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THE p-LAPLACE EQUATION IN A CLASS OF HÖRMANDER VECTOR FIELDS

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Abstract. We find the fundamental solution to the p-Laplace equation in a class of Hörmander vector fields that generate neither a Carnot group nor a Grushin-type space. The singularity occurs at the sub-Riemannian points, which naturally corresponds to finding the fundamental solution of a generalized operator in Euclidean space. We then extend these solutions to a generalization of the p-Laplace equation and use these solutions to find infinite harmonic functions and their generalizations. We also compute the capacity of annuli centered at the singularity.

1. Introduction

The p-Laplace equation is the model equation for nonlinear potential theory [11, 10] and as such, has applications to a variety of topics in mathematics and physics. Of particular interest is exploring the geometric and analytic properties of solutions to the p-Laplace equation in sub-Riemannian spaces, which are spaces with the property that their topological dimension is greater than or equal to the topological dimension of their tangent space. It was shown in [10] that the Euclidean results of [11] can be extended into a class of sub-Riemannian spaces possessing an algebraic group law, called Carnot groups.

One foundational result concerning the p-Laplace equation is finding a formula for the fundamental solution. Finding a closed-form fundamental solution allows for further geometric analysis in the sub-Riemannian environment. Fundamental solutions to generalized operators based on the Euclidean 2-Laplace equation can be found in the seminal works of [2, 3, 4, 5]. However, in these aforementioned papers not all fundamental solutions are closed-form. Closed-form fundamental solutions are difficult to obtain in general spaces. However, closed-form fundamental solutions to the p-Laplace equation in a subclass of Carnot groups, called groups of Heisenberg-type, have been found in [9, 10]. In Hörmander vector fields without an algebraic group law, fundamental solutions to the p-Laplace equation have been found in certain spaces of a subclass called Grushin-type spaces [1, 6, 7], but only those fundamental solutions in [6, 7] are closed-form. In this paper, we find the fundamental solution to the p-Laplace equation for $1 < p < \infty$ in a class of Hörmander vector fields that are not Grushin-type. To our knowledge, this is

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the first instance of a closed-form fundamental solution to the p-Laplace equation outside of groups of Heisenberg-type and Grushin-type spaces. The singularity occurs at the sub-Riemannian points, which naturally corresponds to finding the fundamental solution of a generalized operator in Euclidean space, in the spirit of [2, 3, 4, 5].

In Section 2, we introduce the environment and explore its properties. In Section 3, we recall key technical results needed for Section 4, where the fundamental solution is established. In Section 5, we generalize the p-Laplace equation via the generalization of [8]. In Section 6, we use the fundamental solution to compute capacity of spherical rings centered at the singularity. This article is based on the Ph. D. thesis of the second author at the University of South Florida under the direction of the first author. The second author wishes to thank the Department of Mathematics and Statistics at the University of South Florida for the numerous research opportunities.

2. Environment

Consider $\mathbb{R}^{2n+1}$ and fix a point $x_0 = (a_1, a_2, \dots, a_{2n}, s) \in \mathbb{R}^{2n+1}$ and numbers $0 \neq c \in \mathbb{R}$ and $k \in \mathbb{R}^+$. We begin with the following vector fields:

$$X_i = \begin{cases} \frac{\partial}{\partial x_i} + 2kc(x_{i+n} - a_{i+n})\Big(\sum_{l=1}^{2n}(x_l - a_l)^2\Big)^{k-1}\frac{\partial}{\partial t}, & 1 \le i \le n, \\ \frac{\partial}{\partial x_i} - 2kc(x_{i-n} - a_{i-n})\Big(\sum_{l=1}^{2n}(x_l - a_l)^2\Big)^{k-1}\frac{\partial}{\partial t}, & n+1 \le i \le 2n, \end{cases} \tag{2.1}$$

The Lie bracket $[X_i, X_j]$ for $i < j$, equals

$$\begin{aligned}
&8kc(k-1)\Big(\sum_{l=1}^{2n}(x_l - a_l)^2\Big)^{k-2}\Big((x_{j+n} - a_{j+n})(x_i - a_i) \\
&- (x_{i+n} - a_{i+n})(x_j - a_j)\Big)\frac{\partial}{\partial t} \quad \text{for } i, j \le n; \\
&8kc(k-1)\Big(\sum_{l=1}^{2n}(x_l - a_l)^2\Big)^{k-2}\Big((x_{i-n} - a_{i-n})(x_j - a_j) \\
&- (x_{j-n} - a_{j-n})(x_i - a_i)\Big)\frac{\partial}{\partial t} \quad \text{for } i, j > n; \\
&8kc(k-1)\Big(\sum_{l=1}^{2n}(x_l - a_l)^2\Big)^{k-2}\Big((x_{i+n} - a_{i+n})(x_j - a_j) \\
&- (x_{j-n} - a_{j-n})(x_i - a_i)\Big)\frac{\partial}{\partial t} - 4kc\Big(\sum_{l=1}^{2n}(x_l - a_l)^2\Big)^{k-1}\delta_{i,j-n}\frac{\partial}{\partial t} \quad \text{for } i \le n < j.
\end{aligned}$$

We note that this computation shows that these vector fields are not the tangent space of a Carnot group when $k \neq 1$, as the brackets all vanish at some points, such as x_0, but not at others. In addition, the vector fields do not possess the Grushin-type space form (see, for example, [6, 7]).

