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EXISTENCE OF MULTIPLE BREATHERS FOR DISCRETE NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this article we study discrete nonlinear Schrödinger equations without periodicity assumptions. We show the existence of multiple solutions of the form $u_n e^{-i\omega t}$ (called breathers) by using Clark's Theorem in critical point theory.

1. Introduction

The discrete nonlinear Schrödinger (DNLS) equation is one of the most important inherently discrete models. DNLS equations play a crucial role in the modeling of a great variety of phenomena, ranging from solid state and condensed matter physics to biology [7, 8, 9]. For example, they have been successfully applied to the modeling of localized pulse propagation optical fibers and wave guides, to the study of energy relaxation in solids, to the behavior of amorphous material, to the modeling of self-trapping of vibrational energy in proteins or studies related to the denaturation of the DNA double strand [16].

Below $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For $a, b \in \mathbb{Z}$, we define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a < b$.

This article considers the DNLS equation

$$i\dot{\psi}_n = -\Delta\psi_n + \varepsilon_n\psi_n - f_n(\psi_n), n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is discrete Laplacian operator, ε_n is real valued for each $n \in \mathbb{Z}$, $f_n \in C(\mathbb{R}, \mathbb{R})$, $f_n(0) = 0$ and the nonlinearity $f_n(u)$ is gauge invariant; that is,

$$f(e^{i\theta}u) = e^{i\theta}f(u), \quad \theta \in \mathbb{R}. \tag{1.2}$$

We consider special solutions to (1.1) called breathers. They have the form

$$\psi_n = u_n e^{-i\omega t},$$

where $\omega \in \mathbb{R}$ is the temporal frequency. Note that ψ_n is real valued for each $n \in \mathbb{Z}$, is spatially localized, is time-periodic, and decays to zero at infinity: $\lim_{|n|\to\infty} \psi_n = 0$.

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Then (1.1) becomes

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}, \tag{1.3}$$

$$\lim_{|n|\to\infty} u_n = 0\,. \tag{1.4}$$

Actually, our methods allow us to consider the more general equation

$$\Delta^k(p_{n-k}\Delta^k u_{n-k}) + (-1)^k q_n u_n = (-1)^k f_n(u_n), \quad k \in \mathbb{Z}(1),\ n \in \mathbb{Z}, \tag{1.5}$$

with the same boundary condition (1.4). Here, Δ is the forward difference operator [26] $\Delta u_n = u_{n+1} - u_n$, $\Delta^k u_n = \Delta(\Delta^{k-1}u_n)$, p_n and q_n are real valued for each $n \in \mathbf{Z}$. When $k = 1, p_n \equiv 1$ and $q_n \equiv \varepsilon_n - \omega$, we obtain (1.3). Naturally, if we look for breathers of (1.1), we just need to get the solutions of (1.5) satisfying (1.4).

Peil and Peterson [21] in 1994 studied the asymptotic behavior of solutions of $2k$th-order difference equation

$$\sum_{i=0}^{k} \Delta^i(r_i(n-i)\Delta^i u(n-i)) = 0 \tag{1.6}$$

with $r_i(n) \equiv 0$ for $1 \le i \le k-1$. In 1998, Anderson [1] considered (1.6) for $n \in \mathbf{Z}(a)$, and obtained a formulation of generalized zeros and (k,k)-disconjugacy for (1.6). Cai, Yu [2] in 2007 have obtained some criteria for the existence of periodic solutions of the following difference equation

$$\Delta^k(r_{n-k}\Delta^k u_{n-k}) + f(n, u_n) = 0. \tag{1.7}$$

In 2011, Chen and Tang [4] established some new existence criteria to guarantee the $2k$th-order nonlinear difference equation

$$\Delta^k(r_{n-k}\Delta^k u_{n-k}) + q(n)u(n) = f(n, u_{n+k}, \dots, u_n, \dots, u_{n-k}) \tag{1.8}$$

has at least one or infinitely many homoclinic solutions.

In the past decade, the existence of breathers of the DNLS equations has drawn a great deal of interest [14, 15, 17, 18, 19, 20, 27]. The existence for the periodic DNLS equations with superlinear nonlinearity and with saturable nonlinearity has been studied [17, 18, 19, 20]. And the existence results of breathers of the DNLS equations without periodicity assumptions were established in [14, 15, 27]. As for the existence of the homoclinic orbits of nonlinear Schrödinger equations, we refer to [6, 23, 24]. When the nonlinear term is subquadratic at infinity, as far as the authors are aware, the results on the breathers of (1.5) obtained in the literature are very scarce. The main purpose of this paper is to establish some existence criteria to guarantee that (1.5) has at least multiple breathers by using the critical point theory. The motivation for the present work stems from the recent papers [3, 6, 13, 25].

