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ASYMPTOTIC FORMULAE FOR SOLUTIONS TO IMPULSIVE DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT OF GENERALIZED TYPE

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Abstract. In this article we give some asymptotic formulae for impulsive differential system with piecewise constant argument of generalized type (abbreviated IDEPCAG). These formulae are based on certain integrability conditions, by means of a Grönwall-Bellman type inequality and the Banach's fixed point theorem. Also, we study the existence of an asymptotic equilibrium of nonlinear and semilinear IDEPCAG systems. We present examples that illustrate our the results.

1. Introduction

In the late 70's, Myshkis [35] noticed that there was no theory for differential equations with discontinuous argument of the form $x'(t) = f(t, x(t), x(h(t)))$, where $h(t)$ is a discontinuous argument, for example, $h(t) = [t]$. He called these equations *Differential equations with deviating argument*. The systematic study of problems, related to piecewise constant argument began in the early 80's with the works by Cooke, Wiener and Shah [41]. They called these type of equations *Differential equations with piecewise constant argument* (abbreviated DEPCA). A good source of this type of equations is [45]. Busenberg and Cooke [15] were the first ones to introduce a mathematical model that involved such types of deviated arguments in the study of models of vertically transmitted diseases, reducing their study to discrete equations. Since then, these equations have been studied by many researchers in diverse fields such as biomedicine, chemistry, biology, physics, population dynamics, and mechanical engineering; see [11, 31, 22].

Akhmet [2] considered the equation

$$x'(t) = f(t, x(t), x(\gamma(t))),$$

where $\gamma(t)$ is a piecewise constant argument of generalized type; that is, there exist $(t_k)_{k\in\mathbb{Z}}$ and $(\zeta_k)_{k\in\mathbb{Z}}$ such that $t_k < t_{k+1}$ for all $k \in \mathbb{Z}$ with $\lim_{k\to\pm\infty} t_k = \pm\infty$, $t_k \le \zeta_k \le t_{k+1}$ and $\gamma(t) = \zeta_k$ if $t \in I_k = [t_k, t_{k+1})$. These equations are called *Differential equations with piecewise constant argument of generalized type* (abbreviated DEPCAG). They have continuous solutions, even when $\gamma(t)$ is not. In the end of the constancy intervals they produce a recursive law, i.e., a discrete

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equation. That is the reason why these equations are called *hybrids*, because they combine discrete and continuous dynamics (see [37]).

In the DEPCAG case, when continuity at the endpoints of intervals of the form $I_k = [t_k, t_{k+1})$ is not considered, we have the *impulsive differential equations with piecewise constant argument of generalized type* (abbreviated IDEPCAG)

$$\begin{aligned} x'(t) &= f(t, x(t), x(\gamma(t))), \quad t \neq t_k \\ \Delta x(t_k) &= Q_k(x(t_k^-)), \quad t = t_k \end{aligned} \tag{1.1}$$

where $x(\tau) = x_0$; see [1, 38, 40, 42, 46].

The problem of convergence of solutions and asymptotic equilibrium seems to be studied for the very first time by Bôcher [9]. Wintner [47, 48, 49, 50, 51] and and Brauer [12, 13] studied the asymptotic equilibrium problem for the ODE case. Also, there are important contributions done by Cesari, Hallam, Levinson, Brighland, Trench and Atkinson, see [4, 14, 16, 28, 29, 33, 34, 43, 44] and the references therein. For applications in epidemics (transmission of Gonorrhea), population growth and physics (classical radiating electron) see the works by Cooke, Yorke and Yorke, Kaplan & M. Sorg [20, 30]. Also, the convergence problem has been widely investigated by many researchers for many types of equations. For example, delay functional differential equations were studied in [23, 25, 27, 30, 39], impulsive differential equations in [5, 26], and impulsive delayed and advanced differential equations in [6]. Pinto, Sepúlveda and Torres [38] studied the IDEPCAG system

$$\begin{gathered} y_i'(t) = -a_i(t)y_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}(t)g_j(y_j(\gamma(t))) + d_i(t), \quad t \neq t_k, \\ \Delta y_i(t_k) = -q_{i,k}y_i(t_k^-) + I_{i,k}(y_i(t_k^-)) + e_{i,k}, \quad t = t_k \end{gathered} \tag{1.2}$$

where $\gamma(t) = t_k$, if $t \in [t_k, t_{k+1}]$. The authors obtained some sufficient conditions for the existence, uniqueness, periodicity and stability of solutions for the impulsive Hopfield-type neural network system with piecewise constant arguments (1.2). By means of the Green function associated to (1.2), they established that (1.2) has a unique ω-periodic solution. Assuming some conditions, they also determined that the unique ω-periodic solution of (1.2) is globally asymptotically stable. Hence, a convergence to the unique ω-periodic solution was established.

Akhmet [3] studied the existence, uniqueness and the asymptotic equivalence of the system

$$\begin{gathered} x'(t) = Cx(t), \\ y'(t) = C(t)y(t) + f(t, y(t), y(\gamma(t))), \end{gathered}$$

where $x, y \in \mathbb{C}^n$, $t \in \mathbb{R}$, C is a constant $n \times n$ real valued matrix, $f \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ is a real valued $n \times 1$ function, and $\gamma(t) = \zeta_k$ if $t \in [t_k, t_{k+1})$, where $k \in \mathbb{Z}$. The author actually found an asymptotic formula that relates these two systems, i.e

$$z(t) = e^{Ct}[c + o(1)], \quad \text{as } t \to \infty,$$

where $c \in \mathbb{R}^n$ is a constant vector. Later, Pinto [36] studied the existence, uniqueness and the asymptotic equivalence of the systems

$$\begin{gathered} x'(t) = A(t)x(t), \\ y'(t) = A(t)y(t) + f(t, y(t), y(\gamma(t))), \end{gathered} \tag{1.3}$$

where $x, y \in \mathbb{C}^n$, $t \in \mathbb{R}$, A is a locally integrable $n \times n$ matrix in $\mathbb{R}^+$, $f : \mathbb{R}^+ \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is a continuous function and $\gamma(t) = \zeta_k$ if $t \in [t_k, t_{k+1})$, where $k \in \mathbb{Z}$. The author also found some asymptotic formulae that relates these two systems and the error considered, i.e.

$$y(t) = \Phi(t)[\nu + \epsilon(t)], \quad \text{as } t \to \infty.$$

where $\Phi(t)$ is the fundamental matrix of (1.3), $\nu \in \mathbb{C}^n$ is a constant vector and the error function ϵ is related with some conditions over f. Pinto et al. [21] considered the systems

$$x'(t) = A(t)x(t), \tag{1.4}$$
$$z'(t) = A(t)z(t) + B(t)z(\gamma(t)), \tag{1.5}$$
$$u'(t) = B(t)u(\gamma(t)), \tag{1.6}$$
$$y'(t) = A(t)y(t) + B(t)y(\gamma(t)) + g(t), \tag{1.7}$$
$$w'(t) = A(t)w(t) + B(t)w(\gamma(t)) + f(t, w(t), w(\gamma(t))), \tag{1.8}$$
$$v'(t) = A(t)v(t) + B(t)v(\gamma(t)) + g(t) + f(t, v(t), v(\gamma(t))), , \tag{1.9}$$

and they proved that if the linear DEPCAG system (1.5) has an ordinary dichotomy and in (1.9) f is integrable, then there exists a homeomorphism between the bounded solutions of the linear system (1.7) and the bounded solutions of the quasilinear system (1.9). Moreover, $|y(t) - v(t)| \to 0$, as $t \to \infty$ if $Z(t, 0)P \to 0$ as $t \to \infty$, where $Z(t, s)$ is the fundamental matrix of the DEPCAG linear system (1.5) and P is a projection matrix. Also, (1.8) has an asymptotic equilibrium. Chiu [19], inspired by [36, 37], studied the asymptotic equivalence between the following linear DEPCAG system and its perturbed system

$$\begin{gathered} x'(t) = A(t)x(t) + B(t)x(\gamma(t)), \\ y'(t) = A(t)y(t) + B(t)y(\gamma(t)) + f(t, y(t), y(\gamma(t))), \end{gathered} \tag{1.10}$$

where $x, y \in \mathbb{C}^n$, $t \in \mathbb{R}$, A, B are locally integrable $n \times n$ matrices in $\mathbb{R}^+$, $f : \mathbb{R}^+ \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is a continuous function and $\gamma(t) = \zeta_k$ if $t \in [t_k, t_{k+1})$, where $k \in \mathbb{Z}$. He found the asymptotic formula

$$x(t) = \Psi(t)[\nu + \epsilon(t)] \quad \text{as } t \to \infty,$$

where $\Psi(t)$ is the fundamental matrix of the lineal DEPCAG system (1.10), $\nu \in \mathbb{C}^n$ and $\epsilon(t)$ is the error function. We note that there is no literature about the IDEPCAG case, so this paper tries to fill the gap in this context.

