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MONOTONE ITERATION SCHEME AND ITS APPLICATION TO PARTIAL DIFFERENTIAL EQUATION SYSTEMS WITH MIXED NONLOCAL AND DEGENERATE DIFFUSIONS

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Abstract. A monotone iteration scheme for traveling waves based on ordered upper and lower solutions is derived for a class of nonlocal dispersal system with delay. Such system can be used to study the competition among nonlocally diffusive species and degenerately diffusive species. An example of such system is studied in detail. We show the existence of the traveling wave solutions for this system by this iteration scheme. In addition, we study the minimal wave speed, uniqueness, strict monotonicity and asymptotic behavior of the traveling wave solutions.

1. Introduction

Recently, a lot of attention has been given to the study of nonlocal equations and systems arising from real world applications and theoretical mathematical developments. In [1, 2, 3, 12, 13], nonlocal models from interface of crystal were studied; in [9, 10], the authors handled the nonlocal problems from ecology. In the natural world, some species diffuse locally while others diffuse non-locally or even are non-diffusive. As is well known, the classical diffusion equation can be derived by Brownian motion. By using the position jump method, a rigorous mathematical derivation of the nonlocal diffusion equation was obtained in [16] under various boundary conditions, see also [8]. The nonlocal equations have many similar properties to their classical diffusion counterparts such as the maximum principle and the comparison principle. In [14], a comparison principle based on sliding domain method was derived and it was used to study the uniqueness and asymptotics of the wave solutions of a nonlocal version of Lotka Volterra system. In [15], another nonlocal Lotka Volterra system was set up to study the outcome of the competition between the local and non-local species. It is interesting to ask the question of the outcome of the competition among species without diffusion and species with nonlocal diffusions. A similar problem was treated in [11] for systems with mixed local diffusions and non-diffusions by using spreading speed method. In this article, we study the outcome of the competition between species with nonlocal diffusions and species with no diffusions. In particular, we will focus on the asymptotic decay growth rates of those species as well as the uniqueness of the competition's outcome.

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We consider the traveling wave solutions of the following temporally delayed reaction diffusion system

$$\frac{\partial}{\partial t}U(x,t) = (DU)(x,t) + F(U_t(x)), \tag{1.1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}^+$, $U(x,t) = (u_1, u_2, \dots, u_n)(x,t) \in \mathbb{R}^n$, $F{:}C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$, $\tau \geq 0$ and

$$U_t(x) = U(t+\theta, x) \in C([-\tau,0],\mathbb{R}^n), \quad \theta \in [-\tau,0],\ t \in \mathbb{R}^+,\ x \in \mathbb{R}.$$

The $n \times n$ matrix function $DU = \operatorname{diag}(\dots d_i(J_i * u_i - u_i)\dots 0 \dots)$ is diagonal with $d_i > 0$ for $1 \leq i \leq k \leq n$. The term $J_i * u_i = \int_{\mathbb{R}} J_i(x-y)u_i(y)dy$ is a convolution, and $J_i * u_i - u_i$ represents the nonlocal diffusion. For $1 \leq i \leq k$, the integration kernel J_i satisfies: $J_i(\cdot)$ is even, nonnegative with nontrivial support, $\int_{\mathbb{R}} J(s)ds = 1$, and

$$J_i'(\cdot) \in L^1(\mathbb{R}), \quad \int_{\mathbb{R}} |s|J(s)ds < +\infty.$$

We further assume that J_i decays sufficiently fast at $\pm\infty$ such that $\int_{\mathbb{R}} e^{\lambda s}J(s)ds < +\infty$ for any $\lambda \in \mathbb{R}$. The function $F{:}C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$ satisfies the following conditions [18, 21, 22]:

(H1) $F(\mathbf{0}) = F(\mathbf{K}) = \mathbf{0}$ and $F(W) \neq \mathbf{0}$ for $W \in \mathbb{R}^n$ with $\mathbf{0} < W < \mathbf{K}$.

(H2a) (*Quasi-monotonicity condition*) There exists a positive matrix $\beta = \operatorname{diag}(\beta_1, \beta_2, \dots, \beta_n)$ such that

$$\begin{gathered} F_i(U_t) - F_i(V_t) + (\beta_i - d_i)[U(0) - V(0)]_i \geq 0, \quad i = 1, 2, \dots, k, \\ F_j(U_t) - F_j(V_t) + \beta_j[U(0) - V(0)]_j \geq 0, \quad j = k+1, \dots, n, \end{gathered}$$

for $U, V \in C([-\tau,0],\mathbb{R}^n)$ with $\mathbf{0} \leq V(s) \leq U(s) \leq \mathbf{K}$, $s \in [-\tau,0]$. or

(H2b) (*Exponential quasi-monotonicity condition*) There exists a positive matrix $\beta = \operatorname{diag}(\beta_1, \beta_2, \dots, \beta_n)$ such that

$$\begin{gathered} F_i(U) - F_i(V) + (\beta_i - d_i)[U(0) - V(0)]_i \geq 0, \quad i = 1, 2, \dots, k, \\ F_j(U) - F_j(V) + \beta_j[U(0) - V(0)]_j \geq 0, \quad j = k+1, \dots, n, \end{gathered}$$

for $U, V \in C([-\tau,0],\mathbb{R}^n)$ with $\mathbf{0} \leq V(s) \leq U(s) \leq \mathbf{K}$, $s \in [-\tau,0]$ and $e^{\beta s}[U(s) - V(s)]$ is non-decreasing in $[-\tau,0]$.

(H3) F satisfies uniform Lipschitz condition; that is, there exists a constant $L > 0$ such that

$$|F(U) - F(V)| \leq L|U - V|$$

for $U, V \in C([-\tau,0],\mathbb{R}^n)$ in usual norm in $C([-\tau,0],\mathbb{R}^n)$.

Remark 1.1. The KPP equation with a non-monotonically delayed reaction term (see [21])

$$u_t = d(u) + u[1 - u(t-\theta, x)], \tag{1.2}$$

where $d(u)$ is either the local diffusion term u_{xx}, or the nonlocal diffusion term $J * u - u$, which can be dealt with condition (H2b) but not (H2a).

If $k = n$, (1.1) is the nonlocal reaction diffusion system that has drawn considerable attention recently. The existence of the traveling wave solutions in this case was established in [20, 22, 25, 26] by monotone iteration method. In this paper, we will consider the case $1 < k < n$ which describes the competitions and/or cooperations among non-local diffusive and degenerate diffusive species. As a motivational, we study the system

$$\begin{aligned} u_t &= J * u - u + u(1 - u - rv), \\ v_t &= -buv, \end{aligned} \tag{1.3}$$

where u and v represent population densities of two competing species, and the species u diffuses non-locally while v does not diffuse. This model is a nonlocal analog of the one studied in [24] which describes the competition between the precursor and differentiated cells. We assume that the precursor cells diffuse non-locally. A traveling wave solution connecting the extinction state and coexistence state will provide insight to the outcome in the competition between local species v and nonlocal species u.

