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GROUND STATE, BOUND STATES AND BIFURCATION PROPERTIES FOR A SCHRÖDINGER-POISSON SYSTEM WITH CRITICAL EXPONENT

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Abstract. This article concerns the existence of ground state and bound states, and the study of their bifurcation properties for the Schrödinger-Poisson system

$$-\Delta u + u + \phi u = |u|^4 u + \mu h(x) u, \quad -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3.$$

Under suitable assumptions on the coefficient $h(x)$, we prove that the ground state must bifurcate from zero, and that another bound state bifurcates from a solution, when $\mu = \mu_1$ is the first eigenvalue of $-\Delta u + u = \mu h(x) u$ in $H^1(\mathbb{R}^3)$.

1. Introduction

Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the Dirichlet norm $\|u\|^2_{D^{1,2}} := \int_{\mathbb{R}^3} |\nabla u|^2 dx$ and $H^1(\mathbb{R}^3)$ be the usual Sobolev space with the norm $\|u\|^2 := \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2\right) dx$. In this article, we study the system

$$\begin{gathered} -\Delta u + u + \phi u = |u|^4 u + \mu h(x) u \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.1}$$

where $\mu > 0$ and the real valued function $h(x) \geq 0$ for any $x \in \mathbb{R}^3$. The system (1.1) can be looked on as a non-autonomous version of the system

$$\begin{gathered} -\Delta u + u + \phi u = f(u) \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.2}$$

which has been derived from finding standing waves of the Schrödinger-Poisson system

$$\begin{gathered} i\psi_t - \Delta \psi + \phi \psi = f(\psi) \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi = |\psi|^2 \quad \text{in } \mathbb{R}^3. \end{gathered}$$

A starting point of studying system (1.1) is the following fact. For any $u \in H^1(\mathbb{R}^3)$, there is a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ with

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy$$

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such that $-\Delta\phi_u = u^2$, see e.g. [24] or Section 2 of this article. Inserting this ϕ_u into the first equation of the system (1.1), we obtain

$$-\Delta u + u + \phi_u u = |u|^4 u + \mu h(x)u, \quad u \in H^1(\mathbb{R}^3). \tag{1.3}$$

Denote

$$F(u) = \int_{\mathbb{R}^3} \phi_u(x)|u(x)|^2 dx$$

and introduce the Euler-Lagrange functional

$$I_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \int_{\mathbb{R}^3} \Big(\frac{1}{6}|u|^6 + \frac{\mu}{2}h(x)u^2\Big)dx.$$

We know from [24] and the Sobolev inequality that I_μ is well defined and $I_\mu \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. Moreover, for any $v \in H^1(\mathbb{R}^3)$,

$$\langle I'_\mu(u), v\rangle = \int_{\mathbb{R}^3} \big(\nabla u \nabla v + uv + \phi_u uv - (|u|^4 uv + \mu h(x)uv)\big)\, dx.$$

It is known that there is a one to one correspondence between solutions of (1.3) and critical points of I_μ in $H^1(\mathbb{R}^3)$. Note that if $u \in H^1(\mathbb{R}^3)$ is a solution of (1.3), then (u, ϕ_u) is a solution of the system (1.1). If $u \geq 0$, $u \neq 0$ and u is a solution of (1.3), then (u, ϕ_u) is a nonnegative solution of (1.1) since ϕ_u is always nonnegative. We call $u \in H^1(\mathbb{R}^3)$ a bound state of (1.3) if $I'_\mu(u) = 0$. At this time (u, ϕ_u) is called a bound state of the system (1.1). A $u \in H^1(\mathbb{R}^3)$ is called a ground state of (1.3) if $I'_\mu(u) = 0$ and $I_\mu(u) \leq I_\mu(w)$ for any $w \in \{u \in H^1(\mathbb{R}^3) : I'_\mu(u) = 0\}$. In this case, we call (u, ϕ_u) a ground state of the system (1.1). Hence to study the system (1.1), it suffices to study (1.3). We assume the following condition.

- (H1) The function $h \in L^{3/2}(\mathbb{R}^3)$, $h(x) \geq 0$ and $h(x) \not\equiv 0$ in $\mathbb{R}^3$. There are $\rho_1 > 0$, $\rho'_1 > 0$, $\rho_2 > 0$ and $\beta > 0$ such that $\rho'_1|x|^{-\beta} \leq h(x) \leq \rho_1|x|^{-\beta}$ for any $x \in B_{\rho_2}(0)\backslash\{0\}$.

Under condition (H1), we know that the eigenvalue problem $-\Delta u + u = \mu h(x)u, \quad u \in H^1(\mathbb{R}^3)$ has a first eigenvalue $\mu_1 > 0$ and μ_1 is simple, see Lemma 2.1 of Section 2.

The aim of this article is to prove the existence of a nonnegative ground state and nonnegative bound states of (1.3), and study their bifurcation properties in the case of $\mu \geq \mu_1$. We begin with a result concerning with the case of $0 < \mu \leq \mu_1$.

Theorem 1.1. *Suppose that the function $h(x)$ satisfies the assumption* (H1) *and $1 < \beta < 2$. If $0 < \mu \leq \mu_1$, then equation* (1.3) *has at least one nonnegative bound state in $H^1(\mathbb{R}^3)$.*

The next theorem considers the case when μ in a small right neighborhood of μ_1.

Theorem 1.2. *Suppose that the function $h(x)$ satisfies the assumption* (H1) *and $3/2 < \beta < 2$. Then there exists $\delta > 0$ such that, for $\mu_1 < \mu < \mu_1 + \delta$,*

1. *equation* (1.3) *has at least one nonnegative ground state $u_{0,\mu}$ with $I_\mu(u_{0,\mu}) < 0$, which bifurcates from 0 in the sense that for any $\mu^{(n)} > \mu_1$ and $\mu^{(n)} \to \mu_1$, there exist a sequence of solutions $u_{0,\mu^{(n)}}$ such that $u_{0,\mu^{(n)}} \to 0$ strongly in $H^1(\mathbb{R}^3)$;*
2. *equation* (1.3) *has a nonnegative bound state $u_{2,\mu}$ with $I_\mu(u_{2,\mu}) > 0$. Moreover the $u_{2,\mu}$ bifurcates from a solution of (P_{μ_1}) in the sense that for any $\mu^{(n)} > \mu_1$ and $\mu^{(n)} \to \mu_1$, there exist a sequence of solutions $u_{2,\mu^{(n)}}$ and a*

$u_{\mu_1} \in H^1(\mathbb{R}^3)$ with $I'_{\mu_1}(u_{\mu_1}) = 0$ and $I_\mu(u_{\mu_1}) > 0$ such that $u_{2,\mu^{(n)}} \to u_{\mu_1}$ strongly in $H^1(\mathbb{R}^3)$.

The proofs of Theorem 1.1 and 1.2 are based on critical point theory. There are several difficulties in the road of getting critical points of I_μ in $H^1(\mathbb{R}^3)$ since we are dealing with the problem in the whole space $\mathbb{R}^3$, the power 6 in the term $\int_{\mathbb{R}^3} |u|^6 dx$ reaching the critical Sobolev exponent for $H^1(\mathbb{R}^3)$, the appearance of a nonlocal term $\phi_u u$ and the non coercive linear part. To explain our strategy, we review some related known results. For the system (1.2), under various conditions of f, there are a lot of papers dealing with the existence and nonexistence of positive solutions $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, see for example [2, 24] and the references therein. The lack of compactness from $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ $(2 < p < 6)$ was overcome by restricting the problem in $H_r^1(\mathbb{R}^3)$ which is a subspace of $H^1(\mathbb{R}^3)$ containing only radial functions. The existence of multiple radial solutions and/or multiple non-radial solutions have been obtained in [2, 14]. See also [6, 16, 17, 18, 19, 20, 25, 30, 31] for some other results related to the system (1.2).

While for nonautonomous version of Schrödinger-Poisson system, just a few results are known. Jiang et al. [22] studied the following Schrödinger-Poisson system with non constant coefficient,

$$\begin{gathered} -\Delta u + (1 + \lambda g(x))u + \theta\phi(x)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \quad \lim_{|x|\to\infty} \phi(x) = 0, \end{gathered}$$

in which the authors prove the existence of ground state solution and its asymptotic behavior depending on θ and λ. The lack of compactness was overcome by suitable assumptions on $g(x)$ and λ large enough. The Schrödinger-Poisson system with critical nonlinearity of the form

$$\begin{gathered} -\Delta u + u + \phi u = V(x)|u|^4 u + \mu P(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \quad 2 < q < 6, \mu > 0 \end{gathered}$$

has been studied by Zhao et al. [32]. Zhao et al. [32] assumed that $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, $\lim_{|x|\to\infty} V(x) = V_\infty \in (0, \infty)$ and $V(x) \geq V_\infty$ for $x \in \mathbb{R}^3$ and prove the existence of one positive solution for $4 < q < 6$ and each $\mu > 0$. It is also proven the existence of one positive solution for $q = 4$ and μ large enough. Cerami et al. [12] study the following type of Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + u + L(x)\phi u = g(x, u) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = L(x)u^2 \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.4}$$

Besides some other conditions and the assumption of $L(x) \in L^2(\mathbb{R}^3)$, they prove the existence and nonexistence of ground state solutions. The assumption $L(x) \in L^2(\mathbb{R}^3)$ will imply suitable compactness property of the coupled term $L(x)\phi u$. Huang et al. [21] used this property to study the existence of multiple solutions of (1.4) when $g(x, u) = a(x)|u|^{p-2}u + \mu h(x)u$ and μ stays in a right neighborhood of μ_1. The lack of compactness was overcome by suitable assumptions on the sign changing function $a(x)$. But the case of $L(x) \equiv 1$ and $a(x) \equiv 1$ is still unknown since in this case a global compactness can not be recovered. In [13], the authors considered the system

$$-\Delta u + |x|^2 u + \phi u = |u|^{p-2}u + \mu u \quad \text{in } \mathbb{R}^3,$$

$$-\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3,$$

in which the lack of compactness was overcome with the help of the harmonic potential $|x|^2u$. While for the (1.3), we have to analyze the energy level of the functional I_μ such that the Palais-Smale ((PS) for short) condition may hold at suitable interval. Besides these, another difficulty is to find mountain pass geometry for the functional I_μ in the case of $\mu \geq \mu_1$. We emphasize that for the semilinear elliptic equation

$$-\Delta u = a(x)|u|^{p-2}u + \tilde{\mu}K(x)u, \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

Costa et al. [15] have proven the mountain pass geometry for the related functional of (1.5) when $\tilde{\mu} \geq \tilde{\mu}_1$, where $\tilde{\mu}_1$ is the first eigenvalue of $-\Delta u = \tilde{\mu}K(x)u$ in $D^{1,2}(\mathbb{R}^3)$. Costa et al. have managed to do these with the help of an additional condition $\int_{\mathbb{R}^3} a(x)\tilde{e}_1^p dx < 0$, where $\tilde{e}_1$ is the eigenvalue corresponding to $\tilde{\mu}_1$. In the present paper, no such kind of condition can be used. We will develop further the techniques in [21] to prove the mountain pass geometry. A third difficulty is to look for a ground state of (1.3). A usual method of getting a ground state is by minimizing the functional I_μ over the set $\{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I'_\mu(u), u\rangle = 0\}$. But in the case of $\mu > \mu_1$, one can not do like this because we do not know if 0 belongs to the boundary of this Nehari set. To overcome this trouble, we will investigate minimization problems over the set $\{u \in H^1(\mathbb{R}^3)\backslash\{0\} \ : \ I'_\mu(u) = 0\}$.

