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PERIODIC SOLUTIONS OF SECOND-ORDER NON-AUTONOMOUS DYNAMICAL SYSTEMS WITH VANISHING GREEN'S FUNCTIONS

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Abstract. In this article, we study the existence and multiplicity of positive periodic solutions for second-order non-autonomous dynamical systems when Green's functions are non-negative. The proofs are based on a nonlinear alternative principle of Leray-Schauder and the fixed point theorem in cones. Some recent results in the literature are generalized and improved.

1. Introduction

The main purpose of this paper is to study the existence and multiplicity of positive solutions of the second-order dynamical system

$$\begin{gathered} \ddot{x} + a(t)x = f(t,x), \\ x(0) = x(T), \quad x'(0) = x'(T). \end{gathered} \tag{1.1}$$

where $a(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$ and $f(t,x) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}^N, \mathbb{R}^N)$. The type of perturbation $f(t,x)$ that we are interested in can be not only superlinear and sublinear, but also combinations of them. From the physical explanation, The *sublinearity* of $f(t,x)$ means that for each $i = 1, \dots, N$, it holds

$$f_i^0 = \lim_{|x|\to 0} \frac{f_i(t,x)}{|x|} = +\infty \quad \text{and} \quad f_i^\infty = \lim_{|x|\to+\infty} \frac{f_i(t,x)}{|x|} = 0 \quad \text{uniformly in } t.$$

The *superlinearity* of $f(t,x)$ means that

$$f_i^0 = \lim_{|x|\to 0} \frac{f_i(t,x)}{|x|} = 0 \quad \text{and} \quad f_i^\infty = \lim_{|x|\to+\infty} \frac{f_i(t,x)}{|x|} = +\infty \quad \text{uniformly in } t.$$

During the previous two decades, some classical tools have been used in the study of periodic solutions of equation (1.1), including the method of upper and lower solutions [11, 18], degree theory [7, 22, 23], fixed point theorems in cones for completely continuous operators [9, 19], Schauder's fixed point theorem [3, 8, 20] and a nonlinear alternative principle of Leray-Schauder [4, 5, 12].

In the above mentioned works, when one tried to apply some fixed point theorems in cones, or the nonlinear alternative principle of Leray-Schauder, to study the

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existence of periodic solutions of equation (1.1), one major assumption is that the corresponding Green's function $G_i(t,s)$ for the scalar linear differential equation

$$x'' + a_i(t)x = 0 \tag{1.2}$$

is positive ($i = 1, 2, \dots, N$), which is equivalent to the strict anti-maximum principle for equation (1.2). Such an assumption plays an important role in constructing the following cone

$$K_1 = \{x \in X : \min_{0 \le t \le T} x(t) \ge \sigma_i \parallel x \parallel\}$$

where

$$\sigma_i = \frac{m_i}{M_i}, \quad m_i = \min_{0 \le s,t \le T} G_i(t,s), \quad M_i = \max_{0 \le s,t \le T} G_i(t,s).$$

When the Green's function vanishes, we know that $m = 0$ and K_1 becomes the cone of nonnegative functions, which is not effective in obtaining the desired estimates. For example, when $a_i(t) = k^2$ with $k > 0$ and $k \ne 2n\pi/T$ ($n \in \mathbb{Z}^+$), the Green's function is given as [9, 19]

$$G_i(t,s) = \begin{cases} \frac{\sin k(t-s) + \sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \le s \le t \le T, \\ \frac{\sin k(s-t) + \sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \le t \le s \le T, \end{cases} \tag{1.3}$$

and

$$\frac{1}{2k} \cot \frac{kT}{2} \le G_i(t,s) \le \frac{1}{2k \sin(kT/2)}.$$

Therefore, the positiveness of Green's fuction is equivalent to $0 < k^2 < \lambda_1 = (\pi/T)^2$. Note that λ_1 is the first eigenvalue of the linear problem (1.2) with the Dirichlet condition $x(0) = x(T) = 0$. For the critical case of $k = \frac{\pi}{T}$, the Green's function vanishes at $t = s$, and thus the results in [1, 4, 5, 6, 19] cannot be applied to such a critical case. In this paper, we focus on the case of $k \le \pi/T$, and we assume that the following condition holds

(A1) The associated Green's function $G_i(t,s)$ of (1.2) is non-negative for all $(t,s) \in [0,T] \times [0,T]$.