Let $F_n(u) = \int_0^u f_n(t)dt, t \in \mathbb{R}$. Our main results are the following theorems.

For the next theorem we use the following hypotheses:

- (H1) $f_n(u)$ is odd in u and
  - for any $n \in \mathbb{Z}$, $\underline{p} := \inf_{n\in\mathbb{Z}} p_n > 0$;
  - for any $\mathbb{Z}$, $\underline{q} := \inf_{n\in\mathbb{Z}} q_n > 0$;
- (H2) for each $n \in \mathbb{Z}$, there exist two constants $1 < \nu_1 < \nu_2 < 2$ and two functions $a, b \in l^{\frac{2}{2-\nu_1}}(\mathbb{Z}, [0, +\infty))$ such that

$$|F_n(u)| \le a_n|u|^{\nu_1}, \quad |u| \le 1,$$

$$|F_n(u)| \le b_n|u|^{\nu_2}, \quad |u| > 1;$$

(H3) for each $n \in \mathbb{Z}$, there exist two constants K, $c > 0$ and a function $d \in l^{\frac{2}{2-\nu_1}}(\mathbb{Z}, [0, +\infty))$ such that

$$|f_n(u)| \le K d_n|u|^{\nu_1 - 1}, \quad |u| \le c;$$

(H4) there exist two constants $1 < \nu_3 < 2$, $\eta > 0$ and a set $M \subset \mathbb{Z}$ with $m > 0$ elements such that

$$F_n(u) \ge \eta|u|^{\nu_3}, \quad n \in M, \ |u| \le 1.$$

Theorem 1.1. *Under assumptions* (H1)–(H4)*, equation* (1.5) *has at least m distinct pairs of nontrivial solutions satisfying* (1.4)*.*

Corollary 1.2. *Suppose that* (H1)–(H4) *and the following hypotheses are satisfied:*

(H5) $F_n(u) = e_n\varphi(u)$*, where* $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ *and* $e \in l^{\frac{2}{2-\nu_1}}(\mathbb{Z}, [0, +\infty))$*,* $1 < \nu_1 < 2$ *is a constant, such that* $e_n > 0$ *for all* $n \in \{n_1, n_2, \dots, n_k\} \subset \mathbb{Z}$*;*

(H6) *there exist four positive constants* $\theta_1, \theta_2, \nu_2 \in [\nu_1, 2)$ *and* $1 < \nu_3 < 2$ *such that*

$$\theta_2|u|^{\nu_3} \le \varphi(u) \le \theta_1|u|^{\nu_1}, \ |u| \le 1,$$
$$0 < \varphi(u) \le \theta_1|u|^{\nu_2}, \ |u| > 1.$$

Then (1.5) *has at least m distinct pairs of nontrivial solutions satisfying* (1.4)*.*

As it is well known, critical point theory is a powerful tool to deal with the homoclinic solutions of differential equations [10, 11, 12, 13] and is used to study homoclinic solutions of discrete systems in recent years [3, 4, 5, 25, 28]. The main idea is to transfer the problem of solutions in E (defined in Section 2) of (1.5) into that of critical points of the corresponding functional.

2. Preliminaries

To apply the critical point theory, we establish the variational framework corresponding to (1.3) and give some lemmas which will be of fundamental importance in proving our main results. We start by some basic notation.

Let S be the vector space of real sequences of the form

$$u = (\dots, u_{-n}, \dots, u_{-1}, u_0, u_1, \dots, u_n, \dots) = \{u_n\}_{n=-\infty}^{+\infty},$$

namely $S = \{\{u_n\} : u_n \in \mathbb{R}, \ n \in \mathbb{Z}\}$. Define

$$E = \big\{u \in S : \sum_{n=-\infty}^{+\infty} [p_{n-1}(\Delta^k u_{n-1})^2 + q_n u_n^2] < +\infty\big\}.$$