This paper is organized as follows. In Section 2, we give some useful preliminaries. Special attentions are focused on several lemmas analyzing the Palais-Smale conditions of the functional I_μ, which will play an important role to prove the main results. In Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorem 1.2.

Throughout this paper, $o(1)$ is a generic infinitesimal. The $H^{-1}(\mathbb{R}^3)$ denotes dual space of $H^1(\mathbb{R}^3)$. $L^q(\mathbb{R}^3)$ $(1 \leq q \leq +\infty)$ is a Lebesgue space with the norm denoted by $\|u\|_{L^q}$. The S_6 denotes the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$ defined by

$$S_6 = \inf_{u\in D^{1,2}(\mathbb{R}^3)\backslash\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\big(\int_{\mathbb{R}^3} |u|^6 dx\big)^{1/3}}.$$

For any $\rho > 0$ and $x \in \mathbb{R}^3$, $B_\rho(x)$ denotes the ball of radius ρ centered at x. C or C_j $(j = 1,\ 2,\ \dots)$ denotes various positive constants, whose exact value is not important.

2. Preliminaries

In this section, we give some useful preliminary lemmas, which will be used to analyze the (PS) conditions. First of all, let us recall the variational setting of the problem. For any $u \in H^1(\mathbb{R}^3)$, denoting $L_u(v)$ the linear functional in $D^{1,2}(\mathbb{R}^3)$ by $L_u(v) = \int_{\mathbb{R}^3} u^2 v dx$, one may deduce from the Hölder and the Sobolev inequalities that

$$|L_u(v)| \leq \|u\|^2_{L^{\frac{12}{5}}}\|v\|_{L^6} \leq C\|u\|^2_{L^{\frac{12}{5}}}\|v\|_{D^{1,2}}. \tag{2.1}$$

Hence, for any $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v dx = \int_{\mathbb{R}^3} u^2 v dx \quad \text{for any } v \in D^{1,2}(\mathbb{R}^3),$$

i.e., ϕ_u is the weak solution of $-\Delta\phi = u^2$ in $D^{1,2}(\mathbb{R}^3)$. Moreover it holds that

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy.$$

Clearly $\phi_u(x) \geq 0$ for any $x \in \mathbb{R}^3$. We also have that

$$\|\phi_u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx = \int_{\mathbb{R}^3} \phi_u u^2 dx. \tag{2.2}$$

Using (2.1) and (2.2), we obtain that

$$\|\phi_u\|_{L^6} \leq C\|\phi_u\|_{D^{1,2}} \leq C\|u\|^2_{L^{\frac{12}{5}}} \leq C\|u\|^2. \tag{2.3}$$

Then we deduce that

$$\int_{\mathbb{R}^3} \phi_u(x)u^2(x)dx \leq C\|u\|^4. \tag{2.4}$$

Hence on $H^1(\mathbb{R}^3)$, the functionals

$$F(u) = \int_{\mathbb{R}^3} \phi_u(x)u^2(x)dx, \tag{2.5}$$

$$I_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \int_{\mathbb{R}^3}\Big(\frac{1}{6}|u|^6 + \frac{\mu}{2}h(x)u^2\Big)dx \tag{2.6}$$

are well defined and C^1. Moreover, for any $v \in H^1(\mathbb{R}^3)$,

$$\langle I'_\mu(u), v\rangle = \int_{\mathbb{R}^3}\big(\nabla u\nabla v + uv + \phi_u uv - |u|^4uv - \mu h(x)uv\big)\,dx.$$

The following Lemma is a direct consequence of [29, Lemma 2.13].

Lemma 2.1. *Assume that* (H1) *holds. Then we have the following conclusions.*

1. *The functional $u \in H^1(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} h(x)u^2dx$ is weakly continuous.*
2. *For each $v \in H^1(\mathbb{R}^3)$, the functional $u \in H^1(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} h(x)uvdx$ is weakly continuous.*

Using the spectral theory of compact symmetric operators on Hilbert space, the above lemma implies the existence of a sequence of eigenvalues $(\mu_n)_{n\in\mathbb{N}}$ of

$$-\Delta u + u = \mu h(x)u, \text{ in } H^1(\mathbb{R}^3)$$

with $\mu_1 < \mu_2 \leq \dots$ and each eigenvalue being of finite multiplicity. The associated normalized eigenfunctions are denoted by $e_1, e_2, \dots$ with $\|e_i\| = 1$, $i = 1, 2, \dots$. Moreover, one has $\mu_1 > 0$ with an eigenfunction $e_1 > 0$ in $\mathbb{R}^3$. In addition, we have the following variational characterization of μ_n:

$$\mu_1 = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3} h(x)u^2dx}, \quad \mu_n = \inf_{u\in S^\perp_{n-1}\setminus\{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3} h(x)u^2dx},$$

where $S^\perp_{n-1} = \{\text{span}\{e_1, e_2, \dots, e_{n-1}\}\}^\perp$.

Next we analyze the (PS) condition of the functional I_μ in $H^1(\mathbb{R}^3)$. The following definition is standard.

Definition 2.2. For $d \in \mathbb{R}$, the functional I_μ is said to satisfy $(PS)_d$ condition if for any $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ with $I_\mu(u_n) \to d$ and $I'_\mu(u_n) \to 0$, the $(u_n)_{n\in\mathbb{N}}$ contains a convergent subsequence in $H^1(\mathbb{R}^3)$. The functional I_μ is said to satisfy (PS) conditions if I_μ satisfies $(PS)_d$ condition for any $d \in \mathbb{R}$.

Lemma 2.3. *Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be such that $I_\mu(u_n) \to d \in \mathbb{R}$ and $I'_\mu(u_n) \to 0$, then $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$.*

Proof. For n large enough, we have

$$\begin{aligned} d + 1 + o(1)\|u_n\| &= I_\mu(u_n) - \frac{1}{4}\langle I'_\mu(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 - \frac{\mu}{4}\int_{\mathbb{R}^3} h(x)u_n^2 dx + \frac{1}{12}\int_{\mathbb{R}^3} |u_n|^6 dx. \end{aligned} \tag{2.7}$$

Note that for any $\vartheta > 0$, from $h \in L^{3/2}(\mathbb{R}^3)$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} h(x)u_n^2 dx &\le \Big(\int_{\mathbb{R}^3} |u_n|^6 dx\Big)^{1/3}\Big(\int_{\mathbb{R}^3} |h(x)|^{3/2} dx\Big)^{2/3} \\ &\le \frac{\vartheta}{3}\int_{\mathbb{R}^3} |u_n|^6 dx + \frac{2}{3}\vartheta^{-1/2}\int_{\mathbb{R}^3} |h(x)|^{3/2} dx. \end{aligned}$$

Choosing $\vartheta = 1/\mu$, we obtain

$$\begin{aligned} d + 1 + o(1)\|u_n\| &\ge \frac{1}{4}\|u_n\|^2 - \frac{\mu}{4}\int_{\mathbb{R}^3} h(x)u_n^2 dx + \frac{1}{12}\int_{\mathbb{R}^3} |u_n|^6 dx \\ &\ge \frac{1}{4}\|u_n\|^2 - \frac{1}{6}\mu^{3/2}\int_{\mathbb{R}^3} |h(x)|^{3/2} dx. \end{aligned} \tag{2.8}$$

This implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. □

The following lemma is a variant of Brezis-Lieb lemma. One may find the proof in [32].

Lemma 2.4 ([32])**.** *If a sequence $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ and $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$, then for n large enough*

$$\int_{\mathbb{R}^3} \phi_{u_n}(u_n)^2 dx = \int_{\mathbb{R}^3} \phi_{u_0}(u_0)^2 dx + \int_{\mathbb{R}^3} \phi_{(u_n-u_0)}(u_n - u_0)^2 dx + o(1).$$

Lemma 2.5. *There is a $\delta_1 > 0$ such that for any $\mu \in [\mu_1, \mu_1 + \delta_1)$, any solution u of (1.3) satisfies*

$$I_\mu(u) > -\frac{1}{3}S_6^{3/2}.$$

Proof. Since u is a weak solution of (1.3), we obtain

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\Big(\|u\|^2 - \mu\int_{\mathbb{R}^3} h(x)u^2 dx\Big) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx \\ &= \frac{1}{3}\Big(\|u\|^2 - \mu\int_{\mathbb{R}^3} h(x)u^2 dx\Big) + \frac{1}{12}\int_{\mathbb{R}^3} \phi_u u^2 dx. \end{aligned}$$