Our main motivation comes from the recent works [2, 10, 14, 15], in which the second order systems have been studied in the case where the associated non-negative Green's functions may have zeros. Chu and O'Regan [2] established the multiplicity results for second order non-autonomous singular Dirichlet systems

$$\begin{gathered} \ddot{x} + q(t)f(t,x) + e(t) = 0, \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = 0. \end{gathered}$$

based on a well-known fixed point theorem in cones and Leray-Schauder alternative principle. Especially, we observe that even when the Green's function vanishes, the following fact also holds

$$\nu = \min_{0 \le s \le T} \int_0^T G(t,s)dt > 0.$$

Based on this fact, Graef, Kong and Wang [10] introduced the cone

$$K_2 = \big\{x \in X : x(t) \ge 0 \text{ and } \int_0^T x(t)dt \ge \frac{\nu}{M} \parallel x \parallel \big\}.$$

Using the above cone, it was proved in [10] that equation

$$x'' + a(t)x = g(t)f(x)$$

has at least one nontrivial T-periodic solution for the superlinear or sublinear case. Li and Zhang [14] extended this result to a more general case. They proved some existence and nonexistence results for nonnegative solutions of the following second-order periodic boundary-value problem with a parameter $\lambda \in (0,\infty)$, $i = 1,2,\dots,n$,

$$x_i'' + a_i(t)x_i = \lambda g^i(t) f^i(x), \quad 0 \le t \le T,$$

by using fixed point theorems in a cone under different combinations of superlinearity and sublinearity of functions f^i at zero and infinity for an appropriately chosen parameter λ. Liao [15] showed that the scalar problem (1.1) has at least two positive solution under the given conditions. In this paper, we will construct a new cone and establish the existence and multiplicity of nontrivial T-periodic solutions for equation (1.1) using a nonlinear alternative principle of Leray-Schauder and a fixed point theorem in cones. We emphasize that in this study the corresponding nonnegative Green's function of linear system of (1.1) may have zeros. Hence, we need to overcome this obstacle for system (1.1).

Our goal is to obtain the existence of positive periodic solutions for the system

$$\begin{aligned} \ddot{x} + a_1(t)x &= \sqrt{(x^2+y^2)^\alpha} + \mu\sqrt{(x^2+y^2)^\beta} + e_1(t), \\ \ddot{y} + a_2(t)y &= \sqrt{(x^2+y^2)^\alpha} + \mu\sqrt{(x^2+y^2)^\beta} + e_2(t). \end{aligned}$$

with $a_1, a_2, e_1, e_2 \in C[0,T]$, $0 < \alpha < 1$, $\beta > \alpha$, and $\mu > 0$ is a given parameter. Here we wish to point out that in our results, e_1 and e_2 may not be positive. Thus, we generalize and improve some results presented in [8, 16] and even for the scalar cases in [12].

This articles is organized as follows. In Section 2, some preliminary results and notations will be introduced. In Section 3, by employing a nonlinear alternative principle of Leray-Schauder, we state and prove the existence result for (1.1). In Section 4, we establish the existence result for (1.1) by using the well-known fixed point theorem in cones.

2. Preliminaries

In this article we use the following notation: $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_i \ge 0 \text{ for } i = 1,2,\dots,N\}$ with the norm $|x| = \max_i |x_i|$. For $x = (x_1,\dots,x_N)$, $y = (y_1,\dots,y_N)$, we write $x \ge y$, if $x - y = (x_1 - y_1, \dots, x_N - y_N) \in \mathbb{R}_+^N$. We say that a function $\varphi : \mathbb{R}^N \to \mathbb{R}$ is nondecreasing if $\varphi(x) \ge \varphi(y)$ for $x, y \in \mathbb{R}^N$ with $x \ge y$. Given $\psi \in L^1[0,T]$, we write $\psi \succ 0$ if $\psi \ge 0$ for all $t \in [0,T]$ and it is positive in a set of positive measure. We take $X = \mathbb{C}[0,T] \times \dots \times \mathbb{C}[0,T]$(N copies) with the supremum norm $\|\cdot\|$.