This choice of vector field arises from the fact that it simultaneously generalizes different spaces, each having a closed-form fundamental solution to the p-Laplace equation. When $k = 1$, these vector fields generate the $(2n+1)$-dimensional Heisenberg group. As $c \to 0$, the vector fields approach the standard Euclidean vector fields in $\mathbb{R}^{2n}$ and the $\frac{\partial}{\partial t}$ coefficient is inspired by the Grushin-type space of [6].

We endow $\mathbb{R}^{2n+1}$ with an inner product so that the collection $\{X_i\}_{i=1}^{2n} \cup \{\frac{\partial}{\partial t}\}$ forms an orthonormal basis, producing a sub-Riemannian manifold that we shall call g_n that also forms the tangent space to a sub-Riemannian space, denoted G_n. Points in G_n will also be denoted by $x = (x_1, x_2, \dots, x_{2n}, t)$.

Though G_n is not a Carnot group, it is a metric space with the natural metric being the Carnot-Carathéodory distance, which is defined for points x and y as follows:

$$d_C(x, y) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| \, dt.$$

Here Γ is the set of all curves γ such that $\gamma(0) = x, \gamma(1) = y$ and

$$\gamma'(t) \in \operatorname{span}\{\{X_i(\gamma(t))\}_{i=1}^{2n}\}.$$

Using this metric, we can define a Carnot-Carathéodory ball of radius r centered at a point x_0 by

$$B_C(x_0, r) = \{x \in G_n : d_C(x, x_0) < r\}$$

and similarly, we shall denote a bounded domain in G_n by Ω. Given a smooth function f on G_n, we define the horizontal gradient of f as

$$\nabla_0 f(x) = (X_1 f(x), X_2 f(x), \dots, X_{2n} f(x))$$

and the symmetrized second order (horizontal) derivative matrix by

$$((D^2 f(x))^\star)_{ij} = \frac{1}{2}(X_i X_j f(x) + X_j X_i f(x))$$

for $i, j = 1, 2, \dots n$.

Definition 2.1. The function $f : G_n \to \mathbb{R}$ is said to be $\mathcal{C}^1_{\text{sub}}$ if $X_i f$ is continuous for all $i = 1, 2, \dots, 2n$. Similarly, the function f is $\mathcal{C}^2_{\text{sub}}$ if $X_i X_j f(x)$ is continuous for all $i, j = 1, 2, \dots, 2n$.

Using these derivatives, we consider two main operators on $\mathcal{C}^2_{\text{sub}}$ functions called the p-Laplacian

$$\Delta_p f = \operatorname{div}(\|\nabla_0 f\|^{p-2} \nabla_0 f) = \sum_{i=1}^{2n} X_i(\|\nabla_0 f\|^{p-2} X_i f)$$

defined for $1 < p < \infty$ and the infinite Laplacian

$$\Delta_\infty f = \sum_{i,j=1}^{2n} X_i f X_j f X_i X_j f = \langle \nabla_0 f, (D^2 f)^\star \nabla_0 f \rangle = \frac{1}{2} \nabla_0 f \cdot \nabla_0 \|\nabla_0 f\|^2.$$

We may define Sobolev spaces in the natural way. Namely, for any open set $\mathcal{O} \subset G_n$, the function f is in the horizontal Sobolev space $W^{1,q}(\mathcal{O})$ if the functions $f, X_1 f, \dots, X_{2n} f$ lie in $L^q(\mathcal{O})$. Replacing $L^q(\mathcal{O})$ by $L^q_{loc}(\mathcal{O})$, the space $W^{1,q}_{loc}(\mathcal{O})$ is defined similarly. We may then use these Sobolev spaces to consider the above operators in the usual weak sense.

3. Co-Area formula and measure theory

Let $\Omega \subset G_n$ be a bounded domain, and let $\psi \in \mathcal{C}^1_{\text{sub}}(\Omega)$ be a smooth, real-valued function which extends continuously to $\partial\Omega$. For convenience, we write ∇ for the Euclidean gradient on $G_n = \mathbb{R}^{2n+1}$. In place of Fubini's Theorem for iterated integrals, we will make use of the following Co-Area Formula in the sub-Riemannian case via [12, Theorem 4.2].

Theorem 3.1. *Under the hypotheses as above, for any function $g \in L^1(\Omega)$, we have*

$$\iint_\Omega g\|\nabla\psi\|\, d\mathcal{L}_{2n+1} = \int_0^\infty \int_{\psi^{-1}\{r\}} g\, d\mathcal{H} dr, \tag{3.1}$$

where $d\mathcal{L}_{2n+1}$ denotes Lebesgue $(2n+1)$-measure on Ω, and $d\mathcal{H}$ denotes the Hausdorff $(2n)$-measure on $\psi^{-1}(\{r\})$.

Remark 3.2. As above, the theorem also holds for continuous functions ψ which are smooth everywhere except at isolated points.

We now consider a particular case, where $x_0 \in G_n$ has coordinates

$$x_0 = (a_1, a_2, \dots, a_{2n}, s)$$

and ψ is a non-negative radial function with $\psi(x_0) = 0$. The following notation is suggestive for the inverse images of ψ:

$$\begin{gathered} B_R(x_0) = \psi^{-1}([0,R)) = \{x \in \Omega : \psi(x) < R\} \\ \partial B_R(x_0) = \psi^{-1}(\{R\}) = \{x \in \Omega : \psi(x) = R\}. \end{gathered}$$

The x_0 is omitted when it is clear from the context. Now choose $g(x) := \|\nabla_0\psi\|^p$. Since $\|\nabla_0\psi\| \lesssim \|\nabla\psi\|$ we may apply the Co-Area Formula to the function $g = (g/\|\nabla\psi\|) \cdot \|\nabla\psi\|$ to obtain the following proposition.

