Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 30, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

STABILITY OF NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH CAPUTO FRACTIONAL DERIVATIVE AND BOUNDED DELAYS

SNEZHANA HRISTOVA, CEMIL TUNC

Abstract. We use Lyapunov functions to study stability of the first-order Volterra integro-differential equation with Caputo fractional derivative

$$ {}_{t_0}^{C}D_t^q x(t) = -a(t)f(x(t)) + \int_{t-r}^{t} B(t,s)g(s,x(s))ds + h(t,x(t),x(t-\tau(t)))\,. $$

For the Lyapunov functions, we consider three types of fractional derivatives. By means of these derivatives, we obtain new sufficient conditions for stability and uniformly stability of solutions We consider both constant and time variable bounded delays, and illustrated our results with an example.

1. Introduction

Volterra integral equations and Volterra integro-differential equations were introduced by Vito Volterra, in 1926. Thereafter they have been widely used in sciences and engineering. They appear in physical applications such as: glass forming process, nano-hydrodynamics, heat transfer, diffusion process in general neutron diffusion and in biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. More details about the sources for these equations can be found in physics, biology and engineering books (see for example [8, 18, 25, 26]).

Fractional calculus relates with calculus of integrals and derivatives of orders that may not be integers. It has become very popular in recent years because of its applications to fields of science and engineering, such as mechanics, electricity, chemistry, biology, economics, and control theory. For instance, it has been used in diffusion processes, such as those founded in batteries [19], and heat transfer process [10]. Moreover, these models are often used inside classic control schemes, and for that reason the whole control system results in a fractional order system. In these cases, the stability of the whole control system has to be analyzed using the fractional order techniques [6]. For some recent developments in emerging scientific areas, namely nanoscience, nonlinear science and complexity, symmetries and integrability, and application of fractional calculus in science, engineering, economics and finance, we refer the reader to [7, 11]. Nowadays, there are several definitions of fractional derivatives in the literature, but the three most commonly used are

2010 *Mathematics Subject Classification.* 26A33, 34A08, 34D20, 34K20.
Key words and phrases. Fractional derivative; integro-differential equation; delay; Lyapunov functional; stability.

Submitted January 6, 2019. Published February 19, 2019.

Riemann-Liouville, Caputo, and Grünwald-Letnikov (see, for example, [16]). Each of these definitions has advantages and disadvantages. One of the main advantages of the Caputo fractional derivative is that it requires an initial condition similar to the one in ordinary differential case, and that leads to applications modeling real phenomena.

Time dependent delays appear in many engineering systems such as aircraft, chemical control systems, laser models, internet, biology, and medicine [12, 13].

One of main qualitative properties of solutions of differential equations is their stability. There are many methods for studying stability; one of them is called the Lyapunov direct method and it is applicable in practice to Lyapunov functions and functionals. In applications to delay differential equations there are two different types of the Lyapunov methods. One of them was proposed by Krasovskii [14] and it is based on the application of functionals. The other one was proposed by Razumikhin [17], and it is based on the application of functions and the so called Razumikhin condition. Razumikhin method uses Lyapunov-type functions depending on the current value of the solution. Krasovskii method uses functionals depending on all values of the solution over the whole past time interval. Krasovskii's functionals are applied for various types in integro-differential equations [21, 22, 23]. Nevertheless, it is difficult to apply the Krasovskii theorem to test the stability of a fractional system with delay, because of the non-locality of the fractional derivatives. Razumikhin method for fractional nonlinear time-delay systems has been extended recently [20].

There are many papers that study different types of stability and boundedness of solutions of linear and nonlinear fractional differential equations. However, there are no papers on the stability of solutions of linear or nonlinear Volterra integro-differential equations with Caputo fractional derivative and delays. In this paper, we extend and applied the Razumikhin method to Volterra integro-differential equations with Caputo fractional derivative and two types of delays: constant and a time-variable delays. To the best of our knowledge, this is the first study of stability properties of such kind of equations. The presence of fractional derivatives requires an appropriately defined derivative of Lyapunov functions. In these article we apply three types of derivatives to obtain sufficient conditions for stability and uniformly stability of solutions. An example is given to illustrate the applicability of our results.

In this article we assume $q \in (0,1)$ and we use the Riemann-Liouville fractional derivative [16],

$$ {}_{t_0}^{RL}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^t (t-s)^{-q} m(s) ds, \quad t \geq t_0, $$

where $\Gamma(\cdot)$ denotes the Gamma function. Also we use the Caputo fractional derivative [16],

$$ {}_{t_0}^{C}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^t (t-s)^{-q} m'(s) ds, \quad t \geq t_0. $$

For a wide class of functions we have the following properties: From [16, Section 2.4.1] we have

$$ {}_{t_0}^{C}D^q m(t) = {}_{t_0}^{RL}D^q [m(t) - m(t_0)]. $$

From [16, Section 2.4.1] we have

$$\begin{aligned} {}_{t_0}^{C}D^q m(t) &= {}_{t_0}^{RL}D^q m(t), \quad t > t_0 \text{ if } m(t_0) = 0, \\ {}_{t_0}^{C}D^q m(t) &< {}_{t_0}^{RL}D^q m(t), \quad t > t_0 \text{ if } m(t_0) > 0. \end{aligned} \tag{1.1}$$

2. Statement of the problem

Let t_0 be an arbitrary initial time. The physical meaning of the independent variable t is time in differential equations, so we will assume $t_0 \in \mathbb{R}_+ = [0, \infty)$.