We denote by $a_1, a_2, \dots, a_N$ and $e_1, e_2, \dots, e_N$ the components of given functions $a(t), e(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$, respectively. For each $i = 1,2,\dots,N$, we consider the scalar equation

$$x'' + a_i(t)x = e_i(t). \tag{2.1}$$

with periodic boundary conditions

$$x(0) = x(T), x'(0) = x'(T). \tag{2.2}$$

We assume condition (A1). In other words, the anti-maximum principle holds for (2.1). In this case, the solution of (2.1) is given by

$$x(t) = (\mathcal{L}e_i)(t) := \int_0^T G_i(t,s)e_i(s)ds.$$

Some classes of potentials $a(t)$ for (A1) have been presented in [19]. Let $K(q)$ denote the best Sobolev constant in the inequality

$$C\|u\|_q^2 \le \|u'\|_2^2 \quad \text{for all } u \in H_0^1(0,1).$$

The explicit formula for $K(q)$ is

$$K(q) = \begin{cases} \frac{2\pi}{q}\big(\frac{2}{2+q}\big)^{1-2/q}\big(\frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})}\big)^2 & \text{if } 1 \le q < \infty, \\ 4 & \text{if } q = \infty, \end{cases}$$

where Γ is the Gamma function [24].

Lemma 2.1 ([19])**.** *For $i = 1, 2, \dots, N$ assume that $a_i(t) \succ 0$ and $a_i \in L^p[0,T]$ for some $1 \le p \le \infty$. If*

$$\|a_i\|_p \le K(2\tilde{p}),$$

then (A1) *holds.*

Under assumption (A1), we denote

$$M_i = \max_{0 \le s,t \le T} G_i(t,s), \quad M = \max_{0 \le i \le N} M_i.$$

We also use $w(t)$ to denote the unique periodic solution of (2.1) with $e_i(t) = 1$, $i = 1, 2 \dots, N$; i.e.,

$$w_i(t) = (\mathcal{L}1)(t) := \int_0^T G_i(t,s)\,ds,$$
$$w_*(t) = \min_{i,t} w(t), \quad w^*(t) = \max_{i,t} w(t).$$

Define

$$K = \big\{x \in X : x_i(t) \ge 0 \text{ for all } t, \text{ and } \int_0^T x_i(t)dt \ge \frac{\delta}{M}\|x_i\|\big\},$$

where

$$\delta = \min_{0 \le i \le N} \delta_i, \quad \delta_i = \min_{0 \le s \le T} \int_0^T G_i(t,s)dt.$$

Under condition (A1), we observe that even when the Green's function vanishes, we have $\delta_i > 0$, and $\delta > 0$. One may readily verify that K is a cone in X [17].

Lemma 2.2 ([17])**.** *Assume that Ω is a relatively compact subset of a convex set K in a normed space X. Let $T : \Omega \to K$ be a compact map with $0 \in \Omega$. Then one of the following two statements hold:*

(I) *T has at least one fixed point in Ω.*
(II) *There exist $x \in \partial\Omega$ and $0 < \lambda < 1$ such that $x = \lambda Tx$.*

To obtain a second periodic solution of (1.1), we need the following well known fixed point theorem in cones [13] and Jensen's inequality, see [21]. Let K be a cone in X and D a subset of X, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

Lemma 2.3 ([13, p. 148]). *Let X be a Banach space and K be a cone in X. Assume that Ω^1, Ω^2 are open subsets of X with $\Omega^1_K \neq \emptyset$, $\bar{\Omega}^1_K \subset \Omega^2_K$, and let $S : \bar{\Omega}^2_K \to K$ be a continuous and completely continuous operator such that*

(i) *$x \neq \lambda Sx$ for $\lambda \in [0, 1)$ and $x \in \partial_K \Omega^1$, and*
(ii) *there exists $v \in K \setminus \{0\}$ such that $x \neq Sx + \lambda v$ for all $x \in \partial_K \Omega^2$ and all $\lambda > 0$.*

Then S has a fixed point in $\bar{\Omega}^2_K \backslash \Omega^1_K$.

Lemma 2.4 (Jensen's inequality). *Let m be a (positive) measure and let Ω be a measurable set with $m(\Omega) = 1$. Let I be an interval and suppose that u is a real function in $L^1(dm)$ with $u(t) \in I$ for all $t \in \Omega$. If f is convex on I, then*

$$f\Big(\int_\Omega u(t)dm(t)\Big) \leq \int_\Omega f(u(t))dm(t).$$

3. Existence result (I)

In this section we establish the first existence result by using the nonlinear alternative of Leray-Schauder. Define an operator $T : X \to X$ by $Tx = (T_1 x, T_2 x, \dots, T_1 x)^\top$, where

$$(T_i x)(t) = \int_0^T G_i(t, s) f_i(s, x(s))ds, \quad i = 1, 2, \dots, N.$$

Lemma 3.1. *T is well defined and maps X into K. Moreover, T is continuous and completely continuous.*

It is easy to see that finding a fixed point for the operator T is equivalent to finding a T-periodic solution of system (1.1). For the next theorem we use the following assumptions:

(A2) There exists a continuous function $\phi_i \succ 0$ such that each component f_i of f satisfies $f_i(t, x) \geq \phi_i(t)$ for all $(t, x) \in [0, T] \times \mathbb{R}^N_+$.