Proposition 3.3. *Let $\mathcal{V}$ be an absolutely continuous measure to $\mathcal{L}_{2n+1}$ with Radon-Nikodym derivative $g = [d\mathcal{V}/d\mathcal{L}_{2n+1}]$. Then for sufficiently small $R > 0$,*

$$\mathcal{V}(B_R) = \int_{B_R} d\mathcal{V} = \int_0^R \int_{\partial B_r} \frac{g}{\|\nabla\psi\|}\, d\mathcal{H} dr. \tag{3.2}$$

In light of the equality in (3.2), we see that the measure space $(G_n, \mathcal{V})$ is globally Ahlfors Q-regular with respect to balls centered at x_0. In particular, for $R > 0$,

$$\mathcal{V}(B_R) = \sigma_p R^Q \tag{3.3}$$

where $Q = 2n + 2k$ and $\sigma_p = \mathcal{V}(B_1)$ is a fixed positive constant.

For technical purposes we proceed to study the boundary behavior of precompact domains Ω. This now motivates the following definition.

Definition 3.4. For small values $R \in R_\psi$, define a measure $\mathcal{S}$ on ∂B_R as

$$\mathcal{S}(\partial B_R) = \int_{\partial B_R} d\mathcal{S} = \int_{\partial B_R} \frac{g}{\|\nabla\psi\|}\, d\mathcal{H}.$$

In particular, $\mathcal{S}$ is absolutely continuous with respect to the Hausdorff $(2n)$-measure $\mathcal{H}$. Using previous results, we now conclude:

Corollary 3.5. *(1) $\mathcal{S}$ is locally Ahlfors $(Q-1)$-regular and*

$$\mathcal{S}(\partial B_R) = Q\sigma_1 R^{Q-1}. \tag{3.4}$$

(2) Let φ be a continuous and integrable function on B_R. Then as $R \to 0$,

$$\frac{R^{1-Q}}{Q\sigma_p} \int_{\partial B_R} \varphi\, d\mathcal{S} \;\to\; \varphi(0). \tag{3.5}$$

Remark 3.6. Equation (3.4) follows immediately from differentiating both (3.2) and (3.3). Since $\mathcal{S}$ is absolutely continuous with respect to Hausdorff $(2n)$-measure $\mathcal{H}$, it follows that $\mathcal{S}$ is Borel regular. As a result, (3.5) is the analogue of the Lebesgue Density Theorem.

4. p-Laplace equation

Now, we compute an explicit formula for the fundamental solution of the p-Laplacian for the vector fields defined by Equation (2.1) above and for $1 < p < \infty$. We define the constant $Q = 2k + 2n$.

Theorem 4.1. *Let $x_0 = (a_1, a_2, \dots, a_{2n}, s)$ be an arbitrary fixed point. Let $Q = 2n + 2k$. Consider the following quantities for $1 < p < \infty$:*

$$w = \frac{Q-p}{(1-p)(4k)}, \quad \alpha = \frac{Q-p}{1-p},$$

$$h(x_1, \dots, x_{2n}, t) = c^2 \Big(\sum_{l=1}^{2n} (x_l - a_l)^2 \Big)^{2k} + (t-s)^2 \equiv c^2 \Sigma^{2k} + (t-s)^2,$$

$$\psi(x_1, \dots, x_{2n}, t) = [h(x_1, \dots, x_{2n}, t)]^{\frac{1}{4k}},$$

$$f(x_1, \dots, x_{2n}, t) = [h(x_1, \dots, x_{2n}, t)]^w = \psi(x_1, \dots, x_{2n}, t)^\alpha,$$

$$\sigma_p = \int_{B_1} \|\nabla_0 \psi\|^p \, d\mathcal{L}_{2n+1},$$

where $d\mathcal{L}_{2n+1}$ denotes the Lebesgue $(2n+1)$-measure

$$C_1 = \alpha^{-1} (Q\sigma_p)^{\frac{1}{1-p}} \quad \text{and} \quad C_2 = (Q\sigma_p)^{\frac{1}{1-p}}.$$

Then, for the constants C_1 and C_2, we have

$$\Delta_p C_1 f(x_1, \dots, x_{2n}, t) = \delta_{x_0} \quad \text{when } p \neq Q, \tag{4.1}$$

$$\Delta_p (C_2 \log \psi(x_1, \dots, x_{2n}, t)) = \delta_{x_0} \quad \text{when } p = Q, \tag{4.2}$$

in the sense of distributions.