We consider the following initial value problem (IVP) for the scalar nonlinear Volterra integro-differential equation with Caputo fractional derivative and delays (IFrDDE)

$$\begin{gathered} {}_{t_0}^{C}D_t^q x(t) = -a(t)f(x(t)) + \int_{t-r}^{t} B(t,s)g(s,x(s))ds + h(t,x(t),x(t-\tau(t))) \quad \text{for } t > t_0, \\ x(t_0+\Theta) = \phi_0(\Theta) \quad \text{for } \Theta \in [-r,0], \quad x(t_0+) = \phi_0(0), \end{gathered} \tag{2.1}$$

where $x \in \mathbb{R}$, $a : [0,\infty) \to (0,\infty)$, $r > 0$ is a given constant, $f : \mathbb{R} \to \mathbb{R}$, $g : [-r,\infty) \times \mathbb{R} \to \mathbb{R}$, $\tau : [0,\infty) \to [0,r]$, $h : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ and $\phi_0 \in C([-r,0],\mathbb{R})$.

First we give some basic notation related to (2.1) and used throughout this paper. Let $x(t,t_0,\phi)$ denote the solution of (2.1) on $[t_0 - r, \infty)$. For any $\phi \in C([-r,0],\mathbb{R})$ we denote $\|\phi\|_0 = \max_{t\in[-r,0]} |\phi(t)|$. We use the class

$$\mathcal{K} = \{w \in C(\mathbb{R}_+,\mathbb{R}_+) : w(s) \text{ is strictly increasing and } w(0) = 0\}.$$

Next we give the definition for stability of the zero solution of (2.1) with zero initial function $\phi_0 \equiv 0$. This definition is similar to the one in [1].

Definition 2.1. The zero solution of (2.1) with zero initial function is said to be

- *stable with respect to* t_0, if for any number $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that for any initial function $\phi \in C([-r,0],\mathbb{R}^n) : 0 < \|\phi\|_0 < \delta$ the inequality $\|x(t;t_0,\phi)\| < \varepsilon$ holds for $t \geq t_0$;
- *uniformly stable*, if for any number $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any initial function $\phi \in C([-r,0],\mathbb{R}^n) : 0 < \|\phi\|_0 < \delta$ and any initial time $t_0 \in \mathbb{R}_+$ the inequality $\|x(t;t_0,\phi)\| < \varepsilon$ holds for $t \geq t_0$.

One approach to study various stability properties of solutions of any types of nonlinear differential equations is based on using Lyapunov functions. The first step is to define a Lyapunov function. The second step is to define its derivative along the solution to the studied equation.

3. Lyapunov functions for Caputo fractional differential equations

We will use the well known class of functions called Lyapunov functions [12, 13].

Definition 3.1. Let $J = [t_0 - r, T)$, $T \leq \infty$, be a given interval, and $\Delta \subset \mathbb{R}$, $0 \in \Delta$ be a given set. We will say that the function $V(t,x) : J \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda(J,\Delta)$ if $V(t,x)$ is continuous on $[t_0,T) \times \Delta$ and it is locally Lipschitzian with respect to its second argument.

We will give a brief overview of the main three types of derivatives of Lyapunov functions among Caputo fractional delay differential equations (FrDDE) (for more details see [1]):

$${}_{t_0}^{C}D_t^q x(t) = F(t,x(t),x(t-\tau(t))) \quad \text{for } t > t_0 \tag{3.1}$$

here $x \in \mathbb{R}^n$, $F : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $F(t,0,0) = 0$, i.e. the FrDDE (3.1) has a zero solution for zero initial function.

First type - Caputo fractional derivative. Let $x(t) \in \Delta$, $t \in [t_0, T)$, be a solution of the FrDDE (3.1). Then

$$ {}_{t_0}^{c}D^q V(t,x(t)) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} \frac{d}{ds}\Big(V(s,x(s))\Big) ds, \quad t \in (t_0,T) \tag{3.2} $$

This type of derivative is applicable mainly for quadratic Lyapunov functions to study some stability properties of fractional differential equations (see [15]).