2. Scope

In this work we will conclude the existence of an Asymptotic Equilibrium for the class of IDEPCAG systems of fixed times. In other words, we prove strongly based on certain integrability conditions, Grönwall-Bellman type inequality and the Banach's fixed point theorem, that every solution of (1.1) with initial condition $x(a) = x_0$ where $a \geq \tau$ satisfies

$$\lim_{t \to \infty} x(t) = \xi,$$

for some $\xi \in \mathbb{C}^n$, and has the asymptotic formulae

$$x(t) = \xi + \mathcal{O}\Big(\sum_{i=1}^{3}\int_t^\infty \lambda_i(s)ds + \sum_{t\le t_k<\infty}(\mu_k^1 + \mu_k^2)\Big). \tag{2.1}$$

where λ and μ are Lipschitz constants related to f and Q_k respectively. These results extend the works by González and Pinto [26] for the IDE case, and the one done by Pinto [36] for the DEPCAG case. Indeed, [36] was taken as the principal reference in the subject for the present work. Also, as a consequence of the existence of an asymptotic equilibrium for system (1.1), we will study the existence of an asymptotic equilibrium for the semilinear system

$$\begin{gathered} y'(t) = A(t)y(t) + f(t, y(t), y(\gamma(t))), \quad t \neq t_k \\ \Delta y(t_k) = J_k y(t_k^-) + I_k(y(t_k^-)), \quad t = t_k,\ k \in \mathbb{N} \end{gathered} \tag{2.2}$$

by some conditions on the coefficients involved concluding aymptotic formulae for unbounded solutions. Thus, any solution $y(t)$ of (2.2) satisfies the asymptotic formula

$$x(t) = \Phi(t)(\xi + \epsilon(t)), \quad \text{as } t \to \infty \tag{2.3}$$

where $\Phi(t)$ is the fundamental matrix of the impulsive linear system

$$\begin{gathered} x'(t) = A(t)x(t), \quad t \neq t_k \\ \Delta x(t_k) = J_k x(t_k^-), \quad t = t_k,\ k \in \mathbb{N} \end{gathered} \tag{2.4}$$

$\xi \in \mathbb{C}^n$ is a constant vector and the error $\epsilon(t)$ satisfies

$$\epsilon(t) = \mathcal{O}((\exp(\int_t^\infty \eta(s)ds) - 1) + \sum_{t<t_k}(1 + \eta_3(t_k))),$$

where $\eta(t)$ and $\eta_3(t_k)$ are Lipschitz constants related to f and I_k respectively. Moreover, if $\epsilon_0(t) \to 0$ as $t \to \infty$, where

$$\epsilon_0(t) = \int_t^\infty |\Phi(t,s)\|\Phi(s)|(\lambda_1(s) + |\Phi^{-1}(\gamma(s),s)|\lambda_2(s))ds + \sum_{t<t_k}|\Phi(t,t_k)\|\Phi(t_k^-)|\widetilde{\mu}_k$$

Equations (2.2) and (2.4) are asymptotically equivalent; i.e., they share the same asymptotic behavior, and

$$y(t) = x(t) + \epsilon_0(t), \quad \epsilon_0(t) \to 0 \quad \text{as } t \to \infty.$$

This asymptotic relationship includes the case of unbounded solutions. An example of a second order IDEPCAG will be shown.

3. Main assumptions

In this section we present the main hypothesis that will be used in the rest of this work. Let $|\cdot|$ be a suitable norm, $\|\cdot\|_\infty$ be the supremum norm, $f : [0,\infty[\times\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ and $Q_k : \{t_k\} \to \mathbb{C}^n$ be continuous function satisfying:

(H1) (a) There exist integrable functions $\lambda_i(t), i = 1,2,3$ on $I = [\tau,\infty)$ such that for all $(t, x(t), x(\gamma(t))) \in I \times \mathbb{C}^n \times \mathbb{C}^n$ we have

$$|f(t, x(t), x(\gamma(t)))| \le \lambda_1(t)|x(t)| + \lambda_2(t)|x(\gamma(t))| + \lambda_3(t),$$

(b) There exist a summable sequences of non-negative numbers $(\mu_k^i)_{k=1}^\infty$ with $i = 1,2$ such that for each $x \in \mathbb{C}^n$ we have

$$|Q_k(x(t_k^-))| \le \mu_k^1|x(t_k^-)| + \mu_k^2, \quad \forall k \in \mathbb{N}.$$

(H2) (a) The function $f(t,0,0)$ is integrable on I and there exist integrable functions $\lambda_1(t),\lambda_2(t)$ on I such that for all $(t,x(t),x(\gamma(t)))$, $(t,y(t),y(\gamma(t)))$ in $I\times\mathbb{C}^n\times\mathbb{C}^n$, we have

$$\begin{aligned}&|f(t,x(t),x(\gamma(t)))-f(t,y(t),y(\gamma(t)))|\\&\le \lambda_1(t)|x(t)-y(t)|+\lambda_2(t)|x(\gamma(t))-y(\gamma(t))|,\end{aligned}$$

(b) The function $Q_k(0)$ is summable on I and there exists a summable sequence of non-negative real numbers $(\widetilde{\mu}_k)_{k=1}^{\infty}$ such that for all $x,y\in\mathbb{C}^n$, we have

$$|Q_k(x(t_k^-))-Q_k(y(t_k^-))|\le \widetilde{\mu}_k|x(t_k^-)-y(t_k^-)|,\quad \forall k\in\mathbb{N}.$$

(H3) The functions $\lambda_1(t),\lambda_2(t)$ also satisfy

$$\nu_k=\int_{t_k}^{\zeta_k}(\lambda_1(s)+\lambda_2(s))ds\le \nu:=\sup_{k\in\mathbb{N}}\nu_k<1.$$

(H4) Let the following conditions are satisfied

$$\eta_1(t)=|\Phi(t)|\,|\Phi^{-1}(t)|\lambda_1(t), \tag{3.1}$$

$$\eta_2(t)=|\Phi^{-1}(t,\gamma(t))|\|\Phi^{-1}(t)\|\Phi(t)|\lambda_2(t)\in L_1(I) \tag{3.2}$$

$$\eta_3(t_k)=|\Phi(t_k^-)|\,|\Phi^{-1}(t_k)|\widetilde{\mu}_k\in l_1(I). \tag{3.3}$$

where $\Phi(t)$ is the fundamental matrix of the impulsive linear system (2.4)

4. Preliminaries

In the following, we give the definition of a IDEPCAG solution for (1.1).

Definition 4.1. A function $y(t)$ is a solution of IDEPCAG (1.1) if

(i) $y(t)$ is continuous in every interval of the form $I_k=[t_k,t_{k+1})$ for all $k\in\mathbb{N}$;
(ii) The derivative $y'(t)$ exists at each point $t\in I=[\tau,\infty)$ with the exception of the points t_k, $k\in\mathbb{N}$, where the left derivative exists;
(iii) On each interval I_k, the ordinary differential equation

$$x'(t)=f(t,x(t),x(\zeta_k))$$

is satisfied, where $\gamma(t)=\zeta_k$ for all $t\in I_k$;
(iv) For $t=t_k$, the solution satisfies the jump condition

$$\Delta x(t_k)=x(t_k)-x(t_k^-)=Q_k(x(t_k^-)),$$

where $x(t_k^-)=\lim_{t\to t_k,\,t<t_k}x(t)$ exists for all t_k with $k\in\mathbb{N}$ and $x(t_k^+)=x(t_k)$ is defined by

$$x(t_k)=x(t_k^-)+Q_k(x(t_k^-)).$$

The following lemma is the main tool of the rest of this work; it presents an integral equation associated with (1.1).