Noticing that $\|u\|^2 \ge \mu_1\int_{\mathbb{R}^3} h(x)u^2 dx$ for any $u \in H^1(\mathbb{R}^3)$, we deduce that for $u \ne 0$,

$$I_{\mu_1}(u) \ge \frac{1}{12}\int_{\mathbb{R}^3} \phi_u u^2 dx > 0.$$

Claim: there is a $\delta_1 > 0$ such that for any $\mu \in [\mu_1, \mu_1 + \delta_1)$, any solution u of (1.3) satisfies

$$I_\mu(u) > -\frac{1}{3}S_6^{3/2}.$$

Suppose that this claim is not true, then there are $\mu^{(n)} > \mu_1$ with $\mu^{(n)} \to \mu_1$ and solutions $u_{\mu^{(n)}}$ of (1.3) such that

$$I_{\mu^{(n)}}(u_{\mu^{(n)}}) \le -\frac{1}{3}S_6^{3/2}.$$

Note that $I'_{\mu^{(n)}}(u_{\mu^{(n)}}) = 0$. Then similar to the proof in Lemma 2.3, we deduce that for n large enough,

$$\begin{aligned} I_{\mu^{(n)}}(u_{\mu^{(n)}}) + o(1)\|u_{\mu^{(n)}}\| &\ge I_{\mu^{(n)}}(u_{\mu^{(n)}}) - \frac{1}{4}\langle I'_{\mu^{(n)}}(u_{\mu^{(n)}}), u_{\mu^{(n)}}\rangle \\ &\ge \frac{1}{4}\|u_{\mu^{(n)}}\|^2 - \frac{1}{6}\big(\mu^{(n)}\big)^{3/2}\int_{\mathbb{R}^3}|h(x)|^{3/2}dx. \end{aligned}$$

This implies that $(u_{\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Since for any $n \in \mathbb{N}$, $\|u_{\mu^{(n)}}\|^2 \ge \mu_1 \int_{\mathbb{R}^3} h(x)(u_{\mu^{(n)}})^2dx$, we obtain

$$\|u_{\mu^{(n)}}\|^2 - \mu^{(n)}\int_{\mathbb{R}^3} h(x)(u_{\mu^{(n)}})^2dx \ge \Big(1 - \frac{\mu^{(n)}}{\mu_1}\Big)\|u_{\mu^{(n)}}\|^2 \to 0$$

as $\mu^{(n)} \to \mu_1$ because $(u_{\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Combining this with the fact of

$$I_{\mu^{(n)}}(u_{\mu^{(n)}}) = \frac{1}{3}\Big(\|u_{\mu^{(n)}}\|^2 - \mu^{(n)}\int_{\mathbb{R}^3} h(x)(u_{\mu^{(n)}})^2dx\Big) + \frac{1}{12}\int_{\mathbb{R}^3}\phi_{u_{\mu^{(n)}}}(u_{\mu^{(n)}})^2dx,$$

we deduce that

$$\liminf_{n\to\infty} I_{\mu^{(n)}}(u_{\mu^{(n)}}) \ge \frac{1}{12}\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_{\mu^{(n)}}}(u_{\mu^{(n)}})^2dx \ge 0,$$

which contradicts that

$$I_{\mu^{(n)}}(u_{\mu^{(n)}}) \le -\frac{1}{3}S_6^{3/2}.$$

This proves the claim; the proof is complete. □

Lemma 2.6. *If $\mu \in [\mu_1, \mu_1 + \delta_1)$, then I_μ satisfies $(PS)_d$ condition for any $d < 0$.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be a $(PS)_d$ sequence of I_μ with $d < 0$, i.e., $I_\mu(u_n) \to d$ and $I'_\mu(u_n) \to 0$ as $n \to \infty$. Then for n large enough,

$$d + o(1) = \frac{1}{2}\|u_n\|^2 - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)u_n^2dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx - \frac{1}{6}\int_{\mathbb{R}^3}|u_n|^6dx$$

and

$$\langle I'_\mu(u_n), u_n\rangle = \|u_n\|^2 - \mu\int_{\mathbb{R}^3} h(x)u_n^2dx + \int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx - \int_{\mathbb{R}^3}|u_n|^6dx.$$

Similar to the proof in Lemma 2.3, we can deduce that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Without loss of generality, we may assume that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^3$. Denoting $w_n := u_n - u_0$, we obtain from Brezis-Lieb lemma and Lemma 2.4 that for n large enough,

$$\begin{gathered} \|u_n\|^2 = \|u_0\|^2 + \|w_n\|^2 + o(1), \\ \int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx = \int_{\mathbb{R}^3}\phi_{u_0}u_0^2dx + \int_{\mathbb{R}^3}\phi_{w_n}w_n^2dx + o(1), \\ \|u_n\|_{L^6}^6 = \|u_0\|_{L^6}^6 + \|w_n\|_{L^6}^6 + o(1). \end{gathered}$$

Using Lemma 2.1, we also have that $\int_{\mathbb{R}^3} h(x)u_n^2dx \to \int_{\mathbb{R}^3} h(x)u_0^2dx$ as $n \to \infty$. Therefore

$$\begin{aligned} d + o(1) &= I_\mu(u_n) \\ &= I_\mu(u_0) + \frac{1}{2}\|\nabla w_n\|_{L^2}^2 + \frac{1}{2}\|w_n\|_{L^2}^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |w_n|^6 dx. \end{aligned} \tag{2.9}$$

Since as $n \to \infty$, $\langle I'_\mu(u_n), \psi\rangle \to 0$ for any $\psi \in H^1(\mathbb{R}^3)$, we obtain that $I'_\mu(u_0) = 0$. From which we deduce that

$$\|u_0\|^2 - \mu\int_{\mathbb{R}^3} h(x)u_0^2dx + \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 dx = \int_{\mathbb{R}^3} |u_0|^6 dx. \tag{2.10}$$

Noting that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, we deduce from $I'_\mu(u_n) \to 0$ that

$$o(1) = \|u_n\|^2 - \mu\int_{\mathbb{R}^3} h(x)u_n^2dx + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} |u_n|^6 dx.$$

Combining this with (2.10) as well as Lemma 2.1, we obtain that

$$o(1) = \|w_n\|^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - \int_{\mathbb{R}^3} |w_n|^6 dx. \tag{2.11}$$

Recalling the definition of S_6, we know that

$$\|\nabla u\|_{L^2}^2 \ge S_6\Big(\int_{\mathbb{R}^3} |u|^6 dx\Big)^{1/3} \quad \text{for any } u \in D^{1,2}(\mathbb{R}^3).$$

Now we distinguish two cases:

(i) $\int_{\mathbb{R}^3} |w_n|^6 dx \not\to 0$ as $n \to \infty$;
(ii) $\int_{\mathbb{R}^3} |w_n|^6 dx \to 0$ as $n \to \infty$.

Suppose that the case (i) occurs. Then there exist $\eta_1 > 0$ and a subsequence of $(w_n)_{n\in\mathbb{N}}$, still denoted by $(w_n)_{n\in\mathbb{N}}$, such that $\int_{\mathbb{R}^3} |w_n|^6 dx \ge \eta_1 > 0$. We obtain from (2.11) that

$$\|\nabla w_n\|_{L^2}^2 \ge S_6\Big(\|\nabla w_n\|_{L^2}^2 + \|w_n\|_{L^2}^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - o(1)\Big)^{1/3}.$$

Hence we get that for n large enough,

$$\|\nabla w_n\|_{L^2}^2 \ge S_6^{3/2} + o(1). \tag{2.12}$$

Therefore using (2.9), (2.11) and (2.12), we deduce that for n large enough,

$$\begin{aligned} d + o(1) &= I_\mu(u_n) \\ &= I_\mu(u_0) + \frac{1}{2}\|\nabla w_n\|_{L^2}^2 + \frac{1}{2}\|w_n\|_{L^2}^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |w_n|^6 dx \\ &= I_\mu(u_0) + \frac{1}{3}\|\nabla w_n\|_{L^2}^2 + \frac{1}{3}\|w_n\|_{L^2}^2 + \frac{1}{12}\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx \\ &> -\frac{1}{3}S_6^{3/2} + \frac{1}{3}\|\nabla w_n\|_{L^2}^2 + \frac{1}{3}\|w_n\|_{L^2}^2 + \frac{1}{12}\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx > 0, \end{aligned} \tag{2.13}$$

which contradicts to the condition $d < 0$. This means that the case (i) does not occur. Therefore the case (ii) occurs. Using (2.11), we deduce that $\|w_n\|^2 \to 0$ as $n \to \infty$. Hence we have proven that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$. $\square$

We end this section by a characterization of the mountain pass geometry for the functional I_μ in $H^1(\mathbb{R}^3)$.

Lemma 2.7. *There exist $\delta_2 > 0$ with $\delta_2 \le \delta_1$, $\rho > 0$ and $\alpha > 0$ such that for any $\mu \in [\mu_1, \mu_1 + \delta_2)$,*

$$I_\mu|_{\partial B_\rho} \ge \alpha > 0.$$

Proof. For any $u \in H^1(\mathbb{R}^3)$, there exist $t \in \mathbb{R}$ and $v \in S_1^\perp$ such that

$$u = te_1 + v, \quad \text{where } \int_{\mathbb{R}^3} (\nabla v \nabla e_1 + v e_1)\, dx = 0. \tag{2.14}$$

Hence from direct computations we obtain

$$\|u\| = \left(\|\nabla(te_1 + v)\|_2^2 + \|te_1 + v\|_2^2\right)^{1/2} = \left(t^2 + \|v\|^2\right)^{1/2}, \tag{2.15}$$

$$\mu_2 \int_{\mathbb{R}^3} h(x) v^2 dx \le \|v\|^2, \quad \mu_1 \int_{\mathbb{R}^3} h(x) e_1^2 dx = \|e_1\|^2 = 1, \tag{2.16}$$

$$\mu_1 \int_{\mathbb{R}^3} h(x) e_1 v dx = \int_{\mathbb{R}^3} (\nabla v \nabla e_1 + v e_1)\, dx = 0. \tag{2.17}$$

We first consider the case of $\mu = \mu_1$. Denoting $\theta_1 := \frac{1}{2}\left(1 - \frac{\mu_1}{\mu_2}\right) > 0$, then by the relations from (2.14) to (2.17), we obtain

$$\begin{aligned}
I_{\mu_1}(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x) u^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx \\
&= \frac{1}{2}\|te_1 + v\|^2 + \frac{1}{4}F(te_1 + v) \\
&\quad - \frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)(te_1 + v)^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |te_1 + v|^6 dx \\
&\ge \frac{1}{2}\Big(1 - \frac{\mu_1}{\mu_2}\Big)\|v\|^2 + \frac{1}{4}F(te_1 + v) - \frac{1}{6}\int_{\mathbb{R}^3} |te_1 + v|^6 dx \\
&\ge \theta_1 \|v\|^2 + \frac{1}{4}F(te_1 + v) - C_1|t|^6 - C_2\|v\|^6.
\end{aligned}$$

Next we estimate the term $F(te_1 + v)$. Using the expression of $F(u)$, we have that

$$F(te_1 + v) = \frac{1}{4\pi}\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(te_1(y) + v(y))^2 (te_1(x) + v(x))^2}{|x - y|} dy dx.$$

