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NONEMPTYNESS AND COMPACTNESS OF THE SOLUTION SET FOR FRACTIONAL EVOLUTION INCLUSIONS WITH NON-INSTANTANEOUS IMPULSES

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Abstract. In this article, we consider the nonemptyness and compactness of the solution set for a class of fractional semilinear evolution inclusions with non-instantaneous impulses in Banach spaces. To achieve this we use fixed point theorems with semigroup theory, upper semicontinuous multi-functions and measures of noncompactness.

1. Introduction

The qualitative theory of differential equations and evolution inclusions involving various fractional derivatives was considered in the monographs [8, 10, 15, 40], and in a series of papers [3, 4, 17, 19, 20, 23, 24, 34, 35, 36, 37, 38, 39]. Recently research in mathematical modelling described by differential equations with non-instantaneous impulses was considered in [13, 14, 25, 26, 27], and these equations are suitable to characterize the dynamics of evolution processes in pharmacotherapy. In particular, existence, topological structure, stability and controllability theory in the fractional order case was investigated in [1, 2, 5, 28, 29, 30, 31, 32, 33]. However, there seems to be very little available in the literature concerning the existence of mild solutions to evolution inclusions with not instantaneous impulses (with integer order or fractional order). This is the main motivation in this paper.

Let $J = [0, l]$, $l > 0$, $\alpha \in (0, 1)$ and E be a Banach space. Denote A by the infinitesimal generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ on E. Inspired by the references mentioned above, in this work, we consider the nonemptyness and compactness of the solution set to the following fractional semilinear evolution inclusions with non-instantaneous impulses:

$$\begin{gathered}
{}^{c}\mathbb{D}^{\alpha}_{a_i,t} u(t) \in Au(t) + F(t, u(t)), \\
\text{a.e. } t \in \cup_{i=0}^{m} (a_i, b_{i+1}] \subset J,\ a_0 := 0,\ b_{m+1} := l > 0, \\
u(t) = g_i(t, u(b_i^-)), \quad t \in (b_i, a_i] \subset [0, l],\ i = 1, 2, \dots, m, \\
u(0) = u_0 \in E,
\end{gathered} \tag{1.1}$$

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where ${}^c\mathbb{D}^{\alpha}_{a_i,t}$ denotes the Caputo derivative [15] of order α from the lower limit a_i to the upper limit t, $F : [0,l] \times E \to 2^E - \{\phi\}$ is a multi-function, the sequences $\{a_i\}$ and $\{b_{i+1}\}$ satisfy $a_i < b_{i+1}$, $i = 0, 1, \dots, m$, and moreover, $g_i : [b_i, a_i] \times E \to E, i = 1, 2, \dots, m$, and $u(b_i^-)$ denotes the left limit of u at b_i.

The article is organized as follows. In Section 2, we collect some background material concerning multi-functions and establish necessary lemmas on the operator semigroup and a generalized Cauchy operator. In Section 3, we show that the solution set $\Sigma^F_{u_0}[0,l]$ is nonempty and compact under mild conditions on $\{T(t) : t \geq 0\}$ and F. An example is given in the final section to illustrate our theory.

2. Notation and preliminaries

Denote $L^1(J,E) = \{v : v : J \to E \text{ is Bochner integrable}\}$ endowed with the norm $\|v\|_{L^1(J,E)} = \int_0^l \|v(t)\|dt$. Denote

$$
\begin{gathered}
P_b(E) = \{B \subseteq E : B \text{ is nonempty and bounded}\},\\
P_{\rm cl}(E) = \{B \subseteq E : B \text{ is nonempty and closed}\},\\
P_k(E) = \{B \subseteq E : B \text{ is nonempty and compact}\},\\
P_{\rm cl,cv}(E) = \{B \subseteq E : B \text{ is nonempty,closed and convex}\},\\
P(E) = \{B \subseteq E : B \text{ is nonempty}\},\\
P_{ck}(E) = \{B \subseteq E : B \text{ is nonempty, convex and compact}\}.
\end{gathered}
$$

$\operatorname{conv}(B)$ (respect., $\overline{\operatorname{conv}}(B)$) is the convex hull (respect., convex closed hull in E) of a subset B.

Let $C(J,E) = \{f : f : J \to E \text{ is continuous }\}$ be endowed with the supremum norm. We consider the set of functions $PC(J,E) = \big\{u : J \to E : u_{|_{J_i}} \in C(J_i,E),\ J_i := (b_i, b_{i+1}],\ i = 0, 1, 2, \dots, m \text{ and } u(b_i^+) \text{ and } u(b_i^-) \text{ exist for each } i = 1, 2, \dots, m\big\}$. It is easy to check that $PC(J,E)$ is a Banach space endowed with the Chebyshev PC-norm: $\|u\|_{PC(J,E)} = \max\{\|u(t)\| : t \in J\}$.

Let $G : J \to P(E)$ be a multifunction and $S^1_G = \{z \in L^1(J,E)\text{: } z(t) \in G(t) \text{ a.e.}\}$. This set may be empty. For $P_{\rm cl}(E)$-valued measurable multi-function, it is nonempty if and only if $t \to \inf\{\|x\| : x \in G(t)\} \in L^1(J,\mathbb{R}^+)$. In particular, this is the case if $t \to \sup\{\|x\| : x \in G(t)\} \in L^1(J,\mathbb{R}^+)$ (such a multifunction is said to be integrably bounded). Note that $S^1_G \subseteq L^1(J,E)$ is closed and it is convex if and only if for almost all $t \in J$, $G(t)$ is a convex set in E. The following definitions on multivalued mappings can be found in [6, 11, 16].

Definition 2.1. Let X and Y be two topological spaces. A multifunction $G : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) if $G^{-1}(V) = \{x \in X : G(x) \subseteq V\}$ is an open subset of X for every open $V \subseteq Y$. The map G is said to be closed if its graph $\Gamma_G = \{(x,y) \in X \times Y : y \in G(x)\}$ is closed subset of the topological space $X \times Y$. The map G is said to be compact if $G(B)$ is relatively compact for every bounded subset B of X. The map G is said to be locally compact if for every point $x \in X$ has a neighborhood $V(x)$ such that $\cup\{F(z) : z \in V(x)\}$ is relatively compact. Finally the map G is said to be quasicompact, if its restriction to any compact subset $A \subset X$ is compact.

Remark 2.2. Let X and Y be two topological spaces and $G : X \to P(Y)$. If Y is regular and the multifunction G is u.s.c. with nonempty closed values, then it is closed. If G is closed, quasi-compact and has nonempty compact values, then it is u.s.c. (see [6, 16]).

Definition 2.3. A sequence $\{f_n\}_{n=1}^{\infty} \subseteq L^P(J,E)(P \geq 1)$ is called P-time integrably bounded if there is a function $h \in L^P(J,\mathbb{R})$ such that $\|f_n(t)\| \leq h(t)$, a.e. on J.

