Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 44, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EFFECTS OF CROSS-DIFFUSION FOR A PREY-PREDATOR SYSTEM IN A HETEROGENEOUS ENVIRONMENT

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Abstract. This article concerns the stationary problem of a cross-diffusion prey-predator system with a protection zone for the prey. We first give the necessary condition and sufficient condition for the existence of coexistence states of the two species, by applying the bifurcation theory. Furthermore, the asymptotic behavior of coexistence states is established as the cross-diffusion coefficient of the prey tends to infinity. We also analyze the corresponding limiting system.

1. Introduction and statement of main results

In recent decades, the research of reaction-diffusion equations have made great progress (see, for example, [3, 15, 17, 21, 23, 22] and the references therein). In these equations, the prey-predator model is an important branch. In most prey-predator systems, the prey would become extinct when the predation rate is too high. To human beings, it is necessary to take measures to save the endangered prey species. From this viewpoint, we study the following cross-diffusion prey-predator system with a protection zone for the prey,

$$
\begin{gathered}
u_t = \Delta[(1+k\rho(x)v)u] + u\big(\lambda - u - \frac{b(x)v}{1+mu}\big), \quad x \in \Omega,\ t>0,\\
v_t = \Delta v + v\big(-\mu + \frac{cb(x)u}{1+mu}\big), \quad x \in \Omega\setminus\overline{\Omega}_0,\ t>0,\\
\partial_n u = 0, \quad x\in\partial\Omega,\ t>0,\\
\partial_n v = 0, \quad x\in\partial\Omega\cup\partial\Omega_0,\ t>0,\\
u(x,0) = u_0(x)\ge 0, \quad x\in\Omega,\\
v(x,0) = v_0(x)\ge 0, \quad x\in\Omega\setminus\overline{\Omega}_0,
\end{gathered}
\tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \ge 1$) with smooth boundary $\partial\Omega$ and $\overline{\Omega}_0 \subset\subset \Omega$ with smooth boundary $\partial\Omega_0$; the parameters k, λ, μ, c, m are positive constants; $\rho(x)$ and $b(x)$ are smooth functions, $\rho(x) > 0$ and $b(x) > 0$ in $\Omega\setminus\overline{\Omega}_0$, whereas $\rho(x) = b(x) = 0$ in $\overline{\Omega}_0$, moreover, we assume that $\partial_n\rho(x) = 0$ on $\partial\Omega$, $\rho(x)/b(x)$ and $b(x)/\rho(x)$ are bounded in $\overline{\Omega}\setminus\overline{\Omega}_0$. One can see [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 18, 19, 20] and references therein for more studies on this topic.

2010 *Mathematics Subject Classification.* 35J65, 35B32, 92D25.
Key words and phrases. Prey-predator model; cross-diffusion; protection zone; stationary.

Submitted March 15, 2018. Published March 22, 2019.

In this article, we denote $\Omega_1 = \Omega \backslash \overline{\Omega}_0$, and mainly discuss the stationary problem associated with (1.1):

$$\begin{gathered}
\Delta[(1+k\rho(x)v)u] + u\big(\lambda - u - \frac{b(x)v}{1+mu}\big) = 0, \quad x \in \Omega, \\
\Delta v + v\big(-\mu + \frac{cb(x)u}{1+mu}\big) = 0, \quad x \in \Omega_1, \\
\partial_n u = 0, \quad x \in \partial\Omega, \\
\partial_n v = 0, \quad x \in \partial\Omega_1.
\end{gathered} \tag{1.2}$$

Let O be any bounded domain in $\mathbb{R}^n$ with smooth boundary. Denote the usual norm of $L^p(O)$ for $p \in [1,\infty)$ by $\|\psi\|_{p,O} = \big(\int_O |\psi(x)|^p dx\big)^{1/p}$. For $q(x) \in L^\infty(O)$, we denote by $\lambda_1^N(q(x), O)$ the first eigenvalue of $-\Delta + q(x)$ over a region O, with Neumann boundary condition.

Now we are ready to present our main results. The first result gives the necessary condition and the sufficient condition for the existence of positive solutions of (1.2), and the coexistence region of (1.2) in the $\lambda\mu$-plane is given in Figure 1.

Theorem 1.1. *Let $n \geq 1$. Then*

1. *If $\lambda > 0$ and $\mu \geq -\lambda_1^N\big(-\frac{cb(x)}{m}, \Omega_1\big)$, then (1.2) has no positive solution.*
2. *If $\lambda > \lambda_*(\mu)$ and $0 < \mu < -\lambda_1^N\big(-\frac{cb(x)}{m}, \Omega_1\big)$, then (1.2) has at least one positive solution, where $\lambda_*(\mu)$ is uniquely determined by $\mu = -\lambda_1^N\big(-\frac{cb(x)\lambda_*}{1+m\lambda_*}, \Omega_1\big)$.*

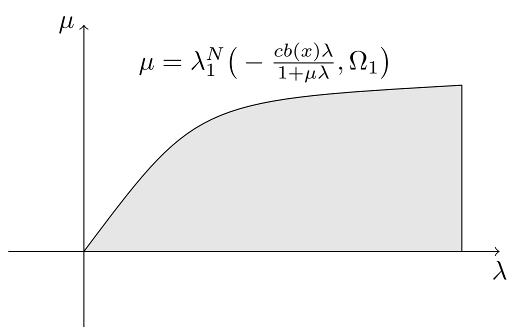

Figure 1. Coexistence region of (1.2).

The following theorem gives the asymptotic behavior of positive solutions of (1.2) as $k \to \infty$.