A traveling wave solution to (1.1) is a $C^2(\mathbb{R})^k \times C^1(\mathbb{R})^{n-k}$ function $U(x,t) = U(x+ct) = U(\xi),\ \xi = x+ct,\ c > 0$, which satisfies

$$\begin{gathered} D(U) - cU' + F^c(U_\xi) = \mathbf{0}, \\ U(-\infty) = \mathbf{0}, \quad U(+\infty) = \mathbf{K}, \end{gathered} \tag{1.4}$$

where $F^c{:}C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$ is defined by

$$F^c(\Psi) = F(\Psi^c), \quad \Psi^c(\theta) = \Psi(c\theta), \quad \theta \in [-\tau, 0].$$

Since system (1.1) is monotone/quasi-monotone, an iteration scheme based on upper and lower solutions can be proposed. In [21], an iteration scheme was developed, and in [18, 22] a fixed point type of argument was applied to show the existence of the traveling wave solutions. A suitably constructed upper and lower solution pairs are the key ingredient in the proof of the existence as well as the asymptotics of the traveling wave solutions.

In section 2, a continuous mapping which maps a compact invariant region into itself is constructed. The profile set is proven to be compact by the Helly selection theorem. We obtain the existence of the traveling wave solution by the Schauder's fixed point theorem.

In Section 3, we apply the monotone iteration scheme developed in the previous section to system (1.3). We use the ideas from [22] to set up the upper solution and use a known traveling wave solution of a nonlocal KPP equation to set up the lower solution. Then we show the orderness of the upper and lower solutions by a generalized sliding domain method. We further show that the traveling wave solution is unique for every wave speed. In addition, we also derive the monotonicity of the traveling wave solutions and their asymptotics. We note that there are few results on asymptotics of the traveling wave solution for nonlocal equations due to the lack of systematic treatment of linear nonlocal equations [5, 17]. We overcome the difficulty by adapting Ikehara's Tauberien Theorem into nonlocal systems. These results are new and have the potential to be used in studies of other models in real world applications. However, we would like to point out that our construction of the upper and lower solutions is different from that in [21, 22].

Throughout this article, the inequality between two vectors is understood componentwise.

2. Monotone iteration scheme

The existence of the traveling wave solution for systems (1.4) under the quasi-monotonicity condition (H2a) is proved in section 2.1. In section 2.2, the corresponding results for the systems satisfying the exponential quasi-monotonicity condition (H2b) are stated without proving.

2.1. **Waves in the mixed diffusion system under quasi-monotone condition.**

We establish the existence of the monotone traveling solutions for system (1.4) using the Schauder fixed point theorem. The following set up is standard [18, 22]. Denote

$$C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)=\{U(\xi):U(\xi)\in C(\mathbb{R},\mathbb{R}^n),\ \mathbf{0}\le U(\xi)\le \mathbf{K},\ \xi\in\mathbb{R}\}, \tag{2.1}$$

a cube in the continuous function space $C(\mathbb{R},\mathbb{R}^n)$.

Let $\Phi=(\phi_1,\phi_2,\dots,\phi_n)$. System (1.4) is written as

$$-c\phi_i'-\beta_i\phi_i+\mathbb{H}_i(\Phi)=0,\quad 1\le i\le n, \tag{2.2}$$

where

$$\mathbb{H}_i(\Phi)=\begin{cases} d_i(J_i*\phi_i-\phi_i)+\beta_i\phi_i+F_i^c(\Phi_\xi), & i=1,2,\dots,k,\\ \beta_i\phi_i+F_i^c(\Phi_\xi), & i=k+1,\dots,n.\end{cases} \tag{2.3}$$

Define the map

$$\mathbb{T}:C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)\to C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n) \tag{2.4}$$

by

$$(\mathbb{T}\Phi)_i(\xi)=\frac{1}{c}e^{-\frac{\beta_i}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{\beta_i}{c}y}\mathbb{H}_i(\Phi)(y)dy,\quad i=1,2,\dots,n. \tag{2.5}$$

We collect the properties of $\mathbb{H}$ and $\mathbb{T}$.

Lemma 2.1. *For functions $\underline{\Phi}$ and $\bar\Phi$ with $\mathbf{0}\le\underline{\Phi}\le\bar\Phi\le\mathbf{K}$ and any $\Phi\in[\mathbf{0},\mathbf{K}]$ we have*

1. $\mathbb{H}(\underline{\Phi})(\xi)\le\mathbb{H}(\bar\Phi)(\xi)$, $\mathbb{T}(\underline{\Phi})(\xi)\le\mathbb{T}(\bar\Phi)(\xi)$ *for* $\xi\in\mathbb{R}$;
2. $\mathbf{0}\le\mathbb{H}(\Phi)(\xi)\le\beta\mathbf{K}$, $\mathbf{0}\le\mathbb{T}(\Phi)(\xi)\le\mathbf{K}$ *for* $\xi\in\mathbb{R}$;
3. $\mathbb{H}(\Phi)(\xi)$ *and* $\mathbb{T}(\Phi)(\xi)$ *are non-decreasing provided* $\Phi(\xi)$ *is nondecreasing on* $\mathbb{R}$.

Proof. (1) The first part of conclusion comes from (H2a), and for $1\le i\le n$, we have

$$\mathbb{T}_i(\bar\Phi)(\xi)-\mathbb{T}_i(\underline{\Phi})(\xi)=\frac{1}{c}e^{-\frac{\beta_i}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{\beta_i}{c}y}(\mathbb{H}_i(\bar\Phi)-\mathbb{H}_i(\underline{\Phi}))(y)dy\ge0.$$

(2) For any $\mathbf{0}\le\Phi\le\mathbf{K}$, by (1), we have

$$\mathbf{0}=\mathbb{H}(\mathbf{0})\le\mathbb{H}(\Phi)\le\mathbb{H}(\mathbf{K})=\beta\mathbf{K},$$

and the second half of the conclusion comes from direct integration.

(3) If for $\zeta\ge0$ we have $\Phi(\xi+\zeta)\ge\Phi(\xi)$ for $\xi\in\mathbb{R}$, then the rest of the conclusion follows from conclusions (1) and (2). □

Let ρ be chosen such that

$$0 < \rho < \min_{1\le i\le n} \{\frac{\beta_i}{c}\}, \tag{2.6}$$

then the space

$$C^w(\mathbb{R},\mathbb{R}^n) = \{\Phi(\xi) \in C(\mathbb{R},\mathbb{R}^n) | \sup_{\xi\in\mathbb{R}} |\Phi(\xi)|e^{-\rho|\xi|} < \infty\} \tag{2.7}$$

equipped with norm

$$|\Phi|_\rho = \sup_{\xi\in\mathbb{R}} |\Phi(\xi)|e^{-\rho|\xi|} \tag{2.8}$$

is a weighted Banach space.

