential operator on $\mathbb{R}^n$ of order $\nu > 0$, then the (dyadic) Triebel-Lizorkin space $F^r_{p,q,A}(\mathbb{R}^n)$ associated with A is defined by the norm

$$\|f\|_{F^r_{p,q,A}} := \|\{2^{kr/\nu} \|\psi_k(A) f\|_{L^p}\}\|_{\ell^q}\,, \tag{2.10}$$

where $r \in \mathbb{R}$ and $0 < p, q \leq \infty$. For $A = H$ or $A = \Delta_x$, it is well known the Littlewood-Paley theorem [7] which states that $F^0_{p,2,A} = L^p$ for all $1 < p < \infty$. If $A = \Delta_x$, one also has

$$\Big\| \Big(\sum_k |1_{(k,k+1)}(\Delta_x) f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p}, \quad 2 < p < \infty, \tag{2.11}$$

with C depending only on p. However, such inequality is false for $1 < p < 2$, $\ell \in 2\mathbb{N} + n$, $P_\ell = 1_{[\ell,\ell+1)}(H)$ and

$$\|f\|_{F^0_{p,2}} = \Big\| \Big(\sum_\ell |1_{[\ell,\ell+1)}(H)f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)}. \tag{2.12}$$

In Remark 3.2, we shall explain in detail that we have not a Littlewood-Paley theorem for $F^0_{p,2}$, in the proof of our main theorem we obtain the following estimate for $1 \le p \le 2$ (see equation (3.18))

$$\|f\|_{F^0_{p',2}} = \Big\| \Big(\sum_\ell |1_{[\ell,\ell+1)}(H)f|^2 \Big)^{1/2} \Big\|_{L^{p'}(\mathbb{R}^n)} \le C\|f\|_{L^p} \tag{2.13}$$

provided that $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n}$. Such inequality is indeed, an analogue of (2.11). An immediate consequence is the estimate

$$\|f\|_{F^s_{p',2}} = \|H^s f\|_{F^0_{p',2}} \le C\|H^s f\|_{L^p} =: C\|f\|_{W^{2s,p,H}} \tag{2.14}$$

provided that $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n}$.

3. Regularity properties

To analyze the mixed norms of solutions of the Schödinger equation we need the following multiplier theorem. The space $L^2_f(\mathbb{R}^n)$ consists of those finite linear combinations of Hermite functions on $\mathbb{R}^n$.

Theorem 3.1. *Let us assume that $m \in L^\infty(\mathbb{N}_0)$ is a bounded function. Then the multiplier $m(H)$ extends to a bounded operator on $F^0_{p,q}(\mathbb{R}^n)$ for all $0 < p \le \infty$ and $0 < q \le \infty$. Moreover*

$$\|m(H)\|_{\mathscr{B}(F^0_{p,q})} = \|m\|_{L^\infty}. \tag{3.1}$$

In particular if $m := 1_{[0,\ell']}$, then $S_{\ell'} = 1_{[0,\ell']}(H)$, $\|S_{\ell'}\|_{\mathscr{B}(F^0_{p,q})} = 1$ and

$$\lim_{\ell' \to \infty} \|S_{\ell'} f - f\|_{F^0_{p,q}} = 0 \tag{3.2}$$

uniformly on the $F^0_{p,q}$-norm.

Proof. Let us consider $f \in F^0_{p,q}$. Then, $P_\ell(m(H)f) = m(\ell)P_\ell f$ and

$$\|m(H)f\|_{F^0_{p,q}} = \Big\| \Big(\sum_\ell |m(\ell)|^q |P_\ell f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \le \sup_\ell |m(\ell)| \|f\|_{F^0_{p,q}}. \tag{3.3}$$

As a consequence,

$$\|m(H)\|_{\mathscr{B}(F^0_{p,q})} \le \|m\|_{L^\infty}. \tag{3.4}$$

Now, for the reverse inequality, let us choose $f = \phi_\nu$, $\ell' = 2|\nu| + n$. Then $\|m(H)f\|_{(F^0_{p,q})} = |m(\ell')| \|f\|_{(F^0_{p,q})}$ and as consequence $\|m(H)\|_{\mathscr{B}(F^0_{p,q})} \ge \sup_\ell |m(\ell)|$. The second part is consequence of the uniform boundedness principle. □