Since

$$\begin{aligned}
&(te_1(y) + v(y))^2 \, (te_1(x) + v(x))^2 \\
&= t^4 (e_1(y))^2 (e_1(x))^2 + (v(y))^2 (v(x))^2 \\
&\quad + 2t^3 \left(e_1(y)(e_1(x))^2 v(y) + e_1(x)(e_1(y))^2 v(x)\right) \\
&\quad + 2t \left(e_1(x) v(x) (v(y))^2 + e_1(y) v(y) (v(x))^2\right) \\
&\quad + t^2 \left((e_1(x))^2 (v(y))^2 + 4 e_1(y) e_1(x) v(y) v(x) + (e_1(y))^2 (v(x))^2\right),
\end{aligned}$$

we know that

$$\Big|\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{e_1(y)(e_1(x))^2 v(y) + e_1(x)(e_1(y))^2 v(x)}{|x - y|} dy\, dx\Big| \le C\|v\|, \tag{2.18}$$

$$\Big|\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{2(e_1(x))^2 (v(y))^2 + 4 e_1(y) e_1(x) v(y) v(x)}{|x - y|} dy dx\Big| \le C\|v\|^2, \tag{2.19}$$

$$\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{e_1(x)v(x)(v(y))^2 + e_1(y)v(y)(v(x))^2}{|x-y|}dydx\Big| \le C\|v\|^3. \tag{2.20}$$

Hence

$$\begin{aligned} I_{\mu_1}(u) &\ge \theta_1\|v\|^2 + \theta_2|t|^4 - C_1|t|^6 - C_2\|v\|^6 - C_3|t|^3\|v\| \\ &\quad - C_4|t|^2\|v\|^2 - C_5|t|\|v\|^3 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_v v^2 dx, \end{aligned}$$

where $\theta_2 = \frac{1}{4}\int_{\mathbb{R}^3}\phi_{e_1}e_1^2dx$. Note that

$$\begin{gathered} t^2\|v\|^2 \le \frac{1}{3}|t|^6 + \frac{2}{3}\|v\|^3, \\ |t|\|v\|^3 \le \frac{1}{6}|t|^6 + \frac{5}{6}\|v\|^{18/5} \end{gathered}$$

and for some q_0 with $2 < q_0 < 4$, we also have that

$$|t|^3\|v\| \le \frac{1}{q_0}\|v\|^{q_0} + \frac{q_0-1}{q_0}|t|^{\frac{3q_0}{q_0-1}}.$$

Therefore,

$$\begin{aligned} I_{\mu_1}(u) &\ge \theta_1\|v\|^2 + \theta_2|t|^4 - \frac{C_3}{q_0}\|v\|^{q_0} - \frac{C_3(q_0-1)}{q_0}|t|^{\frac{3q_0}{q_0-1}} - \frac{C_4}{3}|t|^6 \\ &\quad - \frac{2C_4}{3}\|v\|^3 - \frac{C_5}{6}|t|^6 - \frac{5C_5}{6}\|v\|^{18/5} - C|t|^6 - C\|v\|^6. \end{aligned} \tag{2.21}$$

From $q_0 > 2$ and $\frac{3q_0}{q_0-1} > 4$ (since $q_0 < 4$), we know that there are positive constants θ_3, θ_4 and $\tilde\theta_3$, $\tilde\theta_4$ such that

$$I_{\mu_1}(u) \ge \theta_3\|v\|^2 + \theta_4|t|^4$$

provided that $\|v\| \le \tilde\theta_3$ and $|t| \le \tilde\theta_4$. Hence there are positive constants θ_5 and $\tilde\theta_5$ such that

$$I_{\mu_1}(u) \ge \theta_5\|u\|^4 \quad \text{for } \|u\|^2 \le \tilde\theta_5^2. \tag{2.22}$$

Set $\bar\delta := \min\{\frac{\mu_1}{2}\theta_5\tilde\theta_5^2,\, \mu_2-\mu_1\} > 0$ and $\delta_2 := \min\{\bar\delta, \delta_1\}$. Then for any $\mu \in [\mu_1, \mu_1+\delta_2)$, we deduce from (2.22) that

$$\begin{aligned} I_\mu(u) &= I_{\mu_1}(u) + \tfrac{1}{2}(\mu_1-\mu)\textstyle\int_{\mathbb{R}} h(x)u^2dx \\ &\ge \theta_5\|u\|^4 - \tfrac{\mu-\mu_1}{2\mu_1}\|u\|^2 \\ &= \|u\|^2\Big(\theta_5\|u\|^2 - \tfrac{\mu-\mu_1}{2\mu_1}\Big) \\ &\ge \|u\|^2\Big(\tfrac{1}{2}\theta_5\tilde\theta_5^2 - \tfrac{1}{4}\theta_5\tilde\theta_5^2\Big) \\ &= \tfrac{1}{4}\theta_5\tilde\theta_5^2\|u\|^2 \end{aligned}$$

for $\frac{1}{2}\tilde\theta_5^2 \le \|u\|^2 \le \tilde\theta_5^2$. Choosing $\rho := \|u\| = \big(t^2 + \|v\|^2\big)^{1/2}$ and $\alpha := \frac{1}{4}\theta_5\tilde\theta_5^2\rho^2$, we complete the proof. □

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. Since for $0 < \mu < \mu_1$, it is standard to prove the existence of one bound state of (1.3), which corresponds to a mountain pass type critical point of the functional I_μ. In the following we will focus our attention to the case of $\mu = \mu_1$. As we have seen in Lemma 2.7, with the help of the competing between the Poisson term $\phi_u u$ and the nonlinear term, the 0 is a local minimizer of the functional I_{μ_1} and I_{μ_1} contains mountain pass geometry. We will use mountain pass lemma to prove Theorem 1.1.

Lemma 3.1 ([3]). *Let E be a Banach space and a functional $I \in C^1(E, \mathbb{R})$. Suppose that $I(0) = 0$ and*

(1) *there are constants $\rho, \alpha > 0$ such that $I_{|\partial B_\rho} \geq \alpha$; and*
(2) *there is a $\bar{u} \in E \backslash \bar{B}_\rho$ such that $I(\bar{u}) < 0$.*

Define $c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t))$ with

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \, \gamma(1) = \bar{u}\}.$$

If I satisfies $(PS)_c$ condition, then I possesses a critical value $c \geq \alpha$.

We will use Lemma 3.1 by choosing $I = I_\mu$ and $E = H^1(\mathbb{R}^3)$. Since the problem contains critical nonlinearity, we need an extremal function u_ε of the embedding from $D^{1,2}(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, where

$$u_\varepsilon(x) = C \frac{\varepsilon^{1/4}}{(\varepsilon + |x|^2)^{1/2}}, \quad \varepsilon > 0$$

and C is a normalizing constant. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be such that $0 \leq \varphi \leq 1, \varphi|_{B_{R_2}} \equiv 1$ and $\operatorname{supp} \varphi \subset B_{2R_2}$ for some $R_2 > 0$. Set $v_\varepsilon = \varphi u_\varepsilon$ and then $v_\varepsilon \in H^1(\mathbb{R}^3)$ with $v_\varepsilon(x) \geq 0$ for any $x \in \mathbb{R}^3$. The following asymptotic estimates hold for ε small enough (see [11]):

$$\|\nabla v_\varepsilon\|_{L^2}^2 = K_1 + O(\varepsilon^{1/2}), \quad \|v_\varepsilon\|_{L^6}^2 = K_2 + O(\varepsilon) \tag{3.1}$$

and

$$\|v_\varepsilon\|_{L^\alpha}^\alpha = \begin{cases} O(\varepsilon^{\alpha/4}) & \alpha \in [2,3), \\ O(\varepsilon^{\alpha/4} |\ln \varepsilon|) & \alpha = 3, \\ O(\varepsilon^{\frac{6-\alpha}{4}}) & \alpha \in (3,6), \end{cases} \tag{3.2}$$

with $\frac{K_1}{K_2} = S_6$. Hence $\int \phi_{v_\varepsilon} v_\varepsilon^2 dx \leq C \|v_\varepsilon\|_{L^{\frac{12}{5}}}^4 = C\varepsilon$. Since $h(x)$ satisfies (H1), we also have that for ε small enough,

$$\begin{aligned} \int_{\mathbb{R}^3} h(x) |v_\varepsilon|^2 dx &\geq C\rho_1 \int_{|x|<\rho_2} \frac{|x|^{-\beta} \varepsilon^{1/2}}{\varepsilon + |x|^2} dx + \int_{|x| \geq \rho_2} h(x) |v_\varepsilon|^2 dx \\ &\geq C\rho_1 \varepsilon^{1/2} \int_0^{\rho_2} \frac{r^2}{r^\beta (\varepsilon + r^2)} dr \\ &= C\rho_1 \varepsilon^{1-\frac{\beta}{2}} \int_0^{\rho_2 \varepsilon^{-\frac{1}{2}}} \frac{\rho^2}{\rho^\beta (1+\rho^2)} d\rho \\ &\geq C\rho_1 \varepsilon^{1-\frac{\beta}{2}} \int_0^1 \frac{\rho^2}{2\rho^\beta} d\rho = C\varepsilon^{1-\frac{\beta}{2}}. \end{aligned} \tag{3.3}$$

Next, we define the following minimax value

$$d_{\mu_1} = \inf_{\gamma \in \Gamma_1} \sup_{t \in [0,1]} I_{\mu_1}(\gamma(t))$$

with

$$\Gamma_1 = \left\{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \; I_{\mu_1}(\gamma(1)) < 0\right\}.$$

Lemma 3.2. *If $h(x)$ satisfies the assumption (H1) with $1 < \beta < 2$, then the d_{μ_1} defined in Lemma 3.1 satisfies $d_{\mu_1} < \frac{1}{3} S_6^{3/2}$.*

Proof. It suffices to find a path $\gamma(t)$ starting from 0 such that

$$\sup_{t\in[0,1]} I_{\mu_1}(\gamma(t)) < \frac{1}{3}S_6^{3/2}.$$

Note that for $t>0$, $\frac{\partial}{\partial t}I_{\mu_1}(tv_\varepsilon) = tg(t)$ with

$$g(t) = \|v_\varepsilon\|^2 - \mu_1\int_{\mathbb{R}^3} h(x)v_\varepsilon^2dx + t^2\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx - t^4\int_{\mathbb{R}^3}v_\varepsilon^6dx.$$