Definition 2.4. A sequence $\{f_n\}_{n=1}^{\infty} \subseteq L^P(J,E)(P \geq 1)$ is called P-time semicompact if it is P-time integrably bounded and the set $\{f_n(t) : n \geq 1\}$ is relatively compact for a.e. $t \in J$.

Next, we recall the definition of a mild solution. For any fixed $t \geq 0$, define

$$K_1(t) = \int_0^{\infty} \xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \qquad K_2(t) = \alpha\int_0^{\infty} \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

where $\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}w_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0$, and

$$w_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\ \alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty).$$

Definition 2.5. A function $u \in PC(J,E)$ is called a PC-mild solution of (1.1) if

$$u(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [0,b_1], \\ g_i(t,u(b_i^-)), & t \in (b_i,a_i],\ i=1,2,\dots,m, \\ K_1(t-a_i)g_i(a_i,u(b_i^-)) & \\ +\int_{a_i}^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [a_i,b_{i+1}],\ i=1,2,\dots,m, \end{cases}$$

where $f \in S_G^P = \{f \in L^P(J,E) : f(t) \in F(t,u(t)) \text{ a.e.}\}$, $P > 1/\alpha$.

The Hausdorff measure of noncompactness on E is defined on bounded subsets as

$$\chi(B) = \inf\{\epsilon > 0 : B \text{ can be covered by finitely many balls of radius} \leq \epsilon\}.$$

Next, the map $\chi_{PC} : P_b(PC(J,E)) \to [0,\infty)$ is defined as

$$\chi_{PC}(B) = \max_{i=0,1,2,\dots,m} \chi_i(B|_{\overline{J_i}}),\ \overline{J_i} := [b_i,b_{i+1}],$$

where χ_i is the Hausdorff measure of noncompactness on the Banach space $C(\overline{J_i},E)$ and $B_{|}\overline{J_i} = \{u^* : \overline{J_i} \to E : u^*(t) = u(t), t \in J_i \text{ and } u^*(b_i) = u(b_i^+), u \in B\}$, $i = 0,1,\dots,m$. Of course $B|_{\overline{J_0}} = \{u_{\overline{J_0}} : u \in B\}$. It is easily seen that χ_{PC} is the Hausdorff measure of noncompactness on $PC(J,E)$.

We assume the following conditions:

- (A1) $A : D(A) \subseteq E \to E$ is a linear closed (not necessarily bounded) operator generating a C_0-semigroup $\{T(t) : t \geq 0\}$ of bounded linear operators and there exists a $M \geq 1$ such that $\sup_{t\geq 0}\|T(t)\| \leq M$.
- (A2) for every $u \in E$, $t \longrightarrow F(t,u)$ is strongly measurable, and for almost every $t \in J$, $u \longrightarrow F(t,u)$ is upper semicontinuous.
- (A3) for any bounded subset Ω there exists a function $\varphi_\Omega \in L^P(J,\mathbb{R}^+)(P \geq 1)$ such that for any $u \in E$,

$$\|F(t,u)\| \leq \varphi_\Omega(t), \quad \forall u \in \Omega \text{ and for a.e. } t \in J.$$

(A4) there exists a function $\beta \in L^P(J,\mathbb{R}^+)(P \geq 1)$ satisfying
$$\chi(F(t,D)) \leq \beta(t)\chi(D), \quad \text{for a.e. } t \in J,$$
for every bounded subset $D \subseteq E$.

Lemma 2.6 ([24]). *Assume* (A1) *holds. Then we have*

- (i) *$K_1(t)$ and $K_2(t)$ are linear bounded operators for $t \geq 0$.*
- (ii) $\int_0^\infty \theta^\gamma \xi_\alpha(\theta)d\theta = \frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)}, \gamma \in [0,1]$.
- (iii) *$\|K_1(t)u\| \leq M\|u\|$ and $\|K_2(t)u\| \leq \frac{M}{\Gamma(\alpha)}\|u\|$ for any $u \in E$.*
- (iv) *$K_1(t)$ and $K_2(t)$ are strongly continuous for any $t \geq 0$.*
- (v) *$K_1(t)$ and $K_2(t)$ are compact if $T(t),\ t > 0$ is compact.*

Lemma 2.7. *Assume that* (A1) *holds. Let K be a compact subset of E. For any $s \in (0,\infty)$ and $t \in (s,\infty)$, one has*
$$\lim_{t\to s} \sup_{x\in K} \|T(t)x - T(s)x\| = 0.$$

Proof. Let $\epsilon > 0$. Note from the compactness of K, there exist $x_1, x_2, \dots, x_r$ such that $K \subseteq \cup\{B(x_i, \frac{\epsilon}{4M^2}) : i = 1,2,\dots,r\}$. Next, the strong continuity of $\{T(t) : t \geq 0\}$ implies that for any $x_i,\ i = 1,2,\dots,r$ there exits $\delta_i > 0$ such that for any $|t| < \delta_i$, $i = 1,2,\dots,r$ we have $\|T(t)x_i - T(0)x_i\| < \epsilon/(2M)$.

Put $\delta = \min\{\delta_i : i = 1,2,\dots,r\}$ and set $x \in K$. There exists $x_i,\ i = 1,2,\dots,r$ such that $\|x - x_i\| < \frac{\epsilon}{4M^2}$. For any $s \in (0,\infty)$ and $t \in (s, s+\delta)$ we have
$$\begin{aligned}
\|T(t)x - T(s)x\| &= \|T(s)(T(t-s)x - T(0)x))\| \\
&\leq \|T(s)\|\|T(t-s)x - T(0)x\| \\
&\leq M(\|T(t-s)x_i - T(0)x_i\| + \|T(t-s)x - T(t-s)x_i\| \\
&\quad + \|T(0)x_i - T(0)x\|) \\
&\leq \frac{\epsilon}{2} + 2M^2\|x - x_i\| < \epsilon.
\end{aligned}$$
The proof is complete. □

Lemma 2.8 ([16, Lemma 5.1.1, for $P > 1$], [22, for $P = 1$])). *The conditions* (A2)–(A4) *imply that the superposition multi-operator $P_F : C(J,E) \to P(L^P(J,E)),\ P \in [1,\infty),\ P_F(x) = S^P_{F(\cdot,x(\cdot))}$, generated by F is well defined, and is weakly closed in the following sense: if $\{x_n\}_{n=1}^\infty \subset C(J,E),\ \{f_n\}_{n=1}^\infty \subset L^P(J,E),\ f_n \in P_F(x_n)$, $n \geq 1$ are such that $x_n \to x$, $f_n \rightharpoonup f$ (weakly), then $f \in P_F(x)$.*

Lemma 2.9 ([18, Lemmas 3.4,3.5]). *Let $P \in [1,\infty)$ and $S : L^P(J,X) \to C(J,X)$ be an operator satisfying the conditions:*

(A5) *there exists a $\zeta \geq 0$ such that*
$$\|Sf(t) - Sh(t)\|_E \leq \zeta\Big(\int_0^t \|f(s) - h(s)\|^P ds\Big)^{1/p}, \quad t \in J, \tag{2.1}$$
for every $f, h \in L^P(J,E)$.