Note that we may choose other weight functions such that $C^w(\mathbb{R},\mathbb{R}^n)$ is a Banach space and all the proofs in the sequel hold.

Lemma 2.2. $\mathbb{T} : C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n) \to C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)$ *is continuous with respect to the norm* (2.8).

Proof. Firstly, we will show that $\mathbb{H} : C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n) \to C^w(\mathbb{R},\mathbb{R}^n)$ is continuous. For any $\Phi, \Psi \in C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)$ and $1 \le i \le k$,

$$\begin{aligned}
|\mathbb{H}_i(\Phi) - \mathbb{H}_i(\Psi)|e^{-\rho|\xi|} &\le d_i|J_i * (\Phi-\Psi)_i - (\Phi-\Psi)_i|_\rho \\
&\quad + \beta_i|(\Phi-\Psi)_i|_\rho + |(F^c(\Phi) - F^c(\Psi))_i|_\rho \\
&\le (2d_i + \beta_i + L)|\Phi-\Psi|_\rho,
\end{aligned}$$

and for $k+1 \le j \le n$,

$$\begin{aligned}
|\mathbb{H}_j(\Phi) - \mathbb{H}_j(\Psi)|e^{-\rho|\xi|} &\le \beta_j|\Phi-\Psi|_\rho + |F^c(\Phi) - F^c(\Psi)|_\rho \\
&\le (\beta_j + L)|\Phi-\Psi|_\rho.
\end{aligned}$$

The continuity of $\mathbb{H}$ follows.

Next, we show that $\mathbb{T}$ is a continuous mapping into $C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)$. That $\mathbb{T}$ maps $C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)$ into itself follows Lemma 2.1. For $1 \le i \le n$,

$$|(\mathbb{T}\Phi)_i(\xi) - (\mathbb{T}\Psi)_i(\xi)|e^{-\rho|\xi|} \le \frac{1}{c}e^{-\frac{\beta_i}{c}\xi-\rho|\xi|}\int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}y+\rho|y|}dy|\mathbb{H}_i(\Phi) - \mathbb{H}_i(\Psi)|_\rho, \tag{2.9}$$

if $\xi \le 0$, then (2.9) is

$$\frac{1}{c}e^{-\frac{\beta_i}{c}\xi+\rho\xi}\int_{-\infty}^{\xi} e^{(\frac{\beta_i}{c}-\rho)y}dy|\mathbb{H}_i(\Phi) - \mathbb{H}_i(\Psi)|_\rho \le C_1(\beta_i, c)|\mathbb{H}_i(\Phi) - \mathbb{H}_i(\Psi)|_\rho,$$

if $\xi > 0$, then

$$\begin{aligned}
&\frac{1}{c}e^{-\frac{\beta_i}{c}\xi-\rho\xi}[\int_{-\infty}^{0} e^{(\frac{\beta_i}{c}+\rho)y}dy + \int_0^{\xi} e^{(\frac{\beta_i}{c}-\rho)y}dy]|\mathbb{H}_i(\Phi) - \mathbb{H}_i(\Psi)|_\rho \\
&\le C_2(\beta_i, c)|\mathbb{H}_i(\Phi) - \mathbb{H}_i(\Psi)|_\rho,
\end{aligned}$$

where C_1, C_2 are two positive numbers depending on β_i and c. Therefore, $\mathbb{T}$ is continuous following easily from the continuity of $\mathbb{H}$. $\square$

We next define the upper and lower solutions for system (1.4).

Definition 2.3. We say $\bar{U}(\xi) \in C^2(\mathbb{R})^k \times C^1(\mathbb{R})^{n-k}$, $\xi \in \mathbb{R}$, is an upper solution for system (1.4) if it satisfies

$$\begin{gathered} D(\bar{U}) - c\bar{U}' + F^c(\bar{U}_\xi) \le \mathbf{0}, \\ \bar{U}(-\infty) = \mathbf{0}, \quad \bar{U}(+\infty) = \mathbf{K}. \end{gathered} \tag{2.10}$$

A function $\underline{U}(\xi) \in C^2(\mathbb{R})^k \times C^1(\mathbb{R})^{n-k}$, $\xi \in \mathbb{R}$, is called the lower solution for system (1.4) if it satisfies

$$\begin{gathered} D(\underline{U}) - c\underline{U}' + F^c(\underline{U}_\xi) \ge \mathbf{0}, \\ \underline{U}(-\infty) = \mathbf{0}, \quad \underline{U}(+\infty) \le \mathbf{K}. \end{gathered}$$

Remark 2.4. If $\bar{U}_1(\xi) = (\bar{u}_{11}, \bar{u}_{12}, \dots, \bar{u}_{1n})$ and $\bar{U}_2(\xi) = (\bar{u}_{21}, \bar{u}_{22}, \dots, \bar{u}_{2n})$, $\xi \in \mathbb{R}$ are two upper solutions for (1.4), so is the vector function $(\min(\bar{u}_{11}(\xi), \bar{u}_{21}(\xi)), \dots, \min(\bar{u}_{1n}(\xi), \bar{u}_{2n}(\xi)))$. The same conclusion also applies to two lower solutions of (1.4) but we should change the minimum value of the two components into maximum of the two [12].

Based on the upper and lower solutions of (1.4), we next define the profile set:

$$\begin{aligned} \Gamma = \Big\{ &\Phi(\xi) \in C(\mathbb{R}, \mathbb{R}^n) : (1)\Phi(\xi) \text{ is nondecreasing}, \\ &(2)\underline{U}(\xi) \le \Phi(\xi) \le \bar{U}(\xi),\ (3)|\Phi(\xi) - \Phi(\zeta)| \le L_1|\xi - \zeta| \Big\}, \end{aligned}$$

where $L_1 = \max_{1 \le i \le n}\{\frac{2k_i\beta_i}{c}\}$, and k_i $(1 \le i \le n)$ is the i-th component of $\mathbf{K}$.