Proof. Note that, for the sake of rigor, we should invoke the regularization of h given by

$$h_\varepsilon(x_1, \dots, x_{2n}, t) = c^2 \Big(\sum_{l=1}^{2n} (x_l - a_l)^2 + \varepsilon^2 \Big)^{2k} + (t-s)^2$$

for $\varepsilon > 0$ and let $\varepsilon \to 0$. Instead, we proceed formally. We will need some calculations for the proof that we will compute first. For $p \neq Q$, we have:

$$X_i f = \alpha h^{w-1} c^2 \Sigma^{2k-1} (x_i - a_i) + \alpha c h^{w-1} \Sigma^{k-1} (x_{i+n} - a_{i+n})(t-s) \quad \text{when } i \leq n,$$

$$X_j f = \alpha h^{w-1} c^2 \Sigma^{2k-1} (x_j - a_j) - \alpha c h^{w-1} \Sigma^{k-1} (x_{j-n} - a_{j-n})(t-s) \quad \text{when } j > n,$$

so that

$$\|\nabla_0 f\|^2 = \alpha^2 c^2 h^{2w-1} \Sigma^{2k-1},$$

$$\|\nabla_0 f\|^{p-2} = |c\alpha|^{p-2} h^{(w-\frac{1}{2})(p-2)} \Sigma^{(k-\frac{1}{2})(p-2)}.$$

Then for $i \leq n$, we have

$$\begin{aligned} \|\nabla_0 f\|^{p-2} X_i f &= \alpha |\alpha|^{p-2} |c|^p h^{(w(p-1)-\frac{p}{2})} \Sigma^{(kp-\frac{p}{2})} (x_i - a_i), \\ &\quad + \alpha |\alpha|^{p-2} c |c|^{p-2} h^{(w(p-1)-\frac{p}{2})} \Sigma^{(k(p-1)-\frac{p}{2})} (x_{i+n} - a_{i+n})(t-s) \end{aligned}$$

and for $j > n$, we have

$$\begin{aligned} \|\nabla_0 f\|^{p-2} X_j f &= \alpha |\alpha|^{p-2} |c|^p h^{(w(p-1)-\frac{p}{2})} \Sigma^{(kp-\frac{p}{2})} (x_j - a_j) \\ &\quad - \alpha |\alpha|^{p-2} c |c|^{p-2} h^{(w(p-1)-\frac{p}{2})} \Sigma^{(k(p-1)-\frac{p}{2})} (x_{j-n} - a_{j-n})(t-s). \end{aligned}$$

Letting $\zeta = w(p-1) - \frac{p}{2}$ and $\chi = kp - \frac{p}{2}$, we employ routine calculations to compute the p-Laplacian:

$$
\begin{aligned}
&(\alpha|\alpha|^{p-2}|c|^p)^{-1}\Delta_p f \\
&= (\alpha|\alpha|^{p-2}|c|^p)^{-1}\Big(\sum_{i=1}^{n} X_i(\|\nabla_0 f\|^{p-2}\nabla_0 f) + \sum_{j=n+1}^{2n} X_j(\|\nabla_0 f\|^{p-2}\nabla_0 f)\Big) \\
&= (2n + 2k + 2\chi + 4k\zeta)h^{\zeta}\Sigma^{\chi} \\
&= (Q + 2kp - p + (p - Q) - 2kp)h^{\zeta}\Sigma^{\chi} = 0\,.
\end{aligned}
$$

Note that these computations are valid wherever the function f is smooth and, in particular, these are valid away from the point x_0. We note that by our computations above, $\|\nabla_0 f\|^{p-1}$ is locally integrable on G_n. We then consider $\phi \in C_0^{\infty}$ with compact support in the ball

$$B_R = \{y : \psi(y) < R\}.$$

Let $0 < r < R$ be given so that $B_r \subset B_R$. In the annulus $\mathcal{A} := B_R \setminus \overline{B_r}$ we have, via the Leibniz rule,

$$
\begin{aligned}
\operatorname{div}(\phi\|\nabla_0 f\|^{p-2}\nabla_0 f) &= \phi\operatorname{div}(\|\nabla_0 f\|^{p-2}\nabla_0 f) + \|\nabla_0 f\|^{p-2}\langle\nabla_0 f, \nabla_0\phi\rangle \\
&= 0 + \|\nabla_0 f\|^{p-2}\langle\nabla_0 f, \nabla_0\phi\rangle.
\end{aligned}
$$

Let $\mathcal{L}_{2n+1}$ and $\mathcal{H}$ be the measures from (3.1). Applying Stokes' Theorem,

$$
\begin{aligned}
&\int_{\mathcal{A}} \|\nabla_0 f\|^{p-2}\langle\nabla_0 f, \nabla_0\phi\rangle d\mathcal{L}_{2n+1} \\
&= \int_{\mathcal{A}} \operatorname{div}(\phi\|\nabla_0 f\|^{p-2}\nabla_0 f) d\mathcal{L}_{2n+1} \\
&= \int_{\mathcal{A}} \sum_{l=1}^{2n} X_l[\phi\|\nabla_0 f\|^{p-2}X_l f] d\mathcal{L}_{2n+1} \\
&= \int_{\mathcal{A}} \sum_{l=1}^{2n} \frac{\partial}{\partial x_l}[\phi\|\nabla_0 f\|^{p-2}X_l f] + \sum_{i=1}^{n} 2kc(x_{i+n} - a_{i+n})\Sigma^{k-1}\frac{\partial}{\partial t}[\phi\|\nabla_0 f\|^{p-2}X_i f] \\
&\quad - \sum_{j=n+1}^{2n} 2kc(x_{j-n} - a_{j-n})\Sigma^{k-1}\frac{\partial}{\partial t}[\phi\|\nabla_0 f\|^{p-2}X_j f] d\mathcal{L}_{2n+1} \\
&= \int_{\mathcal{A}} \operatorname{div}_{\rm eucl}[\xi] d\mathcal{L}_{2n+1}
\end{aligned}
$$

where the $(2n+1)$-vector ξ is defined by

$$
\xi = \begin{bmatrix}
\phi\|\nabla_0 f\|^{p-2}X_1 f \\
\phi\|\nabla_0 f\|^{p-2}X_2 f \\
\dots \\
\phi\|\nabla_0 f\|^{p-2}X_{2n} f \\
2kc\Sigma^{k-1}\phi\|\nabla_0 f\|^{p-2}\Big(\sum_{i=1}^{n}(x_{i+n} - a_{i+n})X_i f - \sum_{j=n+1}^{2n}(x_{j-n} - a_{j-n})X_j f\Big)
\end{bmatrix}.
$$