This space is a Hilbert space with the inner product

$$\langle u, v\rangle = \sum_{n=-\infty}^{+\infty} (p_{n-1}\Delta^k u_{n-1}\Delta^k v_{n-1} + q_n u_n v_n), \quad \forall u, v \in E, \tag{2.1}$$

and the corresponding norm

$$\|u\| = \Big(\sum_{n=-\infty}^{+\infty} \big[p_{n-1}(\Delta^k u_{n-1})^2 + q_n u_n^2\big]\Big)^{1/2}, \quad \forall u \in E. \tag{2.2}$$

On the other hand, we define the space of real sequences,

$$l^s = \big\{u \in S : \|u\|_s = \Big(\sum_{n=-\infty}^{+\infty} |u_n|^s\Big)^{1/s} < +\infty\big\}, \quad 1 \le s < +\infty, \tag{2.3}$$

with $\|u\|_\infty = \sup_{n\in\mathbb{Z}} |u_n|$ when $s = +\infty$.

Since $u \in E$, it follows that $\lim_{|n|\to\infty} |u_n| = 0$. Hence, there exists $n^* \in \mathbb{Z}$ such that

$$\|u\|_\infty = |u_{n^*}| = \max_{n\in\mathbb{Z}} |u_n|.$$

By (H1) and (2.2), we have

$$\|u\|^2 \ge \sum_{n\in\mathbb{Z}} q_n u_n^2 \ge \underline{q} \sum_{n\in\mathbb{Z}} u_n^2 \ge \underline{q}\|u\|_\infty^2.$$

Thus,

$$\underline{q}\|u\|_\infty^2 \le \underline{q}\|u\|_2^2 \le \|u\|^2. \tag{2.4}$$

For a function $f : \mathbb{Z} \to \mathbb{R}$ and $a \in \mathbb{R}$, we define

$$\mathbb{Z}(f_n \ge a) = \{n \in \mathbb{Z} : f_n \ge a\}, \quad \mathbb{Z}(f_n \le a) = \{n \in \mathbb{Z} : f_n \le a\}.$$

For $u \in E$, we define the functional J on E as follows

$$\begin{aligned} J(u) :=& \frac{1}{2} \sum_{n=-\infty}^{+\infty} \big[p_{n-1}(\Delta^k u_{n-1})^2 + q_n u_n^2\big] - \sum_{n=-\infty}^{+\infty} F_n(u_n) \\ =& \frac{1}{2}\|u\|^2 - \sum_{n=-\infty}^{+\infty} F_n(u_n)\,. \end{aligned} \tag{2.5}$$

Now we can prove that $J \in C^1(E, \mathbb{R})$.

Firstly, we can prove that $J : E \to \mathbb{R}$. For any $u \in E$, there exists an integer $D > 0$ such that $|u_n| < 1$ for $|n| > D$. By (H2), (2.4) and Hölder inequality, we have

$$\begin{aligned} \sum_{|n|>D} |F_n(u_n)| &\le \sum_{|n|>D} a_n |u_n|^{\nu_1} \\ &\le \Big(\sum_{|n|>D} |a_n|^{\frac{2}{2-\nu_1}}\Big)^{(2-\nu_1)/2} \Big(\sum_{|n|>D} u_n^2\Big)^{\nu_1/2} \\ &\le \underline{q}^{-\nu_1} \|a\|_{(2-\nu_1)/2} \|u\|^{\nu_1}, \end{aligned}$$

which implies that $J : E \to \mathbb{R}$.

Secondly, we can prove that $J \in C^1(E, \mathbb{R})$. Let $u^{(j)} \to u$ in E. For any $\epsilon \in (0, \sqrt{\underline{q}})$, we can choose an integer D_ϵ such that

$$\Big[\sum_{|n|>D_\epsilon} \big(p_{n-1}|\Delta^k u_{n-1}^{(j)}|^2 + q_n|u_n^{(j)}|^2\big)\Big]^{1/2} < \epsilon, \quad j \in \mathbb{N}, \tag{2.6}$$

$$\Big[\sum_{|n|>D_\epsilon} \big(p_{n-1}|\Delta^k u_{n-1}|^2 + q_n|u_n|^2\big)\Big]^{1/2} < \epsilon. \tag{2.7}$$