(A6) *for any compact $K \subseteq E$ and sequence $\{f_n\}_{n=1}^{+\infty} \subset L^P(J,E)$ such that for alln ≥ 1, $f_n(t) \in K$, a.e. $t \in J$, the weak convergence $f_n \rightharpoonup f_0$ in $L^P(J,E)$ implies the convergence $Sf_n \to Sf_0$.*

Then for every P-time semicompact set $\{f_n\}_{n=1}^{+\infty} \subset L^P(J,E)$ the set $\{Sf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(J,E)$. Moreover, if $(f_n)_{n\geq 1}$ converges weakly to f_0 in $L^P(J,E)$ then $Sf_n \to Sf_0$ in $C(J,E)$.

Definition 2.10. Let $P > 1/\alpha$, $\alpha \in (0,1)$. Then the operator $G : L^P(J,X) \to C(J,X)$ defined by

$$Gf(t) = \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s) ds, \tag{2.2}$$

is called the generalized Cauchy operator.

Lemma 2.11. *Let $P > 1/\alpha$, $\alpha \in (0,1)$. Then*

(i) *G satisfies property* (A5), *with $\zeta = \frac{M}{\Gamma(\alpha)} (\frac{P-1}{P\alpha-1})^{\frac{P-1}{P}} l^{\alpha-\frac{1}{P}}$.*
(ii) *if condition* (A1) *holds then G satisfies* (A6).

Proof. (i) The proof of the first assertion is exactly as in [18, Lemma 3.6]. (ii) To prove (A6) we note that for every compact $K \subseteq E$ the set

$$Q_t = \int_0^t (t-s)^{\alpha-1} K_2(t-s) K ds, \quad t \in J,$$

is relatively compact. Let $\{f_n\}_{n=1}^{+\infty} \subset L^P(J,E)$ be a sequence such that $\forall n \geq 1$, $f_n(t) \in K$, a.e. $t \in J$ and $f_n \rightharpoonup f_0$ in $L^P(J,E)$. Note that for $t \in J$,

$$\{Gf_n(t) : n \geq 1\} \subseteq Q_t.$$

Then $\{Gf_n(t) : n \geq 1\}$ is relatively compact for every $t \in J$. In order to apply the Arzela-Ascoli theorem we show that the set of functions $\{Gf_n(t) : n \geq 1\}$ is equi-continuous. Since $f_n(t) \in K$, for $n \geq 1$, and a.e. $t \in J$, there is a $N > 0$ such that

$$\|f_n(t)\| \leq N, \quad \text{for } n \geq 1 \text{ and a.e. } t \in J.$$

Let $n \geq 1$ be fixed and t_1, t_2 ($t_1 < t_2$) be two points in J. Then

$$\begin{aligned}
&\|Gf_n(t_2) - Gf_n(t_1)\| \\
&\leq \Big\| \int_0^{t_2} (t_2-s)^{\alpha-1} K_2(t_2-s) f_n(s) ds - \int_0^{t_1} (t_1-s)^{\alpha-1} K_2(t_1-s) f_n(s) ds \Big\| \\
&\leq I_1 + I_2 + I_3,
\end{aligned}$$

where

$$I_1 = \int_0^{t_1} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}| \|K_2(t_2-s) f_n(s)\| ds,$$

$$I_2 = \int_0^{t_1} (t_1-s)^{\alpha-1} \|K_2(t_1-s) f_n(s) - K_2(t_2-s) f_n(s)\| ds,$$

$$I_3 = \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \|K_2(t_2-s) f_n(s)\| ds.$$

Note that

$$\lim_{t_2 \to t_1} I_1 \leq \lim_{t_2 \to t_1} \frac{NM}{\Gamma(\alpha)} \int_0^{t_1} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}| ds = 0.$$

Then $\lim_{t_2 \to t_1} I_1 = 0$. Similarly, $\lim_{t_2 \to t_1} I_3 = 0$.

For I_2, it follows from Lemma 2.7, for any $s \in (0,\infty)$ and $t \in (s,\infty)$, that
$$\lim_{t\to s} \|T(t)x - T(s)x\| = 0,$$
independently of $x \in K$. Then, by applying the Lebesgue dominated convergence theorem, we obtain
$$\begin{aligned}
\lim_{t_2\to t_1} I_2 &\le \int_0^{t_1}\int_0^{\infty} \theta(t_1-s)^{\alpha-1}\zeta_\alpha(\theta)\Big[\lim_{t_2\to t_1}\|(T((t_2-s)^\alpha\theta) \\
&\quad - T(t_1-s)^\alpha\theta))f_n(s)\|\Big]d\theta ds \\
&= \int_0^{t_1}\int_0^{\infty} \theta(t_1-s)^{\alpha-1}\zeta_\alpha(\theta)\Big[\lim_{t_2\to t_1}\|T((t_1-s)^\alpha\theta)[(T((t_2-s)^\alpha\theta) \\
&\quad - (t_1-s)^\alpha\theta)) - T(0))f_n(s)\|\Big]d\theta ds \\
&\le M\int_0^{t_1}\int_0^{\infty} \theta(t_1-s)^{\alpha-1}\zeta_\alpha(\theta)\Big[\lim_{t_2\to t_1}\|(T((t_2-s)^\alpha\theta) \\
&\quad - (t_1-s)^\alpha\theta)) - T(0))f_n(s)\|\Big]d\theta ds = 0.
\end{aligned}$$
Then $\{Gf_n : n \ge 1\}$ is equicontinuous. The relative compactness of the set $\{Gf_n : n \ge 1\}$ follows from the Arzela-Ascoli theorem.

Now, condition (A5) implies that $G : L^P(J,X) \to C(J,X)$ is a linear bounded operator. Hence $f_n \rightharpoonup f_0$ in $L^P(J,E)$ implies the convergence $Sf_n \rightharpoonup Sf_0$ in $C(J,E)$. However the relative compactness of $\{Gf_n : n \ge 1\}$ implies that this last convergence is in the norm of the space $C(J,E)$. □

Applying Lemmas 2.9 and 2.11 we have the following corollary.