Thus,

$$
\int_{\mathcal{A}} \|\nabla_0 f\|^{p-2}\langle\nabla_0 f, \nabla_0\phi\rangle d\mathcal{L}_{2n+1} = \int_{\partial\mathcal{A}} \frac{1}{\|\nu\|}\sum_{l=1}^{2n} \phi\|\nabla_0 f\|^{p-2}X_l f\nu_l + \xi_{2n+1}\nu_{2n+1} d\mathcal{H}
$$

$$= -\int_{\partial B_r} \frac{1}{\|\nu\|} \sum_{l=1}^{2n} \phi \|\nabla_0 f\|^{p-2} X_l f \nu_l + \xi_{2n+1}\nu_{2n+1} d\mathcal{H}$$

where ν is the outward Euclidean normal. Recalling that

$$\psi(x_1, x_2, \dots, x_{2n}, t) = [h(x_1, x_2, \dots, x_{2n}, t)]^{\frac{1}{4k}},$$

we proceed with the computation

$$\begin{aligned}
&\int_{\mathcal{A}} \|\nabla_0 f\|^{p-2} \langle \nabla_0 f, \nabla_0 \phi\rangle d\mathcal{L}_{2n+1} \\
&= -\int_{\partial B_r} \frac{1}{\|\nu\|} \Big(\sum_{l=1}^{2n} \phi \|\nabla_0 f\|^{p-2} X_l f \nu_l + \xi_{2n+1}\nu_{2n+1} \Big) d\mathcal{H} \\
&= -\int_{\partial B_r} \frac{\alpha \psi^{\alpha-1}}{\|\nu\|} \phi \|\nabla_0 \psi^\alpha\|^{p-2} \Big(\sum_{l=1}^{2n} X_l \psi \frac{\partial \psi}{\partial x_l} + 2kc\Sigma^{k-1} \\
&\quad \times \Big(\sum_{i=1}^{n} (x_{i+n} - a_{i+n}) X_i \psi - \sum_{j=n+1}^{2n} (x_{j-n} - a_{j-n}) X_j \psi \Big) \frac{\partial \psi}{\partial t} \Big) d\mathcal{H} \\
&= -\int_{\partial B_r} \frac{\alpha \psi^{\alpha-1}}{\|\nu\|} \phi \|\nabla_0 \psi\|^{p-2} |\alpha|^{p-2} \psi^{(p-2)(\alpha-1)} \Big(\|\nabla_0 \psi\|^2 \Big) d\mathcal{H} \\
&= -\int_{\partial B_r} \frac{|\alpha|^{p-2} \alpha \psi^{(p-1)(\alpha-1)}}{\|\nu\|} \phi \|\nabla_0 \psi\|^p d\mathcal{H}.
\end{aligned}$$

Recall that by definition, $\psi \equiv r$ on ∂B_r. We then have

$$\int_{\mathcal{A}} \|\nabla_0 f\|^{p-2} \langle \nabla_0 f, \nabla_0 \phi\rangle d\mathcal{L}_{2n+1} = -|\alpha|^{p-2} \alpha r^{1-Q} \int_{\partial B_r} \frac{\phi \|\nabla_0 \psi\|^p}{\|\nu\|} d\mathcal{H}.$$

Letting $r \to 0$, we apply (3.5) and obtain

$$\int_{\mathcal{A}} \|\nabla_0 f\|^{p-2} \langle \nabla_0 f, \nabla_0 \phi\rangle d\mathcal{L}_{2n+1} \to -|\alpha|^{p-2} \alpha (Q\sigma_p) \phi(x_0). \tag{4.3}$$

We then obtain the case for $p \neq Q$. The case of $p = Q$ is similar and left to the reader. □

Corollary 4.2. *The function ψ, as defined above, is infinite harmonic in the space $G_n \setminus \{x_0\}$.*

Proof. We use the formula that for a smooth function u,

$$2\Delta_\infty u = \nabla_0 u \cdot \nabla_0 \|\nabla_0 u\|^2.$$

Computing as in the Theorem, we have

$$\|\nabla_0 \psi\|^2 = c^2 \Sigma^{2k-1} h^{\frac{1-2k}{2k}}.$$

Thus for $i = 1$ to n we obtain

$$\begin{aligned}
X_i \|\nabla_0 \psi\|^2 = 2c^2 \Sigma^{2k-2} h^{\frac{1-4k}{2k}} (2k-1) \Big(& h(x_i - a_i) - c^2 (x_i - a_i) \Sigma^{2k} \\
& - c\Sigma^k (x_{i+n} - a_{i+n})(t - s) \Big),
\end{aligned}$$

and for $j = n+1$ to $2n$, we obtain

$$X_j \|\nabla_0 \psi\|^2 = 2c^2 \Sigma^{2k-2} h^{\frac{1-4k}{2k}} (2k-1) \Big(h(x_j - a_j) - c^2 (x_j - a_j) \Sigma^{2k}$$

$$+ c\Sigma^k(x_{j-n} - a_{j-n})(t-s)\Big).$$

Calculations then give us

$$\Delta_\infty \psi = c^2\Sigma^{2k-2}h^{\frac{1-4k}{2k}}(2k-1)h^{\frac{1-4k}{4k}}\Big(c^2h\Sigma^{2k} - c^4\Sigma^{4k} - c^2\Sigma^{2k}(t-s)^2\Big) = 0.$$