It is easy to see that there is a unique T_ε such that $g(T_\varepsilon)=0$. We claim that there are constants $\tilde{C}$ and $\hat{C}$ such that $0<\tilde{C}\le T_\varepsilon\le\hat{C}$. Indeed if up to a subsequence (still denoted by T_ε) such that $T_\varepsilon\to 0$ as $\varepsilon\to 0$, then

$$\begin{aligned}
0 = g(T_\varepsilon) &= \|v_\varepsilon\|^2 - \mu_1\int_{\mathbb{R}^3} h(x)v_\varepsilon^2dx + T_\varepsilon^2\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx - T_\varepsilon^4\int_{\mathbb{R}^3}v_\varepsilon^6dx\\
&= K_1 + O(\varepsilon^{1/2}) - O(\varepsilon^{1-\frac{\beta}{2}}) + o(1)
\end{aligned}$$

which is a contradiction since $K_1>0$. If there is a subsequence (still denoted by T_ε) such that $T_\varepsilon\to\infty$ as $\varepsilon\to 0$, then we have that

$$0 = \frac{1}{T_\varepsilon^2}\Big(\|v_\varepsilon\|^2 - \mu_1\int_{\mathbb{R}^3}h(x)v_\varepsilon^2dx\Big) + O(\varepsilon) - T_\varepsilon^2\int_{\mathbb{R}^3}v_\varepsilon^6dx,$$

which is also a contradiction since $\int_{\mathbb{R}^3}v_\varepsilon^6dx = (K_2+O(\varepsilon))^3$ and $K_2>0$. Hence we only need to estimate $I_{\mu_1}(tv_\varepsilon)$ for t in a finite interval. Therefore we may have that

$$I_{\mu_1}(tv_\varepsilon) \le g_1(t) + C\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx,$$

where

$$g_1(t) = \frac{t^2}{2}\Big(\|v_\varepsilon\|^2 - \mu_1\int_{\mathbb{R}^3}h(x)v_\varepsilon^2dx\Big) - \frac{t^6}{6}\int_{\mathbb{R}^3}v_\varepsilon^6dx.$$

It is deduced from (3.1), (3.2) and (3.3) that

$$\begin{aligned}
\sup_{t>0} I_{\mu_1}(tv_\varepsilon) &\le \sup_{t>0} g_1(t) + O(\varepsilon)\\
&\le \frac{1}{3}\Big(\|v_\varepsilon\|^2 - \mu_1\int_{\mathbb{R}^3}h(x)v_\varepsilon^2dx\Big)^{3/2}\Big(\int_{\mathbb{R}^3}v_\varepsilon^6dx\Big)^{-3/2} + O(\varepsilon)\\
&\le \frac{1}{3}\Big(K_1 + O(\varepsilon^{1/2}) - C\varepsilon^{1-\frac{\beta}{2}}\Big)^{3/2}(K_2+O(\varepsilon))^{-3/2} + O(\varepsilon)\\
&= \frac{1}{3}K_1^{3/2}\Big(1+O(\varepsilon^{1/2}) - C\varepsilon^{1-\frac{\beta}{2}}\Big)K_2^{-3/2}(1-O(\varepsilon)) + O(\varepsilon)\\
&= \frac{1}{3}S_6^{3/2}\Big(1+O(\varepsilon^{1/2}) - C\varepsilon^{1-\frac{\beta}{2}}\Big) + O(\varepsilon)\\
&< \frac{1}{3}S_6^{3/2}
\end{aligned}$$

for ε small enough since $1<\beta<2$. □

Lemma 3.3. *If the function $h(x)$ satisfies the assumption (H1), then the I_μ satisfies $(PS)_d$ condition for any $d<\frac{1}{3}S_6^{3/2}$ in the case of $\mu=\mu_1$.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be a $(PS)_d$ sequence of I_{μ_1} with $d < \frac{1}{3}S_6^{3/2}$. Then we have that for n large enough,

$$d + o(1) = \frac{1}{2}\|u_n\|^2 - \frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)u_n^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6 dx,$$

$$\langle I'_{\mu_1}(u_n), u_n\rangle = \|u_n\|^2 - \mu_1\int_{\mathbb{R}^3} h(x)u_n^2 dx + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} |u_n|^6 dx.$$

Similar to the proof in Lemma 2.3, we can deduce that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, we may assume that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^3$. Denote $w_n := u_n - u_0$. We then obtain from Brezis-Lieb lemma and Lemma 2.4 that for n large enough,

$$\|u_n\|^2 = \|u_0\|^2 + \|w_n\|^2 + o(1),$$

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 dx + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx + o(1),$$

$$\|u_n\|_{L^6}^6 = \|u_0\|_{L^6}^6 + \|w_n\|_{L^6}^6 + o(1).$$

Since $\int_{\mathbb{R}^3} h(x)u_n^2 dx \to \int_{\mathbb{R}^3} h(x)u_0^2 dx$ as $n \to \infty$, we obtain that

$$\begin{aligned} d + o(1) &= I_{\mu_1}(u_n) \\ &= I_{\mu_1}(u_0) + \frac{1}{2}\|\nabla w_n\|_{L^2}^2 + \frac{1}{2}\|w_n\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |w_n|^6 dx. \end{aligned} \tag{3.4}$$

Using $\langle I'_{\mu_1}(u_n), \psi\rangle \to 0$ for any $\psi \in H^1(\mathbb{R}^3)$, one may deduce that $I'_{\mu_1}(u_0) = 0$. Hence we have

$$\|u_0\|^2 - \mu_1\int_{\mathbb{R}^3} h(x)u_0^2 dx + \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 dx = \int_{\mathbb{R}^3} |u_0|^6 dx$$

and then

$$I_{\mu_1}(u_0) \ge \frac{1}{3}\Big(\|u_0\|^2 - \mu_1\int_{\mathbb{R}^3} h(x)u_0^2 dx\Big) + \frac{1}{12}\int_{\mathbb{R}^3} \phi_{u_0} u_0^2 dx \ge 0.$$

Now using an argument similar to the proof of (2.11), we have

$$o(1) = \|\nabla w_n\|_{L^2}^2 + \|w_n\|_{L^2}^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - \int_{\mathbb{R}^3} |w_n|^6 dx. \tag{3.5}$$

Using the relation

$$\|\nabla u\|_{L^2}^2 \ge S_6\Big(\int_{\mathbb{R}^3} |u|^6 dx\Big)^{1/3} \quad \text{for any } u \in D^{1,2}(\mathbb{R}^3),$$

we proceed our discussion according to the following two cases:

(I) $\int_{\mathbb{R}^3} |w_n|^6 dx \not\to 0$ as $n \to \infty$;
(II) $\int_{\mathbb{R}^3} |w_n|^6 dx \to 0$ as $n \to \infty$.

Suppose that case (I) occurs. Then going if necessary to a subsequence, still denoted by $(w_n)_{n\in\mathbb{N}}$, we may obtain from (3.5) that

$$\|\nabla w_n\|_{L^2}^2 \ge S_6\Big(\|\nabla w_n\|_{L^2}^2 + \|w_n\|_{L^2}^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - o(1)\Big)^{1/3},$$

which implies that for n large enough,

$$\|\nabla w_n\|_{L^2}^2 \ge S_6^{3/2}.$$

It is deduced from this and (3.4) that $d \geq \frac{1}{3}S_6^{3/2}$, which is a contradiction. Therefore the case (II) must occur, i.e., $\int_{\mathbb{R}^3} |w_n|^6 dx \to 0$ as $n \to \infty$. This and (3.5) imply that $\|w_n\| \to 0$. Hence we have proven that I_μ satisfies $(PS)_d$ condition for any $d < \frac{1}{3}S_6^{3/2}$ and $\mu = \mu_1$. □

Proof of Theorem 1.1. By Lemma 3.1–Lemma 3.3, the d_{μ_1} is a critical value of I_{μ_1} and $d_{\mu_1} > 0$. The proof of nonnegativity for at least one of the corresponding critical point is inspired by the idea of [1]. In fact, since $I_{\mu_1}(u) = I_{\mu_1}(|u|)$ for any $u \in H^1(\mathbb{R}^3)$, for every $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma_1$ with $\gamma_n(t) \geq 0$ (a.e. in $\mathbb{R}^3$) for all $t \in [0,1]$ such that

$$d_{\mu_1} \leq \max_{t\in[0,1]} I_{\mu_1}(\gamma_n(t)) < d_{\mu_1} + \frac{1}{n}. \tag{3.6}$$

Consequently, by means of Ekeland's variational principle [5], there exists $\gamma_n^* \in \Gamma_1$ with the following properties:

$$\begin{gathered} d_{\mu_1} \leq \max_{t\in[0,1]} I_{\mu_1}(\gamma_n^*(t)) \leq \max_{t\in[0,1]} I_{\mu_1}(\gamma_n(t)) < d_{\mu_1} + \frac{1}{n}; \\ \max_{t\in[0,1]} \|\gamma_n(t)) - \gamma_n^*(t))\| < \frac{1}{\sqrt{n}}; \\ \text{there exists } t_n \in [0,1] \text{ such that } z_n = \gamma_n^*(t_n) \text{ satisfies} \\ I_{\mu_1}(z_n) = \max_{t\in[0,1]} I_{\mu_1}(\gamma_n^*(t)), \text{ and } \|I'_{\mu_1}(z_n)\| \leq 1/\sqrt{n}. \end{gathered} \tag{3.7}$$

By Lemma 3.2 and Lemma 3.3 we get a convergent subsequence (still denoted by $(z_n)_{n\in\mathbb{N}}$). We may assume that $z_n \to z$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. On the other hand, by (3.7), we also arrive at $\gamma_n(t_n) \to z$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Since $\gamma_n(t) \geq 0$, we conclude that $z \geq 0$, $z \not\equiv 0$ in $\mathbb{R}^3$ with $I_{\mu_1}(z) > 0$ and it is a nonnegative bound state of (1.3) in the case of $\mu = \mu_1$. □

4. Ground state and bound states for $\mu > \mu_1$

In this section, we assume condition (H1). We will prove the existence of ground state and bound states of (1.3) as well as their bifurcation properties with respect to μ. As we have pointed out in the introduction, usually to study the existence of ground state, one considers a minimization problem like

$$\inf\{I_\mu(u) : u \in \mathcal{M}\}, \quad \mathcal{M} = \{u \in H^1(\mathbb{R}^3) : \langle I'_\mu(u), u\rangle = 0\}.$$

But in the present paper, we can not do like this because for $\mu > \mu_1$, we can not deduce $0 \notin \partial\mathcal{M}$. To overcome this difficulty, we define the set of all nontrivial critical points of I_μ in $H^1(\mathbb{R}^3)$:

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : I'_\mu(u) = 0\}.$$

Then we consider the minimization problem

$$c_{0,\mu} = \inf\{I_\mu(u) : u \in \mathcal{N}\}. \tag{4.1}$$

To show that (4.1) is well defined, we have to prove that $\mathcal{N} \neq \emptyset$ for suitable $\mu > \mu_1$.