Lemma 4.2. *A function $x(t)=x(t,\tau,x_0)$, where τ is a fixed real number, is a solution of* (1.1) *on $\mathbb{R}^+$ if and only if it satisfies the integral equation*

$$x(t)=x_0+\int_{\tau}^{t}f(s,x(s),x(\gamma(s)))ds+\sum_{\tau\le t_k<t}Q_k(x(t_k^-)),\quad \text{on } \mathbb{R}^+. \tag{4.1}$$

Proof. Consider the interval $I_n = [t_n, t_{n+1}]$. If we integrate (4.1) on this interval it follows that

$$x(t) = x(t_n) + \int_{t_n}^{t} f(s, x(s), x(\zeta_n))ds, \tag{4.2}$$

where $\gamma(t) = \zeta_n$ for all $t \in I_n = [t_n, t_{n+1})$. Then, evaluating in $t = t_{n+1}$ we obtain

$$x(t_{n+1}^-) = x(t_n) + \int_{t_n}^{t_{n+1}} f(s, x(s), x(\zeta_n))ds$$

Applying the impulsive condition $\Delta x(t_{n+1}) = x(t_{n+1}) - x(t_{n+1}^-) = Q_{n+1}(x(t_{n+1}^-))$ it follows that

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(s, x(s), x(\zeta_n))ds + Q_{n+1}(x(t_{n+1}^-)).$$

Then, solving the finite difference equation we obtain

$$x(t_n) = x_0 + \sum_{k=i[\tau]}^{n-1} \int_{t_k}^{t_{k+1}} f(s, x(s), x(\zeta_k))ds + \sum_{k=i[\tau]}^{n} Q_k(x(t_k^-)),$$

where $i[t] = n \in \mathbb{Z}$ is the only integer such that $t \in I_n = [t_n, t_{n+1}[$. Next, applying last expression in (4.2) we obtain

$$\begin{aligned} x(t) = x_0 &+ \sum_{k=i[\tau]}^{n-1} \int_{t_k}^{t_{k+1}} f(s, x(s), x(\zeta_k))ds \\ &+ \sum_{\tau \le t_k < t} Q_k(x(t_k^-)) + \int_{t_n}^{t} f(s, x(s), x(\zeta_n))ds. \end{aligned}$$

Finally, defining

$$\int_{\tau}^{t} f(s, x(s), x(\gamma(s)))ds = \sum_{k=i[\tau]}^{n-1} \int_{t_k}^{t_{k+1}} f(s, x(s), x(\zeta_k))ds + \int_{t_n}^{t} f(s, x(s), x(\zeta_n))ds,$$

and replacing it in the last expression we obtain (4.1), so the proof is complete. □

The next lemma provides a Grönwall-Bellman type inequality for the IDEPCAG case. Its proof of is almost identical to the proof of [36, Lemma 2.2] with slight changes because of the impulsive effect and can be found in [42].

Lemma 4.3. *Let I be an interval and u, η_1, η_2 be three functions from $I \subset \mathbb{R}$ to $\mathbb{R}_0^+$ such that u is continuous; η_1, η_2 are locally integrable and $\eta_3 : \{t_k\} \to \mathbb{R}_0^+$. Let $\gamma(t)$ be a piecewise constant argument of generalized type, i.e. a step function such that $\gamma(t) = \zeta_k$ for all $t \in I_k = [t_k, t_{k+1})$, with $t_k \le \zeta_k < t_{k+1}$ for all $k \in \mathbb{N}$ satisfying* (H3) *and*

$$u(t) \le u(\tau) + \int_{\tau}^{t} (\eta_1(s)u(s) + \eta_2(s)u(\gamma(s)))ds + \sum_{\tau \le t_k < t} \eta_3(t_k)u(t_k^-). \tag{4.3}$$

Then

$$u(t) \le \Big(\prod_{\tau \le t_k < t} (1 + \eta_3(t_k))\Big) \exp\Big(\int_{\tau}^{t} \eta(s)ds\Big) u(\tau), \tag{4.4}$$

$$u(\zeta_k) \le (1 - \nu)^{-1} u(t_k), \tag{4.5}$$

$$u(\gamma(t)) \leq (1-\nu)^{-1}\Big(\prod_{\tau \leq t_k < t}(1+\eta_3(t_k))\Big)\exp\Big(\int_\tau^t \eta(s)ds\Big)u(\tau), \tag{4.6}$$

where $\eta(t) = \eta_1(t) + \eta_2(t)(1-\nu)^{-1}$ *for* $t \geq \tau$.

Proof. To prove (4.4), we denote the right-hand side of (4.3) by $v(t)$. Then we have $u(\tau) \leq v(\tau)$, so $u(t) \leq v(t)$, for $t \geq \tau$, because $v(t)$ is increasing. Now, differentiating $v(t)$ we obtain

$$v'(t) = \eta_1(t)u(t) + \eta_2(t)u(\gamma(t)).$$

Then, we have

$$v'(t) \leq \eta_1(t)v(t) + \eta_2(t)v(\gamma(t)).$$

Integrating the last expression between τ and t we obtain

$$v(t) - v(\tau) \leq \int_\tau^t \eta_1(s)v(s) + \eta_2(s)v(\gamma(s))ds, \tag{4.7}$$

Now, when we consider $\tau = t_k$ and $t = \zeta_k$ in (4.7), we have

$$\begin{aligned} v(\zeta_k) - v(t_k) &\leq \int_{t_k}^{\zeta_k} \eta_1(s)v(s) + \eta_2(s)v(\gamma(s))ds \\ &\leq v(\zeta_k)\int_{t_k}^{\zeta_k}(\eta_1(s)+\eta_2(s))ds. \end{aligned}$$

Then, because $\nu < 1$, we have

$$v(\zeta_k) \leq (1-\nu)^{-1}v(t_k), \tag{4.8}$$

so, (4.5) is proved. Applying (4.8) in (4.7) for $\tau = t_k$ and $t \in I_k$, we have

$$v(t) - v(t_k) \leq \int_{t_k}^t \eta_1(s)v(s) + (1-\nu)^{-1}\eta_2(s)v(t_k)ds \leq \int_{t_k}^t \eta(s)v(s)ds$$

Hence,

$$v(t) \leq v(t_k) + \int_{t_k}^t \eta(s)v(s)ds. \tag{4.9}$$

Now, applying the classical Bellman-Grönwall lemma to the last inequality we have

$$v(t) \leq v(t_k)\exp\Big(\int_{t_k}^t \eta(s)ds\Big).$$

Next, evaluating the above expression for $t = t_{k+1}^-$, we have

$$v(t_{k+1}^-) \leq v(t_k)\exp\Big(\int_{t_k}^{t_{k+1}} \eta(s)ds\Big). \tag{4.10}$$

Now, applying the impulsive condition we obtain

$$\begin{aligned} v(t_{k+1}) &\leq (1+\eta_3(t_{k+1}))v(t_{k+1}^-) \\ &\leq (1+\eta_3(t_{k+1}))v(t_k)\exp\Big(\int_{t_k}^{t_{k+1}} \eta(s)ds\Big). \end{aligned}$$

This expression defines a finite difference inequality, which has solution satisfying

$$\nu(t_k) \leq \Big(\prod_{j=i[\tau]}^{i[k]-1}(1+\eta_3(t_{j+1}))\exp\Big(\int_{t_j}^{t_{j+1}} \eta(s)ds\Big)\Big)\nu(\tau) \tag{4.11}$$

Now, from $u(t) \leq \nu(t)$, $\forall t \geq \tau$, we have

$$u(t_k) \leq \Big(\prod_{j=i[\tau]}^{i[k]-1} (1+\eta_3(t_{j+1})) \exp\Big(\int_{t_j}^{t_{j+1}} \eta(s)ds \Big) \Big) u(\tau).$$

This inequality represents a discrete Grönwall-Bellman inequality. Then, applying (4.11) in (4.9) we obtain

$$u(t) \leq \Big(\prod_{\tau \leq t_k < t} (1+\eta_3(t_{k+1})) \exp\Big(\int_{t_k}^{t_{k+1}} \eta(s)ds \Big) \Big) \exp\Big(\int_{t_{i[t]}}^{t} \eta(s)d \Big) u(\tau).$$

Therefore,

$$u(t) \leq \Big(\prod_{\tau \leq t_k < t} (1+\eta_3(t_{k+1})) \Big) \exp\Big(\int_{\tau}^{t} \eta(s)ds \Big) u(\tau). \tag{4.12}$$

So (4.4) holds. Inequality (4.6) follows from (4.4) and (4.5). □

Corollary 4.4. *Let I be an interval, $h(t)$ an increasing function on $I \subset \mathbb{R}$ to $\mathbb{R}_0^+$, and u, η_1, η_2 be three functions from $I \subset \mathbb{R}$ to $\mathbb{R}_0^+$ and $\eta_3 : \{t_k\} \to \mathbb{R}_0^+$ satisfying the hypothesis described in Lemma 4.3. Consider the step function defined as $\gamma(t) = t_k$ for all $t \in I_k = [t_k, t_{k+1})$ and all $k \in \mathbb{N}$. If*

$$u(t) \leq h(t) + \int_{\tau}^{t} \eta_1(s)u(s) + \eta_2(s)u(\gamma(s))ds + \sum_{\tau \leq t_k < t} \eta_3(t_k)u(t_k^-)$$

holds, then

$$u(t) \leq \Big(\Big(\prod_{\tau \leq t_k < t} (1+\eta_3(t_{k+1})) \Big) \exp\Big(\int_{\tau}^{t} \eta(s)ds \Big) u(\tau) \Big) h(t) \quad \forall t \geq \tau. \tag{4.13}$$

4.1. **Existence and uniqueness.** In this section, we prove existence and uniqueness of solutions for the nonlinear IDEPCAG

$$\begin{gathered} u'(t) = g(t, u(t), u(\gamma(t))), \quad t \neq t_k \\ \Delta u(t_k) = Q_k(u(t_k^-)), \quad t = t_k. \end{gathered} \tag{4.14}$$

on $[\tau, \infty)$, by an inductive argument over each interval of the form $I_r = [t_r, t_{r+1})$ and using Grönwall-Bellman type IDEPCAG inequality showed in Lemma 4.2.