Second type - Dini fractional derivative. Let the initial point $t_0 \in \mathbb{R}_+$ and $\psi \in C([-r,0],\Delta)$ be given. Then

$$ \begin{aligned} D^{+}_{(3.1)} V(t,\psi(0);t_0) = \limsup_{h\to 0} \frac{1}{h^q} \Big[& V(t,\psi(0)) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} {}_qC_r V(t-rh,\psi(0)) \\ & - h^q F(t,\psi(0),\psi(-\tau(0))) \Big], \quad \text{for } t \in (t_0,T). \end{aligned} \tag{3.3} $$

This derivative has a memory. The idea of the definition (3.3) is given in [9].

Third type - Caputo fractional Dini derivative. Let the initial data (t_0,ϕ_0) belong to $\mathbb{R}_+ \times C([-r,0],\Delta)$ for (3.1) and let the function $\psi \in C([-r,0],\Delta)$ be given. Then

$$ \begin{aligned} {}_{(3.1)}^{c}D^q_+ V(t,\psi(0);t_0,\phi_0(0)) = \limsup_{h\to 0^+} \frac{1}{h^q} \Big\{ & V(t,\psi(0)) - V(t_0,\phi_0(0)) \\ & - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} {}_qC_r \Big(V(t-rh,\psi(0) \\ & - h^q F(t,\psi(0),\psi(-\tau(0))) - V(t_0,\phi_0(0)) \Big) \Big\} \end{aligned} \tag{3.4} $$

or its equivalent form

$$ \begin{aligned} & {}_{(3.1)}^{c}D^q_+ V(t,\psi(0);t_0,\phi_0(0)) \\ & = \limsup_{h\to 0^+} \frac{1}{h^q} \Big\{ V(t,\psi(0)) + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r} {}_qC_r V(t-rh,\psi(0) \\ & \quad - h^q F(t,\psi(0),\psi(-\tau(0))) \Big\} - \frac{V(t_0,\phi_0(0))}{(t-t_0)^q \Gamma(1-q)}. \end{aligned} \tag{3.5} $$

The Caputo fractional Dini derivative was applied to study different types of stability of Caputo fractional differential equations without delay in [2]-[5] and for Caputo delay differential equations in [1] .

Note the number T in the above definitions could be ∞.

Remark 3.2 ([1])**.** For any initial data $(t_0,\phi_0) \in \mathbb{R}_+ \times C([-r,0],\Delta)$ of the IVP for IFrDDE (2.1) the relation between the Dini fractional derivative and the Caputo fractional Dini derivative is given by

$$ {}_{(3.1)}^{c}D^q_+ V(t,\psi(0);t_0,\phi_0(0)) = D^{+}_{(3.1)} V(t,\psi(0);t_0) - \frac{V(t_0,\phi_0(0))}{(t-t_0)^q \Gamma(1-q)}, $$

for $t > t_0$, $\psi \in C([-r,0],\Delta)$; or

$$ {}^{c}_{(3.1)}D^q_+ V(t,\psi(0);t_0,\phi_0(0)) = D^+_{(3.1)}V(t,\psi(0);t_0) - {}^{RL}_{t_0}D^q\big(V(t_0,\phi_0(0))\big), $$

for $t > t_0$, $\phi \in C([-r,0],\Delta)$ (compare with (1.1)).

In our proofs we will use Razumikhin method combined with the above defined fractional derivatives of Lyapunov functions to study some qualitative properties of IFrDDE (2.1). For this purpose, we will use some stability results for Caputo fractional differential equations with delays obtained in [1]. In the case of Caputo fractional derivative we will use the following result.

Lemma 3.3 ([1]). *Assume there exists a function $V \in \Lambda([-r,\infty),\mathbb{R}_+)$ such that*

(i) *$a(\|x\|) \le V(t,x) \le b(\|x\|)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}$, where $a,b \in \mathcal{K}$;*
(ii) *for any initial data $(t_0,\phi) \in \mathbb{R}_+ \times C([-r,0],\mathbb{R}$ and any point $s > t_0$ such that $V(s+\Theta, x(s+\Theta)) < V(s,x(s))$ for $\Theta \in [-r,0)$ we have*
$$ {}^{c}_{t_0}D^q V(t,x(t)) \le 0, \quad t \in (t_0,s], \tag{3.6} $$
where $x(t) = x(t;t_0,\phi)$ is the corresponding solution of (3.1).

Then the zero solution of (3.1) with zero initial function is uniformly stable.

Corollary 3.4. *Let the conditions in Lemma 3.3 be satisfied with (ii) replaced by*

(ii*) *there exists an initial time $t_0 \ge 0$ such that for any initial function $\phi \in C([-r,0],\mathbb{R})$ and any point $s > t_0$ such that $V(s+\Theta, x(s+\Theta)) < V(s,x(s))$ for $\Theta \in [-r,0)$, inequality (3.6) holds.*

Then the zero solution of (3.1) with zero initial function is stable with respect to t_0.

If the Dini fractional derivative is applied then we will use the following result.