(A3) There exist continuous, non-negative functions $g_i(x)$ and $h_i(x)$ such that
$$f_i(t, x) \leq g_i(x) + h_i(x) \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^N_+,$$
and $g_i(x) > 0$ and $h_i(x)/g_i(x)$ are non-decreasing in $x \in \mathbb{R}^N_+$.

(A4) There exist a positive number r such that
$$g_i(r, \dots, r) + h_i(r, \dots, r) < \frac{\delta r}{M \omega^* T}$$
for $i = 1, 2, \dots, N$.

Theorem 3.2. *Under assumptions* (A1)–(A4)*, equation* (1.1) *has at least one T-periodic solution with $0 < |x| < r$.*

Proof. Consider a family of equations
$$\ddot{x} + a_i(t)x = \lambda f_i(t, x(t)), \tag{3.1}$$
where $\lambda \in [0, 1]$. Problem (3.1) is equivalent to the following fixed point problem
$$x_i(t) = \lambda (T_i x)(t) = \lambda \int_0^T G_i(t, s) f_i(s, x(s))ds. \tag{3.2}$$
We claim that for any $\lambda \in [0, 1]$, equation (3.2) has no fixed point x with $x \in \partial\Omega$, where
$$\Omega = \{x \in X : |x| < r\},$$

and $X = \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^N)$ is a Banach space with the norm $|x| = \max_i |x_i|$.

Otherwise, assume that x is a solution of (3.2) for some $\lambda_0 \in [0,1]$ such that $|x| = r$. Without loss of generality, we assume that $|x_j| = r$ for some $j = 1, 2, \dots, N$. Thus we have

$$\begin{aligned}
\int_0^T x_j(t)dt &= \lambda_0 \int_0^T \int_0^T G_j(t,s) f_j(s, x(s))\, ds\, dt \\
&= \lambda_0 \int_0^T f_j(s, x(s)) \int_0^T G_j(t,s)\, dt\, ds \\
&\geq \lambda_0 \delta_j \int_0^T f_j(s, x(s)) ds \\
&= \frac{\delta_j}{M_j} M_j \lambda_0 \int_0^T f_j(s, x(s)) ds \\
&\geq \frac{\delta}{M} \max_t \Big\{ \lambda_0 \int_0^T G_j(t,s) f_j(s, x(s)) ds \Big\} \\
&= \frac{\delta}{M} \|x_j\|.
\end{aligned}$$

Thus, for all t we have

$$\int_0^T x_j(t)dt \geq \frac{\delta}{M}\|x_j\| = \frac{\delta}{M} r.$$

On the other hand, for all t, it follows from condition (A3) that

$$\begin{aligned}
\int_0^T x_j(t)dt &= \lambda \int_0^T \int_0^T G_j(t,s) f_j(s, x(s))\, ds\, dt \\
&\leq \int_0^T \int_0^T G_j(t,s) f_j(s, x(s))\, ds\, dt \\
&\leq \int_0^T \int_0^T G_j(t,s) g_j(x(s)) \big\{ 1 + \frac{h_j(x(s))}{g_j(x(s))} \big\}\, ds\, dt \\
&\leq \int_0^T \int_0^T G_j(t,s) g_j(r, \dots, r)) \big\{ 1 + \frac{h_j(r, \dots, r)}{g_j(r, \dots, r)} \big\}\, ds\, dt \\
&\leq \omega^* T g_j(r, \dots, r)) \big\{ 1 + \frac{h_j(r, \dots, r)}{g_j(r, \dots, r)} \big\}.
\end{aligned}$$

Hence,

$$\frac{\delta}{M} r \leq \omega^* T (g_j(r, \dots, r) + h_j(r, \dots, r)).$$

This is a contradiction to condition (A4); thus the claim is proved.

From this claim, the nonlinear alternative principle of Leray-Schauder guarantees that (3.2) (with $\lambda = 1$) has a fixed point, denoted by x, in Ω, that is, equation (3.1) (with $\lambda = 1$) has a periodic solution x with $|x| < r$.