Uniqueness.

Theorem 4.5. *Consider the initial value problem for* (4.14) *with $u(t, \tau, u_0)$. Under conditions* (H1)–(H3) *there exists a unique solution u of* (4.14) *on $[\tau, \infty)$. Moreover, every solution is stable.*

Proof. Let u_1, u_2 be two solutions of (4.14) $[\tau, \infty)$. Then by Lemma 4.3, (H1) and (H2) we have

$$r(t) \leq r(\tau) + \int_{\tau}^{t} \eta_1(s)r(s) + \eta_2(s)r(\gamma(s))ds + \sum_{\tau \leq t_k < t} \eta_3(t_k)r(t_k^-) \tag{4.15}$$

where $r(t) = \|u_1(t) - u_2(t)\|$. Now applying Lemma 4.3 to the above expression, stability is proved. If $r(\tau) = 0$, then $r(t) = 0, \forall t \in [\tau, \infty)$. Hence, the uniqueness is proved. □

Existence of solution to (4.14) **in** $[\tau, t_r)$**.**

Lemma 4.6. *Consider the initial value problem for* (4.14) *with* $u(t,\tau,u_0)$*. Let conditions* (H1)–(H3) *and Lemma 4.3 be satisfied. Then for each* $u_0 \in \mathbb{C}^n$ *and* $\zeta_r \in [t_{r-1}, t_r)$*, there exists a solution* $u(t) = u(t,\tau,u_0)$ *of* (4.14) *on* $[\tau, t_r)$ *such that* $u(\tau) = u_0$*.*

Proof. On the interval $[\tau, t_r)$, by Lemma 4.3, system (4.14) can be written as

$$u(t) = u_0 + \int_\tau^t g(s, u(s), u(\gamma(s)))ds. \tag{4.16}$$

We prove the existence by using successive approximations method. Consider the sequence of functions $\{u_n(t)\}_{n\in\mathbb{N}}$ such that $u_0(t) = u_0$ and

$$u_{n+1}(t) = u_0 + \int_\tau^t g(s, u_n(s), u_n(\gamma(s))ds, \quad n \in \mathbb{N}. \tag{4.17}$$

We can see that

$$\begin{aligned}
\|u_1 - u_0\|_\infty &\le \int_\tau^t |g(s, u_0(s), u_0(\gamma(s))|ds \\
&\le \|u_0\|_\infty \int_\tau^t \eta_1(s) + \eta_2(s)ds \\
&= \|u_0\|_\infty \nu,
\end{aligned}$$

where ν is defined by (H3), and

$$\begin{aligned}
\|u_{n+1} - u_n\|_\infty &\le \int_\tau^t \eta_1(s)|u_n(s) - u_{n-1}(s)| + \eta_2(s)|u_n(\gamma(s)) - u_{n-1}(\gamma(s))|ds \\
&\le \|u_n - u_{n-1}\|_\infty \int_\tau^t \eta_1(s) + \eta_2(s)ds \\
&= \|u_n - u_{n-1}\|_\infty \nu.
\end{aligned}$$

So, by mathematical induction we deduce that

$$\|u_{n+1} - u_n\|_\infty \le \|u_0\|_\infty \nu^{n+1}.$$

Hence, by (H3), the sequence $\{u_n(t)\}_{n\in\mathbb{N}}$ is convergent and its limit u satisfies the (4.16) on $[\tau, t_r]$, so the existence is proved. □

We are able to extend above lemma to $[\tau, \infty)$, to obtain the existence and uniqueness of solutions for (4.14) on $[\tau, \infty)$.

Theorem 4.7. *Assume that conditions* (H1)–(H3) *and Lemma 4.3 are fulfilled. Then, for* $(\tau, u_0) \in \mathbb{R}_0^+ \times \mathbb{C}^n$*, there exists* $u(t) = u(t,\tau,u_0)$ *for* $t \ge \tau$*, a unique solution for* (4.14) *such that* $u(\tau) = u_0$*.*

Proof. Evaluating $t = t_r$ in (4.16) we have

$$u(t_r^-) = u_0 + \int_\tau^{t_r} g(s, u(s), u(\gamma(s)))ds. \tag{4.18}$$

Now, from the impulsive condition

$$\Delta u(t_r) = Q_r(u(t_r^-)),$$

we have

$$\begin{aligned} u(t_r) &= u(t_r^-) + Q_r(u(t_r^-)) \\ &= u_0 + \int_\tau^{t_r} g(s, u(s), u(\gamma(s)))ds + Q_r(u(t_r^-)), \end{aligned} \tag{4.19}$$

because $u(t_r)$ is uniquely defined, we apply Lemma 4.6 to the system $u(t) = u(t, t_r, u(t_r))$ defined in $[t_r, t_{r+1})$. Hence, the existence over the last interval is proved. So, by mathematical induction, the existence of the unique solution of (4.14) over $[\tau, \infty)$ is proved. □

5. Asymptotic equilibrium for an IDEPCAG system

In this section we prove the existence of an asymptotic equilibrium for the class of IDEPCAG systems of fixed times (1.1); i.e., the asymptotic equivalence of (1.1) with the system $x'(t) = 0$.

Definition 5.1. We say that the IDEPCAG system (1.1),

$$\begin{gathered} x'(t) = f(t, x(t), x(\gamma(t))), \quad t \neq t_k \\ \Delta x(t_k) = Q_k(x(t_k^-)), \quad t = t_k \\ x(\tau) = x_0 \quad t = \tau \end{gathered}$$

defined in $[\tau, \infty)$ has an *asymptotic equilibrium* if:

(i) For each $a \geq \tau$, equation (1.1) with initial condition $x(a) = x_0$ has a solution $x(t)$ defined in $[a, \infty)$ that satisfies

$$\lim_{t\to\infty} x(t) = \xi, \tag{5.1}$$

for some $\xi \in \mathbb{C}^n$;

(ii) for all $\xi \in \mathbb{C}^n$ there exists $a \in I$ and a solution $x(t)$ of (1.1) defined in $[a, \infty)$ that satisfies (5.1). (See [36, 5, 47, 49, 33, 32])

6. Main results

Theorem 6.1. *Suppose* (H1) *holds. Then every solution of* (1.1) *with initial condition $x(a) = x_0$ where $a \geq \tau$ satisfies* (5.1) *for some $\xi \in \mathbb{C}^n$, with error*

$$x(t) = \xi + \mathcal{O}\Big(\sum_{i=1}^{3} \int_t^\infty \lambda_i(s)ds + \sum_{t \leq t_k < \infty} (\mu_k^1 + \mu_k^2)\Big). \tag{6.1}$$

Proof. Suppose that $x(t)$ is a solution of (1.1) with initial condition $x(a) = x_0$ where $a \geq \tau$, defined on a finite subinterval $J \subset [\tau, \infty)$. Then $x(t)$, by Lemma 4.2, satisfies $\forall t \in J$

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_\tau^t |f(s, x(s), x(\gamma(s)))|ds + \sum_{\tau \leq t_k < t} |Q_k(x(t_k^-))| \\ &\leq |x_0| + \int_\tau^t \lambda_3(s)ds + \sum_{\tau \leq t_k < t} \mu_k^2 + \int_\tau^t (\lambda_1(s)|x(s)| + \lambda_2(s)|x(\gamma(s))|)ds \\ &\quad + \sum_{\tau \leq t_k < t} \mu_k^1 |x(t_k^-)|. \end{aligned}$$

Then, by Corollary 4.4, we have

$$|x(t)| \le \Big(|x_0| + \int_\tau^t \lambda_3(s)ds + \sum_{\tau \le t_k < t} \mu_k^2\Big)\Big(\prod_{\tau \le t_k < t}(1+\mu_k^1)\Big)\exp\Big(\int_\tau^t \lambda(s)ds\Big)$$

where $\lambda(t) = \lambda_1(t) + \lambda_2(t)(1-\nu)^{-1}$. As a consequence of the coefficients integrability, the solution of (1.1) is bounded, so it can be extended beyond $\sup J$.