Theorem 1.2. *Let $n \leq 3$. For any given $\lambda > \lambda_*(\mu)$ and $0 < \mu < -\lambda_1^N\big(-\frac{cb(x)}{m}, \Omega_1\big)$. Let (u_k, v_k) be any positive solution of (1.2) for each $k > 0$. Then*

$$\lim_{k\to\infty} u_k = \overline{u} \text{ uniformly in } \overline{\Omega}, \quad \lim_{k\to\infty} (v_k, kv_k) = (0, \overline{w}) \text{ in } C^1(\overline{\Omega}_1) \times C^1(\overline{\Omega}_1),$$

where $(\overline{u},\overline{w})$ *is a positive solution of*

$$
\begin{gathered}
\Delta[(1+\rho(x)\overline{w})\overline{u}]+\overline{u}(\lambda-\overline{u})=0, \quad x\in\Omega,\\
\Delta\overline{w}+\overline{w}\Big(-\mu+\frac{cb(x)\overline{u}}{1+m\overline{u}}\Big)=0, \quad x\in\Omega_1,\\
\partial_n\overline{u}=0, \quad x\in\partial\Omega,\\
\partial_n\overline{w}=0, \quad x\in\partial\Omega_1.
\end{gathered}
\tag{1.3}
$$

Theorem 1.3. *Let $n\le 3$. For any fixed $\lambda>0$, the set of positive solutions of (1.3) forms an unbounded connected set which joins the semitrivial solution branch $\{(\mu,\overline{u},\overline{w})=(\mu,\lambda,0):\mu>0\}$ at $(\mu_*(\lambda),\lambda,0)$ and remains bounded until μ approaches 0 where it blows up, where $\mu_*(\lambda)=-\lambda_1^N\big(-\frac{cb(x)\lambda}{1+m\lambda},\Omega_1\big)$. Moreover,*

$$
\lim_{\mu\to 0}\overline{u}_\mu=\lambda \text{ uniformly in } \overline{\Omega}_0, \quad \lim_{\mu\to 0}(\overline{u}_\mu,\overline{w}_\mu)=(0,\infty) \text{ uniformly in } \Omega_1.
$$

This article is organized as follows. In Section 2, we establish some preliminary results, including a priori estimates of any positive solution and local bifurcation result. In Section 3, we complete the proof of main results. Our mathematical approach is based on elliptic estimates, bifurcation theory and elliptic regularity theory.

2. Preliminary results

In this section, we establish a priori estimates of any positive solution and the local bifurcation from semitrivial solution. We define a new unknown function

$$
U=(1+k\rho(x)v)u, \tag{2.1}
$$

and denote

$$
\begin{gathered}
f_1(U,v)=\frac{U}{1+k\rho(x)v}\Big(\lambda-\frac{U}{1+k\rho(x)v}-\frac{b(x)v(1+k\rho(x)v)}{1+k\rho(x)v+mU}\Big),\\
f_2(U,v)=v\Big(-\mu+\frac{cb(x)U}{1+k\rho(x)v+mU}\Big).
\end{gathered}
$$

Then (1.2) can be written as

$$
\begin{gathered}
\Delta U+f_1(U,v)=0, \quad x\in\Omega,\\
\Delta v+f_2(U,v)=0, \quad x\in\Omega_1,\\
\partial_n U=0, \quad x\in\partial\Omega,\\
\partial_n v=0, \quad x\in\partial\Omega_1.
\end{gathered}
\tag{2.2}
$$

By the maximum principle [17, Proposition 2.2] and Harnack inequality [15, Lemma 4.3], we derive the following a priori estimates of any positive solution of (2.2) for any given $\mu>0$ and $k>0$.

Proposition 2.1. *For any given $\mu>0$ and $k>0$, there exists a positive constant C such that any positive solution (U,v) of (2.2) satisfies*

$$
\|U\|_{C^{1,\theta}(\overline{\Omega})}\le C \quad \text{and } \|v\|_{C^{1,\theta}(\overline{\Omega}_1)}\le C,
$$

where $\theta\in(0,1)$.

Proof. Suppose that (U, v) is any positive solution of (2.2). Denote $U(x_0) = \max_{\overline{\Omega}} U$ with $x_0 \in \overline{\Omega}$. When $x_0 \in \overline{\Omega}_0$, we apply the maximum principle due to Lou and Ni [17] to obtain

$$U(x_0) \le \lambda. \tag{2.3}$$

Here we use the assumption that $\rho(x) = b(x) = 0$ in $\overline{\Omega}_0$. When $x_0 \in \Omega \setminus \overline{\Omega}_0$, we apply the maximum principle [17] again to obtain

$$\lambda - \frac{U(x_0)}{1 + k\rho(x_0)v(x_0)} - \frac{b(x_0)v(x_0)(1 + k\rho(x_0)v(x_0))}{1 + k\rho(x_0)v(x_0) + mU(x_0)} \ge 0. \tag{2.4}$$

This implies

$$U(x_0) \le \lambda(1 + k\rho(x_0)v(x_0)) = \lambda\Big((1 + k\frac{\rho(x_0)}{b(x_0)}b(x_0)v(x_0)\Big).$$

Since $\rho(x)/b(x)$ is bounded in $\overline{\Omega} \setminus \overline{\Omega}_0$, we only need to check that $b(x_0)v(x_0)$ is bounded. By some calculations, (2.4) implies

$$\begin{aligned} &m\Big(U(x_0) + \frac{(1 - m\lambda)(1 + k\rho(x_0)v(x_0))}{2m}\Big)^2 \\ &+ \Big(b(x_0)v(x_0) - \lambda - \frac{(1 - m\lambda)^2}{4m}\Big)(1 + k\rho(x_0)v(x_0))^2 \le 0. \end{aligned}$$

Therefore

$$b(x_0)v(x_0) \le \lambda + \frac{(1 - m\lambda)^2}{4m},$$

and so

$$U(x_0) \le \lambda\Big(1 + k\frac{\rho(x_0)}{b(x_0)}\Big(\lambda + \frac{(1 - m\lambda)^2}{4m}\Big)\Big) \quad \text{for } x_0 \in \Omega \setminus \overline{\Omega}_0.$$

Therefore, there exists a positive constant C_1 independent of μ such that

$$U(x_0) = \max_{\overline{\Omega}} U \le C_1.$$

By (1.2), we have

$$\mu \int_{\Omega_1} v dx = \int_{\Omega} c(\lambda u - u^2) dx \le c\lambda \int_{\Omega} u dx \le c\lambda \int_{\Omega} U dx \le c\lambda \max_{\overline{\Omega}} U|\Omega|,$$

where $|\Omega|$ denotes the volume of Ω. This implies

$$\|v\|_{1,\Omega_1} \le \frac{cC_1\lambda|\Omega|}{\mu}.$$

We apply Harnack inequality (Lemma 4.3 of [15]) to the v-equation of (1.2) and obtain

$$\max_{\overline{\Omega}_1} v \le C_2 \min_{\overline{\Omega}_1} v \le C_2 \frac{\int_{\Omega} v dx}{|\Omega|} \le C_2 \frac{cC_1\lambda}{\mu} =: C_3.$$

Consequently, we show that $\|U\|_{\infty,\Omega}$ and $\|v\|_{\infty,\Omega_1}$ are bounded. As a result, by elliptic regularity theory and Sobolev embedding theorem, we obtain the conclusion. □

The following proposition gives a priori estimates of any positive solution of (2.2) for large $k > 0$.