The proof is complete. □

5. Generalizing the p-Laplace equation

In this section, we extend the results of the previous section to include the generalization of [8]. As in Section 2, we fix a point $x_0 = (a_1, a_2, \dots, a_{2n}, s) \in \mathbb{R}^{2n+1}$ and numbers $0 \neq c \in \mathbb{R}$ and $k \in \mathbb{R}^+$. We then let $L \in \mathbb{R}$ and consider the $2n \times 1$ vector Υ with components

$$\Upsilon_l u = \begin{cases} X_l u + iLX_{n+l}u & \text{when } l = 1, 2, \dots, n \\ X_l u - iLX_{l-n}u & \text{when } l = n+1, n+2, \dots, 2n. \end{cases}$$

Then, for $1 < p < \infty$, we consider the equation

$$\begin{aligned} \overline{\Delta_p} u &= \operatorname{div}(\|\Upsilon\|^{p-2}\Upsilon) = \sum_{l=1}^{2n} X_l\big(\|\Upsilon u\|^{p-2}\Upsilon_l u\big) \\ &= \sum_{l=1}^{2n} \frac{1}{2}(p-2)\|\Upsilon u\|^{p-4}\Upsilon_l u X_l \|\Upsilon u\|^2 + \sum_{l=1}^{2n} \|\Upsilon u\|^{p-2} X_l \Upsilon_l u \qquad (5.1) \\ &= \|\Upsilon u\|^{p-4}\Big(\sum_{l=1}^{2n} \frac{1}{2}(p-2)\Upsilon_l u X_l\|\Upsilon u\|^2 + \|\Upsilon u\|^2 \sum_{l=1}^{2n} X_l \Upsilon_l u\Big) = 0 \end{aligned}$$

in $G_n \setminus \{x_0\}$. We observe that if $L = 0$, we have the classic p-Laplace equation and when $p = 2$, we have the generalization of [2].

Recalling the definitions of Q, ω, and Σ from Theorem 4.1, we define the following functions and parameters:

$$\begin{gathered} u = u(x_1, x_2, \dots, x_{2n}, t) = c\Sigma^k + i(t-s), \\ v = v(x_1, x_2, \dots, x_{2n}, t) = c\Sigma^k - i(t-s), \\ \eta = \omega(1-L), \quad \beta = \omega(1+L). \end{gathered}$$

We also note that by their construction, we have that

$$uv = h = h(x_1, x_2, \dots, x_{2n}, t)$$

where $h(x_1, x_2, \dots, x_{2n}, t)$ is the function from Theorem 4.1.

We define the function

$$\begin{aligned} &g_{p,L}(x_1, x_2, \dots, x_{2n}, t) \\ &= \begin{cases} u(x_1, x_2, \dots, x_{2n}, t)^\eta v(x_1, x_2, \dots, x_{2n}, t)^\beta & p \neq Q \\ \log\big(u^{1-L}v^{1+L}\big) & p = Q. \end{cases} \qquad (5.2) \end{aligned}$$

Before we prove our theorem, we will establish a slightly more general lemma building on the case when $p \neq Q$. We consider the function

$$\begin{aligned} &w(x_1, x_2, \dots, x_{2n}, t) \\ &= \begin{cases} u(x_1, x_2, \dots, x_{2n}, t)^\mu v(x_1, x_2, \dots, x_{2n}, t)^\nu & p \neq Q = 2n + 2k \\ g_{Q,L}(x_1, x_2, \dots, x_{2n}, t) = \log\big(u^{1-L}v^{1+L}\big) & p = Q = 2n + 2k \end{cases} \end{aligned}$$

where $\mu = A(1-L)$ and $\nu = A(1+L)$ with $A \in \mathbb{R}$ to be determined later.

Lemma 5.1. *Using the definition of the functions $g_{Q,L}, u, v, w, h$ and the parameters μ, ν, A above, we have: (1)*

$$\|\Upsilon w\|^2 = \begin{cases} 16c^2k^2\Sigma^{2k-1}A^2h^{2A-1}(L^2-1)^2 & p \neq Q \\ 16c^2k^2\Sigma^{2k-1}h^{-1}(L^2-1)^2 & p = Q. \end{cases}$$

(2)

$$\begin{aligned} &2\|\Upsilon w\|^2 \sum_{i=1}^{2n} X_i \Upsilon_i w \\ &= \begin{cases} -256A^3c^4k^3(n-1+k+2Ak)\Sigma^{4k-2}(L^2-1)^3h^{2A-1}u^{\mu-1}v^{\nu-1} & p \neq Q \\ -256c^4k^3(n-1+k)\Sigma^{4k-2}(L^2-1)^3h^{-2} & p = Q. \end{cases} \end{aligned}$$

(3)

$$\begin{aligned} &\sum_{i=1}^{2n} \Upsilon_i w X_i \|\Upsilon w\|^2 \\ &= \begin{cases} -128A^3c^4k^3(4Ak-1)(L^2-1)^3\Sigma^{4k-2}u^{A(3-L)-2}v^{A(3+L)-2} & p \neq Q \\ 128c^4k^3(L^2-1)^3\Sigma^{4k-2}h^{-2} & p = Q. \end{cases} \end{aligned}$$