Corollary 2.12. *For every P-time semicompact set $\{f_n\}_{n=1}^{+\infty} \subset L^P(J,E)$ the set $\{Sf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(J,E)$. Moreover, if $(f_n)_{n\ge1}$ converges weakly to f_0 in $L^P(J,E)$ then $Sf_n \to Sf_0$ in $C(J,E)$.*

Lemma 2.13 ([7, Lemma 4])**.** *Let $\{f_n : n \in \mathbb{N}\} \subset L^P(J,E)$, $P \ge 1$ be an integrably bounded sequence such that*
$$\chi\{f_n : n \ge 1\} \le \gamma(t), \quad a.e.\ t \in J,$$
where $\gamma \in L^1(J,\mathbb{R}^+)$. Then for each $\epsilon > 0$ there exists a compact $K_\epsilon \subseteq E$, a measurable set $I_\epsilon \subset J$, with measure less than ϵ, and a sequence of functions $\{g_n^\epsilon\} \subset L^P(J,E)$ such that: $\{g_n^\epsilon(t) : n \ge 1\} \subseteq K_\epsilon$, for all $t \in J$ and
$$\|f_n(t) - g_n^\epsilon(t)\| < 2\gamma(t) + \epsilon, \quad \text{for every } n \ge 1 \text{ and every } t \in J - J_\epsilon.$$

Lemma 2.14 ([18, Lemma 3.9])**.** *Let the set of function $\{f_n\}_{n=1}^{\infty}$ be integrably bounded in $L^P(J,E)$ with the property $\chi(\{f_n(t) : n \ge 1\}) \le \eta(t)$, for a.e. $t \in J$, where $\eta(\cdot) \in L_+^P(J,\mathbb{R}^+)$. Then*
$$\chi(\{Gf_n(t) : n \ge 1\}) \le 2^{1+\frac{1}{P}}\zeta \int_0^t \eta(s)ds,$$
where ζ is the constant in relation (2.1).

The following fixed point theorems for multi-functions are crucial in the proof of our results.

Lemma 2.15 (Kakutani-Glicksberg-Fan theorem [16]). *Let W be a nonempty compact and convex subset of a locally convex topological vector space. If $\Phi : W \to P_{\mathrm{cl,cv}}(W)$ is an u.s.c. multi-function, then it has a fixed point.*

Lemma 2.16 ([16, Prop.3.5.1]). *Let W be a closed subset of E and $\Phi : W \to P_k(E)$ be a closed multi-function which is γ-condensing on every bounded subset of W, where γ is a monotone measure of noncompactness defined on E. If the set of fixed points for Φ is a bounded subset of E then it is compact.*

3. Nonemptyness and compactness of solution set

By the symbol $\Sigma_{u_0}^{F}[0,l]$, we denote the set of mild solutions to (1.1). In this section, we show that $\Sigma_{u_0}^{F}[0,l]$ is nonempty and compact in $PC(J,E)$. First we prove that $\Sigma_{u_0}^{F}[0,l]$ is nonempty.

Theorem 3.1. *Assume* (A1), (A2), (A4) *and the following conditions:*

- (A7) *there exists a function $\varphi \in L^P(J,\mathbb{R}^+)(P > \frac{1}{\alpha})$ such that for any $u \in E$*
$$\|F(t,u)\| \le \varphi(t)(1+\|u\|) \quad a.e.\ t \in J.$$
- (A8) *for every $i = 1,2,\dots,m$, $g_i : [b_i,a_i]\times E \to E$ is continuous and there exists a positive constant h_i such that*
$$\|g_i(t,u)\| \le h_i\|u\|, \quad t \in [b_i,a_i],\ u \in E.$$

Then the solution set of mild solutions of (1.1) *is nonempty provided that*
$$Mh + \zeta\|\varphi\|_{L^P(J,\mathbb{R}^+)} < 1, \quad h = \sum_{i=1}^{m} h_i. \tag{3.1}$$

Proof. From Lemma 2.8, the superposition multi operator
$$P_F : C(J,E) \to P(L^P(J,E)),$$
$$P_F(u) = S^P_{F(\cdot,u(\cdot))},$$
generated by F is well defined. Therefore, we can define a multi operator $\Phi : PC(J,E) \to P(PC(J,E))$ as follows: let $u \in PC(J,E)$, and a function $y \in \Phi(u)$ if and only if
$$y(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [0,b_1],\\ g_i(t,u(b_i^-)), & t \in (b_i,a_i],\ i = 1,2,\dots,m,\\ K_1(t-a_i)g_i(a_i,u(b_i^-)) & \\ +\int_{a_i}^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [a_i,b_{i+1}],\ i = 1,2,\dots,m, \end{cases} \tag{3.2}$$
where $f \in S^P_{F(\cdot,u(\cdot))}$.

It is clear that any fixed point for Φ is a mild solution for (1.1). We prove using Lemma 2.15 that Φ has a fixed point. We divide the proof into several steps.

Step 1. Φ is closed with compact values. Let $\{u_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ be two sequences in $PC(J,E)$ such that $u_n \to u$, $y_n \to y$ and
$$y_n(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f_n(s)ds, & t \in [0,b_1],\\ g_i(t,u_n(b_i^-)), & t \in (b_i,a_i],\ i = 1,2,\dots,m,\\ K_1(t-a_i)g_i(a_i,u_n(b_i^-)) & \\ +\int_{a_i}^t (t-s)^{\alpha-1}K_2(t-s)f_n(s)ds, & t \in [a_i,b_{i+1}],\ i = 1,2,\dots,m, \end{cases}$$

where $f_n \in S^P_{F(\cdot,u_n(\cdot))}$.

For $i = 0,1,2,\dots,m$, consider $G_i : L^p([a_i,b_{i+1}],E) \to C([a_i,b_{i+1}],E)$ defined by

$$G_i f(t) = \int_{s_i}^{t} (t-s)^{\alpha-1} K_2(t-s) f_n(s)ds. \tag{3.3}$$

As in Lemma 2.11 we see that $G_i (i = 0,1,2,\dots,m)$ satisfies (A5) and (A6). Because $u_n \to u$ in $PC(J,E)$ we can find a positive constant ω such that $\|u_n\|_{PC(J,E)} \le \omega$. Therefore, (A7) implies

$$\|f_n(t)\| \le \varphi(t)(1+\omega), \quad \text{a.e. } t \in J.$$

Then $\{f_n : n \ge 1\}$ is bounded in $L^P(J,E)$, and hence it is weakly compact and we may assume without generality that $f_n \rightharpoonup f_0$ in $L^P(J,E)$. Moreover, (A4) implies

$$\chi\{f_n(t) : n \ge 1\} \le \beta(t)\chi\{u_n(t) : n \ge 1\} = 0, \quad \text{a.e. } t \in J. \tag{3.4}$$

Lemma 2.9 implies that $G_i f_n \to G_i f_0$ in $C([a_i,b_{i+1}],E)$. Moreover the continuity of $g_i(t,u_n(b_i^-))$ implies that

$$\lim_{n\to\infty} g_i(t,u_n(b_i^-)) = g_i(t,u(b_i^-)), \quad t \in (b_i,a_i].$$

Then $y_n \to z$ in $PC(J,E)$, where

$$z(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f_0(s)ds, & t \in [0,b_1], \\ g_i(t,u(b_i^-)), & t \in (b_i,a_i],\ i = 1,2,\dots,m, \\ K_1(t-a_i) g_i(a_i,u(b_i^-)) & \\ + \int_{a_i}^t (t-s)^{\alpha-1} K_2(t-s) f_0(s)ds, & t \in [a_i,b_{i+1}],\ i = 1,2,\dots,m. \end{cases} \tag{3.5}$$

It follows from this and the fact that P_F is weakly closed that $f_0 \in S^P_{F(\cdot,u(\cdot))}$. Therefore, $z \in \Phi(u)$. Since $y_n \to z$ in $PC(J,E)$ then $z = y$. This shows that Φ is closed.