Proposition 2.2. *Let $n \le 3$. For any given $\mu > 0$ and large $k(> M)$, there exists a positive constant C independent of k such that any positive solution (U, v) of (2.2) satisfies*
$$\|U\|_{C^{1,\theta}(\overline{\Omega})} \le C \text{ and } \|v\|_{C^{1,\theta}(\overline{\Omega}_1)} \le C,$$
where $\theta \in (0, 1)$.

Proof. Integrating (2.2), and applying the Hölder inequality, we obtain
$$\int_\Omega u^2 dx = \int_\Omega u\Big(\lambda - \frac{b(x)v}{1+mu}\Big)dx \le \lambda \int_\Omega u dx \le \lambda|\Omega|^{1/2}\|u\|_{2,\Omega}.$$
Thus $\|u\|_{2,\Omega} \le \lambda|\Omega|^{1/2}$. Further we have
$$|\Omega_0|^{1/2} \inf_{\Omega_0} u \le \|u\|_{2,\Omega_0} \le \|u\|_{2,\Omega} \le \lambda|\Omega|^{1/2}.$$
This implies
$$\inf_{\Omega_0} u \le \lambda(|\Omega|/|\Omega_0|)^{1/2}.$$
Denote
$$m(x) = \frac{1}{1 + k\rho(x)v}\Big(\lambda - \frac{U}{1 + k\rho(x)v} - \frac{b(x)v(1 + k\rho(x)v)}{1 + k\rho(x)v + mU}\Big).$$
Then
$$|m(x)| \le |\frac{\lambda}{1 + k\rho(x)v}| + |\frac{u}{1 + k\rho(x)v}| + |\frac{b(x)v}{(1 + k\rho(x)v)(1 + mu)}| \le \lambda + u + \frac{b(x)}{M\rho(x)}.$$
Thus $\|m(x)\|_{2,\Omega} \le C_1$. Then we apply Harnack inequality to obtain
$$\max_{\overline{\Omega}} U \le C_2 \min_{\overline{\Omega}} U \le C_2 \inf_{\Omega_0} u \le C_2\lambda(|\Omega|/|\Omega_0|)^{1/2} =: C_3,$$
where C_3 is independent of μ and k.

The upper bound of v in Ω_1 can be obtained by the same argument as in Proposition 2.1. Hence, by elliptic regularity theory and Sobolev embedding theorem, we have the conclusion. $\square$

For $p > n$, we define
$$X_1 = W_n^{2,p}(\Omega) \times W_n^{2,p}(\Omega_1), \quad X_2 = L^p(\Omega) \times L^p(\Omega_1),$$
where $W_n^{2,p}(O) = \{w \in W^{2,p}(O) : \partial_n w = 0 \text{ on } \partial O\}$. We also define
$$E = C_n^1(\overline{\Omega}) \times C_n^1(\overline{\Omega}_1),$$
where $C_n^1(\overline{O}) = \{w \in C^1(\overline{O}) : \partial_n w = 0 \text{ on } \partial O\}$. Hence, it follows from the Sobolev embedding theorem that $X_1 \subset E$. For any $\lambda > 0$, system (2.2) has a semitrivial solution: $(\lambda, 0)$. Therefore, system (2.2) has a curve of semitrivial solution:
$$\Gamma_U = \{(\lambda, U, v) = (\lambda, \lambda, 0) : \lambda > 0\}.$$
Then the following local bifurcation property holds.

Proposition 2.3. *For any fixed $\mu > 0$, a branch of positive solutions of (2.2) bifurcates from Γ_U if and only if $\lambda = \lambda_*(\mu)$, moreover, positive solutions of (2.2) near $(\lambda_*, \lambda_*, 0) \in \mathbb{R} \times X_1$ can be expressed as*
$$\Gamma_\delta = \{(\lambda_*, U, v) = (\lambda_*(s), s(\phi_* + U(s)), s(\psi_* + v(s))) : s \in (0, \delta)\}$$

for some $\delta > 0$, where

$$\phi_* = -(\Delta - \lambda I)^{-1}\Big[\lambda\Big(k\rho(x)\lambda - \frac{b(x)}{1+m\lambda}\Big)\psi_*\Big]$$

and ψ_* is a positive solution of

$$-\Delta\psi_* - \frac{cb(x)\lambda_*}{1+m\lambda_*}\psi_* = -\mu\psi_* \ \text{in } \Omega_1, \quad \partial_n\psi_* = 0 \ \text{on } \partial\Omega_1.$$

Here $(\lambda_*(s), U(s), v(s))$ is a smooth function with respect to s and satisfies

$$(\lambda_*(0), U(0), v(0)) = (\lambda_*, 0, 0)$$

and $\int_{\Omega_1} v(s)\psi_* dx = 0$.

Proof. Denote $z := U - \lambda$ in (2.2) and define an operator $\Phi : \mathbb{R} \times X_1 \to X_2$ by

$$\Phi(\lambda, z, v) = \begin{pmatrix} \Delta z + f_1(z+\lambda, v) \\ \Delta v + f_2(z+\lambda, v) \end{pmatrix},$$

where

$$\begin{gathered} f_1(z+\lambda, v) = \frac{z+\lambda}{1+k\rho(x)v}\Big(\lambda - \frac{z+\lambda}{1+k\rho(x)v} - \frac{b(x)v(1+k\rho(x)v)}{1+k\rho(x)v+m(z+\lambda)}\Big), \\ f_2(z+\lambda, v) = v\Big(-\mu + \frac{cb(x)(z+\lambda)}{1+k\rho(x)v+m(z+\lambda)}\Big). \end{gathered}$$

By direct calculations, we obtain

$$\Phi_{(z,v)}(\lambda, 0, 0)[\phi, \psi] = \begin{pmatrix} \Delta\phi - \lambda\phi + \lambda\left(k\rho(x)\lambda - \frac{b(x)}{1+m\lambda}\right)\psi \\ \Delta\psi - \left(\mu - \frac{cb(x)\lambda}{1+m\lambda}\right)\psi \end{pmatrix}.$$