Finally, by condition (A2), we obtain

$$\begin{aligned}
\int_0^T x_j(t)dt &= \int_0^T \int_0^T G_j(t,s) f_j(s, x(s))\, ds\, dt \\
&\geq \int_0^T \int_0^T G_j(t,s) \phi_i(s)\, ds\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \phi_i(s) \int_0^T G_j(t,s)\,dt\,ds \\
&\ge \delta_j \int_0^T \phi_i(s) ds > 0.
\end{aligned}
$$

This implies that x is a nontrivial T-periodic solution. $\square$

Example 3.3. Suppose that $a_1(t), a_2(t)$ satisfy (A1) and consider the differential equations

$$
\begin{aligned}
\ddot{x} + a_1(t)x &= \sqrt{(x^2+y^2)^\alpha} + \mu\sqrt{(x^2+y^2)^\beta} + e_1(t), \\
\ddot{y} + a_2(t)y &= \sqrt{(x^2+y^2)^\alpha} + \mu\sqrt{(x^2+y^2)^\beta} + e_2(t).
\end{aligned}
\tag{3.3}
$$

where $a_i, e_i \in X$, $e_i \succ 0$, $i = 1, 2$, $0 < \alpha < 1$, $\beta > \alpha$, and $\mu > 0$ is a positive parameter. Then we have

- (i) if $\beta < 1$, (3.3) has at least one nontrivial T-periodic solution for each $\mu > 0$;
- (ii) if $\beta \ge 1$, (3.3) has at least one nontrivial periodic solution for each $0 < \mu < \mu_*$, where μ_* is some positive constant.

Proof. To apply Theorem 3.2, we take

$$
\begin{gathered}
\phi_i(t) = e_i(t), \quad i = 1, 2, \\
g_1(x,y) = g_2(x,y) = \sqrt{(x^2+y^2)^\alpha} + e^*, \\
h_1(x,y) = h_2(x,y) = \mu\sqrt{(x^2+y^2)^\beta}
\end{gathered}
$$

where $e^* = \max_t\{e_1(t), e_2(t)\}$. Clearly, (A2) is satisfied since $e \succ 0$. Moreover, since $0 < \alpha < \beta$, it is easy to verify that $g_i(x,y) (i = 1, 2)$ and

$$
\frac{h_i(x,y)}{g_i(x,y)} = \frac{\mu\sqrt{(x^2+y^2)^\beta}}{\sqrt{(x^2+y^2)^\alpha} + e^*}
$$

is non-decreasing in $x \in \mathbb{R}_+^2$. Then (A3) is satisfied. Now the existence condition (A4) becomes

$$
\mu < \frac{\delta r - 2^{\frac{\alpha}{2}} TM\omega^* r^\alpha - e^* TM\omega^*}{2^{\frac{\beta}{2}} TM\omega^* r^\beta}, \quad i = 1, 2,
$$

for some $r > 0$. So equation (3.3) has at least one T-periodic solution for

$$
0 < \mu < \mu_* := \sup_{r>0} \frac{\delta r - 2^{\frac{\alpha}{2}} TM\omega^* r^\alpha - e^* TM\omega^*}{2^{\frac{\beta}{2}} TM\omega^* r^\beta}.
$$

Note that $\mu_* = \infty$ if $\beta < 1$ and $\mu_* < \infty$ if $\beta \ge 1$. We have the desired results (i) and (ii). $\square$

4. Existence result (II)

In this section, we consider system (1.1) by the well-known fixed point theorem in cones (i.e. Lemma 2.3). For the next theorem we use the following assumptions:

- (A5) There exist continuous, non-negative functions $g_i^1(x)$ and $h_i^1(x)$ on $\mathbb{R}_+^N$ such that
$$
f_i(t,x) \ge g_i^1(x) + h_i^1(x) \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}_+^N,
$$
and $g_i^1(x) > 0$ is non-decreasing and convex, and $h_i^1(x)/g_i^1(x)$ is non-increasing in $x \in \mathbb{R}_+^N$;

(A6) There exists a positive number $R > r$ such that
$$\frac{R}{\delta} < g_i^1\Big(0,\dots,\frac{\delta R}{MT},\dots,0\Big)\Big\{1+\frac{h_i^1(R,\dots,R)}{g_i^1(R,\dots,R)}\Big\},$$
where δ and M are the same as in Section 2.