To show that the values of Φ are compact let $u \in PC(J,E)$ and $y_n \in \Phi(u)$, $n \ge 1$. The same argument as above implies that $\{y_n : n \ge 1\}$ has a convergent subsequence. Thus, $\Phi(u)$ is relatively compact. Notice arguing as above we see that $\Phi(u)$ is closed. Then $\Phi(u)$ is compact.

Step 2. Φ is upper semicontinuous. Since Φ is closed with compact values, it is enough to show that Φ is quasicompact (see Definition 2.1). Let U be a compact subset in $PC(J,E)$ and $\{y_n\}_{n=1}^\infty$ be a sequence in $\Phi(U)$. Then there exists a sequence $\{u_n\}_{n=1}^\infty$ in U such that $y_n \in \Phi(u_n)$, $n \ge 1$. The compactness of U implies that we can assume without loss of generality that $u_n \to u$ in U. Let $f_n \in S^P_{F(\cdot,u_n(\cdot))}$ be such that

$$y_n(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f_n(s)ds, & t \in [0,b_1], \\ g_i(t,u_n(b_i^-)), & t \in (b_i,a_i],\ i = 1,2,\dots,m, \\ K_1(t-a_i) g_i(a_i,u_n(b_i^-)) & \\ + \int_{a_i}^t (t-s)^{\alpha-1} K_2(t-s) f_n(s)ds, & t \in [a_i,b_{i+1}],\ i = 1,2,\dots,m. \end{cases}$$

The same argument as above implies that we can assume without loss of generality that $f_n \rightharpoonup f_0$ in $L^P(J,E)$ and $y_n \to z$ in $PC(J,E)$, where z is given in (3.5) and $f_0 \in S^P_{F(\cdot,u(\cdot))}$. Therefore $\{y_n : n \ge 1\}$ converges to an element in $\Phi(u)$.

Step 3. Let $B_r = B(0,r) = \{u \in PC(J,E) : \|u\| \le r\}$, where

$$r = \frac{M\|u_0\| + \zeta\|\varphi\|_{L^p(J,\mathbb{R}^+)}}{1 - [Mh + \zeta\|\varphi\|_{L^p(J,\mathbb{R}^+)}]}. \tag{3.6}$$

Note that from (3.1), r is well defined. Obviously, B_r is a bounded, closed and convex subset of $PC(J,E)$. We claim that $\Phi(B_r) \subseteq B_r$. Let $u \in B_r$ and $y \in \Phi(u)$. By using (3.2), (A7) and Hölder's inequality we obtain for $t \in [0, b_1]$,

$$\begin{aligned}\|y(t)\| &\le M\|u_0\| + \frac{M}{\Gamma(\alpha)}(1+r)\int_0^t (t-s)^{\alpha-1}\varphi(s)ds \\ &\le M\|u_0\| + \zeta(1+r)\|\varphi\|_{L^p(J,\mathbb{R}^+)}.\end{aligned}$$

If $t \in (b_i, a_i]$, $i = 1, 2, \dots, m$, then

$$\|y(t)\| \le \|g_i(t, u(b_i^-))\| \le h_i\|u(b_i^-)\| \le h_m r \le Mhr.$$

Similarly, we obtain for $t \in [a_i, b_{i+1}]$, $i = 1, 2, \dots, m$,

$$\begin{aligned}\|y(t)\| &\le K_1(t-a_i)g_i(a_i, u(b_i^-)) + \int_{a_i}^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds \\ &\le Mh\ r + \zeta(1+r)\|\varphi\|_{L^p(J,\mathbb{R}^+)}.\end{aligned}$$

Therefore,

$$\|y\|_{PC(J,E)} \le \ M(\|u_0\| + hr) + \zeta(1+r)\|\varphi\|_{L^p(J,\mathbb{R}^+)} \le r.$$

It follows that $\Phi(B_r) \subseteq B_r$. By applying Lemma 2.15, we see that there is a function $u \in PC(J,E)$ such that $u \in \Phi(u)$. Clearly the function u is a solution for (1.1). $\square$

In the following theorem we prove the compactness of $\Sigma_{u_0}^F[0,l]$.

Theorem 3.2. *Replace the conditions* (A1) *and* (A8) *in Theorem 3.1 with the following two conditions:*

(A9) *the C_0-semigroup $\{T(t) : t \ge 0\}$ is equicontinuous.*

(A10) *for every $i = 1, 2, \dots, m$, $g_i : [b_i, a_i] \times E \to E$ is uniformly continuous on bounded sets and for any $t \in [b_i, a_i]$, $g_i(t, \cdot)$ maps any bounded subset of E into a relatively compact subset of E and there exists a positive constant h_i such that*

$$\|g_i(t,u)\| \le h_i\|u\|, \quad t \in [b_i, a_i],\ u \in E.$$

Then $\Sigma_{u_0}^F[0,l]$ is compact provided that (3.1) *holds and*

$$4\zeta\|\beta\|_{L^p(J,\mathbb{R}^+)} < 1. \tag{3.7}$$

Proof. From Theorem 3.1, Φ is a closed multifunction from B_r to $P_{ck}(B_r)$, where r is given in (3.6). We will use Lemma 2.16 and we divide the proof into two steps.

Step 1. Set $B_1 = \overline{\text{conv}}\Phi(B_r)$ and $B_n = \overline{\text{conv}}\Phi(B_{n-1})$, $n \ge 2$. From Theorem 3.1, B_n is a nonempty, closed and convex subset of $PC(J,E)$. Moreover, $B_1 = \overline{\text{conv}}\Phi(B_r) \subseteq B_r$. Also $B_2 = \overline{\text{conv}}\Phi(B_1) \subseteq \overline{\text{conv}}\Phi(B_r) = B_1$. By induction, the sequence (B_n), $n \ge 1$ is a decreasing sequence of nonempty, closed and bounded subsets of $PC(J,E)$. Set $B = \cap_{n=1}^{\infty} B_n$. Notice that every B_n being bounded, closed and convex, B is also bounded closed and convex. We now show $\Phi(B) \subseteq B$.

Indeed, $\Phi(B) \subseteq \Phi(B_n) \subseteq \overline{\operatorname{conv}}\Phi(B_n) = B_{n+1}$, for every $n \geq 1$. Therefore, $\Phi(B) \subset \cap_{n=2}^{\infty} B_n$. On the other hand $B_n \subset B_1$ for every $n \geq 1$. Thus,

$$\Phi(B) \subset \cap_{n=2}^{\infty} B_n = \cap_{n=1}^{\infty} B_n = B \subseteq B_r.$$

We now show B is compact. According to the generalized Cantor's intersection property (see [16]) to ensure the compactness of B, it is enough to show that

$$\lim_{n\to\infty} \chi_{PC}(B_n) = 0, \tag{3.8}$$

where χ_{PC} is the Hausdorff measure of noncompactness on $PC(J, E)$.