Now taking in account the integrability of the coefficients, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $t, s > N$ then

$$\begin{aligned}
|x(t) - x(s)| &\le \int_s^t |f(u, x(u), x(\gamma(u)))|du + \sum_{t_{k(s)} \le t_k < t} |Q_k(x(t_k^-))| \\
&\le \int_s^t (\lambda_1(u)|x(u)| + \lambda_2(u)|x(\gamma(u))|)du + \int_s^t \lambda_3(u)du \\
&\quad + \sum_{t_{k(s)} \le t_k < t} \mu_k^1 |x(t_k^-)| + \sum_{t_{k(s)} \le t_k < t} \mu_k^2;
\end{aligned}$$

i.e.,

$$\begin{aligned}
|x(t) - x(s)| &\le C\Big(\int_s^t (\lambda_1(u) + \lambda_2(u))ds + \sum_{t_{k(s)} \le t_k < t} \mu_k^1\Big) + \int_s^t \lambda_3(u)du \\
&\quad + \sum_{t_{k(s)} \le t_k < t} \mu_k^2 < \varepsilon.
\end{aligned}$$

In this way, by the Cauchy's criterion, $x(t)$ converges to some $\xi \in \mathbb{C}^n$. I.e. we obtain condition (i) of the asymptotic equilibrium definition. □

To satisfy condition (ii) of the asymptotic equilibrium definition we use the Banach's fixed point theorem together with (H2), so we have the following theorem.

Theorem 6.2. *Suppose that condition* (H2) *holds. Then for each $\xi \in \mathbb{C}^n$ there exists $a \ge \tau$ and a solution $x(t)$ of* (1.1) *defined on $[a, \infty)$ satisfying* (5.1).

Proof. Using (H2), we can choose a sufficiently large real number $a \ge \tau$ such that

$$L = \int_a^\infty (\lambda_1(s) + \lambda_2(s))ds + \sum_{a \le t_k} \widetilde{\mu}_k < 1.$$

We consider the Banach space $\mathcal{B}$ consisting of bounded functions defined on $[a, \infty)$ with values on $\mathbb{C}^n$ endowed by the norm

$$|f| = \sup\{|f(t)| : t \in [a, \infty)\},$$

and the operator $T : \mathcal{B} \longrightarrow \mathcal{B}$ defined by

$$(Tx)(t) = \xi - \int_t^\infty f(s, x(s), x(\gamma(s)))ds - \sum_{t \le t_k} Q_k(x(t_k^-)).$$

Easily we verify that $T(\mathcal{B}) \subset \mathcal{B}$, since

$$\begin{aligned}
|(Tx)(t)| &\le |\xi| + \int_t^\infty |f(s, x(s), x(\gamma(s))) - f(s, 0, 0)|ds + \int_t^\infty |f(s, 0, 0)|ds \\
&\quad + \sum_{t \le t_k} |Q_k(x(t_k^-)) - Q_k(0)| + \sum_{t \le t_k} |Q_k(0)|
\end{aligned}$$

$$
\begin{aligned}
&\le |\xi| + \int_a^\infty \lambda_1(s)|x(s)| + \lambda_2(s)|x(\gamma(s))|ds + \sum_{a\le t_k} \widetilde{\mu}_k |x(t_k^-)| \\
&\quad + \int_a^\infty |f(s,0,0)|ds + \sum_{a\le t_k} |Q_k(0)|.
\end{aligned}
$$

Thus, by the integrability of the coefficients, $f(t,0,0)$ and $Q_k(0)$, we obtain the desired result. Now, we have to check if T defines a contraction. Since

$$
\begin{aligned}
&|(Tx)(t) - (Ty)(t)| \\
&\le \int_t^\infty |f(s,x(s),x(\gamma(s))) - f(s,y(s),y(\gamma(s)))|ds + \sum_{t\le t_k} |Q_k(x(t_k^-)) - Q_k(y(t_k^-))| \\
&\le \int_t^\infty \lambda_1(s)|x(s)-y(s)| + \lambda_2(s)|x(\gamma(s)) - y(\gamma(s))|ds + \sum_{t<t_k} \widetilde{\mu}_k |x(t_k^-) - y(t_k^-)| \\
&\le |x-y|_\infty \Big(\int_t^\infty \lambda_1(s) + \lambda_2(s)ds + \sum_{t<t_k} \widetilde{\mu}_k \Big) \\
&= L|x-y|_\infty.
\end{aligned}
$$

Thus, there exists a unique fixed point for T in $\mathcal{B}$. Then

$$
x(t) = \xi - \int_t^\infty f(s,x(s),x(\gamma(s)))ds - \sum_{t\le t_k} Q_k(x(t_k^-)), \quad \forall t \ge a.
$$

Now, defining

$$
\xi^{'} = \xi - \int_a^\infty f(s,x(s),x(\gamma(s)))ds - \sum_{a\le t_k} Q_k(x(t_k^-)),
$$

we have that $x(t)$ satisfies (1.1). Since

$$
x(t) = \xi^{'} + \int_a^t f(s,x(s),x(\gamma(s)))ds + \sum_{a\le t_k<t} Q_k(x(t_k^-)), \quad \forall t \ge a,
$$

and it satisfies $\lim_{t\to\infty} x(t) = \xi \in \mathbb{C}^n$, i.e.,

$$
x(t) = \xi + \mathcal{O}\Big(\sum_{i=1}^3 \int_t^\infty \lambda_i(s)ds + \sum_{t\le t_k<\infty} (\mu_k^1 + \mu_k^2) \Big),
$$

where $\lambda_3 = |f(t,0,0)|$, $\mu_k^2 = |Q_k(0)|$ and $\xi \in \mathbb{C}^n$. So, condition (ii) of the asymptotic equilibrium definition is proved. $\square$

As a consequence of the previous theorems we have the following corollary.

Corollary 6.3. *Let conditions* (H1) *and* (H2) *hold. Then there exists a global asymptotic equilibrium, i.e., for any $a \ge \tau$ sufficiently large, every solution of the IDEPCAG system*

$$
\begin{gathered}
x'(t) = f(t,x(t),x(\gamma(t))), \quad t \ne t_k \\
\Delta x(t_k) = Q_k(x(t_k^-)), \quad t = t_k \\
x(a) = x_0
\end{gathered}
$$

converges to some $\xi \in \mathbb{C}^n$.

Remark 6.4. It is important to notice that the above result holds for any $r > 0$ such that $|x|_\infty \le r$.

7. Asymptotic equilibrium for a quasilinear IDEPCAG system

In this section we study the existence of an asymptotic equilibrium of IDEPCAG system (2.2). We will conclude that as a consequence of the existence of an asymptotic equilibrium of the solutions of an equivalent system obtained by the variation of constants formula.

Consider the quasilinear system (2.2)

$$\begin{gathered} y'(t) = A(t)y(t) + f(t, y(t), y(\gamma(t))), \quad t \neq t_k \\ \Delta y(t_k) = J_k y(t_k^-) + I_k(y(t_k^-)), \quad t = t_k, \quad k \in \mathbb{N} \end{gathered}$$

and the linear system (2.4),

$$\begin{gathered} x'(t) = A(t)x(t), \quad t \neq t_k \\ \Delta x(t_k) = J_k x(t_k^-), \quad t = t_k, \quad k \in \mathbb{N}\,. \end{gathered}$$

Theorem 7.1. *Assume that* (H1)–(H4) *hold. Then, for any* $(\tau, y_0) \in I \times \mathbb{C}^n$ *there exists a unique solution* $y(t) = y(t, \tau, y_0)$ *of* (2.2) *on all of* $[\tau, \infty)$.