Lemma 2.5. *The set Γ is a compact and convex subset of $C^w_{[\mathbf{0},\mathbf{K}]}(\mathbb{R}, \mathbb{R}^n)$ and $\mathbb{T}$ maps Γ into Γ.*

Proof. The process for verifying that Γ is convex is straightforward. We next show that Γ is compact. Let $\{\Phi_n(\xi)\}_{n=1}^\infty$ be a sequence in Γ. Then Φ_n is uniformly bounded and nondecreasing. It follows from Helly's selection theorem [23] that there is a subsequence $\{\Phi_{n_i}(\xi)\}_{n=1}^\infty$ and a function $\Phi(\xi)$, $\xi \in \mathbb{R}$ such that for each $\xi \in \mathbb{R}$, $\{\Phi_{n_i}(\xi)\} \to \Phi(\xi)$ pointwise as $i \to +\infty$. Then it follows that $\Phi(\xi)$ is bound by $\underline{U}(\xi)$ and $\bar{U}(\xi)$ and is nondecreasing.

Next we show that $\Phi(\xi)$ is continuous in the topology induced by the weighted norm. For any $\xi_1, \xi_2 \in \mathbb{R}$, we have

$$|\Phi(\xi_1) - \Phi(\xi_2)| \le |\Phi(\xi_1) - \Phi_{n_i}(\xi_1)| + |\Phi_{n_i}(\xi_1) - \Phi_{n_i}(\xi_2)| + |\Phi_{n_i}(\xi_2) - \Phi(\xi_2)| \to 0$$

as $\xi_2 \to \xi_1$ and $i \to +\infty$. This shows that $\Phi(\xi)$ is a continuous function in $C(\mathbb{R}, \mathbb{R}^n)$.

Since $e^{-\rho|\xi|} \le 1$ for all $\xi \in \mathbb{R}$, the continuity of $\Phi(\xi)$ in the weighted Banach C^w follows from the continuity of Φ in C. This leads to the convergence of $\{\Phi_{n_i}(\xi)\}_{n=1}^\infty$ in C^w. Then $\Phi(\xi) \in \Gamma$ follows from that $C^w_{[\mathbf{0},\mathbf{K}]}$ is a Banach space.

We next show that $\mathbb{T}$ maps Γ into itself. First for $\Phi_n \in \Gamma$ and any $\zeta \ge 0$, we fix $1 \le i \le n$,

$$\begin{aligned} &(\mathbb{T}\Phi)_i(\xi + \zeta) - (\mathbb{T}\Phi)_i(\xi) \\ &= \frac{1}{c}e^{-\frac{\beta_i}{c}(\xi+\zeta)} \int_{-\infty}^{\xi+\zeta} e^{\frac{\beta_i}{c}y}\mathbb{H}_i(\Phi)(y)dy - \frac{1}{c}e^{-\frac{\beta_i}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}y}\mathbb{H}_j(\Phi)(y)dy \\ &= \frac{1}{c}e^{-\frac{\beta_i}{c}(\xi+\zeta)} \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}(y+\zeta)}\mathbb{H}_i(\Phi)(y+\zeta)dy - \frac{1}{c}e^{-\frac{\beta_i}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}y}\mathbb{H}_i(\Phi)(y)dy \end{aligned}$$

$$= \frac{1}{c} e^{-\frac{\beta_i}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta_j}{c} y} (\mathbb{H}_i(\Phi)(y+\zeta) - \mathbb{H}_i(\Phi)(y)) dy \geq 0.$$

This shows that $\mathbb{T}(\Phi)(\xi)$ is nondecreasing for $\xi \in \mathbb{R}$.

We next show that $\mathbb{T}$ satisfies the second condition in Γ. By Lemma 2.1, this can be reduced to show that

$$\underline{U}(\xi) \leq (\mathbb{T}\underline{U})(\xi) \leq (\mathbb{T}\bar{U})(\xi) \leq \bar{U}(\xi), \quad \xi \in \mathbb{R}.$$

For $1 \leq i \leq n$, we have $(\mathbb{T}\underline{U})_i(-\infty) = 0$ and

$$\begin{aligned} \lim_{\xi \to +\infty} (\mathbb{T}\underline{U})_i(\xi) &= \lim_{\xi \to +\infty} \frac{\int_{-\infty}^{\xi} e^{\beta_i y/c} \mathbb{H}_i(\underline{U})(y) dy}{e^{\beta_i \xi/c}} \\ &= \underline{U}_i(+\infty) + \frac{1}{\beta_i} F_i(\underline{U}(+\infty)) \geq \underline{U}_i(+\infty). \end{aligned}$$

We can also verify that $(\mathbb{T}\underline{U})_i(\xi)$ satisfies

$$-c(\mathbb{T}\underline{U})_i' - \beta_i(\mathbb{T}\underline{U})_i + \mathbb{H}_i(\underline{U}) = 0, \quad \xi \in \mathbb{R}.$$

Since $\underline{U}(\xi)$ is a lower solution of (1.4),

$$-c(\underline{U})_i' - \beta_i(\underline{U})_i + \mathbb{H}_i(\underline{U}) \geq 0.$$

We can set $W(\xi) = \mathbb{T}\underline{U}(\xi) - \underline{U}(\xi)$. Then $W_i(\xi)$ satisfies

$$cW_i' + \beta_i W_i \geq 0, \quad \xi \in \mathbb{R}$$

or equivalently, $c(W_i(\xi)e^{\beta_i\xi/c})' \geq 0$, which means $W_i(\xi)e^{\beta_i\xi/c}$ is increasing for $\xi \in \mathbb{R}$. In particular,

$$W_i(\xi)e^{\beta_i\xi/c} \geq \lim_{\xi \to -\infty} W_i(\xi)e^{\beta_i\xi/c} = 0.$$

Therefore, $(\mathbb{T}\underline{U})_i(\xi) \geq \underline{U}_i(\xi)$, $\xi \in \mathbb{R}$. In the same way, we have $\mathbb{T}\bar{U}(\xi) \leq \bar{U}(\xi)$, $\xi \in \mathbb{R}$.