First we verify that $Z|_{\overline{J_i}}$ is equicontinuous for every $i = 0, 1, 2, \dots, m$, where $Z = \Phi(B_r)$ and

$$Z|_{\overline{J_i}} = \{y^* \in C(\overline{J_i}, E) : y^*(t) = y(t), \quad t \in J_i,\ y^*(b_i) = y(b_i^+),\ y \in Z\}.$$

Let $y \in Z$. Then there is a $u \in B_r$ and $f \in S^P_{F(\cdot,u(\cdot))}$ such that

$$y(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s)ds, & t \in [0, b_1],\\ g_i(t, u(b_i^-)),\ t \in (b_i, a_i], & i = 1, 2, \dots, m,\\ K_1(t-a_i) g_i(a_i, u(b_i^-)) & \\ + \int_{a_i}^t (t-s)^{\alpha-1} K_2(t-s) f(s)ds, & t \in [a_i, b_{i+1}],\ i = 1, 2, \dots, m. \end{cases}$$

We consider the following cases:

Case 1. Let $i = 0$ and t, $t+\delta$ be two points in $\overline{J_0} = [0, b_1]$. Then

$$\begin{aligned} &\|y^*(t+\delta) - y^*(t)\|\\ &= \|y(t+\delta) - y(t)\|\\ &\leq \|K_1(t+\delta)(u_0) - K_1(t)(u_0)\| + \Big\| \int_0^{t+\delta} (t+\delta-s)^{\alpha-1} K_2(t+\delta-s) f(s))ds\\ &\quad - \int_0^t (t-s)^{\alpha-1} K_2(t-s) f(s)ds \Big\|\\ &= G_1 + G_2 + G_3 + G_4, \end{aligned}$$

where

$$\begin{gathered} G_1 = \|K_1(t+\delta)u_0 - K_1(t)u_0\|,\\ G_2 = \| \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} K_2(t+\delta-s) f(s)ds\|,\\ G_3 = \| \int_0^t \left[(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] K_2(t+\delta-s) f(s)ds\|,\\ G_4 = \| \int_0^t (t-s)^{\alpha-1} [K_2(t+\delta-s) - K_2(t-s)] f(s)ds\|. \end{gathered}$$

By Lemma 2.6, $K_1(t)$, $t \in J$ is strongly continuous, and hence $\lim_{\delta\to 0} G_1 = 0$. For G_2, note that by Lemma 2.6, (A7) and Hölder's inequality it follows that

$$\begin{aligned} \lim_{\delta\to 0} G_2 &= \lim_{\delta\to 0} \Big\| \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} K_2(t+\delta-s) f(s)ds \Big\|\\ &\leq \frac{M}{\Gamma(\alpha)} \lim_{\delta\to 0} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} \|f(s)\|ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M}{\Gamma(\alpha)}(r+1)\lim_{\delta\to 0}\int_t^{t+\delta}(t+\delta-s)^{\alpha-1}\varphi(s)ds\\
&\leq \frac{M}{\Gamma(\alpha)}(r+1)\lim_{\delta\to 0}\Big[\int_t^{t+\delta}(t+\delta-s)^{\frac{P}{P-1}}ds\Big]^{\frac{P-1}{P}}\|\varphi\|_{L^P_{(J,\mathbb{R}^+)}}=0.
\end{aligned}
$$

It is easy to show that

$$
\lim_{\delta\to 0}G_3\leq \frac{M(r+1)}{\Gamma(\alpha)}\lim_{\delta\to 0}\Big\|\int_0^t[(t+\delta-s)^{\alpha-1}-(t-s)^{\alpha-1}]\varphi(s)ds\Big\|=0.
$$

For G_4, the equicontinuity of $\{T(t):t\geq 0\}$ means that

$$
\lim_{t_2\to t_1}\|T(t_2)-T(t_1)\|=0,\ t_2,t_1\in(0,\infty).
$$

Therefore by the Lebesgue dominated convergence theorem,

$$
\begin{aligned}
\lim_{\delta\to 0}G_4\leq \int_0^t\int_0^\infty &\theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)\\
&\times\lim_{\delta\to 0}\|T((t+\delta-s)^\alpha\theta)-T((t-s)^\alpha\theta)\|[(1+r)\varphi(s)]d\theta ds=0.
\end{aligned}
$$

Case 2. Let $i=1$. If t and $t+\delta$ are two points in $(b_1,a_1]$ then invoking the definition of Φ and by the uniform continuity of g_1 on bounded sets it follows that

$$
\begin{aligned}
\lim_{\delta\to 0}\|y^*(t+\delta)-y^*(t)\|&=\lim_{\delta\to 0}\|y(t+\delta)-y(t)\|\\
&\leq\lim_{\delta\to 0}\|g_1(t+\delta,u(b_1^-))-g_1(t,u(b_1^-))\|=0,
\end{aligned}
$$

independently of x.

If t and $t+\delta$ are two points in $(a_1,b_2]$ then invoking the definition of Φ we have

$$
\begin{aligned}
\lim_{\delta\to 0}\|y^*(t+\delta)-y^*(t)\|&=\lim_{\delta\to 0}\|y(t+\delta)-y(t)\|\\
&\leq\|K_1(t+\delta-a_1)g_1(a_1,u(b_1^-))-K_1(t-a_1)g_1(a_1,u(b_1^-))\|\\
&\quad+\Big\|\int_{a_1}^{t+\delta}(t+\delta-s)^{\alpha-1}K_2(t+\delta-s)f(s)ds\\
&\quad-\int_{a_1}^{t+\delta}(t-s)^{\alpha-1}K_2(t+\delta-s)f(s)ds\Big\|.
\end{aligned}
$$

By arguing as in the first case we obtain $\lim_{\delta\to 0}\|y^*(t+\delta)-y^*(t)\|=0$.

If $t=b_1$ and $t+\delta\in(b_1,a_1]$ then

$$
\begin{aligned}
\|y^*(b_1+\delta)-y^*(b_1)\|&=\lim_{\sigma\to b_1^+}\|y(b_1+\delta)-y(\sigma)\|\\
&=\lim_{\sigma\to b_1^+}\|g_1(b_1+\delta,u(b_1^-))-g_1(\sigma,u(b_1^-))\|.
\end{aligned}
$$

Again by the uniform continuity of g_1, we obtain

$$
\lim_{\delta\to 0,\,\sigma\to b_1^+}\|y(b_1+\delta)-y(\sigma)\|=0.
$$

Therefore, $Z|_{\overline{J_1}}$ is equicontinuous. Similarly $Z|_{\overline{J_1}}$ is equicontinuous for $i=2,\dots,m$.