Proof. If we make $y(t) = \Phi(t)u(t)$, where $\Phi(t)$ is the fundamental matrix of (2.4) in (2.2), we have

$$\Phi(t_k)u(t_k) - \Phi(t_k^-)u(t_k^-) = J_k\Phi(t_k^-)u(t_k^-) + I_k(\Phi(t_k^-)u(t_k^-)).$$

Adding and subtracting the term $\Phi(t_k)u(t_k^-)$ to the left side, we have

$$\Delta u(t_k) = (\Phi^{-1}(t_k)\Phi(t_k^-) - I)u(t_k^-) + \Phi^{-1}(t_k)J_k\Phi(t_k^-)u(t_k^-) + \Phi^{-1}(t_k)I_k(\Phi(t_k^-)u(t_k^-));$$

i.e.,

$$\begin{aligned} \Delta u(t_k) &= (\Phi^{-1}(t_k)\Phi(t_k^-) - I + \Phi^{-1}(t_k)J_k\Phi(t_k^-))u(t_k^-) + \Phi^{-1}(t_k)I_k(\Phi(t_k^-)u(t_k^-)) \\ &= (\Phi^{-1}(t_k)(I + J_k)\Phi(t_k^-))u(t_k^-) - u(t_k^-) + \Phi^{-1}(t_k)I_k(\Phi(t_k^-)u(t_k^-)) \\ &= (\Phi^{-1}(t_k)\Phi(t_k))u(t_k^-) - u(t_k^-) + \Phi^{-1}(t_k)I_k(\Phi(t_k^-)u(t_k^-)) \\ &= \Phi^{-1}(t_k)I_k(\Phi(t_k^-)u(t_k^-)). \end{aligned}$$

So, $u(t)$ satisfies

$$\begin{gathered} u'(t) = \hat{g}(t, u(t), u(\gamma(t))), \quad t \neq k, \\ \Delta u(t_k) = \hat{h}(t_k^-, u(t_k^-)), \quad t = t_k, \end{gathered} \tag{7.1}$$

where

$$\hat{g}(t, u(t), u(\gamma(t))) = \Phi^{-1}(t)f(t, \Phi(t)u(t), \Phi(\gamma(t))u(\gamma(t))), \tag{7.2}$$

$$\hat{h}(t_k^-, u(t_k^-)) = \Phi^{-1}(t_k)I_k(\Phi(t_k^-)u(t_k^-)). \tag{7.3}$$

From (H4), the functions $\hat{g}$ and $\hat{h}$ satisfy

$$\begin{gathered} |\hat{g}(t, u_1(t), u_1(\gamma(t))) - \hat{g}(t, u_2(t), u_2(\gamma(t)))| \\ \le \eta_1(t)|u_1(t) - u_2(t)| + \eta_2(t)|u_1(\gamma(t)) - u_2(\gamma(t))| \\ |\hat{h}(t_k^-, \Phi(t_k^-)u_1(t_k^-)) - \hat{h}(t_k^-, \Phi(t_k^-)u_2(t_k^-))| \le \eta_3(t_k)|u_1(t_k^-) - u_2(t_k^-)|, \end{gathered} \tag{7.4}$$

where η_1, η_2 and η_3 are given by (H4). Hence, the existence and uniqueness of solutions for (7.1) hold by Lemmas 4.2 and 4.3 and Theorem 4.7. $\square$

7.1. **Asymptotic equilibrium for system** (2.2)**.** The following result establishes the existence of an asymptotic equilibrium for system (2.2), as a consequence of the existence of an asymptotic equilibrium for system (7.1).

Theorem 7.2. *If* (H1)–(H4) *are fulfilled, then each solution of* (2.2) *is defined on* $I_\tau = [\tau, \infty)$. *Furthermore, solutions of systems* (2.2) *and* (2.4) *are related by the asymptotic formula*

$$y(t) = \Phi(t)(\xi + \epsilon(t)), \quad \text{as } t \to \infty, \tag{7.5}$$

where $\xi \in \mathbb{C}^n$ *is a constant vector,* Φ *is the fundamental matrix of* (2.4) *and the error has the following estimation*

$$\epsilon(t) = \mathcal{O}\Big(\Big(\exp(\int_t^\infty \eta(s)ds) - 1\Big) + \sum_{t_k > t} \eta_3(t_k)\Big) \tag{7.6}$$

where $\eta(t) = \eta_1(t) + \frac{\eta_2(t)}{1-\nu}$. *Moreover,* (2.2) *and* (2.4) *have the same asymptotic behavior if* $\epsilon_0(t) \to 0$ *as* $t \to \infty$, *where*

$$\epsilon_0(t) = \int_t^\infty |\Phi(t,s)\|\Phi(s)|(\lambda_1(s) + \lambda_2(s)|\Phi^{-1}(s, \gamma(s))|)ds + \sum_{t_k > t} |\Phi(t, t_k)\|\Phi(t_k^-)|\widetilde{\mu}_k, \tag{7.7}$$

and we have the asymptotic formula

$$y(t) = \Phi(t)\xi + \mathcal{O}(\epsilon_0(t)), \quad \xi \in \mathbb{C}^n, \quad \text{as } t \to \infty,$$

i.e.,

$$y(t) = x(t) + \mathcal{O}(\epsilon_0(t)), \quad \xi \in \mathbb{C}^n, \quad \text{as } t \to \infty, \tag{7.8}$$

where $x(t)$ *is a solution of* (2.4).

Proof. By Lemma 4.3 and (H1)–(H3), the solution (7.1) satisfies

$$|u(t)| \le |u(\tau)| + \int_\tau^t \eta_1(s)|u(s)| + \eta_2(s)|u(\gamma(t))|ds + \sum_{\tau \le t_k < t} \eta_3(t_k)|u(t_k^-)|. \tag{7.9}$$

Also, this expression satisfies the hypothesis of Lemma 4.2. Then, by applying the Grönwall-Bellman inequality and by the summability of the coefficients we have that u is bounded and $u(t) \in L^1(I)$; i.e.,

$$|u(t)| \le |u(\tau)| \prod_{\tau \le t_k} (1 + \eta_3(t_{k+1})) \exp\Big(\int_\tau^t \eta(s)ds\Big) < \infty \tag{7.10}$$

and

$$|u(\gamma(t))| \le |u(\tau)|(1-\nu)^{-1}\Big(\prod_{\tau \le t_k < t} (1 + \eta_3(t_k))\Big) \exp\Big(\int_\tau^t \eta(s)ds\Big) < \infty, \tag{7.11}$$

so we conclude that g and $Q_k \in L^1(I)$ and $l^1(I)$ respectively; i.e.,

$$u_\infty = u_0 + \int_\tau^\infty \hat{g}(s, u(s), u(\gamma(s))ds + \sum_{\tau \le t_k} \hat{h}(t_k^-, u(t_k^-)) \tag{7.12}$$

exists. Now we can write u as

$$u(t) = u_\infty - \int_t^\infty \hat{g}(s, u(s), u(\gamma(s))ds - \sum_{t_k > t} \hat{h}(t_k^-, u(t_k^-)). \tag{7.13}$$

By using (7.13) and by making the change of variables $y(t) = \Phi(t)u(t)$, we obtain

$$y(t) = \Phi(t)\Big[u_\infty - \int_t^\infty \hat{g}(s, u(s), u(\gamma(s)))ds - \sum_{t_k>t} \hat{h}(t_k^-, u(t_k^-))\Big], \tag{7.14}$$

i.e.,

$$y(t) = \Phi(t)u_\infty - \int_t^\infty \Phi(t)\hat{g}(s, u(s), u(\gamma(s)))ds - \sum_{t_k>t} \Phi(t)\hat{h}(t_k^-, u(t_k^-)), \tag{7.15}$$

We note that $x(t) = \Phi(t)u_\infty$ is a solution of (2.4). Now, we can estimate

$$\begin{aligned}
&\int_t^\infty |\Phi(t)\hat{g}(s, u(s), u(\gamma(s)))|ds \\
&= \int_t^\infty |\Phi(t,s)f(s, \Phi(s)u(s), \Phi(\gamma(s))u(\gamma(s))|ds \\
&\le \int_t^\infty |\Phi(t)|(|\Phi^{-1}(s)\|\Phi(s)|\lambda_1(s)|u(s)| \\
&\quad + |\Phi^{-1}(s,\gamma(s))\|\Phi^{-1}(s)\|\Phi(s)|\lambda_2(s)|u(\gamma(s))|)ds \\
&\le |\Phi(t)|\int_t^\infty \eta_1(s)|u(s)| + \eta_2(s)|u(\gamma(s))|ds,
\end{aligned} \tag{7.16}$$

and

$$\begin{aligned}
\sum_{t_k>t} |\Phi(t)\hat{h}(t_k^-, u(t_k^-))| &= \sum_{t_k>t} |\Phi(t)\Phi^{-1}(t_k)I_k(\Phi(t_k^-)u(t_k^-))| \\
&\le |\Phi(t)|\sum_{t_k>t} |\Phi(t_k^-)\|\Phi^{-1}(t_k)|\widetilde{\mu}_k|u(t_k^-)|, \\
&\le |\Phi(t)|\sum_{t_k>t} \eta_3(t_k)|u(t_k^-)|,
\end{aligned} \tag{7.17}$$

where $\eta_1(t), \eta_2(t)$ and $\eta_3(t_k)$ given in (H_4). Now, from (7.15), (7.16) and (7.17) we have

$$\begin{aligned}
&|y(t) - \Phi(t)u_\infty| \\
&\le |\Phi(t)|\Big(\int_t^\infty \eta_1(s)|u(s)| + \eta_2(s)|u(\gamma(s))|ds + \sum_{t_k>t} \eta_3(t_k)|u(t_k^-)|\Big).
\end{aligned} \tag{7.18}$$