For sufficient large j, by (H3), (2.4)-(2.7), and Hölder inequality, we have

$$|\langle J'(u^{(j)}) - J'(u), v\rangle|$$

$$
\begin{aligned}
&\le \Big|\sum_{n=-\infty}^{+\infty}[p_{n-1}(|\Delta^k u_{n-1}|^{p-2}\Delta^k u_{n-1}-|\Delta^k v_{n-1}|^{p-2}\Delta^k v_{n-1})\Delta^k v_{n-1}\\
&\quad + q_n(|u_n|^{p-2}u_n-|v_n|^{p-2}v_n)v_n]\Big| + \sum_{n=-\infty}^{+\infty}|[f_n(u_n^{(j)})-f_n(u_n)]v_n|\\
&\le |\langle u^{(j)}-u,v\rangle| + \sum_{n=-\infty}^{+\infty}|f_n(u_n^{(j)})-f_n(u_n)||v_n|\\
&\le \|u^{(j)}-u\|\|v\| + \sum_{|n|\le D_\epsilon}|f_n(u_n^{(j)})-f_n(u_n)||v_n| + \sum_{|n|>D_\epsilon}[|f_n(u_n^{(j)})|+|f_n(u_n)|]|v_n|\\
&\le \|u^{(j)}-u\|\|v\| + \sum_{|n|\le D_\epsilon}|f_n(u_n^{(j)})-f_n(u_n)||v_n|\\
&\quad + K\sum_{|n|>D_\epsilon}d_n[|u_n^{(j)}|^{\nu_1-1}+|u_n|^{\nu_1-1}]|v_n|\\
&\le o(1) + K\underline{q}^{-1/2}\Big(\sum_{|n|>D_\epsilon}|d_n|^{\frac{2}{2-\nu_1}}\Big)^{\frac{2-\nu_1}{2}}\Big(\sum_{|n|>D_\epsilon}(u_n^{(j)})^2\Big)^{\frac{\nu_1-1}{2}}\|v\|\\
&\quad + K\underline{q}^{-1/2}\Big(\sum_{|n|>D_\epsilon}|d_n|^{\frac{2}{2-\nu_1}}\Big)^{\frac{2-\nu_1}{2}}\Big(\sum_{|n|>D_\epsilon}u_n^2\Big)^{\frac{\nu_1-1}{2}}\|v\|\\
&\le o(1) + 2K\underline{q}^{-\nu_1/2}\|d\|_{\frac{2}{2-\nu_1}}\|v\|\epsilon^{\nu_1-1}, \quad \forall v\in E,
\end{aligned}
$$

which implies that $J\in C^1(E,\mathbb{R})$.

For the derivative of J we have the formula

$$
\langle J'(u),v\rangle = \sum_{n=-\infty}^{+\infty}[(-1)^k p_{n-1}\Delta^k u_{n-1}\Delta^k v_{n-1} + q_n u_n v_n - f_n(u_n)v_n], \tag{2.8}
$$

for all $u,v\in E$. Finally, the critical points of J in E are solutions of (1.5) satisfying (1.4).

For $u,v\in E$, there exists an integer $D_1>0$ such that $|u_n|+|v_n|<1$ for $|n|>D_1$. For any sequence $\{\theta_n\}_{n\in\mathbb{Z}}$ with $|\theta_n|<1$, $n\in\mathbb{Z}$ and any number $h\in(0,1)$, it follows from (H3) and (2.4) that