It follows from the Krein-Rutman theorem [24] that $\Phi_{(z,v)}(\lambda, 0, 0)[\phi, \psi] = (0, 0)$ has a solution with $\psi > 0$ if and only if $\lambda = \lambda_*(\mu)$. Hence, by further calculations, we obtain

$$\begin{gathered} \ker \Phi_{(z,v)}(\lambda_*, 0, 0) = \operatorname{span}\{(\phi_*, \psi_*)\}, \\ \operatorname{range} \Phi_{(z,v)}(\lambda_*, 0, 0) = \big\{(\phi, \psi) \in X_2 : \int_{\Omega_1} \psi \cdot \psi_* dx = 0\big\}, \end{gathered}$$

which imply that $\dim\ker \Phi_{(z,v)}(\lambda_*, 0, 0) = \operatorname{codim}\operatorname{range} \Phi_{(z,v)}(\lambda_*, 0, 0) = 1$. Moreover,

$$\begin{aligned} &\Phi_{\lambda(z,v)}(\lambda_*, 0, 0)[\phi_*, \psi_*] \\ &= \begin{pmatrix} -\phi_* + 2k\rho(x)\lambda\psi_* - \frac{b(x)}{(1+m\lambda)^2}\psi_* \\ \frac{cb(x)}{(1+m\lambda)^2}\psi_* \end{pmatrix} \notin \operatorname{range} \Phi_{(z,v)}(\lambda_*, 0, 0). \end{aligned}$$

By applying the local bifurcation theorem [1] to Φ at $(\lambda_*, 0, 0)$, we can obtain the result stated in Proposition 2.3. $\square$

3. Proof of main results

First, we apply the global bifurcation theorem [16, Theorem 6.4.3] for proving Theorem 1.1.

Proof of Theorem 1.1. We first establish the necessary condition for the existence of positive solutions of (1.2). Suppose that (u, v) is any positive solution of (1.2). Then v is a positive solution of the equation

$$\begin{gathered} -\Delta v - \frac{cb(x)u}{1+mu}v = -\mu v, \quad x \in \Omega_1, \\ \partial_n v = 0, \quad x \in \partial\Omega_1. \end{gathered}$$

Then

$$-\mu = \lambda_1^N\Big(-\frac{cb(x)u}{1+mu}, \Omega_1\Big) > \lambda_1^N\Big(-\frac{cb(x)}{m}, \Omega_1\Big) \Longleftrightarrow \mu < -\lambda_1^N\Big(-\frac{cb(x)}{m}, \Omega_1\Big).$$

Therefore, (1.2) has no positive solution when $\mu \geq -\lambda_1^N\big(-\frac{cb(x)}{m}, \Omega_1\big)$.

Next, we establish the sufficient condition for the existence of positive solutions of (1.2). Define an operator

$$F(\lambda, U, v) = \begin{pmatrix} U \\ v \end{pmatrix} - \begin{pmatrix} (-\Delta + I)_{\Omega}^{-1}[U + f_1(U, v)] \\ (-\Delta + I)_{\Omega_1}^{-1}[v + f_2(U, v)] \end{pmatrix}.$$

For any fixed $\lambda > 0$ and $\mu > 0$, the elliptic regularity theory ensures that the second term of F is a compact operator.

By the similar argument to [10, Theorem 3.2], we can verify that the conditions of [16, Theorem] hold. Consequently, it follows from [16, Theorem 6.4.3] that the local bifurcation branch Γ_δ obtained in Proposition 2.3 is contained in Γ_M which is a component (i.e., maximal connected subset) of S where $S = \{(\lambda, U, v) \in \mathbb{R} \times E : F(\lambda, U, v) = 0, (\lambda, U, v) \neq (\lambda_*, \lambda_*, 0)\}$; that is,

$$\Gamma_\delta \subset \Gamma_M \subset \{(\lambda, U, v) \in (\mathbb{R} \times E)\backslash\{(\lambda_*, \lambda_*, 0)\} : F(\lambda, U, v) = 0\}. \tag{3.1}$$

Moreover, by [16, Theorem 6.4.3], Γ_M satisfies one of the following three alternatives:

(1) Γ_M is unbounded in $\mathbb{R} \times E$;
(2) Γ_M contains a point $(\overline{\lambda}, \overline{\lambda}, 0)$ and $\overline{\lambda} \neq \lambda_*$;
(3) Γ_M contains a point $(\hat{\lambda}, \hat{\phi}, \hat{\psi})$ and $(\hat{\lambda}, \hat{\phi}, \hat{\psi}) \in \mathbb{R} \times (Y \setminus \{(\lambda, 0)\})$, where

$$Y = \{(\phi, \psi) \in E : \int_{\Omega_1} \psi \cdot \psi^* = 0\}. \tag{3.2}$$

We next claim that only case (1) can occur. Define $P_O = \{w \in C_n^1(\overline{O}) : w > 0 \text{ in } \overline{O}\}$. We first prove that

$$\Gamma_M \subset \mathbb{R} \times P_\Omega \times P_{\Omega_1}. \tag{3.3}$$

Assume that (3.3) is not true. Then there exist a point

$$(\lambda_\infty, U_\infty, v_\infty) \in \Gamma_M \cap (\mathbb{R} \times \partial(P_\Omega \times P_{\Omega_1})) \tag{3.4}$$

and a sequence $\{(\lambda_i, U_i, v_i)\}_{i=1}^\infty \subset \Gamma_M \cap (\mathbb{R} \times P_\Omega \times P_{\Omega_1})$ such that

$$\lim_{i\to\infty} (\lambda_i, U_i, v_i) = (\lambda_\infty, U_\infty, v_\infty) \quad \text{in } \mathbb{R} \times E.$$

It follows from the maximum principle that (U_∞, v_∞) satisfies one of the following three alternatives:

(a) $U_\infty \equiv 0$ in $\overline{\Omega}$, $v_\infty \equiv 0$ in $\overline{\Omega}_1$;
(b) $U_\infty > 0$ in $\overline{\Omega}$, $v_\infty \equiv 0$ in $\overline{\Omega}_1$;
(c) $U_\infty \equiv 0$ in $\overline{\Omega}$, $v_\infty > 0$ in $\overline{\Omega}_1$.

By integrating the second equation of (2.2) with $(U, v) = (U_i, v_i)$, we obtain

$$\int_{\Omega_1} v_i\Big(-\mu + \frac{cb(x)U_i}{1 + k\rho(x)v_i + mU_i}\Big)dx = 0 \quad \text{for all } i \in N. \tag{3.5}$$

Suppose that (a) or (c) occurs. Then for sufficiently large $i \in N$, we have

$$-\mu + \frac{cb(x)U_i}{1 + k\rho(x)v_i + mU_i} < 0 \quad \text{in } \Omega_1$$

because of $\mu > 0$. This contradicts (3.5). Suppose that (b) occurs. Then

$$\Delta U_\infty = U_\infty(\lambda_\infty - U_\infty) \text{ in } \Omega, \quad \partial_n U_\infty = 0 \text{ on } \partial\Omega,$$

and thus $U_\infty = \lambda_\infty$ in $\overline{\Omega}$. As a result, we must have $(\lambda_\infty, U_\infty, v_\infty) = (\lambda_*, \lambda_*, 0)$ by Proposition 2.3. This is a contradiction with (3.1) and (3.4). Consequently, (3.3) is true.