Theorem 4.1. *Suppose that* (A1) *and* (A3)–(A6) *are satisfied. Then, besides the periodic solution x constructed in Theorem 3.2, equation* (1.1) *has another positive periodic solution $\tilde{x}$ with $r < |\tilde{x}| \le R$.*

Proof. Define the open sets as
$$\Omega^1 = \{x \in X : |x| < r\}, \quad \Omega^2 = \{x \in X : |x| < R\}$$
and the operator $S : \bar{\Omega}_K^2 \backslash \Omega_K^1 \to K$ is defined by $Sx = (S_1x, S_2x, \dots, S_1x)^T$, where
$$(S_ix)(t) = \int_0^T G_i(t,s)f_i(s,x(s))ds, \quad i = 1,2,\dots,N.$$
Using Lemma 3.1, one may readily verify that $S : \bar{\Omega}_K^2 \backslash \Omega_K^1 \to K$ is well defined. Next we claim that:

(i) $x \ne \lambda Sx$ for $\lambda \in [0,1)$ and $x \in \partial_K\Omega^1$, and
(ii) there exists $v \in K \setminus \{0\}$ such that $x \ne Sx + \lambda v$ for all $x \in \partial_K\Omega^2$ and all $\lambda > 0$.

We start with (i). Suppose there exist $x \in \partial_K\Omega^1$ and $\lambda \in [0,1)$ such that $x = \lambda Sx$, We can assume that $\lambda \ne 0$. Now since $x = \lambda Sx$ we have
$$\begin{gathered} \ddot{x} + a(t)x = \lambda f(t,x(t)), \\ x(0) = x(T), \quad x'(0) = x'(T). \end{gathered}$$
Since $x \in \partial_K\Omega^1$, then $|x| = r$. Without loss of generality, we assume that $|x_j| = r$ for some $j = 1,2,\dots,N$. Thus we have
$$\begin{aligned} \int_0^T x_j(t)dt &= \lambda \int_0^T \int_0^T G_j(t,s)f_j(s,x(s))\,ds\,dt \\ &= \lambda \int_0^T f_j(s,x(s)) \int_0^T G_j(t,s)\,dt\,ds \\ &\ge \lambda\delta_j \int_0^T f_j(s,x(s))ds \\ &= \frac{\delta_j}{M_j} M_j \lambda \int_0^T f_j(s,x(s))ds \\ &\ge \frac{\delta}{M} \max_t \Big\{\lambda \int_0^T G_j(t,s)f_j(s,x(s))ds\Big\} \\ &= \frac{\delta}{M}\|x_j\| \end{aligned}$$
Thus, for all t we have
$$\int_0^T x_j(t)dt \ge \frac{\delta}{M}\|x_j\| = \frac{\delta}{M}r.$$

On the other hand, for all t, it follows from condition (A3) that

$$
\begin{aligned}
\int_0^T x_j(t)dt &= \lambda \int_0^T G_j(t,s) f_j(s,x(s))\,ds\,dt \\
&\le \int_0^T \int_0^T G_j(t,s) f_j(s,x(s))\,ds\,dt \\
&\le \int_0^T \int_0^T G_j(t,s) g_j(x(s))\big\{1+\frac{h_j(x(s))}{g_j(x(s))}\big\}\,ds\,dt \\
&\le \int_0^T \int_0^T G_j(t,s) g_j(r,\dots,r))\big\{1+\frac{h_j(r,\dots,r)}{g_j(r,\dots,r)}\big\}\,ds\,dt \\
&\le \omega^* T g_j(r,\dots,r))\big\{1+\frac{h_j(r,\dots,r)}{g_j(r,\dots,r)}\big\}.
\end{aligned}
$$

Therefore,

$$
\frac{\delta}{M} r \le \omega^* T (g_j(r,\dots,r) + h_j(r,\dots,r)).
$$

This is a contradiction to the condition (A4) and the claim is proved.

Next we consider (ii). Let $v(t) \equiv 1$, so $v \in K\backslash\{0\}$. Next, suppose that there exist $x \in \partial_K \Omega^2$ and $\lambda > 0$ such that $x = Sx + \lambda v$. Since $x \in \partial_K \Omega^2$, then $|x| = R$. Without loss of generality, we assume that $|x_j| = R$ for some $j = 1, 2, \dots, N$. Thus we have

$$
\int_0^T x_j(t)dt \ge \frac{\delta}{M}\|x_j\| = \frac{\delta R}{M}, \quad \text{and} \quad x_j(t) = (S_j x)(t) + \lambda.
$$