Finally, we show that $\mathbb{T}$ satisfies the third condition on Γ. For $\Phi \in \Gamma$, $1 \leq i \leq n$ and $\zeta \leq \xi$, we have

$$\begin{aligned} &|(\mathbb{T}\Phi)_i(\xi) - (\mathbb{T}\Phi)_i(\zeta)| \\ &= \frac{1}{c} |e^{-\frac{\beta_i}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c} y} \mathbb{H}_i(\Phi)(y) dy - e^{-\frac{\beta_i}{c}\zeta} \int_{-\infty}^{\zeta} e^{\frac{\beta_i}{c} y} \mathbb{H}_i(\Phi)(y) dy| \\ &= \frac{1}{c} |e^{-\frac{\beta_i}{c}\xi} \int_{-\infty}^{\zeta} e^{\frac{\beta_i}{c} y} \mathbb{H}_i(\Phi)(y) dy - e^{-\frac{\beta_i}{c}\zeta} \int_{-\infty}^{\zeta} e^{\frac{\beta_i}{c} y} \mathbb{H}_i(\Phi)(y) dy \\ &\quad + e^{-\frac{\beta_i}{c}\xi} \int_{\zeta}^{\xi} e^{\frac{\beta_i}{c} y} \mathbb{H}_i(\Phi)(y) dy| \\ &\leq \frac{1}{c} |e^{-\frac{\beta_i}{c}\xi} - e^{-\frac{\beta_i}{c}\zeta}| \int_{-\infty}^{\zeta} e^{\frac{\beta_i}{c} y} \mathbb{H}_i(\Phi)(y) dy + \frac{1}{c} e^{-\frac{\beta_i}{c}\xi} \int_{\zeta}^{\xi} e^{\frac{\beta_i}{c} y} \mathbb{H}_i(\Phi)(y) dy \\ &\leq k_i |e^{-\frac{\beta_i}{c}(\xi - \zeta)} - 1| + k_i |1 - e^{-\frac{\beta_i}{c}(\xi - \zeta)}| \\ &\leq \frac{2k_i\beta_i}{c} |\xi - \zeta| \leq L_1 |\xi - \zeta|. \end{aligned}$$

The proof is complete. □

2.2. **Waves in mixed diffusion systems under exponential quasi monotone conditions.** We use the same framework as in Section 2.1. Hence, (2.1) through (2.5) will be carried to this section. Let $\bar{U}(\xi)$ and $\underline{U}(\xi),\ \xi\in\mathbb{R}$ be defined as in Definition 2.3, we introduce the profile set

$$
\begin{aligned}
\Gamma_1=\Big\{&\Phi(\xi)\in C(\mathbb{R},\mathbb{R}^n):(1)\underline{U}(\xi)\le\Phi(\xi)\le\bar{U}(\xi),\ \Phi(\xi)\text{ is nondecreasing,}\\
&(2)|\Phi(\xi)-\Phi(\zeta)|\le L_1|\xi-\zeta|,\ (3)e^{\beta_i\xi}(\Phi_i(\xi+s)-\Phi_i(\xi)),\ e^{\beta_i\xi}(\bar{U}_i(\xi)-\Phi_i(\xi))\\
&\text{and } e^{\beta_i\xi}(\Phi_i(\xi)-\underline{U}_i(\xi))\text{ are nondecreasing, } i=1,2,\dots,k;\\
&(4)e^{\frac{\beta_j}{c}\xi}(\Phi_j(\xi+s)-\Phi_j(\xi)),\ e^{\frac{\beta_j}{c}\xi}(\bar{U}_j(\xi)-\Phi_j(\xi))\text{ and}\\
&e^{\frac{\beta_j}{c}\xi}(\Phi_j(\xi)-\underline{U}_j(\xi))\text{ are nondecreasing, } j=k+1,\dots,n.\Big\}
\end{aligned}
$$

The following properties of $\mathbb{H}$ and $\mathbb{T}$ can be proved similarly as in Section 2.1.

Lemma 2.6. *For functions $\underline{\Phi}$ and $\bar{\Phi}$ with $\mathbf{0}\le\underline{\Phi}(\xi)\le\bar{\Phi}(\xi)\le\mathbf{K}$ and any $\Phi\in C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)$, we have*

1. *$\mathbb{H}(\underline{\Phi})(\xi)\le\mathbb{H}(\bar{\Phi})(\xi)$, $\mathbb{T}(\underline{\Phi})(\xi)\le\mathbb{T}(\bar{\Phi})(\xi)$ for $\xi\in\mathbb{R}$;*
2. *$\mathbf{0}\le\mathbb{H}(\Phi)(\xi)\le\beta\mathbf{K}$, $\mathbf{0}\le\mathbb{T}(\Phi)(\xi)\le\mathbf{K}$ for $\xi\in\mathbb{R}$;*
3. *$e^{\beta\xi}(\Phi(\xi+s)-\Phi(\xi))$ is non-decreasing, and $\mathbb{H}(\Phi)(\xi)$ and $\mathbb{T}(\Phi)(\xi)$ are non-decreasing provided $\Phi(\xi)$ is nondecreasing on $\mathbb{R}$.*

Lemma 2.7. *$\mathbb{T}$:$C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)\to C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)$ is continuous with respect to the norm* (2.8).

Lemma 2.8. *The set Γ_1 is a compact and convex subset of $C^w_{[\mathbf{0},\mathbf{K}]}(\mathbb{R},\mathbb{R}^n)$, and $\mathbb{T}$ maps Γ_1 into Γ_1.*

Proof. The proof of this first part is the same as that of Lemma 2.5. Next, we will show that $\mathbb{T}$ maps Γ_1 into itself. Since $\underline{U}(\xi)\le\mathbb{T}\Phi(\xi)\le\bar{U}(\xi)$, the nondecreasing of $\mathbb{T}\Phi(\xi)$, and $|(\mathbb{T}\Phi)(\xi)-(\mathbb{T}\Phi)(\zeta)|\le L_1|\xi-\zeta|$ can be proved exactly as that in Lemma 2.5, we will skip this process. We show that $\mathbb{T}\Phi$ satisfies the third and fourth conditions in the definition of Γ_1.

For $1\le i\le n$, we can verify that

$$
\begin{aligned}
&\frac{d}{d\xi}(e^{\beta_i\xi/c}(\mathbb{T}\Phi)_i(\xi+s)-(\mathbb{T}\Phi)_i(\xi))\\
&=\frac{d}{d\xi}(e^{\beta_i\xi/c}(e^{-\frac{\beta_i}{c}(\xi+s)}\int_{-\infty}^{\xi+s}e^{\frac{\beta_i}{c}y}(\mathbb{H}\Phi)_i(y)dy-e^{-\frac{\beta_i}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{\beta_i}{c}y}(\mathbb{H}\Phi)_i(y)dy)\\
&=\frac{d}{d\xi}(\int_{-\infty}^{\xi}e^{\frac{\beta_i}{c}y}(\mathbb{H}\Phi)_i(y+s)dy-\int_{-\infty}^{\xi}e^{\frac{\beta_i}{c}y}(\mathbb{H}\Phi)_i(y)dy)\\
&=e^{\beta_i\xi/c}((\mathbb{H}\Phi)_i(\xi+s)-(\mathbb{H}\Phi)_i(\xi))\ge 0.
\end{aligned}
$$

Next we show that $e^{\beta_i\xi/c}((\mathbb{T}\bar{U})_i(\xi)-(\mathbb{T}\Phi)_i(\xi))$ for $1\le i\le n$ are nondecreasing in ξ.