Lemma 3.5 ([1]). *Assume there exists a function $V \in \Lambda([-r,\infty),\mathbb{R}_+)$ such that*

(i) *$a(\|x\|) \le V(t,x) \le b(\|x\|)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}$, where $a,b \in \mathcal{K}$;*
(ii) *for any initial time $t_0 \ge 0$ and any function $\psi \in C([-r,0],\mathbb{R}_+)$ such that if for a point $t \ge t_0$ we have $V(t+\Theta,\psi(\Theta)) < V(t,\psi(0))$ for $\Theta \in [-r,0)$, then we have*
$$ D^+_{(3.1)}V(t,\psi;t_0) \le 0\,. \tag{3.7} $$

Then the zero solution of (3.1) with zero initial function is uniformly stable.

Corollary 3.6. *Let the conditions in Lemma 3.5 be satisfied with (ii) replaced by*

(ii**) *there exists an initial time $t_0 \ge 0$ such that for any function $\phi \in C([-r,0],\mathbb{R}_+)$: if for a point $t \ge t_0$ we have $V(t+\Theta,\phi(\Theta)) < V(t,\phi(0))$ for $\Theta \in [-r,0)$, then (3.7) holds.*

Then the zero solution of (3.1) with zero initial function is stable with respect to t_0.

In the case of the Caputo fractional Dini derivative the following result will be applied.

Lemma 3.7 ([1]). *Assume there exists a function $V \in \Lambda([\tau,\infty),\mathbb{R}_+)$ such that $V(t,0)=0$ for $t \in \mathbb{R}_+$ and*

(i) *$a(\|x\|) \le V(t,x) \le b(\|x\|)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n : |x| \le \rho$, where $a,b \in \mathcal{K}$;*

(ii) *for any initial data* $(t_0, \phi) \in \mathbb{R}_+ \times C([-r, 0], \mathbb{R}))$ *and any function* $\psi \in C([-r, 0], \mathbb{R})$ *such that if for a point* $t \geq t_0$ *we have* $V(t + \Theta, \psi(\Theta)) < V(t, \psi(0))$ *for* $\Theta \in [-r, 0)$*, then*

$$
{}^{c}_{(3.1)}D^q_+ V(t, \psi(0); t_0, \phi(0)) \leq 0\,. \tag{3.8}
$$

Then the zero solution of (3.1) *with zero initial function is uniformly stable.*

Corollary 3.8. *Let the conditions of Lemma 3.7 be satisfied with (ii) replaced by*

(ii***) *there exists an initial time* $t_0 \geq 0$ *such that for any initial function* $\phi \in C([-r, 0], \mathbb{R})$ *and any function* $\psi \in C([-r, 0], \mathbb{R})$ *such that if for a point* $t \geq t_0$ *we have* $V(t + \Theta, \psi(\Theta)) < V(t, \psi(0))$ *for* $\Theta \in [-r, 0)$*, then* (3.8) *holds.*

Then the zero solution of (3.1) *with zero initial function is stable with respect to* t_0*.*

4. Stability of the Caputo fractional integro-differential equations with delays

In this section we will use Razumikhin method combined with the above defined fractional derivatives of Lyapunov functions to study some qualitative properties of (2.1). We will obtain several types of explicit conditions for stability with respect to a given initial time as well as uniform stability of zero solution. We use the following assumptions:

(H1) $a \in C([0, \infty), (0, \infty))$ and there exists a positive constant a_0 such that $a(t) \geq a_0$ for $t \geq 0$.
(H2) $f, g \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $g(0) = 0$ and there exist positive constants m_0, M_0 such that $xf(x) \geq M_0 x^2$ and $g^2(x) \leq m_0 x^2$ for $x \in \mathbb{R}$.
(H3) $B \in C([-r, \infty) \times [-r, \infty), \mathbb{R})$ and $B(t, s) \geq 0$ for $s \leq t$.
(H4) τ belongs to $C([0, \infty), [0, r])$.
(H5) $h \in C([0, \infty) \times \mathbb{R}^2, \mathbb{R})$ is such that $h(t, 0, 0) = 0$ for $t \geq 0$ and there exists a function $\lambda \in C([0, \infty), [0, \infty))$ such that for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$
|h(t, x_1, y_1) - h(t, x_2, y_2)| \leq \lambda(t)(]x_1 - x_2| + |y_1 - y_2|), \quad t \geq 0\,.
$$

Theorem 4.1. *Assume* (H1)–(H5) *are satisfied and*

$$
2M_0 a(t) \geq (1 + m_0) \int_{t-r}^{t} B(t, s)ds + 4\lambda(t), \quad t \geq 0\,. \tag{4.1}
$$