Lemma 4.1. *Let δ_2 and ρ be as in Lemma 2.7 and $\mu \in (\mu_1, \mu_1 + \delta_2)$. Define the minimization problem*

$$d_{0,\mu} = \inf_{\|u\|<\rho} I_\mu(u).$$

Then the $d_{0,\mu}$ is achieved by a nonnegative $w_{0,\mu} \in H^1(\mathbb{R}^3)$. Moreover this $w_{0,\mu}$ is a nonnegative solution of the (1.3).

Proof. In the first place, we prove that for $\mu \in (\mu_1, \mu_1+\delta_2)$, it holds $-\infty < d_{0,\mu} < 0$. Keeping the definition of $I_\mu(u)$ in mind, we obtain from the Sobolev inequality that

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\|u\|^2 - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\mu}{2\mu_1}\|u\|^2 - C\|u\|^6 > -\infty \quad \text{for } \|u\| < \rho. \end{aligned}$$

Next, for any $t > 0$, we have

$$I_\mu(te_1) = \frac{t^2}{2}\|e_1\|^2 - \frac{\mu t^2}{2}\int_{\mathbb{R}^3} h(x)e_1^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_{e_1} e_1^2 dx - \frac{t^6}{6}\int_{\mathbb{R}^3} |e_1|^6 dx.$$

It is now deduced from $\mu_1 \int_{\mathbb{R}^3} h(x)e_1^2 dx = \|e_1\|^2$ that

$$I_\mu(te_1) = \frac{t^2}{2}\Big(1 - \frac{\mu}{\mu_1}\Big)\|e_1\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_{e_1} e_1^2 dx - \frac{t^6}{6}\int_{\mathbb{R}^3} |e_1|^6 dx.$$

Since $\mu > \mu_1$, we obtain that $I_\mu(te_1) < 0$ for t small enough. Thus we have proven that $-\infty < d_{0,\mu} < 0$ for $\mu \in (\mu_1, \mu_1 + \delta_2)$.

In the second place, let $(v_n)_{n\in\mathbb{N}}$ be a minimizing sequence, that is, $\|v_n\| < \rho$ and $I_\mu(v_n) \to d_{0,\mu}$ as $n \to \infty$. By the Ekeland's variational principle, we can obtain that there is a sequence $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ with $\|u_n\| < \rho$ such that as $n \to \infty$,

$$I_\mu(u_n) \to d_{0,\mu} \quad \text{and} \quad I'_\mu(u_n) \to 0.$$

Then similar to the proof of Lemma 2.3, we can prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Using Lemma 2.6, we obtain that $(u_n)_{n\in\mathbb{N}}$ contains a convergent subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, such that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$. Noticing the fact that if $(v_n)_{n\in\mathbb{N}}$ is a minimizing sequence, then $(|v_n|)_{n\in\mathbb{N}}$ is also a minimizing sequence, we may assume that for each $n \in \mathbb{N}$, the $u_n \geq 0$ in $\mathbb{R}^3$. Therefore we may assume that $u_0 \geq 0$ in $\mathbb{R}^3$. The $I'_\mu(u_n) \to 0$ and $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$ imply that $I'_\mu(u_0) = 0$. Hence choosing $w_{0,\mu} \equiv u_0$, we know that $w_{0,\mu}$ is a nonnegative solution of the (1.3). The proof is complete. □

We emphasize that the above lemma does NOT mean that $w_{0,\mu}$ is a ground state of (1.3). But it does imply that $\mathcal{N} \neq \emptyset$ for any $\mu \in (\mu_1, \mu_1 + \delta_2)$. Now we are in a position to prove that the $c_{0,\mu}$ defined in (4.1) can be achieved.

Lemma 4.2. *For $\mu \in (\mu_1, \mu_1 + \delta_2)$, the $c_{0,\mu}$ is achieved by a non negative $v_{0,\mu} \in H^1(\mathbb{R}^3)$, which is a nonnegative critical point of I_μ. Moreover, this $v_{0,\mu}$ is a nonnegative ground state of the* (1.3).

Proof. Noting that from Lemma 4.1, we know that $\mathcal{N} \neq \emptyset$ for $\mu \in (\mu_1, \mu_1 + \delta_2)$. Hence we have that $c_{0,\mu} < 0$. Next we prove that the $c_{0,\mu} > -\infty$.

For any $u \in \mathcal{N}$, since $I'_\mu(u) = 0$, then $\langle I'_\mu(u), u\rangle = 0$. Similar to the proof of Lemma 2.3, we can obtain that

$$I_\mu(u) = I_\mu(u) - \frac{1}{4}\langle I'_\mu(u), u\rangle \geq \frac{1}{4}\|u\|^2 - \frac{1}{6}\mu^{3/2}\int_{\mathbb{R}^3} |h(x)|^{3/2} dx.$$

Therefore $c_{0,\mu} > -\infty$.

Now let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ be a sequence such that

$$I_\mu(u_n) \to c_{0,\mu} \quad \text{and} \quad I'_\mu(u_n) = 0.$$

Since $-\infty < c_{0,\mu} < 0$, we know from Lemma 2.6 that $(u_n)_{n\in\mathbb{N}}$ contains a convergent subsequence in $H^1(\mathbb{R}^3)$ and then we may assume without loss of generality that $u_n \to v_0$ strongly in $H^1(\mathbb{R}^3)$. Therefore we have that $I_\mu(v_0) = c_{0,\mu}$ and $I'_\mu(v_0) = 0$. Choosing $v_{0,\mu} \equiv v_0$ and we finish the proof of the Lemma 4.2. □

Next, to analyze further the $(PS)_d$ condition of the functional I_μ, we have to prove a relation between the minimizer $w_{0,\mu}$ obtained in Lemma 4.1 and the minimizer $v_{0,\mu}$ obtained in Lemma 4.2.

Lemma 4.3. *There exists $\delta_3 > 0$ and $\delta_3 \le \delta_2$ such that for any $\mu \in (\mu_1, \mu_1 + \delta_3)$, the $v_{0,\mu}$ obtained in Lemma 4.2 coincides the $w_{0,\mu}$ obtained in Lemma 4.1.*

Proof. The proof is divided into two steps. In the first place, for $u \neq 0$ and u is a critical point of I_μ with $\mu = \mu_1$, we have that

$$\|u\|^2 - \mu_1 \int_{\mathbb{R}^3} h(x)u^2dx + \int_{\mathbb{R}^3} \phi_u u^2dx = \int_{\mathbb{R}^3} |u|^6dx$$

and hence

$$I_{\mu_1}(u) = \frac{1}{3}\Big(\|u\|^2 - \mu_1 \int_{\mathbb{R}^3} h(x)u^2dx\Big) + \frac{1}{12}\int_{\mathbb{R}^3} \phi_u u^2dx.$$

Since $\|u\|^2 \ge \mu_1 \int_{\mathbb{R}^3} h(x)u^2dx$ for any $u \in H^1(\mathbb{R}^3)$, we obtain that

$$I_{\mu_1}(u) \ge \frac{1}{12}\int_{\mathbb{R}^3} \phi_u u^2dx > 0.$$

In the second place, denoted by $u_{0,\mu}$ a ground state obtained in Lemma 4.2. For any sequence $\mu^{(n)} > \mu_1$ and $\mu^{(n)} \to \mu_1$ as $n \to \infty$, we have that $u_{0,\mu^{(n)}}$ satisfies

$$\begin{gathered} I'_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = 0, \\ c_{0,\mu^{(n)}} = I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) < 0. \end{gathered}$$

Similar to the proof of Lemma 2.3, we may deduce that $(u_{0,\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Since $I'_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = 0$, one also has that

$$\begin{aligned} I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) &= \frac{1}{3}\Big(\|u_{0,\mu^{(n)}}\|^2 - \mu^{(n)}\int_{\mathbb{R}^3} h(x)(u_{0,\mu^{(n)}})^2dx\Big) \\ &\quad + \frac{1}{12}\int_{\mathbb{R}^3} \phi_{u_{0,\mu^{(n)}}}(u_{0,\mu^{(n)}})^2dx. \end{aligned}$$

Using the definition of μ_1, we may deduce that, as $n \to \infty$,

$$\|u_{0,\mu^{(n)}}\|^2 - \mu^{(n)}\int_{\mathbb{R}^3} h(x)(u_{0,\mu^{(n)}})^2dx \ge \Big(1 - \frac{\mu^{(n)}}{\mu_1}\Big)\|u_{0,\mu^{(n)}}\|^2 \to 0$$

because the $(u_{0,\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Next from $(u_{0,\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, we may assume without loss of generality that $u_{0,\mu^{(n)}} \rightharpoonup \tilde{u}_0$ weakly in $H^1(\mathbb{R}^3)$.

Claim: As $n \to \infty$, the $u_{0,\mu^{(n)}} \to \tilde{u}_0$ strongly in $H^1(\mathbb{R}^3)$ and $\tilde{u}_0 = 0$.

Proof of this claim. From $u_{0,\mu^{(n)}} \rightharpoonup \tilde{u}_0$ weakly in $H^1(\mathbb{R}^3)$, we may assume that $u_{0,\mu^{(n)}} \to \tilde{u}_0$ a.e. in $\mathbb{R}^3$. Using these and the fact of $I'_{\mu^{(n)}}(u_{0,\mu^{(n)}}) = 0$, we deduce that $I'_{\mu_1}(\tilde u_0)=0$. Then similar to the proof in Lemma 2.6, we obtain

$$\begin{aligned} o(1)+I_{\mu^{(n)}}(u_{0,\mu^{(n)}}) &= I_{\mu^{(n)}}(\tilde u_0)+\frac{1}{2}\|\nabla \tilde w_n\|_{L^2}^2+\frac{1}{2}\|\tilde w_n\|_{L^2}^2 \\ &\quad +\frac{1}{4}\int_{\mathbb{R}^3}\phi_{\tilde w_n}(\tilde w_n)^2dx-\frac{1}{6}\int_{\mathbb{R}^3}|\tilde w_n|^6dx, \end{aligned} \tag{4.2}$$

where $\tilde w_n := u_{0,\mu^{(n)}}-\tilde u_0$. Now we distinguish two cases:

(i) $\int_{\mathbb{R}^3}|\tilde w_n|^6dx\not\to 0$ as $n\to\infty$;
(ii) $\int_{\mathbb{R}^3}|\tilde w_n|^6dx\to 0$ as $n\to\infty$.