Remark 3.2. As an important consequence of the previous result, $L^2_f(\mathbb{R}^n)$ is a dense subspace of every space $F^r_{p,q}$, in fact, for every $f \in F^r_{p,q}$, the sequence $\{S_{\ell'} f\}_{\ell'}$ lies in $L^2_f(\mathbb{R}^n)$ and $S_{\ell'} f \to f$ in norm. For $n = 1$, it is well known that the sequence of operators $\{S_{\ell'}\}_{\ell'}$ is uniformly bounded on L^p if and only if $4/3 < p < 4$, so the spaces $F^0_{p,2}$ does not coincide necessarily with Lebesgue spaces and we have not a general Littlewood-Paley Theorem. Nevertheless, this disadvantag is compensated by the efficiency of such spaces when we want to estimate solutions of the Schrödinger equation.

We shall use the first part of this remark in the following result.

Lemma 3.3. *If $f \in F^0_{p,2}(\mathbb{R}^n)$, then for all $0 < p \leq \infty$,*

$$\|u(t,x)\|_{L^p_x[\mathbb{R}^n, L^2_t[0,2\pi]]} = \sqrt{2\pi}\|f\|_{F^0_{p,2}(\mathbb{R}^n)}\,. \tag{3.5}$$

Proof. In view of (3.2), by denseness, we consider $f \in L^2_f(\mathbb{R}^n)$. The solution of (1.1) is given by

$$u(t,x) = \sum_{\nu\in\mathbb{N}_0^n} e^{-it(2|\nu|+n)}\widehat{f}(\phi_\nu)\phi_\nu(x). \tag{3.6}$$

Then, we have (see [9])

$$\|u(t,x)\|^2_{L^2_t[0,2\pi]} = \sum_{\ell} 2\pi \cdot |P_\ell f(x)|^2$$

which can be proved using the orthogonality of trigonometric polynomials. So, we conclude that

$$\|u(t,x)\|_{L^2_t[0,2\pi]} = \Big(\sum_{\ell} 2\pi \cdot |P_\ell f(x)|^2\Big)^{1/2}, \quad f \in L^2_f(\mathbb{R}^n). \tag{3.7}$$

Consequently,

$$\|u(t,x)\|_{L^p_x(\mathbb{R}^n, L^2_t[0,2\pi])} = \sqrt{2\pi}\|f\|_{F^0_{p,2}(\mathbb{R}^n)}. \tag{3.8}$$

□

Lemma 3.4. *Let $0 < p \leq \infty$, $2 \leq q < \infty$ and $s_q := \frac{1}{2} - \frac{1}{q}$. Then*

$$C'_p\|f\|_{F^0_{p,2}} \leq \|u(t,x)\|_{L^p_x(\mathbb{R}^n, L^q_t[0,2\pi])} \leq C_{p,s}\|f\|_{F^s_{p,2}}, \tag{3.9}$$

for every $s \geq s_q$.

Proof. By a denseness argument, we consider $f \in L^2_f(\mathbb{R}^n)$. By following the approach in [3], to estimate the norm $\|u(t,x)\|_{L^p_x[\mathbb{R}^n, L^q_t[0,2\pi]]}$ we use the Wainger Sobolev embedding Theorem,

$$\Big\|\sum_{\ell\in\mathbb{Z},\ell\neq 0} |\ell|^{-\alpha}\widehat{F}(\ell)e^{-i\ell t}\Big\|_{L^q[0,2\pi]} \leq C\|F\|_{L^r[0,2\pi]}, \quad \alpha := \frac{1}{r} - \frac{1}{q}. \tag{3.10}$$

For $s_q := \frac{1}{2} - \frac{1}{q}$ we have

$$\begin{aligned}
\|u(t,x)\|_{L^q[0,2\pi]} &= \Big\|\sum_{\nu\in\mathbb{N}_0^n} e^{-it(2|\nu|+n)}\widehat{f}(\phi_\nu)\phi_\nu(x)\Big\|_{L^q[0,2\pi]}\\
&= \Big\|\sum_{\ell} e^{-it\ell}P_\ell f(x)\Big\|_{L^q[0,2\pi]}\\
&\leq C\Big\|\sum_{\ell} \ell^{s_q}e^{-it\ell}P_\ell f(x)\Big\|_{L^2[0,2\pi]}\\
&= C\Big\|\sum_{\ell} e^{-it\ell}P_\ell[H^{s_q}f(x)]\Big\|_{L^2[0,2\pi]}\\
&= C\Big(\sum_{\ell} |P_\ell[H^{s_q}f(x)|^2\Big)\\
&:= T'(H^{s_q}f)(x).
\end{aligned}$$