Applying (7.10) and (7.11) in (7.18), we have

$$\begin{aligned}
|y(t) - \Phi(t)u_\infty| &\le |u(\tau)|\prod_{\tau\le t_k}(1+\eta_3(t_{k+1}))|\Phi(t)|\Big\{\int_t^\infty \eta(s)\exp(\int_\tau^s \eta(u)du)ds\Big\} \\
&\quad + |u(\tau)|\prod_{\tau\le t_k}(1+\eta_3(t_{k+1}))\sum_{t_k>t}\eta_3(t_k)\exp\Big(\int_\tau^t \eta(s)ds\Big) \\
&\le |\Phi(t)|\Big[|u(\tau)|\prod_{\tau\le t_k}(1+\eta_3(t_{k+1}))\exp\Big(\int_\tau^t \eta(u)du\Big)\Big] \\
&\quad \times \Big\{\Big(\exp(\int_t^\infty \eta(u)du) - 1\Big) + \sum_{t_k>t}\eta_3(t_k)\Big\}.
\end{aligned}$$

So, (7.5) is proved.

In a similar way, we can see that

$$
\begin{aligned}
&\int_t^{\infty} |\Phi(t)\hat{g}(s,u(s),u(\gamma(s)))|ds \\
&= \int_t^{\infty} |\Phi(t,s)f(s,\Phi(s)u(s),\Phi(\gamma(s))u(\gamma(s))|ds \\
&\leq \int_t^{\infty} |\Phi(t,s)(\lambda_1(s)|\Phi(s)u(s)| + \lambda_2(s)|\Phi(\gamma(s))\Phi^{-1}(s)\Phi(s)u(\gamma(s))|)ds \\
&\leq \int_t^{\infty} |\Phi(t,s)\|\Phi(s)|(\lambda_1(s)|u(s)| + \lambda_2(s)|\Phi^{-1}(s,\gamma(s))\|u(\gamma(s))|)ds
\end{aligned}
\tag{7.19}
$$

and

$$
\begin{aligned}
\sum_{t_k>t} |\Phi(t)\hat{h}(t_k^-,u(t_k^-))| &= \sum_{t_k>t} |\Phi(t)\Phi^{-1}(t_k)I_k(\Phi(t_k^-)u(t_k^-))| \\
&\leq \sum_{t_k>t} |\Phi(t,t_k)I_k(\Phi(t_k^-)u(t_k^-))| \\
&\leq \sum_{t_k>t} |\Phi(t,t_k)\|\Phi(t_k^-)|\widetilde{\mu}_k|u(t_k^-)|.
\end{aligned}
\tag{7.20}
$$

By the boundedness of $u(t), u(\gamma(t))$, (H4) and (7.19)-(7.20), from (7.15) we have

$$
\begin{aligned}
&|y(t) - \Phi(t)\xi| \\
&\leq K\Big(\int_t^{\infty} |\Phi(t,s)\|\Phi(s)|(\lambda_1(s) + \lambda_2(s)|\Phi^{-1}(s,\gamma(s))|)ds + \sum_{t_k>t} |\Phi(t,t_k)\|\Phi(t_k^-)|\widetilde{\mu}_k\Big) \\
&\leq K\epsilon_0(t),
\end{aligned}
$$

where $K = \sup_{t\in[\tau,\infty)} |u(t)|$ and $\xi = u_\infty$. So, (7.8) holds and the proof is complete. $\square$

7.2. **Consequences of Theorem 7.2.** Consider the homogeneous linear IDEPCAG

$$
\begin{gathered}
y'(t) = A(t)y(t) + B(t)y(\gamma(t)), \quad t \neq k, \\
\Delta y(t_k) = J_k y(t_k^-), \quad t = t_k
\end{gathered}
\tag{7.21}
$$

and define

$$
\epsilon_0(t) = \int_t^{\infty} |\Phi(t,s)B(s)\Phi(\gamma(s))|ds. \tag{7.22}
$$

As a direct application of above Theorem 7.2, we have the following result.

Theorem 7.3. *Suppose that $J_k \in l^1$ and $\eta(t) = |\Phi^{-1}(t)B(t)\Phi(\gamma(t))|$ satisfy hypothesis* (H3)*, where $\Phi(t)$ is the fundamental matrix of system* (2.4)*. Then the linear IDEPCAG* (7.21) *is equivalent to the IDE* (2.4) *and for every solution y of* (7.21) *there exists $\xi \in \mathbb{C}^n$ such that*

$$
y = \Phi(t)(\xi + \epsilon(t)), \quad \text{as } t \to \infty, \tag{7.23}
$$

where

$$
\epsilon(t) = \mathcal{O}\Big(\int_t^{\infty} |\Phi^{-1}(s)\|B(s)\|\Phi(\gamma(s))|ds\Big). \tag{7.24}
$$

Moreover, if $\epsilon_0(t) \to 0$, as $t \to \infty$, then the linear IDEPCAG (7.21) *is asymptotically equivalent to the IDE* (2.4) *and for any solution $y(t)$ of* (7.21) *there exists a unique solution $x(t)$ of* (2.4) *such that*

$$
y(t) = x(t) + \mathcal{O}(\epsilon_0(t)), \tag{7.25}
$$

where ϵ_0 is given by (7.22).

Proof. Let

$$\begin{aligned}\hat{g}(t,u(t),u(\gamma(t))) &= \Phi^{-1}(t)B(t)\Phi(\gamma(t))u(\gamma(t)),\\ \hat{h}(t_k^-,u(t_k^-)) &= \Phi^{-1}(t_k)\cdot 0\,.\end{aligned} \tag{7.26}$$

Proceeding as in Theorem 7.2, we see that

$$\begin{aligned}\int_t^\infty |\Phi(t)\hat{g}(s,u(s),u(\gamma(s)))|ds &= \int_t^\infty |\Phi(t,s)B(s)\Phi(\gamma(s))u(\gamma(s))|ds.\\ &\le \int_t^\infty |\Phi(t,s)B(s)\Phi(\gamma(s))\|u(\gamma(s))|ds.\end{aligned} \tag{7.27}$$

So, by (7.27), we have the desired result. □

8. Examples and applications

8.1. **Linear Systems.** In this section we give some examples that illustrate the effectiveness of our results.

(i) Consider the almost constant scalar lineal IDEPCAG

$$\begin{aligned}y'(t) &= ay + b(t)y(\gamma(t)), \quad t \neq t_k\\ \Delta y(t_k) &= q_k y(t_k^-), \quad t = t_k\end{aligned} \tag{8.1}$$

with $a > 0$ a constant, $b(t) \in L^1(I), q_k \in l^1(I)$ and $\tilde{b}(t) = \mathcal{O}(e^{-at})$ where

$$\tilde{b}(t) = \int_t^\infty \Big(\prod_{i[t]}^{i[s]}(1+|q_k|)\Big)|b(s)|e^{a\gamma(s)}ds$$

Then all solutions $y(t)$ of (8.1) have the asymptotic formula

$$y(t) = \Big(\prod_{i[\tau]}^{i[t]}(1+q_k)\Big)e^{at}\big(\xi + \tilde{b}(t)\big), \quad \text{as } t \to \infty,$$

where $\xi \in \mathbb{R}$, and

$$y(t) = \Big(\prod_{i[\tau]}^{i[t]}(1+q_k)\Big)e^{at}\xi + \epsilon(t), \quad \epsilon(t) = e^{at}\tilde{b}(t).$$