Suppose that the case (i) occurs. Using an argument similar to the proof in Lemma 2.6, we deduce that

$$I_{\mu^{(n)}}(u_{0,\mu^{(n)}})+o(1)\ge I_{\mu_1}(\tilde u_0)+\frac{1}{3}S_6^{3/2},.$$

This is a contradiction because $I_{\mu_1}(\tilde u_0)>-\frac{1}{3}S_6^{3/2}$ by Lemma 2.5 and $I_{\mu^{(n)}}(u_{0,\mu^{(n)}})<0$. Hence the case (i) does not occur. Therefore the case (ii) occurs, which implies that $u_{0,\mu^{(n)}}\to\tilde u_0$ strongly in $H^1(\mathbb{R}^3)$ (the proof is similar to those in Lemma 2.6). From this we also have that $\int_{\mathbb{R}^3}\phi_{\tilde w_n}(\tilde w_n)^2dx\to\int_{\mathbb{R}^3}\phi_{\tilde u_0}(\tilde u_0)^2dx$.

Next we prove that $\tilde u_0=0$. Arguing by a contradiction, if $\tilde u_0\ne 0$, then we know that

$$\liminf_{n\to\infty}I_{\mu^{(n)}}(u_{0,\mu^{(n)}})\ge\frac{1}{12}\int_{\mathbb{R}^3}\phi_{\tilde u_0}(\tilde u_0)^2dx>0,$$

which is also a contradiction since $I_{\mu^{(n)}}(u_{0,\mu^{(n)}})<0$. Therefore $\tilde u_0=0$.

Hence there is $\delta_3>0$ and $\delta_3\le\delta_2$ such that for any $\mu\in(\mu_1,\mu_1+\delta_3)$, $\|u_{0,\mu}\|<\rho$, which implies that $c_{0,\mu}=d_{0,\mu}$. Using Lemma 4.1, we get a nonnegative ground state of (1.3), called $w_{0,\mu}$ and $c_{0,\mu}=d_{0,\mu}=I_\mu(w_{0,\mu})$. □

Remark 4.4. The proof of Lemma 4.3 implies that the ground state $w_{0,\mu}$ bifurcates from zero.

Next we prove the existence of another nonnegative bound state of (1.3). To obtain this goal, we have to characterize further the $(PS)_d$ condition of the functional I_μ.

Lemma 4.5. *If $h(x)$ satisfies* (H1) *and $\mu\in(\mu_1,\mu_1+\delta_3)$, then I_μ satisfies $(PS)_d$ condition for any $d<c_{0,\mu}+\frac{1}{3}S_6^{3/2}$.*

Proof. Let $(u_n)_{n\in\mathbb{N}}\subset H^1(\mathbb{R}^3)$ be a $(PS)_d$ sequence of I_μ with $d<c_{0,\mu}+\frac{1}{3}S_6^{3/2}$, that is $I_\mu(u_n)\to d$ and $I'_\mu(u_n)\to 0$ as $n\to\infty$. Then for n large enough,

$$\begin{gathered} d+o(1)=\frac{1}{2}\|u_n\|^2-\frac{\mu}{2}\int_{\mathbb{R}^3}h(x)u_n^2dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx-\frac{1}{6}\int_{\mathbb{R}^3}|u_n|^6dx, \\ \langle I'_\mu(u_n),u_n\rangle=\|u_n\|^2-\mu\int_{\mathbb{R}^3}h(x)u_n^2dx+\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx-\int_{\mathbb{R}^3}|u_n|^6dx. \end{gathered}$$

Similar to the proof in Lemma 2.3, we can deduce that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, we may assume that $u_n\rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ and $u_n\to u_0$ a.e. in $\mathbb{R}^3$. Denote $w_n:=u_n-u_0$. We then obtain from Brezis-Lieb lemma and Lemma 2.4 that for n large enough,

$$\|u_n\|^2=\|u_0\|^2+\|w_n\|^2+o(1),$$

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 dx + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx + o(1),$$
$$\|u_n\|_{L^6}^6 = \|u_0\|_{L^6}^6 + \|w_n\|_{L^6}^6 + o(1).$$

Using Lemma 2.1, $\int_{\mathbb{R}^3} h(x)u_n^2 dx \to \int_{\mathbb{R}^3} h(x)u_0^2 dx$ as $n \to \infty$. Therefore

$$\begin{aligned} d + o(1) &= I_\mu(u_n) \\ &= I_\mu(u_0) + \frac{1}{2}\|\nabla w_n\|_{L^2}^2 + \frac{1}{2}\|w_n\|_{L^2}^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |w_n|^6 dx. \end{aligned} \tag{4.3}$$

Since $\langle I'_\mu(u_n), \psi\rangle \to 0$ for any $\psi \in H^1(\mathbb{R}^3)$, we get that $I'_\mu(u_0) = 0$. Moreover $I_\mu(u_0) \geq c_{0,\mu}$ and

$$\|u_0\|^2 - \mu \int_{\mathbb{R}^3} h(x)u_0^2 dx + \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 dx = \int_{\mathbb{R}^3} |u_0|^6 dx.$$

Note that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. The Brezis-Lieb lemma, Lemma 2.4 and

$$o(1) = \|u_n\|^2 - \mu \int_{\mathbb{R}^3} h(x)u_n^2 dx + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} |u_n|^6 dx$$

imply

$$o(1) = \|\nabla w_n\|_{L^2}^2 + \|w_n\|_{L^2}^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - \int_{\mathbb{R}^3} |w_n|^6 dx. \tag{4.4}$$

Using the relation

$$\|\nabla u\|_{L^2}^2 \geq S_6 \Big(\int_{\mathbb{R}^3} |u|^6 dx\Big)^{1/3} \quad \text{for any } u \in D^{1,2}(\mathbb{R}^3),$$

we distinguish two cases:

(I) $\int_{\mathbb{R}^3} |w_n|^6 dx \not\to 0$ as $n \to \infty$;
(II) $\int_{\mathbb{R}^3} |w_n|^6 dx \to 0$ as $n \to \infty$.

Suppose that the case (I) occurs. Using a proof similar to Lemma 2.6, we obtain

$$\|\nabla w_n\|_{L^2}^2 \geq S_6 \Big(\|\nabla w_n\|_{L^2}^2 + \|w_n\|_{L^2}^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - o(1)\Big)^{1/3}$$

and for n large enough,

$$\|\nabla w_n\|_{L^2}^2 \geq S_6^{3/2} + o(1). \tag{4.5}$$

Therefore using (4.3) and (4.5), we deduce that for n large enough,

$$\begin{aligned} & d + o(1) \\ &= I_\mu(u_n) \\ &= I_\mu(u_0) + \frac{1}{2}\|\nabla w_n\|_{L^2}^2 + \frac{1}{2}\|w_n\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |w_n|^6 dx. \\ &= I_\mu(u_0) + \frac{1}{3}\|\nabla w_n\|_{L^2}^2 + \frac{1}{3}\|w_n\|_{L^2}^2 + \frac{1}{12}\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx \\ &\geq c_{0,\mu} + \frac{1}{3}\|\nabla w_n\|_{L^2}^2 + \frac{1}{3}\|w_n\|_{L^2}^2 + \frac{1}{12}\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx \\ &> c_{0,\mu} + \frac{1}{3} S_6^{3/2}, \end{aligned} \tag{4.6}$$

which contradicts to the assumption $d < c_{0,\mu} + \frac{1}{3}S_6^{3/2}$. Therefore the case (II) must occur, i.e., $\int_{\mathbb{R}^3} |w_n|^6 dx \to 0$ as $n \to \infty$. This and (4.4) imply that $\|w_n\| \to 0$. Hence we have proven that I_μ satisfies $(PS)_d$ condition for any $d < c_{0,\mu} + \frac{1}{3}S_6^{3/2}$. $\square$

For the functional I_μ, we define the following minimax value

$$d_{2,\mu} = \inf_{\gamma\in\Gamma_2} \sup_{t\in[0,1]} I_\mu(\gamma(t))$$

with

$$\Gamma_2 = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = w_{0,\mu},\ I_\mu(\gamma(1)) < c_{0,\mu}\}.$$

Lemma 4.6. *If $\mu \in (\mu_1, \mu_1 + \delta_3)$ and $h(x)$ satisfies* (H1) *with $\frac{3}{2} < \beta < 2$, then the $d_{2,\mu}$ defined in the above satisfies*

$$d_{2,\mu} < c_{0,\mu} + \frac{1}{3}S_6^{3/2}.$$

Proof. It suffices to find a path starting from $w_{0,\mu}$ and the maximum of the energy functional in this path is strictly less than $c_{0,\mu} + \frac{1}{3}S_6^{3/2}$. To simplify the notation, we denote $w_0 := w_{0,\mu}$, which corresponds to the critical value $c_{0,\mu}$. We will prove that there is a T_0 such that the path $\gamma(t) = w_0 + tT_0v_\varepsilon$ is what we need. Note that as $s \to \infty$, $I_\mu(w_0 + sv_\varepsilon) \to -\infty$. Similar to the proof in Lemma 3.2, we only need to estimate $I_\mu(w_0 + tv_\varepsilon)$ for t in a finite interval. By direct calculation, we have

$$\begin{aligned} I_\mu(w_0 + tv_\varepsilon) &= \frac{1}{2}\Big(\|\nabla w_0 + t\nabla v_\varepsilon\|^2 - \mu\int_{\mathbb{R}^3} h(x)|w_0 + tv_\varepsilon|^2 dx\Big) \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{(w_0+tv_\varepsilon)}(w_0 + tv_\varepsilon)^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |w_0 + tv_\varepsilon|^6 dx \\ &= I_\mu(w_0) + I_\mu(tv_\varepsilon) + A_1 + A_2 + A_3, \end{aligned}$$

where

$$A_1 = t\int_{\mathbb{R}^3} (\nabla w_0 \nabla v_\varepsilon + w_0 v_\varepsilon)\, dx - \mu t\int_{\mathbb{R}^3} h(x) w_0 v_\varepsilon dx,$$

$$A_2 = \frac{1}{16\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{B}{|x-y|}\, dx\, dy$$

with

$$B = |w_0(y) + tv_\varepsilon(y)|^2|w_0(x) + tv_\varepsilon(x)|^2 - |w_0(x)|^2|w_0(y)|^2 - |tv_\varepsilon(y)|^2|tv_\varepsilon(x)|^2,$$

$$A_3 = \frac{1}{6}\int_{\mathbb{R}^3} \big(|w_0 + tv_\varepsilon|^6 - |w_0|^6 - |tv_\varepsilon|^6\big)\, dx.$$