So, we have

$$\begin{aligned}\|u(t,x)\|_{L^p_x[\mathbb{R}^n,L^q_t[0,2\pi]]} &\le C\|T'(H^{s_q}f)\|_{L^p(\mathbb{R}^n)}\\ &\le C_p\|H^{s_q}f\|_{F^0_{p,2}(\mathbb{R}^n)} = C_p\|f\|_{F^{s_q}_{p,2}(\mathbb{R}^n)}\,.\end{aligned} \tag{3.11}$$

We complete the proof by taking into account the embedding $F^s_{p,2} \hookrightarrow F^{s_q}_{p,2}$ for every $s > s_q$ and the following inequality for $2 \le q < \infty$,

$$\begin{aligned}\|f\|_{F^0_{p,2}} &= \frac{1}{\sqrt{2\pi}}\|T'f\|_{L^p}\\ &= \frac{1}{\sqrt{2\pi}}\|u(t,x)\|_{L^p_x[\mathbb{R}^n,L^2_t[0,2\pi]]}\\ &\lesssim \|u(t,x)\|_{L^p_x[\mathbb{R}^n,L^q_t[0,2\pi]]}.\end{aligned} \tag{3.12}$$

□

Theorem 3.5. *Let $n > 2$, $2 \le q < \infty$ and $1 \le p \le 2$, satisfy $|\frac{1}{2} - \frac{1}{p}| < \frac{1}{2n}$. Then*

$$\|u(t,x)\|_{L^{p'}_x[\mathbb{R}^n,L^q_t[0,2\pi]]} \le C\|f\|_{W^{2s,p,H}(\mathbb{R}^n)} \tag{3.13}$$

for every $s \ge s_q := \frac{1}{2} - \frac{1}{q}$. In particular, if $q = 2$ we have

$$\|u(t,x)\|_{L^{p'}_x[\mathbb{R}^n,L^2_t[0,2\pi]]} \le C\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.14}$$

Moreover, for $n > 2$, $1 \le p \le 2$, and $1 \le q \le p'$, we have

$$\|u(t,x)\|_{L^{p'}_x[\mathbb{R}^n,L^q_t[0,2\pi]]} \le C\|f\|_{L^p(\mathbb{R}^n)}, \tag{3.15}$$

provided that $|\frac{1}{p} - \frac{1}{2}| < 1/(nq)$.

Proof. First, we want to proof the case $q = 2$ and later we extend the proof for $2 < q < \infty$ by using a suitable embedding. Our main tool will be the dispersive inequality [15, p. 114]

$$\|u(t,x)\|_{L^{p'}_x(\mathbb{R}^n)} \le C|t|^{-n|\frac{1}{p}-\frac{1}{2}|}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \le p \le 2. \tag{3.16}$$

Consequently,

$$\|u(t,x)\|_{L^2_t([0,2\pi],L^{p'}_x(\mathbb{R}^n))} \le C\|\,|\cdot|^{-n|\frac{1}{p}-\frac{1}{2}|}\|_{L^2[0,2\pi]}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \le p \le 2. \tag{3.17}$$

We need $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n}$ in order for $\|\,|\cdot|^{-n|\frac{1}{p}-\frac{1}{2}|}\|_{L^2[0,2\pi]} < \infty$. Because $p' \ge 2$ we can use Minkowski integral inequality to obtain

$$\begin{aligned}\|f\|_{F^0_{p',2}} &= \|u(t,x)\|_{L^{p'}_x(\mathbb{R}^n,L^2_t([0,2\pi]))}\\ &\le \|u(t,x)\|_{L^2_t([0,2\pi],L^{p'}_x(\mathbb{R}^n))} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{3.18}$$