$$
\begin{aligned}
&\frac{d}{d\xi}(e^{\beta_i\xi/c}(\mathbb{T}\bar{U})_i(\xi)-(\mathbb{T}\Phi)_i(\xi))\\
&=\frac{d}{d\xi}(e^{\beta_i\xi/c}(e^{-\frac{\beta_i}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{\beta_i}{c}y}(\mathbb{H}\bar{U})_i(y)dy-e^{-\frac{\beta_i}{c}\xi}\int_{-\infty}^{\xi}e^{\frac{\beta_i}{c}y}(\mathbb{H}\Phi)_i(y)dy)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d}{d\xi}\Big(\int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}y}(\mathbb{H}\bar{U})_i(y)dy - \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}y}(\mathbb{H}\Phi)_i(y)dy\Big) \\
&= e^{\beta_i\xi/c}((\mathbb{H}\bar{U})_i(\xi) - (\mathbb{H}\Phi)_i(\xi)) \geq 0.
\end{aligned}
$$

We can prove similarly that $e^{\beta_i\xi}(\Phi_i(\xi) - \underline{U}_i(\xi))$ and $e^{\beta_i\xi/c}(\Phi_i(\xi) - \underline{U}_i(\xi))$ are both nondecreasing in ξ. □

2.3. **Existence results.**

Theorem 2.9. *Assume that the conditions on F hold. If* (1.4) *has an upper solution $\bar{U}(\xi)$ and a lower solution $\underline{U}(\xi)$ for some $c > 0$ and $\underline{U}(\xi) \leq \bar{U}(\xi)$, $\xi \in \mathbb{R}$, then* (1.4) *has a monotone solution, i.e., there exists a traveling wave solution for $c > 0$.*

Proof. By Lemma 2.5 and the Schauder fixed point theorem, $\mathbb{T}$ has a fixed point $\Phi^* \in \Gamma$. Then $\lim_{t\to\pm\infty} \Phi^*(t)$ exists, and we denote it by $\Phi^*_\pm$. The assumption (H2a) further implies that $\mathbb{T}(\Phi^*_\pm) = \mathbf{0}$. Combining these with the assumptions (H2) and (H3), we have $\Phi^*_- = \mathbf{0}$ and $\Phi^*_+ = \mathbf{K}$. This completes the proof. □

3. Applications

Consider the mixed reaction diffusion model:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= J * u - u + u(1 - u - rv), \\
\frac{\partial v}{\partial t} &= -buv,
\end{aligned}
\tag{3.1}
$$

for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. This model is used in population dynamics. In system (3.1), $u(x,t)$ is the population density of the invasive species and $v(x,t)$ is the population density of the local species. The invasive species u diffuses non-locally with diffusion measured by $J * u - u$ while the native species v is non-diffusive. The interaction constants r and b satisfy the conditions

$$
0 < b < 1 - r, \quad r > 0. \tag{3.2}
$$

Lemma 3.1. *If* (3.2) *holds, then the equilibrium* $(0,1)$ *is unstable and the equilibrium* $(1,0)$ *is stable for system* (3.1).

Proof. The Jacobian matrix of system (3.1) is

$$
\begin{pmatrix} 1 - 2u - rv & -ru \\ -bv & -bu \end{pmatrix}
$$

which is

$$
\begin{pmatrix} 1 - r & 0 \\ -b & 0 \end{pmatrix}
$$

at the equilibrium $(0,1)$. Therefore, if $1 - r > 0$, then $(0,1)$ is an unstable equilibrium of the corresponding ODE. It follows that it is also unstable for system (3.1). Next, we show that the equilibrium $(1,0)$ is locally asymptotically stable. The linearized equation at $(1,0)$ is

$$
\begin{aligned}
\frac{\partial w_1}{\partial t} &= J * w_1 - 2w_1 - rw_2, \\
\frac{\partial w_2}{\partial t} &= -bw_2, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+.
\end{aligned}
$$

It is easy to verify that the above system admits exponential dichotomy [19]. Hence, the corresponding linear operator does not have eigenvalues in $L^2(\mathbb{R})\times L^2(\mathbb{R})$. Next, we study its essential spectrum. Let

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{\lambda t + i\eta x}, \quad \lambda \in \mathbb{C},\ \eta \in \mathbb{R}\,.$$

Inserting them in the above linearized system at $(1,0)$, we have

$$\lambda \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \int_{\mathbb{R}} J(s)e^{i\eta s}ds - 2 & -r \\ 0 & -b \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

So, the essential spectra satisfy $\operatorname{Re}\lambda = \operatorname{Re}(\int_{\mathbb{R}} J(s)e^{i\eta s}ds - 2) < 0$ and $\lambda = -b < 0$. So, $(1,0)$ is stable [3]. $\square$

Let $\xi = x + ct$. Then the traveling wave solution of (3.1) connecting $(0,1)$ with $(1,0)$ is the solution of the system

$$\begin{gathered} J * u - u - cu' + u(1-u-rv) = 0, \\ cv' + buv = 0, \\ (u,v)(-\infty) = (0,1), \quad (u,v)(+\infty) = (1,0). \end{gathered} \tag{3.3}$$

The existence of a traveling wave solution to system (3.3) implies the successful invasion of the nonlocal species u. By the transformation

$$\bar{u} = u, \quad \bar{v} = 1 - v,$$

changes system (3.3) into the monotone system

$$\begin{gathered} J * u - u - cu' + u(1-r-u+rv) = 0, \\ -cv' + bu(1-v) = 0, \\ (u,v)(-\infty) = (0,0), \quad (u,v)(+\infty) = (1,1), \end{gathered} \tag{3.4}$$

where we drop the bars over u and v for convenience. Consider the function

$$\Delta_1(\lambda) = \int_{\mathbb{R}} J(s)e^{\lambda s}ds - 1 - c\lambda + (1-r). \tag{3.5}$$

According to [26], there exists a positive constant

$$c^* = \min_{\lambda>0} \frac{1}{\lambda}\Big\{\Big(\int_{\mathbb{R}} J(s)e^{\lambda s}ds - 1\Big) + (1-r)\Big\} \tag{3.6}$$

such that for $c = c^*$, $\Delta_1(\lambda)$ has one double zero $\lambda(c^*)$ and for any $c > c^*$, Δ_1 has two positive zeros $\lambda_1(c) < \lambda_2(c)$.

Similar to Definition 2.3, we can define the upper and lower solutions for (3.4) as follows.

Definition 3.2. A smooth function $(u(\xi), v(\xi))^T$, $\xi \in \mathbb{R}$ is an upper solution of (3.4) if it satisfies

$$\begin{gathered} J * u - u - cu' + u(1-r-u+rv) \le 0, \\ -cv' + bu(1-v) \le 0, \end{gathered} \tag{3.7}$$

as well as the boundary conditions

$$(u,v)(-\infty) \ge (0,0), \quad (u,v)(+\infty) \ge (1,1). \tag{3.8}$$

A lower solution of (3.4) is defined similarly by reversing the inequalities in (3.7) and (3.8).