Since w_0 is a solution of (1.3), we have that

$$A_1 = \int_{\mathbb{R}^3} (w_0)^5 tv_\varepsilon dx - \int_{\mathbb{R}^3} \phi_{w_0} w_0 tv_\varepsilon dx.$$

From an elementary inequality

$$|a+b|^p - |a|^p - |b|^p \le C(|a|^{p-1}|b| + |a||b|^{p-1}), \quad p > 1,\ a, b \in \mathbb{R},$$

we know that

$$|A_3| \le C\int_{\mathbb{R}^3} \big(|w_0|^5|tv_\varepsilon| + |w_0||tv_\varepsilon|^5\big)\, dx.$$

For the estimate of A_2, using the symmetry property with respect to x and y, we can obtain

$$\begin{aligned} |A_2| \leq C \int_{\mathbb{R}^3} \phi_{w_0} w_0 v_\varepsilon dx + \int_{\mathbb{R}^3} \phi_{w_0} (v_\varepsilon)^2 dx + \int_{\mathbb{R}^3} \phi_{v_\varepsilon} w_0 v_\varepsilon dx \\ + C \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{w_0(x) w_0(y) v_\varepsilon(x) v_\varepsilon(y)}{|x-y|} \, dx \, dy. \end{aligned}$$

Noticing that

$$\int_{\mathbb{R}^3} \phi_{w_0} w_0 v_\varepsilon dx \leq \|\phi_{w_0}\|_{L^6} \|w_0\|_{L^{\frac{12}{5}}} \|v_\varepsilon\|_{L^{\frac{12}{5}}}$$

and $\|v_\varepsilon\|_{L^{\frac{12}{5}}} = O(\varepsilon^{\frac{1}{4}})$ for ε small enough, we obtain that

$$\int_{\mathbb{R}^3} \phi_{w_0} w_0 v_\varepsilon dx = O(\varepsilon^{\frac{1}{4}}).$$

Similarly we obtain that for ε small enough,

$$\begin{gathered} \int_{\mathbb{R}^3} \phi_{w_0} (v_\varepsilon)^2 dx = O(\varepsilon^{1/2}), \quad \int_{\mathbb{R}^3} \phi_{v_\varepsilon} w_0 v_\varepsilon dx = O(\varepsilon^{\frac{3}{4}}), \\ \int_{\mathbb{R}^3} |w_0|^5 |v_\varepsilon| dx = O(\varepsilon^{\frac{1}{4}}), \quad \int_{\mathbb{R}^3} |w_0| |v_\varepsilon|^5 dx = O(\varepsilon^{\frac{1}{4}}), \\ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{w_0(x) w_0(y) v_\varepsilon(x) v_\varepsilon(y)}{|x-y|} \, dx \, dy = O(\varepsilon^{1/2}). \end{gathered}$$

Hence we deduce that for ε small enough,

$$I_\mu(w_0 + t v_\varepsilon) \leq I_\mu(w_0) + I_\mu(t v_\varepsilon) + O(\varepsilon^{\frac{1}{4}}).$$

Since

$$\max_{t>0} I_\mu(t v_\varepsilon) < \frac{1}{3} S_6^{\frac{3}{2}} + O(\varepsilon^{1/2}) + O(\varepsilon) - C \varepsilon^{1-\frac{\beta}{2}},$$

from $\frac{3}{2} < \beta < 2$ we obtain that there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,

$$\max_{t>0} I_\mu(w_0 + t v_\varepsilon) < \frac{1}{3} S_6^{\frac{3}{2}} + I_\mu(w_0) = \frac{1}{3} S_6^{\frac{3}{2}} + c_{0,\mu}.$$

The proof is complete. □

Lemma 4.7. *Let $\mu \in (\mu_1, \mu_1 + \delta_3)$ and $w_{0,\mu}$ be the minimizer obtained in Lemma 4.3. Then the $d_{2,\mu}$ is a critical value of I_μ.*

Proof. Since for $\mu \in (\mu_1, \mu_1 + \delta_3)$, from Lemmas 4.1 and 4.3, the quantity $w_{0,\mu}$ is a local minimizer of I_μ. Moreover, $I_\mu(w_{0,\mu} + s v_\varepsilon) \to -\infty$ as $s \to +\infty$. Therefore Lemma 4.5 and the mountain pass lemma imply that $d_{2,\mu}$ is a critical value of I_μ. □

Proof of Theorem 1.2. The conclusion (i) of Theorem 1.2 follows from Lemma 4.3 and Remark 4.4. It remains to prove (ii) of Theorem 1.2. By Lemma 4.7, Lemma 4.6 and Lemma 2.7, the $d_{2,\mu}$ is a critical value of I_μ and $d_{2,\mu} > 0$. The proof of nonnegativity for at least one of the corresponding critical point is inspired by the idea of [1]. In fact, since $I_\mu(u) = I_\mu(|u|)$ for any $u \in H^1(\mathbb{R}^3)$, for every $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma_2$ with $\gamma_n(t) \geq 0$ (a.e. in $\mathbb{R}^3$) for all $t \in [0,1]$ such that

$$d_{2,\mu} \leq \max_{t \in [0,1]} I_\mu(\gamma_n(t)) < d_{2,\mu} + \frac{1}{n}. \tag{4.7}$$

Consequently, by Ekeland's variational principle, there exists $\gamma_n^* \in \Gamma_2$ with the following properties:

$$\begin{gathered} d_{2,\mu} \leq \max_{t\in[0,1]} I_\mu(\gamma_n^*(t)) \leq \max_{t\in[0,1]} I_\mu(\gamma_n(t)) < d_{2,\mu} + \frac{1}{n}; \\ \max_{t\in[0,1]} \|\gamma_n(t)) - \gamma_n^*(t))\| < \frac{1}{\sqrt{n}}; \\ \text{there exists } t_n \in [0,1] \text{ such that } z_n = \gamma_n^*(t_n) \text{ satisfies} \\ I_\mu(z_n) = \max_{t\in[0,1]} I_\mu(\gamma_n^*(t)), \text{ and } \|I'_\mu(z_n)\| \leq \frac{1}{\sqrt{n}}. \end{gathered} \tag{4.8}$$

From Lemma 4.6 we get a convergent subsequence (still denoted by $(z_n)_{n\in\mathbb{N}}$). We may assume that $z_n \to z$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$. On the other hand, by (4.8), we also arrive at $\gamma_n(t_n) \to z$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Since $\gamma_n(t) \geq 0$, we conclude that $z \geq 0$, $z \not\equiv 0$ in $\mathbb{R}^3$ with $I_\mu(z) > 0$ and it is a nonnegative solution of problem (1.3).

Next, we prove the bifurcation property. Let $u_{2,\mu}$ be the nonnegative solution given by the above proof, that is, $I'_\mu(u_{2,\mu}) = 0$ and $I_\mu(u_{2,\mu}) = d_{2,\mu}$. We claim that for any sequence $\mu^{(n)} > \mu_1$ and $\mu^{(n)} \to \mu_1$, there exist a sequence of solution $u_{2,\mu^{(n)}}$ of (1.3) in the case of $\mu = \mu^{(n)}$ and a $u_{\mu_1} \in H^1(\mathbb{R}^3)$ with $I'_{\mu_1}(u_{\mu_1}) = 0$ and $I_{\mu_1}(u_{\mu_1}) > 0$, such that $u_{2,\mu^{(n)}} \to u_{\mu_1}$ strongly in $H^1(\mathbb{R}^3)$. In fact, by the definition of $d_{2,\mu}$ and the proof of Lemma 4.6, we deduce that for n large enough,

$$0 < \alpha \leq d_{2,\mu^{(n)}} \leq \max_{s>0} I_{\mu^{(n)}}(w_{0,\mu^{(n)}} + sv_\varepsilon),$$

$$I_{\mu^{(n)}}(w_{0,\mu^{(n)}} + sv_\varepsilon) \leq I_{\mu^{(n)}}(w_{0,\mu^{(n)}}) + I_{\mu^{(n)}}(sv_\varepsilon) + O(\varepsilon^{\frac{1}{4}}).$$

Then as $n \to \infty$ (consequently $\mu^{(n)} \to \mu_1$),

$$\limsup_{n\to\infty} d_{2,\mu^{(n)}} \leq \max_{s>0} I_{\mu_1}(sv_\varepsilon) + O(\varepsilon^{\frac{1}{4}}) < \frac{1}{3} S_6^{3/2}. \tag{4.9}$$

Next, similar to the proof in Lemma 2.3, we can deduce that $(u_{2,\mu^{(n)}})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, we may assume that $u_{2,\mu^{(n)}} \rightharpoonup \tilde{u}_2$ weakly in $H^1(\mathbb{R}^3)$ and $u_{2,\mu^{(n)}} \to \tilde{u}_2$ a.e. in $\mathbb{R}^3$. Then we have that $I'_{\mu_1}(\tilde{u}_2) = 0$. Moreover $I_{\mu_1}(\tilde{u}_2) \geq 0$. If $(u_{2,\mu^{(n)}})_{n\in\mathbb{N}}$ does not converge strongly to $\tilde{u}_2$ in $H^1(\mathbb{R}^3)$, then using an argument similar to the proof of Lemma 4.5, we deduce that

$$I_{\mu^{(n)}}(u_{2,\mu^{(n)}}) \geq I_{\mu_1}(\tilde{u}_2) + \frac{1}{3} S_6^{3/2},$$

which contradicts to (4.9). Hence $u_{2,\mu^{(n)}} \to \tilde{u}_2$ strongly in $H^1(\mathbb{R}^3)$ and $I_{\mu_1}(\tilde{u}_2) > 0$. The claim holds and the proof is complete. $\square$

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