As a result, it follows from (A5) that

$$
\begin{aligned}
\int_0^T x_j(t)dt &= \int_0^T \int_0^T G_j(t,s) f_j(s,x(s))\,ds\,dt + \lambda T \\
&= \int_0^T f_j(s,x(s)) \int_0^T G_j(t,s)\,dt\,ds + \lambda T \\
&\ge \delta_j \int_0^T f_j(s,x(s))ds + \lambda T \\
&\ge \delta_j \int_0^T g_j^1(x(s))\Big\{1+\frac{h_j^1(x(s))}{g_j^1(x(s))}\Big\}ds + \lambda T \\
&\ge \delta_j \Big\{1+\frac{h_j^1(R,\dots,R)}{g_j^1(R,\dots,R)}\Big\} \int_0^T g_j^1(x(s))ds + \lambda T.
\end{aligned}
$$

Since g_j^1 is non-decreasing and convex, using the Jensen's inequality, we have

$$
\begin{aligned}
\int_0^T g_j^1(x(s))ds &\ge T g_j^1\Big(\frac{1}{T}\int_0^T x(s)ds\Big) \\
&\ge T g_j^1\Big(0,0,\dots,\frac{1}{T}\int_0^T x(s)ds, 0,\dots,0\Big) \\
&\ge T g_j^1\Big(0,0,\dots,\frac{\delta R}{TM}, 0\dots,0\Big).
\end{aligned}
$$

Therefore,

$$\begin{aligned}|x_j| &\geq \frac{1}{T}\int_0^T x_j(t)dt\\ &\geq \frac{\delta_j}{T}\Big\{1+\frac{h_j^1(R,\dots,R)}{g_j^1(R,\dots,R)}\Big\}\int_0^T g_j^1(x(s))ds+\lambda\\ &\geq \delta g_j^1\Big\{0,0,\dots,\frac{\delta R}{TM},0\dots,0\Big\}\Big\{1+\frac{h_j^1(R,\dots,R)}{g_j^1(R,\dots,R)}\Big\}+\lambda.\end{aligned}$$

This contradicts with (A6) and so (ii) is proved. $\square$

Example 4.2. Suppose that $a_1(t), a_2(t)$ satisfy (A1) and consider the differential equations

$$\begin{aligned}\ddot{x}+a_1(t)x &= \sqrt{(x^2+y^2)^\alpha}+\mu\sqrt{(x^2+y^2)^\beta}+e_1(t),\\ \ddot{y}+a_2(t)y &= \sqrt{(x^2+y^2)^\alpha}+\mu\sqrt{(x^2+y^2)^\beta}+e_2(t),\end{aligned}\tag{4.1}$$

where $a_i, e_i \in X$, $e_i, i=1,2$ are nonnegative. $0<\alpha<1<\beta$, and $\mu>0$ is a positive parameter. Then (4.1) has at least one nontrivial T-periodic solution for each $0<\mu<\mu_*$; where μ_* is the constant as in Example 3.3.

Proof. To apply Theorem 4.1, we take

$$\begin{gathered}g_1(x,y)=g_2(x,y)=\sqrt{(x^2+y^2)^\alpha}+e^*,\\ h_1(x,y)=h_2(x,y)=g_1^1(x,y)=g_2^1(x,y)=\mu\sqrt{(x^2+y^2)^\beta},\\ h_1^1(x,y)=h_2^1(x,y)=\mu\sqrt{(x^2+y^2)^\alpha},\end{gathered}$$

where $e^*=\max_t\{e_1(t),e_2(t)\}$. As in Example 3.3, we know that (A3) and (A4) are satisfied for all $0<\mu<\mu_*$. Moreover, since $\beta>1$, it is easy to see that (A5) is satisfied and (A6) becomes

$$\mu \geq \frac{(\sqrt{2}MT)^\beta R-\sqrt{2}^\alpha\delta^{\beta+1}R^\alpha}{\sqrt{2}^\beta\delta^{\beta+1}R^\beta}\tag{4.2}$$

for some $R>0$. Since $\beta>1$, the right-hand side of (4.2) goes to 0 as $R\to+\infty$. Thus, for any given $0<\mu<\mu_*$, it is always possible to find $R>r$ such that (4.2) is satisfied. Now all conditions of Theorem (4.1) are satisfied. Thus, equation (4.1) has a nontrivial T-periodic solution. $\square$

Remark 4.3. In Theorem 3.2, condition (A2) guarantees that the periodic solution obtained is nontrivial, while (A2) is not required in Theorem 4.1. For system (4.1), we require that the function $e\succ 0$ in Example 3.3, while e is only required to be nonnegative in Example 4.2.

The next multiplicity result is a direct consequence of Theorems 3.2 and 4.1.