In fact, we have

$$\begin{aligned}\|u(t,x)\|_{L^{p'}_x(\mathbb{R}^n,L^2_t([0,2\pi]))} &:= \Big(\int_{\mathbb{R}^n}\Big(\int_0^{2\pi}|u(t,x)|^2dt\Big)^{p'/2}dx\Big)^{\frac{2}{p'}\cdot\frac{1}{2}}\\ &\le \Big(\int_0^{2\pi}\Big(\int_{\mathbb{R}^n}|u(t,x)|^{p'}dx\Big)^{2/p'}dt\Big)^{1/2}\\ &=: \|u(t,x)\|_{L^2_t([0,2\pi],L^{p'}_x(\mathbb{R}^n))}\,.\end{aligned}$$

Now (3.18) can be obtained from (3.17) for $1 \le p \le 2$ and $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n}$. Estimate (3.18) proves the theorem for $q = 2$. The result for $2 < q < \infty$ now follows, as in the proof of Theorem 3.4, by using the Wainger Sobolev embedding Theorem as in (3.11) together with (2.14):

$$\begin{aligned}\|u(t,x)\|_{L^{p'}_x[\mathbb{R}^n,L^q_t[0,2\pi]]} &\leq C\|T'(H^{s_q}f)\|_{L^{p'}(\mathbb{R}^n)} \leq C_{p'}\|H^{s_q}f\|_{F^0_{p',2}(\mathbb{R}^n)}\\ &= C_{p'}\|f\|_{F^{s_q}_{p',2}(\mathbb{R}^n)} \leq C\|f\|_{W^{2s_q,p,H}(\mathbb{R}^n)}.\end{aligned}$$

So, the proof of the first statement is complete.

Now, to proof (3.15) we observe that

$$\|u(t,x)\|_{L^{p'}_x(\mathbb{R}^n)} \leq C|t|^{-n|\frac{1}{p}-\frac{1}{2}|}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1\leq p\leq 2, \tag{3.19}$$

which implies

$$\|u(t,x)\|_{L^q_t[[0,2\pi],L^{p'}_x(\mathbb{R}^n)]} \leq C\cdot I_{p,n,q}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1\leq p\leq 2, \tag{3.20}$$

where

$$I_{p,n,q} = \Big(\int_0^{2\pi}|t|^{-nq|\frac{1}{p}-\frac{1}{2}|}\Big)^{1/q} < \infty$$

for $|1/2-1/p|<1/(nq)$. Since, $q\leq p'$, by using the Minkowski inequality we have

$$\|u(t,x)\|_{L^{p'}_x[\mathbb{R}^n,L^q_t[0,2\pi]]} \leq \|u(t,x)\|_{L^q_t[[0,2\pi],L^{p'}_x(\mathbb{R}^n)]} \tag{3.21}$$

and consequently

$$\|u(t,x)\|_{L^{p'}_x[\mathbb{R}^n,L^q_t[0,2\pi]]} \leq C\|f\|_{L^p}.$$

□

Theorem 3.6. *Let us assume that for some s, $f\in F^s_{p,2}(\mathbb{R}^n)$ is a real function and $u(\cdot,t)=e^{-itH}f(\cdot)$. Let $2\leq p\leq q<\infty$ and $\frac{2}{q}=n(\frac{1}{2}-\frac{1}{p})$. Then*

$$\|e^{it\Delta}f\|_{L^q[(0,\infty),L^p_x(\mathbb{R}^n)]} \asymp \|u(t,x)\|_{L^q_t([0,2\pi],L^p_x(\mathbb{R}^n))} \leq C\|f\|_{F^s_{p,2}(\mathbb{R}^n)}, \tag{3.22}$$

for $s\geq s_q$. Consequently,

$$\|e^{it\Delta}f\|_{L^q[(0,\infty),L^{p'}_x(\mathbb{R}^d)]} \asymp \|u(t,x)\|_{L^q_t([0,2\pi],L^{p'}_x(\mathbb{R}^n))} \leq C\|f\|_{W^{2s,p,H}(\mathbb{R}^n)}, \tag{3.23}$$

for $s\geq s_q$, $|\frac{1}{p}-\frac{1}{2}|<\frac{1}{2n}$, $1<p<2$, $n>2$ and $\frac{2}{q}=n(\frac{1}{p}-\frac{1}{2})$. Moreover, for $2\leq q\leq p<\infty$ and $\frac{2}{q}=n(\frac{1}{2}-\frac{1}{p})$ we have

$$\|f\|_{F^0_{p,2}(\mathbb{R}^n)} \leq C\|e^{it\Delta}f\|_{L^q[(0,\infty),L^p_x(\mathbb{R}^n)]}, C\|u(t,x)\|_{L^q_t([0,2\pi],L^p_x(\mathbb{R}^n))}. \tag{3.24}$$