We set up the upper solutions for (3.4). For each fixed $c > c^*$, let

$$\bar{u}(\xi) = \begin{cases} e^{\lambda_1(c)\xi}, & \xi \le 0, \\ 1, & \xi > 0. \end{cases}$$

Lemma 3.3. *Let condition* (3.2) *hold. Then for each* $c > c^*$, $(\bar{u}, \bar{v})(\xi) = (\bar{u}, \bar{u})(\xi)$, $\xi \in \mathbb{R}$ *defines an upper solution for* (3.4).

Proof. It is easy to see that $(u(\xi), v(\xi)) \equiv (1,1)$ satisfies the inequalities (3.7) and (3.8). For $(u(\xi), v(\xi)) = (e^{\lambda_1(c)\xi}, e^{\lambda_1(c)\xi})$, $\xi \in \mathbb{R}$ we have

$$J * u - u - cu' + u(1-r) - (1-r)u^2 = -(1-r)u^2 \le 0,$$

and

$$-c\lambda_1 e^{\lambda_1 \xi} + be^{\lambda_1 \xi}(1 - e^{\lambda_1 \xi}) = e^{\lambda_1 \xi}(-c\lambda_1 + b) - be^{2\lambda_1 \xi}. \tag{3.9}$$

Let $\Delta_2(\lambda) = -c\lambda + b$. Then by (3.2) we have

$$\begin{aligned}
\Delta_1(\lambda) - \Delta_2(\lambda) &= \int_{\mathbb{R}} J(s)e^{\lambda s} ds - 1 - c\lambda + (1-r) + c\lambda - b \\
&= \int_{\mathbb{R}} J(s)e^{\lambda s} ds - 1 + (1-r) - b \\
&= \int_{\mathbb{R}} J(s)(1 + \lambda s + \frac{\lambda^2 s^2}{2} + \dots) ds - 1 + (1-r) - b \\
&\ge \frac{\lambda^2}{2} \int_{\mathbb{R}} J(s)s^2 ds + (1-r) - b > 0.
\end{aligned}$$

Therefore, $\Delta_2(\lambda_1) < \Delta_1(\lambda_1) = 0$. This means equation (3.9) is negative. Hence, $(u(\xi), v(\xi)) = (e^{\lambda_1(c)\xi}, e^{\lambda_1(c)\xi})$ satisfies inequalities (3.7) and (3.8). So, this conclusion follows by Remark 2.4. □

Next, we define the lower solution for system (3.4). The construction of the lower solution depends on the following information; see [17, 26] for more details.

Lemma 3.4. *Let* c^* *be defined as in* (3.6). *Then for any* $c \ge c^*$, *the nonlocal KPP system*

$$\begin{gathered}
J * u - u - cu' + (1-r)u(1 - \frac{1+l}{1-r}u) = 0, \\
u(-\infty) = 0, \ u(+\infty) = \frac{1-r}{1+l} > 0
\end{gathered} \tag{3.10}$$

has a unique (up to a translation of the origin) monotone solution, and the solution has the following asymptotic behaviors: for the critical front with speed $c = c^*$,

$$\begin{gathered}
w(\xi) = b_w \xi e^{\lambda_1(c^*)\xi} + o(\xi e^{\lambda_1(c^*)\xi}), \quad \xi \to -\infty, \\
w(\xi) = \frac{1-r}{1+l} - d_w e^{\widehat{\lambda}_1(c^*)\xi} + o(e^{\widehat{\lambda}_1(c^*)\xi}), \quad \xi \to +\infty,
\end{gathered}$$

and for the noncritical front with speed $c > c^*$,

$$\begin{gathered}
w(\xi) = a_w e^{\lambda_1(c)\xi} + o(e^{\lambda_1(c)\xi}), \quad \xi \to -\infty; \\
w(\xi) = \frac{1-r}{1+l} - c_w e^{\widehat{\lambda}_1(c)\xi} + o(e^{\widehat{\lambda}_1(c)\xi}), \quad \xi \to +\infty
\end{gathered}$$

where $\widehat{\lambda}_1(c)$ *is the negative root of* $\Delta(\lambda) = \int_{\mathbb{R}} J(s)e^{\lambda s} ds - 1 - c\lambda - (1-r)$, *and* l, a_w, c_w, d_w *are positive constants and* b_w *is negative.*

For any fixed $c > c^*$, let $\underline{u}(\xi)$, $\xi \in \mathbb{R}$, be a solution of (3.10).

Lemma 3.5. *If condition* (3.2) *holds, then for each* $c > c^*$, $(\underline{u}(\xi), \underline{v}(\xi)) = (\underline{u}(\xi), 0)$, $\xi \in \mathbb{R}$, *defines a lower solution for* (3.4).

Proof. At the boundary, we have $(\underline{u}, \underline{v})(-\infty) = (0, 0)$ and $(\underline{u}, \underline{v})(+\infty) = (\frac{1-r}{1+l}, 0)$. For the u component, we have

$$\begin{aligned}
&J * \underline{u} - \underline{u} - c\underline{u}' + \underline{u}(1 - r - \underline{u} + r\underline{v}) \\
&= J * \underline{u} - \underline{u} - c\underline{u}' + (1 - r)\underline{u}(1 - \frac{1 + l}{1 - r}\underline{u}) \\
&\quad - (1 - r)\underline{u}(1 - \frac{1 + l}{1 - r}\underline{u}) + \underline{u}(1 - r - \underline{u}) \\
&= \underline{u}(1 - r) - \underline{u}^2 + \underline{u}[-(1 - r) + (1 + l)\underline{u}] = l\underline{u}^2 \geq 0.
\end{aligned}$$

The verification for v component is trivial, so we have omitted it. This completes the proof. $\square$

For convenience of a later proof, we derive the following version of sliding domain method for the mixed diffusion systems.

Proposition 3.6. *Let N¿0 and for* $1 \leq i \leq k$, $k + 1 \leq j \leq n$, *consider the system*

$$\begin{aligned}
&d_i(J_i * \phi_i - \phi_i) - c\phi_i' + \sum_{m=1}^{n} a_{im}(\xi)\phi_m \leq 0, \\
&-c\phi_j' + \sum_{m=1}^{n} a_{jm}(\xi)\phi_m \leq 0, \quad \xi \in [-N, N]
\end{aligned} \tag{3.11}$$

with boundary conditions:

$$\phi_i(\xi) \geq 0, \quad \xi \in (-\infty, -N] \cup [N, +\infty), \tag{3.12}$$

$$\phi_j(-N) > 0, \quad \phi_j(N) > 0. \tag{3.13}$$

Suppose $a_{lm} \geq 0$ *for* $l \neq m$, $l, m = 1, 2, \dots, n$. *If* $\phi_m(\xi) \geq 0$ *for* $\xi \in [-N, N]$, *then* $\phi_m(\xi) > 0$ *for* $\xi \in (-N, N)$.