$$
\begin{aligned}
&\sum_{n=-\infty}^{+\infty}|f_n(u_n+\theta_n v_n)v_n|\\
&= \sum_{|n|\le D_1}|f_n(u_n+\theta_n v_n)v_n| + \sum_{|n|>D_1}|f_n(u_n+\theta_n v_n)v_n|\\
&\le \sum_{|n|\le D_1}|f_n(u_n+\theta_n v_n)||v_n| + \sum_{|n|>D_1}|f_n(u_n+\theta_n v_n)||v_n|\\
&\le \sum_{|n|\le D_1}\max_{|x|\le\|u\|_\infty+\|v\|_\infty}|f_n(x_n)||v_n| + K\sum_{|n|>D_1}d_n|u_n+v_n|^{\nu_1-1}|v_n|\\
&\le \sum_{|n|>D_1}\max_{|x|\le\|u\|_\infty+\|v\|_\infty}|f_n(x_n)||v_n| + K\sum_{|n|>D_1}d_n(|u_n|^{\nu_1-1}+|v_n|^{\nu_1-1})|v_n|
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{|n|\le D_1} \max_{|x|\le\|u\|_\infty+\|v\|_\infty} |f_n(x_n)||v_n| + K\Big(\sum_{|n|>D_1} d_n^2|u_n|^{2(\nu_1-1)}\Big)^{1/2}\Big(\sum_{|n|>D_1} v_n^2\Big)^{1/2}\\
&\quad + K\Big(\sum_{|n|>D_1} d_n^2|v_n|^{2(\nu_1-1)}\Big)^{1/2}\Big(\sum_{|n|>D_1} v_n^2\Big)^{1/2}\\
&\le \sum_{|n|\le D_1} \max_{|x|\le\|u\|_\infty+\|v\|_\infty} |f_n(x_n)||v_n|\\
&\quad + K\underline{q}^{-1/2}\Big(\sum_{|n|>D_1} |d_n|^{\frac{2}{2-\nu_1}}\Big)^{\frac{2-\nu_1}{2}}\Big(\sum_{|n|>D_1} u_n^2\Big)^{\frac{\nu_1-1}{2}}\|v\|\\
&+ K\underline{q}^{-1/2}\Big(\sum_{|n|>D_1} |d_n|^{\frac{2}{2-\nu_1}}\Big)^{\frac{2-\nu_1}{2}}\Big(\sum_{|n|>D_1} v_n^2\Big)^{\frac{\nu_1-1}{2}}\|v\|\\
&\le \sum_{|n|\le D_1} \max_{|x|\le\|u\|_\infty+\|v\|_\infty} |f_n(x_n)||v_n| + K\underline{q}^{-\nu_1/2}\|d\|_{\frac{2}{2-\nu_1}}(\|u\|^{\nu_1-1}+\|v\|^{\nu_1-1})\|v\|\\
&< +\infty.
\end{aligned}
$$

Combining this with (2.5), we have

$$
\begin{aligned}
\langle J'(u), v\rangle &= \lim_{h\to 0^+} \frac{J(u+hv)-J(u)}{h}\\
&= \lim_{h\to 0^+} \frac{1}{h}\Big\{\frac{\|u+hv\|^2-\|u\|^2}{2} - \sum_{n=-\infty}^{+\infty}[F_n(u_n+hv_n)-F_n(u_n)]\Big\}\\
&= \lim_{h\to 0^+} \Big[\langle u,v\rangle + \frac{h\|v\|^2}{2} - \sum_{n=-\infty}^{+\infty} F_n'(u_n+\theta_n h v_n)v_n\Big]\\
&= \langle u,v\rangle - \sum_{n=-\infty}^{+\infty} f_n(u_n)v_n\\
&= \sum_{n=-\infty}^{+\infty}[p_{n-1}\Delta^k u_{n-1}\Delta^k v_{n-1} + q_n u_n v_n - f_n(u_n)v_n],
\end{aligned}
$$

which implies (2.8). Using

$$
\Delta^k u_{n-1} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} u(n+k-i-1),
$$

we can compute the partial derivative as

$$
\frac{\partial J(u)}{\partial u_n} = (-1)^k\Delta^k(p_{n-k}\Delta^k u_{n-k}) + q_n u_n - f_n(u_n),\ k\in\mathbb{Z}(1),\ n\in\mathbb{Z}.
$$

Thus, the critical points of J in E are solutions of (1.5) satisfying (1.4).

Let E be a real Banach space, $J\in C^1(E,\mathbb{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition ((PS) condition for short) if any sequence $\{u_n\}\subset E$ for which $\{J(u_n)\}$ is bounded and $J'(u_n)\to 0$, as $n\to\infty$, possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Clark's Theorem [22]). *Let E be a Banach space and $J \in C^1(E,\mathbb{R})$ with J even. Suppose that J satisfies the (PS) condition, $J(0)=0$ and*

(1) *there is a set $\tilde{E} \subset E$ such that $\tilde{E}$ is homeomorphic to S^{k-1} by an odd map, $\sup_{\tilde{E}} J < 0$,*

then J possesses at least k distinct pairs of critical points.