Notice that (8.1) is asymptotically equivalent to the IDE

$$\begin{aligned}x'(t) &= ax(t), \quad t \neq t_k\\ \Delta x(t_k) &= q_k x(t_k^-), \quad t = t_k.\end{aligned}$$

Evidently, without the integrability condition $b(t) \in L^1(I)$, even if $a < 0$ the previous results are not true, as is shown by (8.1) with $a = -1$ and $b(t) = 1+\delta$, $\gamma(t) = [t]$, $\delta > 0$, $q_k = \frac{1}{k^2}$ and the unbounded solution

$$y(t) = \Big(\prod_{k=1}^{[t]}(1+\frac{1}{k^2})\Big)\big(1+(1-e^{-(t-[t])})\delta\big)\big(1+(1-e^{-1})\delta\big)^{[t]}, \quad y(0) = 1.$$

(ii) Consider the linear second order IDEPCAG

$$\begin{aligned} y''(t) &= a(t)y + b(t)y(\gamma(t)), \quad t = t_k \\ y'(t_k) &= c_k y'(t_k^-), \quad t = t_k \\ y(t_k) &= d_k y(t_k^-), \quad t = t_k \end{aligned} \tag{8.2}$$

where $c_k, d_k \in l^1$, $a(t) = 2(t+2)^{-2}$, $b(t) = \mathcal{O}(t^{-\delta})$. This equation is equivalent to system (7.21), where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}, \quad B(t) = b(t)\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The ODE $u'' = a(t)u$ has a fundamental system of solutions

$$u_1(t) = (t+2)^2, \quad \text{and} \quad u_2(t) = (t+2)^{-1},$$

and the fundamental matrix Φ of its associated first order system (7.21) satisfies $|\Phi(t)| = |\Phi^{-1}(t)| = O(t^2)$ as $t \to \infty$, since the trace of A is zero. If $\mathcal{O}(\gamma(t)) = \mathcal{O}(t)$, for $\delta > 5$ we have that $\Phi^{-1}(t)B(t)\Phi(\gamma(t)) \in L^1$ and for $\delta > 7$,

$$\epsilon_0(t) = \int_t^\infty |\Phi(t,s)B(s)\Phi(\gamma(s))|ds \to 0, \quad \text{as } t \to \infty.$$

Hence, the conclusions of Theorem 7.3 are true. From (7.23), for any solution $y(t)$ of (8.2), there exist constants $v_1, v_2 \in \mathbb{R}$ such that

$$\begin{aligned} y(t) &= (t+2)^2(v_1 + \epsilon(t)) + (t+2)^{-1}(v_2 + \epsilon(t)), \\ y'(t) &= 2(t+2)(v_1 + \epsilon(t)) - (t+2)^{-2}(v_2 + \epsilon(t)), \end{aligned}$$

where

$$\epsilon(t) = \mathcal{O}(\exp(\eta(t)t^{\delta-5}) - 1), \quad \text{as } t \to \infty.$$

(iii) Consider the linear IDEPCAG case

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)x(\gamma(t)) + C(t), \quad t \neq t_k \\ \Delta x(t_k) &= D_k(x(t_k^-)) + E_k, \quad t = t_k, \end{aligned} \tag{8.3}$$

under the assumption of integrability and summability of the coefficients involved ($A(t), B(t), C(t), D_k$ and E_k). It is easy to verify if

$$\sup_{k\in\mathbb{N}} \int_{t_k}^{\zeta_k} |A(u)| + |B(u)|du < 1, \quad \text{for } k \in \mathbb{N} \text{ sufficiently large,} \tag{8.4}$$

then, by theorem 6.2, (8.3) has an asymptotic equilibrium.

As an application of the last result, Bereketoğlu and Oztepe [8] studied the scalar version of the IDEPCA system

$$\begin{aligned} x'(t) &= A(t)(x(t) - x([t+1])) + g(t,x), \quad t \neq k \\ \Delta x(t_k) &= E_k, \quad t = k, \quad k \in \mathbb{N} \\ x(0) &= x_0, \end{aligned} \tag{8.5}$$

where the fundamental matrix of the linear IDEPCAG system associated to (8.5) is the $n \times n$ identity matrix I (see [37, 42]), the coefficients $A(t), g(t,x)$ and E_k are integrable and summable, respectively. The authors, using some conditions over the adjoint equation related to (8.5), showed that all solutions of this system are convergent to some $\xi \in \mathbb{R}$. To obtain the same conclusions, we only need to apply Theorem 6.2 to (8.5). In this way we obtain an asymptotic equilibrium for system (8.5), which is a stronger result because it implies the convergence of the solutions. Obviously, we can consider (8.5) as a particular case of (8.3) with $B(t) = -A(t)$, $D_k = 0$, $\gamma(t) = [t+1]$ and (8.4). This last condition over the integral is assured for some $k \in \mathbb{N}$ sufficiently large due to integrability of $A(t)$. Thus, as a consequence of Theorem 6.2, (8.5) has an asymptotic equilibrium in $I \subset \mathbb{R}$.

Remark 8.1. Condition (8.4) is of critical importance for existence, uniqueness, boundedness and stability of solutions in the DEPCAG and IDEPCAG context and it was not considered by the authors (see [36, 37]).

(iv) Consider the equation

$$\begin{gathered} x'(t) = \frac{1}{2}e^{-t}\, x(t) + \frac{1}{t^2}\, x([t+1]) + \frac{1}{t^3}, \quad t \neq k, \\ \Delta x(k) = \frac{1}{3^k}, \quad t = k \in \mathbb{N}, \\ x(1) = 2. \end{gathered} \tag{8.6}$$

Here, all hypotheses of theorem 7.2 are satisfied, so (8.6) has an asymptotic equilibrium $\xi = 5/2$.

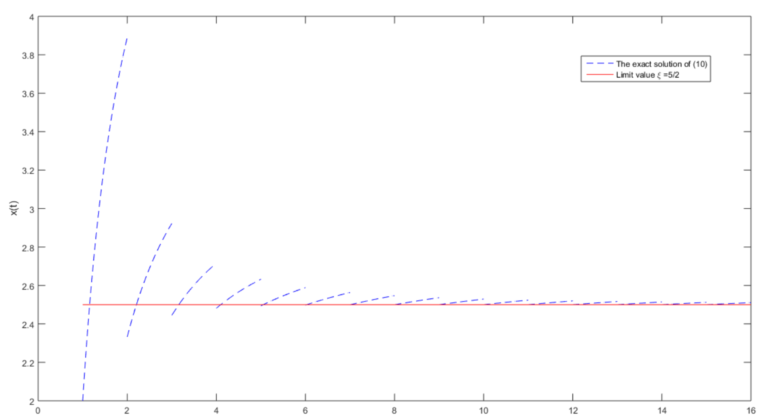

FIGURE 1.

(v) Consider the advanced semilinear IDEPCAG

$$\begin{aligned} y'(t) &= \frac{\sin(1.9(t+1))}{(t+1)^2} y(t) - \frac{1}{(t+1)^2} \tanh(y([t+1])) + e^{-\frac{t}{20}}, \quad t \neq k \\ \Delta y(t_k) &= \big(\frac{9}{10}\big)^k y(t_k^-) + \frac{|y(t_k^-) - 1| - |y(t_k^-) + 1|}{2^k} + \frac{1}{3^k}, \quad t = k,\ k \in \mathbb{N}, \end{aligned} \tag{8.7}$$

with $y(0) = -1.1$. All conditions of Theorem 7.2 are satisfied, so (8.7) has an asymptotic equilibrium, with error

$$\epsilon(t) = \mathcal{O}(e^{\frac{1}{2(t+1)}} - 1 + \frac{1}{2^{i[t]}}),$$

where $i[t] = n \in \mathbb{Z}$ is the only integer such that $t \in I_n = [t_n, t_{n+1}[$.

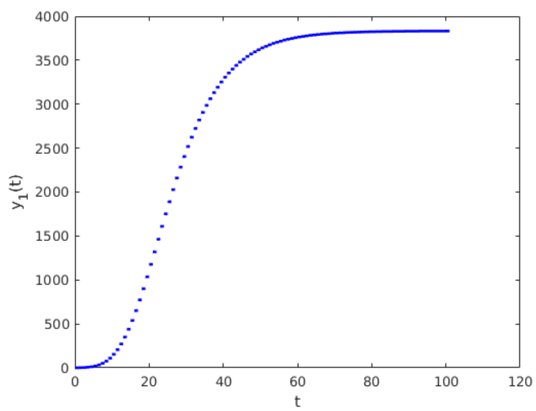

FIGURE 2. Asymptotic equilibrium for (8.7).

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