Then the zero solution of (2.1) *is uniformly stable.*

Proof. We use the Lyapunov function $V(t, x) = x^2$. Consider an arbitrary initial data $(t_0, \phi_0) \in \mathbb{R}_+ \times C([-r, 0], \mathbb{R})$. Let $x(t) = x(t; t_0, \phi_0)$ be the corresponding solution of the IVP for (2.1). Let the point $t > t_0$ be such that $V(t+\Theta, x(t+s)) < V(t, x(t))$ for $s \in [-r, 0)$, i.e. $(x(t + s))^2 < (x(t))^2$ for $s \in [-r, 0)$. Then using

$x^2 + y^2 \geq 2xy$, $x, y \in \mathbb{R}$, we obtain

$$\begin{aligned}
&{}_{t_0}^{c}D^q V(t, x(t)) \\
&\leq 2x(t)\, {}_{t_0}^{c}D^q x(t) \\
&= -2a(t)x(t)f(x(t)) + 2x(t)\int_{t-r}^{t} B(t,s)g(x(s))ds \\
&\quad + 2x(t)h(t, x(t), x(t-\tau(t))) \\
&\leq -2a(t)M_0(x(t))^2 + x^2(t)\int_{t-r}^{t} B(t,s)ds + \int_{t-r}^{t} B(t,s)g^2(x(s))ds \\
&\quad + 2\lambda(t)|x(t)|\Big(|x(t)| + |x(t-\tau(t))|\Big) \\
&\leq -2a(t)M_0(x(t))^2 + x^2(t)\int_{t-r}^{t} B(t,s)ds + m_0\int_{t-r}^{t} B(t,s)(x(s))^2 ds \\
&\quad + 3\lambda(t)x^2(t) + \lambda(t)x^2(t-\tau(t)) \\
&\leq -\Big(2a(t)M_0 - (1+m_0)\int_{t-r}^{t} B(t,s)ds - 4\lambda(t)\Big)(x(t))^2 \leq 0.
\end{aligned} \tag{4.2}$$

From (4.1) and Lemma 3.3, the zero solution of (2.1) is uniformly stable. □

Theorem 4.2. *Assume* (H1)–(H5) *are satisfied and there exists a function* $p \in C(\mathbb{R}_+, (0,\infty))$ *such that* $0 < C_1 \leq p(t) \leq C_2$, C_1, C_2 *are positive constants, and for any* $t_0 \geq 0$,

$$\begin{aligned}
2M_0 a(t) \geq \int_{t-r}^{t} B(t,s)ds + m_0 p(t)\int_{t-r}^{t} \frac{B(t,s)}{p(s)}ds + \frac{{}_{t_0}^{RL}D^q p(t)}{p(t)} \\
+ \lambda(t)\Big(3 + \frac{p(t)}{p(t-\tau(t))}\Big), \quad \text{for } t \geq t_0\,.
\end{aligned} \tag{4.3}$$

Then the zero solution of (2.1) *is uniformly stable.*

Remark 4.3. Note that if the function $p(t)$ in Theorem 4.2 is such that ${}_{t_0}^{RL}D^q p(t) \geq 0$ for any $t_0 \geq 0$ then from (4.3) it follows (4.1). Therefore, if (4.1) is not satisfied, to obtain stability of the zero solution of (2.1) we need a function $p(t)$ such that the ${}_{t_0}^{RL}D^q p(t)$ is not non-negative. Such kind of functions are, for example, $p(t) = 1.1 + \cos(t)$ or $p(t) = 0.1 + \cos^2(t)$.

Proof of Theorem 4.2. Let (4.3) hold but (4.1) is not satisfied. Then we use the Lyapunov function $V(t,x) = p(t)x^2$. In this case the Caputo fractional derivative of the Lyapunov function is difficult to obtain because of Leibniz rule in fractional case. Therefore, we use the Dini fractional derivative. Let $t_0 \geq 0$ be an arbitrary initial time and $\phi \in C([-r,0], \mathbb{R})$. Let $t \geq t_0$ be such that $V(t+s, \phi(s)) < V(t, \phi(0))$, $s \in [-r, 0)$, i.e. $p(t+s)\big(\phi(s)\big)^2 < p(t)\big(\phi(0)\big)^2$ for $s \in [-r, 0)$.

Therefore, using the substitution $s-t=\xi$ we obtain

$$\begin{aligned}
&\int_{t-r}^{t} B(t,s) g^2(\phi(s-t))ds \\
&\leq m_0 \int_{t-r}^{t} B(t,s)\phi^2(s-t)ds = m_0 \int_{-r}^{0} \frac{B(t,t+\xi)}{p(t+\xi)} p(t+\xi)(\phi(\xi))^2 d\xi \\
&\leq m_0 \int_{-r}^{0} \frac{B(t,t+\xi)}{p(t+\xi)} p(t)(\phi(0))^2 d\xi = m_0 p(t)(\phi(0))^2 \int_{-r}^{0} \frac{B(t,t+\xi)}{p(t+\xi)} d\xi \\
&= m_0 p(t)(\phi(0))^2 \int_{t-r}^{t} \frac{B(t,s)}{p(s)} ds
\end{aligned} \tag{4.4}$$