Lemma 2.2. *Suppose that* (H1)–(H4) *are satisfied. Then J is bounded from below.*

Proof. It follows from (2.4), (2.5), (H2) and Hölder inequality that

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \sum_{n=-\infty}^{+\infty} F_n(u_n) \\
&\geq \frac{1}{2}\|u\|^2 - \sum_{\mathbf{Z}(|u_n|\leq 1)} F_n(u_n) - \sum_{\mathbf{Z}(|u_n|>1)} F_n(u_n) \\
&\geq \frac{1}{2}\|u\|^2 - \sum_{\mathbf{Z}(|u_n|\leq 1)} a_n|u_n|^{\nu_1} - \sum_{\mathbf{Z}(|u_n|>1)} b_n|u_n|^{\nu_2} \\
&\geq \frac{1}{2}\|u\|^2 - \Big(\sum_{\mathbf{Z}(|u_n|\leq 1)} |a_n|^{\frac{2}{2-\nu_1}}\Big)^{\frac{2-\nu_1}{2}} \Big(\sum_{\mathbf{Z}(|u_n|\leq 1)} |u_n|^2\Big)^{\nu_1/2} \\
&\quad - \Big(\sum_{\mathbf{Z}(|u_n|>1)} |b_n|^{\frac{2}{2-\nu_1}}\Big)^{\frac{2-\nu_1}{2}} (\sum_{\mathbf{Z}(|u_n|>1)} |u_n|^{\frac{2\nu_2}{\nu_1}})^{\nu_1/2} \\
&\geq \frac{1}{2}\|u\|^2 - \underline{q}^{-\nu_1/2}(\sum_{\mathbf{Z}(|u_n|\leq 1)} |a_n|^{\frac{2}{2-\nu_1}})^{\frac{2-\nu_1}{2}}\|u\|^{\nu_1} \\
&\quad - \Big(\sum_{\mathbf{Z}(|u_n|>1)} |b_n|^{\frac{2}{2-\nu_1}}\Big)^{\frac{2-\nu_1}{2}} \Big(\sum_{\mathbf{Z}(|u_n|>1)} |u_n|^{\frac{2(\nu_2-\nu_1)}{\nu_1}}|u_n|^2\Big)^{\nu_1/2} \\
&\geq \frac{1}{2}\|u\|^2 - \underline{q}^{-\nu_1/2}\|a\|_{\frac{2}{2-\nu_1}}\|u\|^{\nu_1} - \|b\|_{\frac{2}{2-\nu_1}}\|u\|_\infty^{\nu_2-\nu_1}\|u\|_2^{\nu_1} \\
&\geq \frac{1}{2}\|u\|^2 - \underline{q}^{-\nu_1/2}\|a\|_{\frac{2}{2-\nu_1}}\|u\|^{\nu_1} - \underline{q}^{-\frac{\nu_2}{2}}\|b\|_{\frac{2}{2-\nu_1}}\|u\|^{\nu_2}.
\end{aligned} \tag{2.9}$$

From $1<\nu_1<\nu_2<2$ and (2.9) it follows that $J(u)\to+\infty$ as $u\to+\infty$. Thus, J is bounded from below. The proof is complete. □

Lemma 2.3. *Under conditions* (H1)–(H4)*, functional J satisfies the (PS) condition.*

Proof. Let $\{u^{(j)}\}_{j\in\mathbb{N}} \subset E$ be such that $\{J(u^{(j)})\}_{j\in\mathbb{N}}$ is bounded and $J'(u^{(j)})\to 0$ as $j\to\infty$. It follows from (H3) and (2.9) that there exists a constant K_1 such that

$$\|u^{(j)}\|_\infty \leq \underline{q}^{-1/2}\|u^{(j)}\| \leq K_1. \tag{2.10}$$

So passing to a subsequence if necessary, it can be assumed that $u^{(j)} \rightharpoonup u^{(0)}$ in E. It is easy to verify that $u_n^{(j)}$ converges to $u_n^{(0)}$ pointwise for all $n\in\mathbb{Z}$; that is,

$$\lim_{j\to\infty} u_n^{(j)} = u_n^{(0)}, \quad \forall n\in\mathbb{Z}. \tag{2.11}$$

Combining this with (2.10), we have

$$\|u^{(0)}\|_\infty \leq K_1. \tag{2.12}$$

For any given $\epsilon > 0$, by (H3), we can choose an integer $D_2 > 0$ such that

$$\Big(\sum_{|n|>D_2} |d_n|^{\frac{2-\nu_1}{2}}\Big)^{\frac{2}{2-\nu_1}} < \epsilon. \tag{2.13}$$

It follows from (2.12) and the continuity of $f_n(u)$ on u that there exists $j_0 \in \mathbb{N}$ such that

$$\sum_{n=-D_2}^{D_2} |f_n(u_n^{(j)}) - f_n(u_n^{(0)})||u_n^{(j)} - u_n^{(0)}| < \varepsilon, \quad j \geq j_0. \tag{2.14}$$