In view of (3.3), case (2) cannot occur. By (3.2), (3.3) and the fact that $\phi^* > 0$ in $\overline{\Omega}_1$, case (3) cannot occur. Hence, the only possibility is that case (1) occurs; that is, Γ_M is unbounded in $\mathbb{R} \times E$. By Proposition 2.1, for any fixed $\mu > 0$, (2.2) has at least one positive solution if $\lambda > \lambda_*$. □

Proof of Theorem 1.2. We need the following two lemmas.

Lemma 3.1. *Let $n \leq 3$. For any given $\lambda > \lambda_*(\mu)$ and $0 < \mu < -\lambda_1^N\big(-\frac{cb(x)}{m}, \Omega_1\big)$. Let (u_{k_i}, v_{k_i}) be any positive solution of (1.2) with $k = k_i$ and $\lim_{i\to\infty} k_i = \infty$, and denote $U_{k_i} = (1 + k_i\rho(x)v_{k_i})u_{k_i}$. Then there exists some non-negative function $\overline{U} \in C^1(\overline{\Omega})$, by passing to a subsequence if necessary, such that*

$$\lim_{i\to\infty}(U_{k_i}, v_{k_i}) = (\overline{U}, 0) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1).$$

Proof. By Proposition 2.2, the standard elliptic regularity theory ensures that there exists a pair of non-negative functions $(\overline{U}, \overline{v}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1)$, by passing to a subsequence if necessary, such that

$$\lim_{i\to\infty}(U_{k_i}, v_{k_i}) = (\overline{U}, \overline{v}) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1).$$

Recall that $\rho(x) > 0$ for each $x \in \Omega_1$. Then for each $x \in \Omega_1$, we have

$$\lim_{i\to\infty} \frac{u_{k_i}(x)v_{k_i}(x)}{1 + mu_{k_i}(x)} = \lim_{i\to\infty} \frac{U_{k_i}(x)}{1 + k_i\rho(x)v_{k_i}(x)} \cdot \frac{v_{k_i}(x)}{1 + mu_{k_i}(x)} = 0.$$

Hence, we apply the Lebesgue dominated convergence theorem to get

$$0 = \lim_{i\to\infty} \int_{\Omega_1} v_{k_i}\Big(-\mu + \frac{cb(x)u_{k_i}}{1 + mu_{k_i}}\Big)dx = \int_{\Omega_1} -\mu\overline{v}dx.$$

This means that $\overline{v} \equiv 0$ in $\overline{\Omega}_1$. This completes the proof. □

Lemma 3.2. *Let $n \leq 3$. For any given $\lambda > \lambda_*(\mu)$ and $0 < \mu < -\lambda_1^N\big(-\frac{cb(x)}{m}, \Omega_1\big)$. Let (u_{k_i}, v_{k_i}) be any positive solution of (1.2) with $k = k_i$ and $\lim_{i\to\infty} k_i = \infty$, and denote $U_{k_i} = (1 + k_i\rho(x)v_{k_i})u_{k_i}$. If $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^\infty$ is bounded, then by passing to a subsequence if necessary,*

$$\lim_{i\to\infty} u_{k_i} = \overline{u} \text{ uniformly in } \overline{\Omega}_1, \quad \lim_{i\to\infty} k_i v_{k_i} = \overline{w} \text{ uniformly in } C^1(\overline{\Omega}_1),$$

where $(\overline{u},\overline{w})$ *is a positive solution of* (1.3).

Proof. Set $w_{k_i}=k_i v_{k_i}$. Then (u_{k_i},w_{k_i}) satisfies

$$\begin{gathered}
\Delta[(1+\rho(x)w_{k_i})u_{k_i}]+u_{k_i}\Big(\lambda-u_{k_i}-\frac{b(x)v_{k_i}}{1+mu_{k_i}}\Big)=0,\quad x\in\Omega,\\
\Delta w_{k_i}+w_{k_i}\Big(-\mu+\frac{cb(x)u_{k_i}}{1+mu_{k_i}}\Big)=0,\quad x\in\Omega_1,\\
\partial_n u_{k_i}=0,\quad x\in\partial\Omega,\\
\partial_n w_{k_i}=0,\quad x\in\partial\Omega_1.
\end{gathered}\tag{3.6}$$

From the assumption that $\{\max_{\overline{\Omega}_1}k_i v_{k_i}\}_{i=1}^{\infty}$ is bounded, it is clear that $v_{k_i}\to 0$ uniformly in $\overline{\Omega}_1$. Moreover, the elliptic regularity theory and Lemma 3.1 ensure that there exists some non-negative function $\overline{w}\in C^1(\overline{\Omega}_1)$, by passing to a subsequence if necessary, such that

$$\lim_{i\to\infty}(U_{k_i},v_{k_i},w_{k_i})=(\overline{U},0,\overline{w})\quad\text{in } C^1(\overline{\Omega})\times C^1(\overline{\Omega}_1)\times C^1(\overline{\Omega}_1).\tag{3.7}$$

Therefore,

$$\lim_{i\to\infty}u_{k_i}=\frac{\overline{U}}{1+\rho(x)\overline{w}}=:\overline{u}\geq 0\quad\text{in } C^1(\overline{\Omega}).\tag{3.8}$$

By letting $i\to\infty$ in (3.6), together with (3.7) and (3.8), we see that $(\overline{u},\overline{w})$ is a non-negative solution of (1.3).