Next, let $n\geq 1$ be a fixed natural number and $\varepsilon>0$. In view of [9, Lemma 2.9], there exists a sequence (y_k), $k\geq 1$ in $\Phi(B_{n-1})$ such that

$$
\chi_{PC}\Phi(B_{n-1})\leq 2\chi_{PC}\{y_k:k\geq 1\}+\varepsilon.
$$

From the definition of χ_{PC}, the above inequality becomes

$$\chi_{PC}(B_n) = \chi_{PC}\Phi(B_{n-1}) \le 2 \max_{J= 0,1,\dots,m} \chi_i(S_{|\overline{J_i}}) + \varepsilon, \tag{3.9}$$

where $S = \{y_k : k \ge 1\}$ and χ_i is the Hausdorff measure of noncompactness on $C(\overline{J_i}, E)$. Since $B_{n|\overline{J_i}}$, $i = 0, 1, \dots, m$, is equicontinuous, then (see [12]),

$$\chi_i(S|_{\overline{J_i}}) = \sup_{t\in\overline{J_i}} \chi(S(t)).$$

Therefore, by using the nonsingularity of χ, (see [12]) we have

$$\begin{aligned} \chi_{PC}(B_n) &\le 2 \max_{i=0,1,\dots,m} [\sup_{t\in\overline{J_i}} \chi(S(t))] + \epsilon \\ &= 2 \sup_{t\in J} \chi(S(t)) + \varepsilon \\ &= 2 \sup_{t\in J} \chi\{y_k(t) : k \ge 1\} + \varepsilon. \end{aligned} \tag{3.10}$$

Now, since $y_k \in \Phi(B_{n-1})$, $k \ge 1$ there is $x_k \in B_{n-1}$ such that $y_k \in \Phi(x_k)$, $k \ge 1$. By recalling the definition of Φ for every $k \ge 1$ there is $f_k \in S^P_{F(\cdot,x_k(\cdot))}$ such that for every $t \in J$,

$$\begin{aligned} &\chi\{y_k(t) : k \ge 1\} \\ &\le \begin{cases} \chi\Big\{ \int_0^t (t-s)^{\alpha-1} K_2(t-s) f_k(s) ds : k \ge 1\Big\}, & \text{if } t \in [0, b_1], \\ \chi\Big\{ g_i(t, x_k(b_i^-)) : k \ge 1\Big\}, & \text{if } t \in (b_i, a_i],\ i = 1, 2, \dots, m, \\ \chi\{K_1(t-a_i) g_i(a_i, x_k(b_i^-)) : k \ge 1\} \\ +\chi\Big\{ \int_{a_i}^t (t-s)^{\alpha-1} K_2(t-s) f_k(s) ds : k \ge 1\Big\}, & \text{if } t \in [a_i, b_{i+1}],\ i = 1, 2, \dots, m. \end{cases} \end{aligned} \tag{3.11}$$

Note that, by the compactness of g_i, $i = 1, 2, \dots, m$, we obtain

$$\chi\{g_i(t, x_k(b_i^-)) : k \ge 1\} = 0. \tag{3.12}$$

Notice that from (A4) we have for a.e. $t \in J$ that

$$\begin{aligned} \chi(\{f_k(t) : k \ge 1\} &\le \chi\{F(s, x_k(t)) : k \ge 1\} \\ &\le \beta(t)\chi\{x_k(t) : k \ge 1\} \\ &\le \beta(t)\chi(B_{n-1}(t)) \\ &\le \beta(t)\chi_{PC}(B_{n-1}) = \gamma(t). \end{aligned} \tag{3.13}$$

Furthermore, by (A7), for any $k \ge 1$, and for almost $t \in J$, we have $\|f_k(t)\| \le \varphi(t)(r + 1)$. Consequently, $f_k \in L^P(J, E)$, $k \ge 1$, and hence $\gamma \in L^P(J, \mathbb{R}^+)$. Then, from Lemma 2.13, there exist a compact $K_\epsilon \subseteq E$, a measurable set $J_\epsilon \subset J$, with measure less than ϵ, and a sequence of functions $\{z_k^\epsilon\} \subset L^P(J, E)$ such that for all $s \in J$, $\{z_k^\epsilon(s) : k \ge 1\} \subseteq K$ and

$$\|f_k(s) - z_k^\epsilon(s)\| < 2\gamma(s) + \epsilon, \quad \text{for every } k \ge 1 \text{ and every } s \in J - J_\epsilon. \tag{3.14}$$

Let $J_0 = [0, t_1]$. Then, using (3.14) and Hölder's inequality, it follows for $k \geq 1$,

$$\begin{aligned}
&\| \int_{J_0 - J_\epsilon} (t-s)^{\alpha-1} K_2(t-s)(f_k(s) - z_k^\epsilon(s)) ds \| \\
&\leq \frac{M}{\Gamma(\alpha)} \| \int_{J_0 - J_\epsilon} (t-s)^{\alpha-1} (f_k(s) - z_k^\epsilon(s)) ds \| \\
&\leq \frac{M}{\Gamma(\alpha)} \| f_k - z_k^\epsilon \|_{L^P(J_0 - J_\epsilon, E)} \Big(\int_{J_0 - J_\epsilon} (t-s)^{\frac{P}{P-1}} ds \Big)^{\frac{P-1}{P}} \\
&\leq \zeta \| f_k - z_k^\epsilon \|_{L^P(J_0 - J_\epsilon, E)} \\
&\leq \zeta (2 \|\gamma\|_{L^P(J_0 - J_\epsilon, \mathbb{R}^+)} + \epsilon l^{1/p}) \\
&= \zeta (2 \|\beta\|_{L^P(J, \mathbb{R}^+)} \chi_{PC}(B_{n-1}) + \epsilon l^{1/p}),
\end{aligned} \tag{3.15}$$

Also, by Hölder's inequality, for $k \geq 1$, we obtain

$$\begin{aligned}
&\| \int_{J_\epsilon} (t-s)^{\alpha-1} K_2(t-s) f_k(s) ds \| \\
&\leq \frac{M}{\Gamma(\alpha)} (r+1) \int_{J_\epsilon} (t-s)^{\alpha-1} \varphi(s) ds \\
&\leq \frac{M}{\Gamma(\alpha)} (r+1) \|\varphi\|_{L^P(J_\epsilon, \mathbb{R}^+)} \Big(\int_{J_\epsilon} (t-s)^{\frac{P}{P-1}} ds \Big)^{\frac{P-1}{P}}.
\end{aligned}$$