Then using $x(t+s)=\phi(s),\ s\in[-r,0],\ x(t-\tau(t))=\phi(-\tau(t)),\ x(\xi)=\phi(\xi-t)$, $\xi\in[t-r,t]$, the substitution $t-s=\xi$ and (4.4) we obtain

$$\begin{aligned}
&D^{+}_{(2.1)} V(t,\phi;t_0) \\
&= 2\phi(0)p(t)\Big(-a(t)f(\phi(0)) + \int_{t-r}^{t} B(t,s)g(\phi(s-t))ds + h(t,\phi(0),\phi(-\tau(t)))\Big) \\
&\quad + (\phi(0))^2\ {}^{RL}_{t_0}D^q p(t) \\
&\leq -2\phi(0)p(t)a(t)f(\phi(0)) + p(t)\int_{t-r}^{t} B(t,s)\Big(2\phi(0)g(\phi(s-t))\Big)ds \\
&\quad + (\phi(0))^2\ {}^{RL}_{t_0}D^q p(t) + 2\lambda(t)p(t)(\phi(0))^2 + 2\lambda(t)p(t)|\phi(0)||\phi(-\tau(t))| \\
&\leq -2p(t)a(t)M_0(\phi(0))^2 + p(t)\phi^2(0)\int_{t-r}^{t} B(t,s)ds + p(t)\int_{t-r}^{t} B(t,s)g^2(\phi(s-t))ds \\
&\quad + (\phi(0))^2\ {}^{RL}_{t_0}D^q p(t) + 3\lambda(t)p(t)(\phi(0))^2 + \lambda(t)\frac{p(t)}{p(t-\tau(t))}p(t-\tau(t))\phi^2(-\tau(t)) \\
&\leq \Big(-2a(t)M_0 + \int_{t-r}^{t} B(t,s)ds + m_0 p(t)\int_{t-r}^{t}\frac{B(t,s)}{p(s)}ds \\
&\quad + \frac{{}^{RL}_{t_0}D^q p(t)}{p(t)} + \lambda(t)(3+\frac{p(t)}{p(t-\tau(t))})\Big)p(t)(\phi(0))^2.
\end{aligned}$$

From the above inequality, (4.3), and Lemma 3.5 the zero solution of (2.1) is uniformly stable. □

Theorem 4.4. *Assume* (H1)–(H6) *are satisfied and there exist an initial time* t_0 *and a function* $p\in C(\mathbb{R}_+,(0,\infty))$ *such that* $0<C_1\leq p(t)\leq C_2$, C_1,C_2 *are positive constants, and* (4.3) *holds. Then the zero solution of* (2.1) *is stable w.r.t.* t_0.

The proof of the above theorem is similar to that of Theorem 4.2, with the application of Corollary 3.6 instead of Lemma 3.5. We omit it.

Remark 4.5. Note the proofs of Theorems 4.2 and 4.4 could be done by using Caputo fractional Dini derivative of Lyapunov function $V(t,x)=p(t)x^2$ and Lemma 3.7 applying Remark 3.2.

Example 4.6. Consider the IVP for the IFrDDE

$$\begin{aligned} {}_0^C D_t^{0.9} x(t) = &-\Big(\frac{0.7}{1.01 - |\cos(t+0.7)|} - 0.5\Big) \log\Big(\frac{x(t)+0.5}{0.5-x(t)}\Big) \\ &+ 0.05 \int_{t-1}^{t} \cos^2(s) x(s) ds + \frac{0.25}{t+10} x(t - \frac{t}{1+t}) \quad \text{for } t > 0, \\ &x(\Theta) = \phi(\Theta) \quad \text{for } \Theta \in [-1, 0], \end{aligned} \tag{4.5}$$

where $x \in \mathbb{R}$, $\phi \in C([-1,0],\mathbb{R})$, $a(t) = \frac{0.7}{1.01 - |\cos(t+0.7)|} - 0.5$, $f(x) = \log(\frac{x+0.5}{0.5-x})$, $B(t,s) = 0.5\cos^2(s)$, $g(x) = 0.1x$. The IVP for (4.5) with zero initial function has a zero solution. Condition (H1) is satisfied with $a_0 = 0.1$; while (H2) is satisfied with $m_0 = 0.01$, $M_0 = 1$ (see Figure 1). (H4) is satisfied since $\tau(t) = \frac{t}{1+t} \in [0,1]$. (H5) is satisfied with $\lambda(t) = \frac{0.25}{t+10}$. Since $0.5\int_{t-1}^{t} \cos^2(s) ds = \frac{1}{8}(2 + \sin(2-2t) + \sin(2t))$ inequality (4.1) is reduced to $2M_0 a(t) = \varsigma(t) = \frac{1.4}{1.01 - |\cos(t+0.7)|} - 1 \geq 1.01\int_{t-1}^{t} B(t,s)ds + 4\lambda(t) = \xi(t)$ and it is not satisfied for $t \geq 0$ (see Figure 3).