Theorem 4.4. *Suppose* (A1)–(A6) *are satisfied. Then system* (1.1) *has at least two nontrivial T-periodic solutions x and $\tilde{x}$ with* $0<|x|<r\leq|\tilde{x}|\leq R$.

Example 4.5. Let us assume that $a(t)$ satisfy (A1), $0<\alpha<1<\beta$ and $e\succ 0$. Then system (4.1) has at least two nontrivial T-periodic solutions for each $0<\mu<\mu^*$; where μ^* is the constant as in Example 3.3.

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References

[1] F. M. Atici, G. Sh. Guseinov; On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions, *J. Comput. Appl. Math.*, **132** (2001), 341-356.
[2] J. Chu, D. O'Regan; Multiplicity result for second order non-autonomous singular Dirichlet systems, *Acta Appl. Math.*, **105** (2009), 323-338.
[3] J. Chu, P. J. Torres; Applications of Schauder's fixed point theorem to singular differential equations, *Bull. Lond. Math. Soc.*, **39** (2007), 653-660.
[4] J. Chu, P. J. Torres, M. Zhang; Periodic solutions of second order non-autonomous singular dynamical systems, *J. Differential Equations*, **239** (2007), 196-212.
[5] J. Chu, N. Fan, P. J. Torres; Periodic solutions for second order singular damped differential equations, *J. Math. Anal. Appl.*, **388** (2012), 665-675.
[6] L. H. Erbe, H. Wang; On the existence of positive solutions of ordinary differential equations, Proc. *Amer. Math. Soc.*, **120** (1994), 743-748.
[7] A. Fonda, R. Toader; Periodic orbits of radially symmetric Keplerian-like systems: A topological degree approach, *J. Differential Equations*, **244** (2008), 3235-3264.
[8] D. Franco, P. J. Torres; Periodic solutions of singular systems without the strong force condition, *Proc. Amer. Math. Soc.*, **136** (2008), 1229-1236.
[9] D. Franco, J. R. L. Webb; Collisionless orbits of singular and nonsingular dynamical systems, *Discrete Contin. Dyn. Syst.*, **15** (2006), 747-757.
[10] J. R. Graef, L. Kong, H. Wang; A periodic boundary value problem with vanishing Green's function, *Appl. Math. Lett.*, **21** (2008), 176-180.
[11] R. Hakl, P. J. Torres, M. Zamora; Periodic solutions of singular second order differential equations: Upper and lower functions, *Nonlin. Anal.*, **74** (2011), 7078-7093.
[12] D. Jiang, J. Chu, M. Zhang; Multiplicity of positive periodic solutions to superlinear repulsive singular equations, *J. Differential Equations*, **211** (2005), 282-302.
[13] M. A. Krasnosel'skii; *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.
[14] H. X. Li, Y. Zhang; A second order periodic boundary value problem with a parameter and vanishing Green's functions, *Publ. Math. Debrecen*, **85** (2014), 273-283.
[15] F.-F. Liao; Periodic solutions of second order differential equations with vanishing Green's functions, *Electron. J. Qual. Theory Differ. Equ.*, **2017** no. 55 (2017), 1-8.
[16] X. Lin, D. Jiang, D. O'Regan, R. P. Agarwal; Twin positive periodic solutions of second order singular differential systems, Topol. *Methods Nonlinear Anal.*, **25** (2005), 263-273.
[17] D. O'Regan; *Existence Theory for Nonlinear Ordinary Differential Equations*, Kluwer Academic, Dordrecht, 1997.
[18] I. Rachunková, M. Tvrdý, I. Vrkoć; Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems, *J. Differential Equations*, **176** (2001), 445-469.
[19] P. J. Torres; Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, *J. Differential Equations*, **190** (2003), 643-662.
[20] P. J. Torres; Weak singularities may help periodic solutions to exist, *J. Differential Equations*, **232** (2007), 277-284.
[21] J. R. L. Webb; Boundary value problems with vanishing Green's function, *Commun. Appl. Anal.*, **13** (2009), 587-595.
[22] P. Yan, M. Zhang; Higher order nonresonance for differential equations with singularities, *Math. Meth. Appl. Sci.*, **26** (2003), 1067-1074.
[23] M. Zhang; Periodic solutions of damped differential systems with repulsive singular forces, *Proc. Amer. Math. Soc.*, **127** (1999), 401-407.
[24] M. Zhang, W. Li; A Lyapunov-type stability criterion using L ' norms, *Proc. Amer. Math. Soc.*, **130** (2002), 3325-3333.

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