From this inequality, (3.13) and (3.15), for $t \in [0, t_1]$, it follows that

$$\begin{aligned}
&\chi\Big(\Big\{ \int_0^t (t-s)^{\alpha-1} K_2(t-s) f_k(s) ds : k \geq 1 \Big\} \Big) \\
&\leq \chi\Big(\{ \int_{J_0 - J_\epsilon} (t-s)^{\alpha-1} K_2(t-s) f_k(s) ds : k \geq 1 \Big\} \Big) \\
&\quad + \chi\Big(\Big\{ \int_{J_\epsilon} (t-s)^{\alpha-1} K_2(t-s) f_k(s) ds : k \geq 1 \Big\} \Big) \\
&\leq \chi\Big(\Big\{ \int_{J_0 - J_\epsilon} (t-s)^{\alpha-1} (f_k(s) - z_k^\epsilon(s)) ds : k \geq 1 \Big\} \Big) \Big\} \\
&\quad + \chi\Big(\Big\{ \int_{J_0 - J_\epsilon} (t-s)^{\alpha-1} z_k^\epsilon(s) ds : k \geq 1 \Big\} \Big) \Big\} \\
&\quad + \chi\Big(\Big\{ \int_{J_\epsilon} (t-s)^{\alpha-1} f_k(s) ds : k \geq 1 \Big\} \Big) \\
&\leq \zeta \ (2 \|\beta\|_{L^P(J, \mathbb{R}^+)} \ \chi_{PC}(B_{n-1}) + \epsilon l^{1/p}) \\
&\quad + \frac{M}{\Gamma(\alpha)} (r+1) \|\varphi\|_{L^P(J_\epsilon, \mathbb{R}^+)} \Big(\int_{J_\epsilon} (t-s)^{\frac{P}{P-1}} ds \Big)^{\frac{P-1}{P}}.
\end{aligned}$$

Taking into account that ε is arbitrary, the above inequality becomes

$$\chi\Big(\Big\{ \int_0^t (t-s)^{\alpha-1} K_2(t-s) f_k(s) ds : k \geq 1 \Big\} \Big) \leq 2\zeta \|\beta\|_{L^P(J, \mathbb{R}^+)} \chi_{PC}(B_{n-1}). \tag{3.16}$$

Similarly, we can show that if $t \in [a_i, b_{i+1}]$, $i = 1, 2, \dots, m$, then

$$\chi\Big(\Big\{ \int_{a_i}^t (t-s)^{\alpha-1} K_2(t-s) f_k(s) ds : k \geq 1 \Big\} \Big) \leq 2\zeta \|\beta\|_{L^P(J, \mathbb{R}^+)} \chi_{PC}(B_{n-1}). \tag{3.17}$$

Then, by (3.11), (3.12), (3.16) and (3.17) for every $t \in J$,

$$\chi\{y_k(t) : k \geq 1\} \leq 4\zeta\|\beta\|_{L^p(J,\mathbb{R}^+)}\chi_{PC}(B_{n-1}).$$

From this inequality, (3.10) and the fact that ε is arbitrary it follows that

$$\chi_{PC}(B_n) \leq 4\zeta\|\beta\|_{L^p(J,\mathbb{R}^+)}\chi_{PC}(B_{n-1}).$$

From a finite number of steps, we obtain

$$0 \leq \chi_{PC}(B_n) \leq (4\zeta\|\beta\|_{L^p(J,\mathbb{R}^+)})^{n-1}\chi_{PC}(B_1), \ \forall n \geq 1.$$

Since this inequality is true for every $n \in \mathbb{N}$, then by (3.7) and by passing to the limit as $n \to +\infty$, we obtain (3.8). Hence B is a nonempty and compact subset of $PC(J, E)$. Then $\Phi : B \to P_{ck}(B)$ is compact.

Step 2. The set of fixed points of Φ is a bounded subset of $PC(J, E)$. Let $u \in \Phi(u)$, $u \in B$ and $f \in S^P_{F(\cdot,u(\cdot))}$ such that

$$u(t) = \begin{cases} K_1(t)u_0 + \int_0^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [0, b_1], \\ g_i(t, u(b_i^-)), & t \in (b_i, a_i], \ i = 1, 2, \dots, m, \\ K_1(t-a_i)g_i(a_i, u(b_i^-)) \\ + \int_{a_i}^t (t-s)^{\alpha-1}K_2(t-s)f(s)ds, & t \in [a_i, b_{i+1}], \ i = 1, 2, \dots, m. \end{cases}$$

By arguing as in Step 3 in Theorem 3.1 we can show that $\|u\|_{PC(J,E)} \leq r$, where r is given in (3.6). Applying Lemma 2.16 and we conclude that $\Sigma^F_{u_0}[0, l]$ is compact in $PC(J, E)$. □

4. An example

In this section, we give an example to illustrate our results. Let $J = [0, 1]$, K be a non-empty convex compact subset in a Banach space E and $F : J \times E \to P_{ck}(E)$ be a multi-valued function defined by

$$F(t, u) = \frac{e^{-\gamma t}\|u\|}{\lambda(1 + \|u\|)}K, \tag{4.1}$$

where $\gamma \in (1, \infty)$ and λ is a constant such that $\sup\{\|z\| : z \in K\} \leq \lambda$. Clearly for every $u \in E, t \to F(t, u)$ is measurable. Moreover, for any $u, v \in E$ and any $t \in J$, we have

$$H(F(t, u), F(t, v)) \leq e^{-\gamma t}\|u - v\|.$$

Then, for almost all $t \in J$, $u \to F(t, u)$ is upper semicontinuous and so (A2) holds. Moreover, for every bounded subset $D \subseteq E, \chi(F(t, D)) \leq \beta(t)\chi(D)$ for a.e. $t \in J$, holds with $\beta(t) = e^{-\gamma t}$. Therefore (A4) is satisfied. Also, for any $(t, u) \in J \times E$,

$$\|F(t, u)\| \leq e^{-\gamma t} \leq e^{-\gamma t}(1 + \|u\|).$$

Then(A7) is satisfied with $\varphi(t) = e^{-\gamma t}$.

Now, for any $i = 1, 2, \dots, m$, let $g_i : [t_i, s_i] \times E \to E$, defined by

$$g_i(t, u) = tu \tag{4.2}$$

Note that (A8) is satisfied.

Assume that $A : D(A) \subseteq E \to E$ is a linear closed operator generating a C_0-semigroup $\{T(t) : t \geq 0\}$ of bounded linear operator and there is $M \geq 1$, such that $\sup_{t\geq 0} \|T(t)\| \leq M$, see for example, [25, Example 3.12]. By applying Theorem 3.1, problem (1.1), where F and g_i, $i = 1, 2, \dots, m$, are given by (4.1) and (4.2), has a mild solution, provided that

$$Mh + \zeta \|\varphi\|_{L^P(J,\mathbb{R}^+)} < 1,$$

where $h = lm$, $\zeta = \frac{M}{\Gamma(\alpha)}(\frac{P-1}{P\alpha-1})^{\frac{P-1}{P}} l^{\alpha-\frac{1}{P}}$ and $\varphi(t) = e^{-\gamma t}$, $t \in J$.

If, in addition, the C_0-semigroup $\{T(t) : t \geq 0\}$ is equicontinuous then, by Theorem 3.2 the solution set of (1.1) is compact provided that $4\zeta\|\beta\|_{L^P(J,\mathbb{R}^+)} < 1$, where $\beta(t) = e^{-\gamma t}, t \in J$.

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