It remains to prove that $\overline{u}>0$ in $\overline{\Omega}$ and $\overline{w}>0$ in $\overline{\Omega}_1$. In view of (1.3) and (3.8), we have that $\overline{U}$ is a non-negative solution of

$$\Delta\overline{U}+\frac{\overline{U}}{1+\rho(x)\overline{w}}\Big(\lambda-\frac{\overline{U}}{1+\rho(x)\overline{w}}\Big)=0\ \text{in }\Omega,\quad \partial_n\overline{U}=0\ \text{on }\partial\Omega.$$

It follows from the maximum principle that either $\overline{U}>0$ or $\overline{U}\equiv 0$ in $\overline{\Omega}$. If $\overline{U}\equiv 0$ in $\overline{\Omega}$, by (3.8), then $\lim_{i\to\infty}u_{k_i}=0$ uniformly in $\overline{\Omega}_1$. Due to $\lambda>0$, we see that for large i,

$$\int_\Omega u_{k_i}\Big(\lambda-u_{k_i}-\frac{b(x)v_{k_i}}{1+mu_{k_i}}\Big)dx>0.$$

This ia a contradiction. This implies that $\overline{U}>0$ in $\overline{\Omega}$, and thus $\overline{u}>0$ in $\overline{\Omega}$. Similarly, by the maximum principle and the second equation of (1.3), we see that either $\overline{w}>0$ or $\overline{w}\equiv 0$ in $\overline{\Omega}_1$. If $\overline{w}\equiv 0$ in $\overline{\Omega}_1$, then $\overline{u}$ satisfies

$$\Delta\overline{u}+\overline{u}(\lambda-\overline{u})=0\ \text{in }\Omega,\quad \partial_n\overline{u}=0\ \text{on }\partial\Omega,\quad \overline{u}>0\ \text{in }\overline{\Omega}.$$

This implies that $\overline{u}\equiv\lambda$ in $\overline{\Omega}$. Then from the equation of v_{k_i}, we have

$$\begin{aligned}
0&=\lambda_1^N\Big(\mu-\frac{cb(x)u_{k_i}}{1+mu_{k_i}},\Omega_1\Big)\\
&\to\lambda_1^N\Big(\mu-\frac{cb(x)\lambda}{1+m\lambda},\Omega_1\Big)<\lambda_1^N\Big(\mu-\frac{cb(x)\lambda_*(\mu)}{1+m\lambda_*(\mu)},\Omega_1\Big)=0
\end{aligned}$$

by assumption $\lambda>\lambda_*(\mu)$. This is a contradiction, which means that $\overline{w}>0$ in $\overline{\Omega}_1$. Consequently, $(\overline{u},\overline{w})$ is a positive solution of (1.3). $\square$

Proof of Theorem 1.2. Let (u_{k_i}, v_{k_i}) be any positive solution of (1.2) with $k = k_i$ and $\lim_{i\to\infty} k_i = \infty$. We claim that $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is bounded. Otherwise, we assume that $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is unbounded. Note that $k_i v_{k_i}$ satisfies

$$\Delta(k_i v_{k_i}) + (k_i v_{k_i})\Big(-\mu + \frac{cb(x)u_{k_i}}{1+mu_{k_i}}\Big) = 0 \text{ in } \Omega_1, \quad \partial_n(k_i v_{k_i}) = 0 \text{ on } \partial\Omega_1.$$

By Harnack inequality [15], there exists some positive constant C independent of i such that

$$\max_{\overline{\Omega}_1} k_i v_{k_i} \le C \min_{\overline{\Omega}_1} k_i v_{k_i}.$$

This means that $\{\min_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is unbounded. Then by passing to a subsequence if necessary, we assume that

$$\lim_{i\to\infty} \min_{\overline{\Omega}_1} k_i v_{k_i} = \infty. \tag{3.9}$$

By Proposition 2.2 and (3.9), we have

$$\lim_{i\to\infty} u_{k_i} = \lim_{i\to\infty} \frac{U_{k_i}}{1+k_i\rho(x)v_{k_i}} = 0 \quad \text{uniformly in } \Omega_1. \tag{3.10}$$

Let $\tilde{v}_{k_i} = v_{k_i}/\max_{\overline{\Omega}_1} v_{k_i}$. Then $\tilde{v}_{k_i}$ satisfies

$$\begin{gathered}
\Delta\tilde{v}_{k_i} + \tilde{v}_{k_i}\Big(-\mu + \frac{cb(x)u_{k_i}}{1+mu_{k_i}}\Big) = 0 \quad \text{in } \Omega_1,\\
\partial_n \tilde{v}_{k_i} = 0 \quad \text{on } \partial\Omega_1,\\
\max_{\overline{\Omega}_1} \tilde{v}_{k_i} = 1.
\end{gathered}$$

By the elliptic regularity theory, we may assume that

$$\lim_{i\to\infty} \tilde{v}_{k_i} = \tilde{v} \text{ in } C^1(\overline{\Omega}_1), \quad \max_{\overline{\Omega}_1} \tilde{v} = 1,$$

where $\tilde{v} \in C^1(\overline{\Omega}_1)$ is some non-negative function. Thus by (3.10), the maximum principle ensures that $\tilde{v}$ is a positive solution of

$$\Delta\tilde{v} - \mu\tilde{v} = 0 \text{ in } \Omega_1, \quad \partial_n\tilde{v} = 0 \text{ on } \partial\Omega_1.$$

However, due to $\mu > 0$, we must derive $\tilde{v} \equiv 0$ from above equation. This ia a contradiction. This means that $\{\max_{\overline{\Omega}_1} k_i v_{k_i}\}_{i=1}^{\infty}$ is bounded. Consequently, by Lemma 3.2, we complete the proof of Theorem 1.2. □

Proof of Theorem 1.3. Set $\overline{U} = (1+\rho(x)\overline{w})\overline{u}$. Then (1.3) is written as

$$\begin{gathered}
\Delta\overline{U} + g_1(\overline{U}, \overline{w}) = 0, \quad x \in \Omega,\\
\Delta\overline{w} + g_2(\mu, \overline{U}, \overline{w}) = 0, \quad x \in \Omega_1,\\
\partial_n\overline{U} = 0, \quad x \in \partial\Omega,\\
\partial_n\overline{w} = 0, \quad x \in \partial\Omega_1,
\end{gathered} \tag{3.11}$$

where

$$\begin{gathered}
g_1(\overline{U}, \overline{w}) = \frac{\overline{U}}{1+\rho(x)\overline{w}}\Big(\lambda - \frac{\overline{U}}{1+\rho(x)\overline{w}}\Big), \quad x \in \Omega,\\
g_2(\mu, \overline{U}, \overline{w}) = \overline{w}\Big(-\mu + \frac{cb(x)\overline{U}}{1+\rho(x)\overline{w}+m\overline{U}}\Big), \quad x \in \Omega_1.
\end{gathered}$$

For any given $\lambda > 0$, (3.11) has a semitrivial solution: $(\lambda, 0)$. Therefore, (3.11) has a curve of semitrivial solution:

$$\Gamma_{\overline{U}} = \{(\mu, \overline{U}, \overline{w}) = (\mu, \lambda, 0) : \mu > 0\}.$$

By fixing $\lambda > 0$ and regarding μ as a bifurcation parameter, we show the following local bifurcation result.