Proof. From the Minkowski integral inequality applied to $L^{q/p}$, we deduce the inequality

$$\|u(t,x)\|_{L^q_t([0,2\pi],L^p_x(\mathbb{R}^n))} \leq \|u(t,x)\|_{L^p_x[\mathbb{R}^n,L^q_t[0,2\pi]]}. \tag{3.25}$$

In fact,

$$\begin{aligned}\|u(t,x)\|_{L^q_t([0,2\pi],L^p_x(\mathbb{R}^n))} &:= \Big(\int_0^{2\pi}\Big(\int_{\mathbb{R}^n}|u(t,x)|^p dx\Big)^{q/p}dt\Big)^{\frac{p}{q}\cdot\frac{1}{p}}\\ &\leq \Big(\int_{\mathbb{R}^n}\Big(\int_0^{2\pi}|u(t,x)|^q dt\Big)^{p/q}dx\Big)^{1/p}\\ &=: \|u(t,x)\|_{L^p_x[\mathbb{R}^n,L^q_t[0,2\pi]]}.\end{aligned}$$

Now, we only need to apply Lemma 3.4 and the equivalence given by (1.5). Estimate (3.23) is consequence of (2.14) and (3.22) applied to p' instead of p. On the other hand, for $2 \le q \le p < \infty$, by using the Minkowski integral inequality on $L^{p/q}$ we have

$$\begin{aligned} \|f\|_{F^0_{p,2}(\mathbb{R}^n)} &= \|u(t,x)\|_{L^p_x[\mathbb{R}^n, L^2_t[0,2\pi]]} \\ &\lesssim \|u(t,x)\|_{L^p_x[\mathbb{R}^n, L^q_t[0,2\pi]]} \\ &\le \|u(t,x)\|_{L^q_t([0,2\pi], L^p_x(\mathbb{R}^n))}. \end{aligned} \tag{3.26}$$

So, by using the equivalence expressed in (1.5) we obtain

$$\|f\|_{F^0_{p,2}(\mathbb{R}^n)} \le C\|e^{it\Delta}f\|_{L^q[(0,\infty), L^p_x(\mathbb{R}^n)]} \asymp C\|u(t,x)\|_{L^q_t([0,2\pi], L^p_x(\mathbb{R}^n))}.$$

The proof is complete. □

Corollary 3.7. *Let $1 < q \le p < \infty$ and $\frac{1}{q} = \frac{n}{2}(\frac{1}{2} - \frac{1}{p})$. Then*

$$\|u(t,x)\|_{L^p_x(\mathbb{R}^n, L^q_t[0,\pi/4])} \le C_s\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.27}$$

provided that $2 \le p < \infty$ for $n = 1$, $2 \le p < \infty$ for $n = 2$, and $2 \le p < \frac{2n}{n-2}$ for $n \ge 3$.

Proof. As in Theorem 3.6, by using the Minkowski integral inequality on $L^{p/q}$, for $1 < q \le p < \infty$, we have the inequality

$$\|u(t,x)\|_{L^p_x[\mathbb{R}^n, L^q_t[0,\pi/4]]} \le \|u(t,x)\|_{L^q_t([0,\pi/4], L^p_x(\mathbb{R}^n))}. \tag{3.28}$$

Finally (3.27) follows by using (1.6) and the equivalence (1.5). □

Remark 3.8. Note that the compactness of the interval $[0, \pi/4]$ and the embedding $L^q_t[0,\pi/4] \hookrightarrow L^r_t[0,\pi/4]$, for $r \le q$, allow us to obtain the Strichartz estimate

$$\|u(t,x)\|_{L^p_x(\mathbb{R}^n, L^q_t[0,\pi/4])} \le C_s\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.29}$$

provided that $1 < q \le p < \infty$, $\frac{1}{q} \ge \frac{n}{2}(\frac{1}{2} - \frac{1}{p})$, and $n = 1$ for $2 \le p < \infty$, $n = 2$ for $2 \le p < \infty$ and $2 \le p < \frac{2n}{n-2}$ for $n \ge 3$.

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