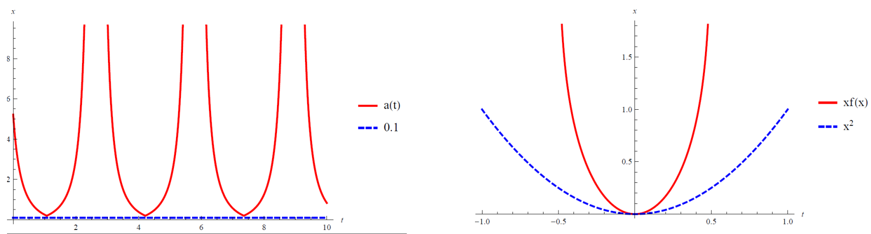

FIGURE 1. Functions $a(t)$ and 0.1 (left); functions $xf(x)$ and $0.01x^2$ (right).

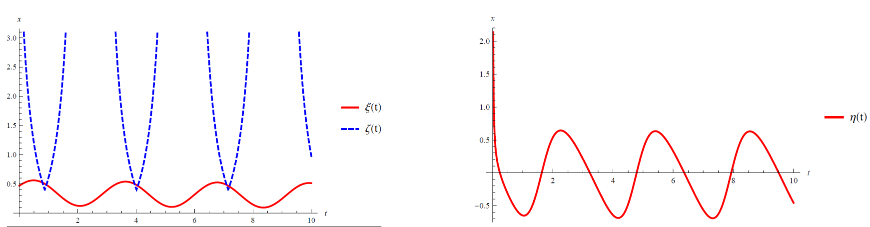

FIGURE 2. Functions $\xi(t)$ and $\varsigma(t)$ (left); function $\eta(t)$ (right).

Now, consider the function $p(t) = (1 + cos^2 t)$. The graph of the fractional derivative $\eta(t) = \frac{{}_{t_0}^{RL}D_t^{0.9}(1+cos^2 t)}{1+cos^2 t}$ with $t_0 = 0$ is given on Figure 2. Also, the inequality

$$\begin{aligned}
&2M_0a(t)\\
&= \varsigma(t) = \frac{1.4}{1.01 - |\cos(t+0.7)|} - 1\\
&\geq \int_{t-1}^{t} B(t,s)ds + m_0(1+cos^2 t)\int_{t-1}^{t}\frac{B(t,s)}{(1+cos^2 s)}ds + \frac{{}_0^{RL}D_t^{0.9}(1+cos^2 t)}{1+cos^2 t}\\
&\quad + \lambda(t)(3+\frac{p(t)}{p(t-\tau(t))}) \qquad (4.6)\\
&= \frac{1}{8}(2+\sin(2-2t)+\sin(2t)) + 0.5m_0(1+\cos^2 t)\int_{t-1}^{t}\frac{\cos^2 s}{(1+\cos^2 s)}ds\\
&\quad + \frac{{}_0^{RL}D_t^{0.9}(1+cos^2 t)}{1+cos^2 t} + \frac{0.25}{t+10}\Big(3+\frac{(1+cos^2 t)}{(1+cos^2(t-\frac{t}{1+t}))}\Big) = \eta(t)
\end{aligned}$$

holds (see Figure 3), i.e. (4.3) is satisfied for $t_0 = 0$.

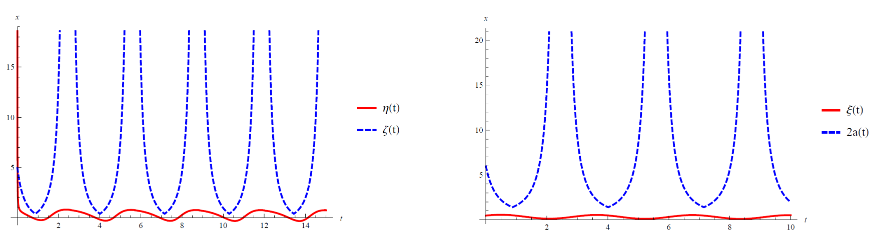

FIGURE 3. Functions $\varsigma(t)$ and $\eta(t)$ (left); functions $2\frac{0.7}{1-|\cos(t+0.7)|}$ and $\xi(t)$ (right).

By Theorem 4.4, the zero solution of (4.5) is stable with respect to the initial time $t_0 = 0$. Note if $a(t) = \frac{0.7}{1.01-|\cos(t+0.7)|}$, then the inequality (4.1) is satisfied (see Figure 3) and by Theorem 4.1 the zero solution of (4.5) is uniformly stable.