On the other hand, it follows from (H3), (2.10), (2.12)-(2.14) and Hölder inequality that

$$\begin{aligned}
&\sum_{|n|\geq D_2} |f_n(u_n^{(j)}) - f_n(u_n^{(0)})|\,|u_n^{(j)} - u_n^{(0)}| \\
&\leq \sum_{|n|\geq D_2} \Big(|f_n(u_n^{(j)})| + |f_n(u_n^{(0)})|\Big)\Big(|u_n^{(j)}| + |u_n^{(0)}|\Big) \\
&\leq K \sum_{|n|\geq D_2} d_n\Big(|u_n^{(j)}|^{\nu_1-1} + |u_n^{(0)}|^{\nu_1-1}\Big)\Big(|u_n^{(j)}| + |u_n^{(0)}|\Big) \\
&\leq 2K \sum_{|n|\geq D_2} d_n\Big(|u_n^{(j)}|^{\nu_1} + |u_n^{(0)}|^{\nu_1}\Big) \\
&\leq 2K\Big(\sum_{|n|>D_2} |d_n|^{\frac{2-\nu_1}{2}}\Big)^{\frac{2}{2-\nu_1}}\Big(\sum_{n=-\infty}^{+\infty} |u_n^{(j)}|^2 + \sum_{n=-\infty}^{+\infty} |u_n^{(0)}|^2\Big)^{\nu_1/2} \\
&\leq 2K\underline{q}^{-\nu_1/2}\Big(\|u^{(j)}\|^2 + \|u^{(0)}\|^2\Big)^{\nu_1/2}\epsilon \\
&\leq 2K\underline{q}^{-\nu_1/2}\Big(\underline{q}K_1^2 + \|u^{(0)}\|^2\Big)^{\nu_1/2}\epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we obtain

$$\sum_{n=-\infty}^{+\infty} \big|f_n(u_n^{(j)}) - f_n(u_n^{(0)})\big|\,\big|u_n^{(j)} - u_n^{(0)}\big| \to 0, \quad j \to \infty. \tag{2.15}$$

From (2.2), (2.4) and (2.8) it follows that

$$\begin{aligned}
&\langle J'(u^{(j)}) - J'(u^{(0)}), u^{(j)} - u^{(0)}\rangle \\
&= \|u^{(j)} - u^{(0)}\|^2 - \sum_{n=-\infty}^{+\infty} \Big(f_n(u_n^{(j)}) - f_n(u_n^{(0)})\Big)(u^{(j)} - u^{(0)}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|u^{(j)} - u^{(0)}\|^2 &\leq \langle J'(u^{(j)}) - J'(u^{(0)}), u^{(j)} - u^{(0)}\rangle \\
&\quad + \sum_{n=-\infty}^{+\infty} \Big(f_n(u_n^{(j)}) - f_n(u_n^{(0)})\Big)(u^{(j)} - u^{(0)}).
\end{aligned}$$

Since $\langle J'(u^{(j)}) - J'(u^{(0)}), u^{(j)} - u^{(0)}\rangle \to 0,\ j \to \infty$. Thus, $u^{(j)} \to u^{(0)}$ in E and the proof of Lemma 2.3 is complete. $\square$

3. Proofs of the main results

In this section, we shall obtain multiple solutions of (1.5) satisfying (1.4) by using the critical point theory.

Proof of Theorem 1.1. We have already known that $J \in C^1(E, \mathbb{R})$, J is bounded from below and J satisfies the (PS) condition. It is easy to see that J is even and $J(0) = 0$. Hence, it suffices to prove that J satisfies the condition (1) of Lemma 2.1. Assume that $\tilde{E} \subset E$ and $u_1, u_2, \dots, u_k$ are the basis of $\tilde{E}$. Let $M = \{n_1, n_2, \dots, n_k\}$, $n_1 < n_2 < \dots < n_k$ and for $i = 1, 2, \dots, k$,

$$u_n^{(i)} = \begin{cases} 1, & n = n_i, \\ 0, & n \neq n_i, \end{cases} \tag{3.1}$$

For $u \in \tilde{E}$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \dots, k$ such that

$$u_n = \sum_{i=1}^{k} \lambda_i u_n^{(i)}, \ \forall n \in \mathbb{Z}. \tag{3.2}$$