Proof. When $p \neq Q$, the definition of Υ gives

$$\Upsilon_l w = \begin{cases} 2ck\Sigma^{k-1}\mu u^{\mu-1}v^{\nu}(1+L)\Big((x_l-a_l)+i(x_{l+n}-a_{l+n})\Big) \\ +2ck\Sigma^{k-1}\nu u^{\mu}v^{\nu-1}(1-L)\Big((x_l-a_l)-i(x_{l+n}-a_{l+n})\Big) \\ \qquad \text{when } 1 \leq l \leq n \\ 2ck\Sigma^{k-1}\mu u^{\mu-1}v^{\nu}(1+L)\Big((x_l-a_l)-i(x_{l+n}-a_{l+n})\Big) \\ +2ck\Sigma^{k-1}\nu u^{\mu}v^{\nu-1}(1-L)\Big((x_l-a_l)+i(x_{l+n}-a_{l+n})\Big) \\ \qquad \text{when } n < l \leq 2n, \end{cases}$$

and when $p = Q$, the definition of Υ gives

$$\Upsilon_l w = \begin{cases} -4ck\Sigma^{k-1}(L^2-1)h^{-1}\Big(c(x_l-a_l)\Sigma^k+(t-s)(x_{l+n}-a_{l+n})\Big) \\ \qquad \text{when } 1 \leq l \leq n \\ -4ck\Sigma^{k-1}(L^2-1)h^{-1}\Big(c(x_l-a_l)\Sigma^k-(t-s)(x_{l-n}-a_{l-n})\Big) \\ \qquad \text{when } n < l \leq 2n. \end{cases}$$

Factoring out the greatest common factor in the $p \neq Q$ case and simplifying produces

$$\Upsilon_l w = \begin{cases} -4Ack(L^2-1)\Sigma^{k-1}u^{\mu-1}v^{\nu-1}\Big(c\Sigma^k(x_l-a_l)+(x_{l+n}-a_{l+n})(t-s)\Big) \\ \qquad \text{when } 1 \leq l \leq n \\ -4Ack(L^2-1)\Sigma^{k-1}u^{\mu-1}v^{\nu-1}\Big(c\Sigma^k(x_l-a_l)-(x_{l-n}-a_{l-n})(t-s)\Big) \\ \qquad \text{when } n < l \leq 2n. \end{cases}$$

The first equation then follows. Using the first equation, routine calculations produce the other two equations. □

We then state the main theorem of this section.

Theorem 5.2. *Let $L \in \mathbb{R}$. If $L \neq \pm 1$, fix $1 < p < \infty$. If $L = \pm 1$, fix $2 \leq p < \infty$. On $G_n \setminus \{x_0\}$, we have $\overline{\Delta_p} g_{p,L} = 0$.*

Proof. By (5.1), we need to show that

$$\sum_{l=1}^{2n} (p-2) \Upsilon_l g_{p,L} X_l \|\Upsilon g_{p,L}\|^2 + 2\|\Upsilon g_{p,L}\|^2 \sum_{l=1}^{2n} X_l \Upsilon_l g_{p,L} = 0. \tag{5.3}$$

In the case $p = Q = 2n + 2k$, via Lemma 5.1, we have

$$\begin{aligned}
&\sum_{l=1}^{2n} (p-2) \Upsilon_l g_{p,L} X_l \|\Upsilon g_{p,L}\|^2 + 2\|\Upsilon g_{p,L}\|^2 \sum_{l=1}^{2n} X_l \Upsilon_l g_{p,L} \\
&= (2n + 2k - 2)(128 c^4 k^3 (L^2 - 1)^3 \Sigma^{4k-2} h^{-2}) \\
&\quad + (-256 c^4 k^3 (n - 1 + k) \Sigma^{4k-2} (L^2 - 1)^3 h^{-2}) \\
&= (128 c^4 k^3 (L^2 - 1)^3 \Sigma^{4k-2} h^{-2}) \big((2n + 2k - 2) - 2(n - 1 + k) \big) = 0.
\end{aligned}$$

In the case $1 < p < \infty$ with $p \neq Q$, we again use Lemma 5.1 and observe that $g_{p,L} = w$ when

$$A = \frac{Q-p}{4k(1-p)}.$$

We then obtain

$$\begin{aligned}
&\sum_{l=1}^{2n} (p-2) \Upsilon_l g_{p,L} X_l \|\Upsilon g_{p,L}\|^2 + 2\|\Upsilon g_{p,L}\|^2 \sum_{l=1}^{2n} X_l \Upsilon_l g_{p,L} \\
&= (p-2)(-128 A^3 c^4 k^3 (4Ak - 1)(L^2 - 1)^3 \Sigma^{4k-2} u^{A(3-L)-2} v^{A(3+L)-2}) \\
&\quad + (-256 A^3 c^4 k^3 (n - 1 + k + 2Ak) \Sigma^{4k-2} (L^2 - 1)^3 h^{2A-1} u^{\mu-1} v^{\nu-1}) \\
&= -128 A^3 c^4 k^3 (L^2 - 1)^3 \Sigma^{4k-2} \Big((p-2)(4Ak - 1) u^{A(3-L)-2} v^{A(3+L)-2}) \\
&\quad + 2(n - 1 + k + 2Ak) h^{2A-1} u^{\mu-1} v^{\nu-1} \Big).
\end{aligned}$$