REFERENCES

[1] R. Agarwal, S. Hristova, D. O'Regan; Lyapunov functions and stability of Caputo fractional differential equations with delays, *Diff. Eq. Dyn.Sys.* (to appear).
[2] R. Agarwal, S. Hristova, D. O'Regan; Lyapunov functions and strict stability of Caputo fractional differential equations, *Adv. Diff. Eq.*, (2015) 2015:346.
[3] R. Agarwal, S. Hristova, D. O'Regan; A survey of Lyapunov functions and impulsive Caputo fractional differential equations, *FCAA*, **19**, 2 (2016), 290-318.
[4] R. Agarwal, D. O'Regan, S. Hristova; Stability of Caputo fractional differential equations by Lyapunov functions, *Appl. Math.*, **60**, 6 (2015), 653-676.
[5] R. Agarwal, D. O'Regan, S. Hristova, M. Cicek; Practical stability with respect to initial time difference for Caputo fractional differential equations, *Commun. Nonl. Sci. Numer. Simul.*, **42** (2017), 106-120.
[6] N. Aguila-Camacho, M. A. Duarte-Mermoud, J. A. Gallegos; Lyapunov functions for fractional order systems, *Commun. Nonlinear Sci. Numer. Simul.*, **19** 9 (2014), 2951-2957.
[7] D. Baleanu, G. C. Wu, S. D. Zeng; Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations, *Chaos Solitons Fractals* , **102** (2017), 99-105.
[8] T. A. Burton; *A first course in integral equations*, Second edition. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[9] J. V. Devi , F. A. Mc Rae, Z. Drici; Variational Lyapunov method for fractional differential equations, *Comput. Math. Appl.*, **64** (2012), 2982-2989.
[10] J. D. Gabano, T. Poinot; Fractional modelling and identification of thermal systems, *Signal Processing*, **91** (3) (2011), 531-541.
[11] Z. B. Guven, D. Baleanu, J. A. Tenreiro Machado; *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, Cham, 2010.
[12] J. K. Hale, S. M. Verduyn Lune; *Introduction to Functional Differential Equations*, Springer-Verlag, New-York, 1993.
[13] V. Kolmanovskii, A. Myshkis; *Applied Theory of Functional Differential Equations*, Kluwer, Dordrecht, 1999.
[14] N. N. Krasovski; On the application of the second method of Lyapunov for equations with time retardations (in Russian), *Prikladnaja Matematika i Mehanika*, **20** (1956), 315-327.
[15] Y. Li, Y. Chen, I. Podlubny; Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Comput. Math. Appl.*, **59** 5 (2010), 1810-1821.
[16] I. Podlubny; *Fractional Differential Equations*, Academic Press: San Diego, 1999.
[17] B. S. Razumikhin; On stability of systems with a delay (in Russian), *Prikladnoi Matematiki i Mekhaniki*, **20** (1956), 500-512.
[18] M. Rahman; *Integral equations and their applications*, WIT Press, Southampton, 2007.
[19] J. Sabatier, M. Aoun, A. Oustaloup , G. Greegoire, F. Ragot, P. Roy; Fractional system identification for lead acid battery state of charge estimation, *Signal Processing*, **86** (10) (2006), 2645-2657.
[20] J. R. Sadati, D. Baleanu, R. Ghaderi, N. A. Ranjbar, T. Abdeljawad (Maraaba), and F. Jarad; Razumikhin stability theorem for fractional systems with delay, *Abstract and Applied Analysis*, **2010**, 2010, Article ID 124812, 9 p.
[21] C. Tunc; Qualitative properties in nonlinear Volterra integro-differential equations with delay, *J. Taibah Univ. Sci.*, **11** (2017), 309-314.
[22] C. Tunc, S. A. Mohammed; On the stability and uniform stability of retarded integro-differential equations, *Alexandria Eng. J.*, (2018) (in press).
[23] C. Tunc, O. Tunc; New results on the stability, integrability and boundedness in Volterra integro-differential equations, *Bull. Comput. Appl. Math.*, **6** 1 (2018), 41-58.
[24] Y. Wang, T. Li; Stability analysis of fractional-order nonlinear systems with delay, *Math. Probl.Eng.*, **2014**, (2014), Art. ID 301235, 8 p.
[25] A. M. Wazwaz; *Linear and nonlinear integral equations. Methods and applications*, Higher Education Press, Beijing; Springer, Heidelberg, 2011.
[26] A. M. Wazwaz; *A first course in integral equations. Second edition*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

Snezhana Hristova
Department of Applied Mathematics and Modeling, University of Plovdiv "Paisii Hilendarski", 4000 Plovdiv, Bulgaria
E-mail address: snehri@gmail.com

Cemil Tunc
Department of Mathematics, Faculty of Sciences, Van Yuzuncu Yil University, 65080 Van, Turkey
E-mail address: cemtunc@yahoo.com