Lemma 3.3. *For any fixed $\lambda > 0$, a branch of positive solutions of* (3.11) *bifurcates from $\Gamma_{\overline{U}}$ if and only if $\mu = \mu_*(\lambda)$, where $\mu_*(\lambda) = -\lambda_1^N\big(-\frac{cb(x)\lambda}{1+m\lambda}, \Omega_1\big)$, moreover, positive solutions of* (3.11) *near $(\mu_*(\lambda), \lambda, 0) \in \mathbb{R} \times X_1$ can be expressed as*

$$\Gamma_{\overline{\delta}} = \{(\mu, \overline{U}, \overline{w}) = (\mu(s), \lambda + s(\overline{\phi} + \overline{U}(s)), s(\psi_* + \overline{w}(s))) : s \in (0, \overline{\delta})\}$$

for some $\overline{\delta} > 0$, where

$$\overline{\phi} = (-\Delta + \lambda I)_{\Omega}^{-1}[\rho(x)\lambda^2\psi_*].$$

Here $(\mu(s), \overline{U}(s), \overline{w}(s))$ is a smooth function with respect to s and satisfies

$$(\mu(0), \overline{U}(0), \overline{w}(0)) = (\mu_*(\lambda), 0, 0)$$

and $\int_{\Omega_1} \psi_*\overline{w}(s)dx = 0$.

The proof of the above lemma is similar to that of Proposition 2.3, we omit it. The following lemma gives further information on the bifurcation curve $\Gamma_{\overline{\delta}}$.

Lemma 3.4. *Let $n \leq 3$. For any fixed $\lambda > 0$, there is an unbounded connected set Γ_M of positive solutions of* (3.11) *in $\mathbb{R} \times E$ which bifurcates from $\{(\mu, \overline{U}, \overline{w}) = (\mu, \lambda, 0) : \mu > 0\}$ at $(\mu_*(\lambda), \lambda, 0)$ and remains bounded until μ approaches 0, where it blows up. Moreover, $(0, \mu_*(\lambda)) \subset \mathrm{Proj}_\mu \Gamma_M \subset \big(0, -\lambda_1^N\big(-\frac{cb(x)}{m}, \Omega_1\big)\big)$, $\overline{U}_\mu$ is bounded in $C^1(\overline{\Omega})$ and $\lim_{\mu\to 0} \overline{w}_\mu = \infty$ in $C^1(\overline{\Omega}_1)$, where $(\mu, \overline{U}_\mu, \overline{w}_\mu) \in \Gamma_M$.*

Proof. Define an operator $\overline{G} : \mathbb{R} \times E \to E$ by

$$\overline{G}(\mu, \overline{U}, \overline{w}) = \begin{pmatrix} \overline{U} - \lambda \\ \overline{w} \end{pmatrix} - \begin{pmatrix} (-\Delta + I)_{\Omega}^{-1}[\overline{U} - \lambda + g_1(\overline{U}, \overline{w})] \\ (-\Delta + I)_{\Omega_1}^{-1}[\overline{w} + g_2(\mu, \overline{U}, \overline{w})] \end{pmatrix}.$$

It is clear that (3.11) is equivalent to $\overline{G}(\mu, \overline{U}, \overline{w}) = 0$. It follows from [16, Theorem 6.4.3] that the local bifurcation branch $\Gamma_{\overline{\delta}}$ is extended into a global curve. Let $\Gamma_M \subset \mathbb{R} \times E$ be the maximal connected set satisfying

$$\Gamma_{\overline{\delta}} \subset \Gamma_M \subset \left\{(\mu, \overline{U}, \overline{w}) \in \mathbb{R} \times E \backslash \{(\mu_*(\lambda), \lambda, 0)\} : \overline{G}(\mu, \overline{U}, \overline{w}) = 0\right\}.$$

Similar to Theorem 1.1, we can show that Γ_M is unbounded in $\mathbb{R} \times E$.

We show that $\|\overline{U}_\mu\|_{C^1(\overline{\Omega})} < C$, where C is independent of μ. Let $(\mu, \overline{U}_\mu, \overline{w}_\mu) \in \Gamma_M$. Integrating the first equation of (3.11) over Ω, we get

$$\int_\Omega \Big(\frac{\overline{U}_\mu}{1 + \rho(x)\overline{w}_\mu}\Big)^2 dx = \lambda \int_\Omega \frac{\overline{U}_\mu}{1 + \rho(x)\overline{w}_\mu} dx \leq \lambda |\Omega|^{1/2} \|\frac{\overline{U}_\mu}{1 + \rho(x)\overline{w}_\mu}\|_{2,\Omega},$$

and thus

$$\|\frac{\overline{U}_\mu}{1 + \rho(x)\overline{w}_\mu}\|_{2,\Omega} \leq \lambda |\Omega|^{1/2}. \tag{3.12}$$

Hence, we apply Harnack inequality [15, Lemma 4.3] with $p = 2$ to the first equation of (3.11) and derive

$$\max_{\overline{\Omega}} \overline{U}_\mu \leq C^* \min_{\overline{\Omega}} \overline{U}_\mu \tag{3.13}$$

for some positive constant C^* independent of μ. Then (3.12) yields

$$|\Omega_0|^{1/2} \min_{\overline{\Omega}_0} \overline{U}_\mu \le \|\overline{U}_\mu\|_{2,\Omega_0} \le \|\frac{\overline{U}_\mu}{1+\rho(x)\overline{w}_\mu}\|_{2,\Omega} \le \lambda|\Omega|^{1/2}.$$

This means that $\min_{\overline{\Omega}_0} \overline{U}_\mu \le \lambda(|\Omega|/|\Omega_0|)^{1/2}$. By (3.13), we have

$$\max_{\overline{\Omega}} \overline{U}_\mu \le C^* \lambda \left(|\Omega|/|\Omega_0|\right)^{1/2}.$$

This shows that $\overline{U}_\mu$ is bounded in $\overline{\Omega}$. Then the boundedness of $\{\|\overline{U}_\mu\|_{C^1(\overline{\Omega})}\}$ is obtained by elliptic regularity theory and Sobolev embedding theorem.