Recalling that $h = uv$, we have

$$u^{A(3-L)-2} v^{A(3+L)-2} = h^{2A-1} u^{A-LA-1} v^{A+LA-1} = h^{2A-1} u^{\mu-1} v^{\nu-1}$$

so that we are left to show that

$$(p-2)(4Ak - 1) + 2(n - 1 + k + 2Ak) = 0.$$

By the choice of A, we have

$$\begin{aligned}
&(p-2)(4Ak - 1) + 2(n - 1 + k + 2Ak) \\
&= (p-2)\Big(\frac{Q-p}{1-p} - 1 \Big) + 2n - 2 + 2k + \frac{Q-p}{1-p} \\
&= \frac{(p-2)(Q-1) + (Q-2)(1-p) + (Q-p)}{1-p} = 0.
\end{aligned}$$

□

We are then able to conclude the following corollary.

Corollary 5.3. *Let $p > Q$. The function $g_{p,L}$, as defined above, is a smooth solution to the Dirichlet problem*

$$\begin{gathered}\overline{\Delta_p} g_{p,L}(x) = 0 \quad x \in \mathbb{G}_n \setminus \{x_0\} \\ 0 \quad x = x_0.\end{gathered}$$

Recalling the definition of $\overline{\Delta_p} u$ in Equation (5.1), and the definition of $g_{p,L}$ in Equation (5.2), letting $p \to \infty$ produces the operator

$$\overline{\Delta_\infty} u = \sum_{l=1}^{2n} \Upsilon_l u X_l \|\Upsilon u\|^2$$

and the function

$$g_{\infty,L} = u^{1-L} v^{1+L}.$$

In light of the previous theorem and corollary, we have the following relationship:

Theorem 5.4. *The function $g_{\infty,L}$ is a smooth solution to the Dirichlet problem*

$$\begin{gathered}\overline{\Delta_\infty} g_{\infty,L}(x) = 0 \quad x \in \mathbb{G}_n \setminus \{x_0\} \\ 0 \quad x = x_0.\end{gathered}$$

Proof. We may prove this theorem by letting $p \to \infty$ and invoking continuity (cf. Corollary 5.3). For completeness, though, we compute formally. Because $g_{\infty,L} = w$ in Lemma 5.1 with $A = (4k)^{-1}$, we have

$$\overline{\Delta_\infty} g_{\infty,L}(x) = -128 A^3 c^4 k^3 (4Ak - 1)(L^2 - 1)^3 \Sigma^{4k-2} u^{A(3-L)-2} v^{A(3+L)-2} = 0.$$

$\square$

Combining this theorem with Theorem 5.2, Theorem 4.1, and Corollary 4.2, we have shown the following diagram commutes in $\mathbb{G}_n \setminus \{x_0\}$:

$$\begin{tikzcd} \overline{\Delta_p} g_{p,L} = 0 \arrow[r, "p\to\infty"'] \arrow[d, "L\to 0"] & \overline{\Delta_\infty} g_{\infty,L} = 0 \arrow[d, "L\to 0"] \\ \Delta_p g_{p,0} = 0 \arrow[r, "p\to\infty"'] & \Delta_\infty g_{\infty,0} = 0 \end{tikzcd}$$

6. Spherical capacity

In this section, we will use previous results to compute the capacity of spherical rings centered at the point $x_0 = (a_1, a_2, \dots, a_{2n}, s)$. We first recall the definition of p-capacity.

Definition 6.1. Let $\Omega \subset G_n$ be a bounded, open set, and $K \subset \Omega$ a compact subset. For $1 \le p < \infty$ we define the p-capacity as

$$\operatorname{cap}_p(K, \Omega) = \inf\Big\{ \int_\Omega \|\nabla_0 u\|^p : u \in C_0^\infty(\Omega),\ u|_K = 1 \Big\}.$$

We note that although the definition is valid for $p = 1$, we will consider only $1 < p < \infty$, as in the previous sections. Because p-harmonic functions are minimizers to the energy integral

$$\int_{G_n} \|\nabla_0 f\|^p$$

it is natural to consider p-harmonic functions when computing the capacity. In particular, an easy calculation similar to the previous section shows

$$u(x)=\begin{cases}\frac{\psi(x)^\alpha-R^\alpha}{r^\alpha-R^\alpha} & \text{when } p\neq Q\\ \frac{\log\psi(x)-\log R}{\log r-\log R} & \text{when } p=Q\end{cases}$$

is a smooth solution to the Dirichlet problem

$$\begin{gathered}\Delta_p u=0 \quad \text{in } B(x_0,R)\setminus B(x_0,r)\\ u=1 \quad \text{on } \partial B(x_0,r)\\ u=0 \quad \text{on } \partial B(x_0,R)\end{gathered}$$

for $1<p<\infty$.

We state the following theorem which follows from the computations of the previous section.

Theorem 6.2. *Let $0<r<R$ and $1<p<\infty$. Then we have*

$$\operatorname{cap}_p\big(B(x_0,r),B(x_0,R)\big)=\begin{cases}\alpha^{p-1}Q\sigma_p\big(r^\alpha-R^\alpha\big)^{1-p} & \text{when } 1<p<Q\\ Q\sigma_Q[\log R-\log r]^{1-Q} & \text{when } p=Q\\ \alpha^{p-1}Q\sigma_p\big(R^\alpha-r^\alpha\big)^{1-p} & \text{when } p>Q.\end{cases}$$

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