Thus,

$$\|u\|_{\nu_3} = \Big(\sum_{n=-\infty}^{+\infty} |u_n|^{\nu_3} \Big)^{1/\nu_3} = \Big(\sum_{i=1}^{k} |\lambda_i|^{\nu_3} |u_n^{(i)}|^{\nu_3} \Big)^{1/\nu_3}. \tag{3.3}$$

Since all norms of a finite dimensional normed space are equivalent, there is a positive constant c_1 such that

$$\|u\| \le c_1 \|u\|_{\nu_3}, \quad \forall u \in \tilde{E}. \tag{3.4}$$

Set $\Theta = \{u \in \tilde{E} : \ \|u\| = 1\}$. For $u \in \Theta$ and $\tau > 0$, it follows from (F_2), (2.5) and (3.2)-(3.4) that

$$\begin{aligned} J(\tau u) &= \frac{1}{2}\|\tau u\|^2 - \sum_{n=-\infty}^{+\infty} F_n(\tau u_n) \\ &\le \frac{\tau^2}{2}\|u\|^2 - \sum_{i=1}^{k} F_n(\tau u_n) \\ &\le \frac{\tau^2}{2}\|u\|^2 - \eta \tau^{\nu_3} \sum_{i=1}^{k} |\lambda_i|^{\nu_3} |u_n^{(i)}|^{\nu_3} \\ &= \frac{\tau^2}{2}\|u\|^2 - \eta \tau^{\nu_3} \|u\|_{\nu_3}^{\nu_3} \\ &\le \frac{\tau^2}{2}\|u\|^2 - \eta (\frac{\tau}{c_1})^{\nu_3} \|u\|^{\nu_3} \\ &= \frac{\tau^2}{2} - \eta (\frac{\tau}{c_1})^{\nu_3}. \end{aligned}$$

Since $1 < \nu_3 < 2$, we can choose τ small enough to ensure that

$$J(\tau u) < -\epsilon_1 < 0, \quad u \in \Theta, \tag{3.5}$$

where ϵ_1 is a positive constant.

For $u \in \tilde{E}$, by (3.2), we have

$$\begin{aligned} \|u\|^2 &= \sum_{n=-\infty}^{+\infty} [p_{n-1}(\Delta^k u_{n-1})^2 + q_n u_n^2] \\ &= \sum_{i=1}^{k} \lambda_i^2 [p_{n-1}(\Delta^k u_{n-1}^{(i)})^2 + q_n (u_n^{(i)})^2] \quad := f(\lambda_1, \lambda_2, \dots, \lambda_k). \end{aligned} \tag{3.6}$$

It is easy to see that $f(\lambda_1, \lambda_2, \dots, \lambda_k)$ is a positive definite quadratic form. Thus, there exists an invertible matrix $P \in \mathbb{R}^{k\times k}$ such that

$$(x_1, x_2, \dots, x_k)^* = P(\lambda_1, \lambda_2, \dots, \lambda_k)^*, \quad f(\lambda_1, \lambda_2, \dots, \lambda_k) = \sum_{i=1}^{k} x_i^2. \tag{3.7}$$

Set

$$\Theta^\tau = \{\tau u : u \in \Theta\}, \quad S^{k-1} = \{(x_1, x_2, \dots, x_k)^* \in \mathbb{R}^k : \sum_{i=1}^{k} x_i^2 = 1\}.$$

By (3.6), we have

$$\Theta^\tau = \big\{ \sum_{i=1}^{k} \lambda_i u^{(i)} : f(\lambda_1, \lambda_2, \dots, \lambda_k) = \tau^2 \big\}.$$

Let $\phi : \Theta^\tau \to S^{k-1}$ and

$$\phi(u) = \tau^{-1}(x_1, x_2, \dots, x_k)^*, \quad u \in \Theta^\tau.$$

It is easy to verify that $\phi : \Theta^\tau \to S^{k-1}$ is an odd homeomorphic map. From (3.5) it follows that

$$J(u) < -\epsilon_1 < 0, \quad u \in \Theta^\tau.$$

Therefore, $\sup_{\Theta^\tau} J \le -\epsilon_1 < 0$. The condition (1) of Lemma 2.1 holds. By Lemma 2.1, J has at least k distinct pairs of critical points, and so (1.5) has at least k distinct pairs of nontrivial solutions satisfying (1.4). The desired results follow. □

Proof of Corollary 1.2. It is easy to see that (H5) and (H6) imply (H2). Thus, by Theorem 1.1, the conclusion of Corollary 1.2 follows. □

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