For any point $(\mu, \overline{U}_\mu, \overline{w}_\mu) \in \Gamma_M \backslash \{(\mu_*(\lambda), \lambda, 0)\}$, $\overline{U}_\mu > 0$ in $\overline{\Omega}$ and $\overline{w}_\mu$ is a positive solution of

$$-\Delta \overline{w}_\mu - \frac{c(x)\overline{U}_\mu}{1+\rho(x)\overline{w}_\mu + m\overline{U}_\mu}\overline{w}_\mu = -\mu\overline{w}_\mu \text{ in } \Omega_1, \quad \partial_n \overline{w}_\mu = 0 \text{ on } \partial\Omega_1. \tag{3.14}$$

Thus

$$\lambda_1^N\Big(-\frac{cb(x)}{m}, \Omega_1\Big) < -\mu = \lambda_1^N\Big(-\frac{cb(x)\overline{U}_\mu}{1+\rho(x)\overline{w}_\mu + m\overline{U}_\mu}, \Omega_1\Big) < 0;$$

that is,

$$0 < \mu < -\lambda_1^N\Big(-\frac{cb(x)}{m}, \Omega_1\Big).$$

As a result, $\operatorname{Proj}_\mu \Gamma_M \subset \big(0, -\lambda_1^N\big(-\frac{cb(x)}{m}, \Omega_1\big)\big)$.

According to the unboundedness of Γ_M in $\mathbb{R} \times E$, we must have $\{\|\overline{w}_\mu\|_{C^1(\overline{\Omega}_1)}\}$ is unbounded. Hence, we apply Harnack inequality [15, Lemma 4.3] to (3.14) and derive that $\{\min_{\overline{\Omega}_1} \overline{w}_\mu\}$ is also unbounded. Thus, there exists some $\mu_\infty \in [0, \overline{\mu}]$ and a sequence $\{\mu_i\}_{i=1}^\infty$ such that

$$\lim_{i\to\infty} \mu_i = \mu_\infty \text{ and } \lim_{i\to\infty} \min_{\overline{\Omega}_1} \overline{w}_{\mu_i} = \infty.$$

Let $\widehat{w}_{\mu_i} = \overline{w}_{\mu_i} / \max_{\overline{\Omega}_1} \overline{w}_{\mu_i}$. Then

$$\begin{gathered}
\Delta \widehat{w}_{\mu_i} + \widehat{w}_{\mu_i}\Big(-\mu_i + \frac{cb(x)\overline{U}_{\mu_i}}{1+\rho(x)\overline{w}_{\mu_i} + m\overline{U}_{\mu_i}}\Big) = 0, \quad x \in \Omega_1, \\
\partial_n \widehat{w}_{\mu_i} = 0, \quad x \in \partial\Omega_1, \\
\max_{\overline{\Omega}_1} \widehat{w}_{\mu_i} = 1.
\end{gathered}$$

The elliptic regularity theory ensures us to obtain $\lim_{i\to\infty} \widehat{w}_{\mu_i} = \widehat{w}$ in $C^1(\overline{\Omega}_1)$, where $\widehat{w}$ is a positive solution of

$$\Delta\widehat{w} - \mu_\infty \widehat{w} = 0 \text{ in } \Omega_1, \quad \partial_n \widehat{w} = 0 \text{ on } \partial\Omega_1, \quad \max_{\overline{\Omega}_1} \widehat{w} = 1.$$

Here we use the fact that $\rho(x) > 0$ in Ω_1. From above equation, we must have $\mu_\infty = 0$, which implies that $(0, \mu_*(\lambda)) \subset \operatorname{Proj}_\mu \Gamma_M$. Thus, $(0, \mu_*(\lambda)) \subset \operatorname{Proj}_\mu \Gamma_M \subset \big(0, -\lambda_1^N\big(-\frac{cb(x)}{m}, \Omega_1\big)\big)$. □

Proof of Theorem 1.3. By Lemma 3.4, it remains to show the convergence result of $\overline{u}_\mu$. Since $\rho(x) > 0$ in Ω_1 and $\lim_{\mu\to 0} \min_{\overline{\Omega}_1} \overline{w}_\mu = \infty$, we have

$$\lim_{\mu\to 0} \overline{u}_\mu = \lim_{\mu\to 0} \frac{\overline{U}_\mu}{1+\rho(x)\overline{w}_\mu} = 0 \quad \text{uniformly in } \Omega_1. \tag{3.15}$$

Dividing the first equation of (3.11) with $(\overline{U}, \overline{v}) = (\overline{U}_\mu, \overline{v}_\mu)$ by $\overline{U}_\mu$ and integrating the resulting equation over Ω, we get

$$\int_\Omega \frac{(\lambda - \overline{u}_\mu)}{1 + \rho(x)\overline{w}_\mu} dx = -\int_\Omega \frac{|\nabla \overline{U}_\mu|^2}{\overline{U}_\mu^2} dx \le 0.$$

Thus

$$\int_{\overline{\Omega}_0} (\lambda - \overline{u}_\mu) dx \le -\int_{\Omega \setminus \overline{\Omega}_0} \frac{\lambda - \overline{u}_\mu}{1 + \rho(x)\overline{w}_\mu} dx.$$

Passing $\mu \to 0$ in above inequality, we obtain

$$\int_{\overline{\Omega}_0} (\lambda - \overline{u}_0) dx \le 0. \tag{3.16}$$

On the other hand, we integrate the first equation of (3.11) with $(\overline{U}, \overline{v}) = (\overline{U}_\mu, \overline{v}_\mu)$ to derive

$$\int_{\overline{\Omega}_0} \overline{u}_\mu(\lambda - \overline{u}_\mu) dx + \int_{\Omega \setminus \overline{\Omega}_0} \overline{u}_\mu(\lambda - \overline{u}_\mu) dx = 0.$$

Letting $\mu \to 0$, together with (3.15), we obtain

$$\int_{\overline{\Omega}_0} \overline{u}_0(\lambda - \overline{u}_0) dx = 0. \tag{3.17}$$

By (3.16) and (3.17), it is obvious that

$$\int_{\overline{\Omega}_0} (\lambda - \overline{u}_0)^2 dx = \lambda \int_{\overline{\Omega}_0} (\lambda - \overline{u}_0) dx - \int_{\overline{\Omega}_0} \overline{u}_0(\lambda - \overline{u}_0) dx \le 0.$$

Hence, $\overline{u}_0 \equiv \lambda$ in $\overline{\Omega}_0$. The proof of Theorem 1.3 is complete. □

Acknowledgments. This work was supported by the Natural Science Foundation of China (11801431, 61872227, 61672021), by the Postdoctoral Science Foundation of China (2018T111014, 2018M631133), by the Natural Science Foundation of Shaanxi Province (2018JQ1004, 2018JQ1017), by the Scientific Research Program Funded by Shaanxi Provincial Education Department (Program No. 18JK0343)

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