Proof. Suppose that the conclusion is not true for some i. If $1 \leq i \leq k$, then there is a $\bar{\xi} \in (-N, N)$ such that $\phi_i(\bar{\xi}) = 0$, then $\phi_i(\xi)$ takes global minimum at $\bar{\xi}$. It then follows that $\phi_i'(\bar{\xi}) = 0$, and

$$(J * \phi_i - \phi_i)(\bar{\xi}) = \int_{\mathbb{R}} J(\bar{\xi} - y)(\phi_i(y) - \phi_i(\bar{\xi}))dy > 0. \tag{3.14}$$

However, by the assumption $a_{lm} \geq 0$ for $l \neq m$, $l, m = 1, 2, \dots, n$, we have for $\xi \in (-N, N)$,

$$d_i(J_i * \phi_i - \phi_i) - c\phi_i' + a_{ii}(\xi)\phi_i \leq -\sum_{m \neq i} a_{im}(\xi)\phi_m \leq 0$$

which leads to a contradiction with (3.14). Hence, we have $\phi_i(\xi) > 0$ for $\xi \in (-N, N)$.

If $k + 1 \leq i \leq n$, then by assumption we have

$$-c\phi_j' + a_{jj}\phi_j \leq \sum_{m \neq j} a_{jm}(\xi)\phi_m \leq 0.$$

Then for $\xi \in (-N, N)$, we have

$$(e^{-\frac{1}{c}\int_{-M}^{\xi} a_{jj}(s)ds}\phi_i(\xi))' \geq 0$$

which means $e^{-\frac{1}{c}\int_{-M}^{\xi} a_{jj}(s)ds}\phi_i(\xi)$ is increasing in $[-N, N]$. In particular,

$$e^{-\frac{1}{c}\int_{-N}^{\bar{\xi}} a_{jj}(s)ds}\phi_i(\bar{\xi}) \geq \phi_i(-N) > 0$$

which leads to a contradiction. This completes the proof. $\square$

Proposition 3.7. *Let two C^2 vector functions $\bar{U}(\xi) = (\bar{u}_1(\xi), \bar{u}_2(\xi), \dots, \bar{u}_n(\xi))$ and $\underline{U}(\xi) = (\underline{u}_1(\xi), \underline{u}_2(\xi), \dots, \underline{u}_n(\xi))$ satisfy the following inequalities*

$$\begin{gathered}
D\bar{U} - c\bar{U}' + F(\bar{U}) \leq 0 \leq D\underline{U} - c\underline{U}' + F(\underline{U}), \quad \xi \in [-N, N], \\
\underline{U}(\xi) < \bar{U}(-N), \quad \xi \in (-\infty, -N], \\
\underline{U}(N) < \bar{U}(\xi), \quad \xi \in [N, +\infty),
\end{gathered}$$

where $DU = \operatorname{diag}(\dots d_i(J_i*u_i - u_i)\dots 0\dots)$, $1 \leq i \leq k$, $F(U) = (F_1(U), \dots, F_n(U))$ is C^1 with respect to its components and $\frac{\partial F_i}{\partial u_j} \geq 0$ for $i \neq j$, $i, j = 1, 2, \dots, n$, then

$$\underline{U}(\xi) < \bar{U}(\xi), \quad \xi \in [-N, N].$$

Proof. We adapt the proof of [4]. Shift $\bar{U}(\xi)$ to the left. For $0 \leq \mu \leq 2N$, consider $\overline{U^\mu}(\xi) := \bar{U}(\xi + \mu)$ on the interval $(-N, N - \mu)$. At both ends of the interval, by (3.7) and (3.7), we have

$$\underline{U}(\xi) < \overline{U^\mu}(\xi). \tag{3.15}$$

Starting from $\mu = 2N$, decreasing μ, for every μ in $0 \leq \mu \leq 2N$, the inequality (3.15) is true at the end points of the respective interval. For decreasing μ, suppose that there is a first μ with $0 \leq \mu \leq 2N$ such that

$$\underline{U}(\xi) < \overline{U^\mu}(\xi), \ \xi \in (-N, \ N - \mu)$$

and there is one component, for example, the $i-th$, such that the equality holds at a point ξ_1 inside the interval. Let $W(\xi) = (w_1(\xi), w_2(\xi), \dots, w_n(\xi)) = \overline{U^\mu}(\xi) - \underline{U}(\xi)$, then $w_i(\xi)$, $i = 1, 2, \dots, n$ satisfies

$$\begin{gathered}
D_i w_i - c w_i' + \frac{\partial F_i}{\partial u_i} w_i \leq D_i w_i - c w_i' + \sum_{j=1}^{n} \frac{\partial F_i}{\partial u_j} w_j \leq 0 \\
w_i(\xi_1) = 0, \ w_j(\xi) \geq 0, \quad \xi \in [-N, N - \mu].
\end{gathered}$$

If $0 \leq i \leq k$, then $w_i \equiv 0$ for $\xi \in [-N, N - \mu]$ by the Maximum principle. This is in contradiction with (3.15) on the boundary points $\xi = -N$ and $\xi = N - \mu$. If $k + 1 \leq i \leq n$, since

$$-c w_i' + \frac{\partial F_i}{\partial u_i} w_i \leq 0,$$

we can have for $\xi \in [-N, N - \mu]$,

$$(e^{-\frac{1}{c}\int_{-N}^{\xi} \frac{\partial F_i}{\partial u_i} ds} w_i(\xi))' \geq 0$$

which means that $e^{-\frac{1}{c}\int_{-N}^{\xi} \frac{\partial F_i}{\partial u_i} ds} w_i(\xi)$ is increasing on $[-N, N - \mu]$. This together with (3.15) implies that $w_i(\xi) > 0$ on $[-N, N - \mu]$ which is in contradiction with $w_i(\xi_1) = 0$. Thus, we can decrease μ all the way to zero. This proves the conclusion. $\square$

Based on the previous two propositions, we show the upper and lower solutions constructed in Lemmas 3.3 and 3.5 are ordered.

Lemma 3.8. *For each fixed $c > c^*$, let $(\bar{u}, \bar{v})(\xi)$, $(\underline{u}, \underline{v})(\xi)$, $\xi \in \mathbb{R}$ be the corresponding upper and lower solutions obtained in Lemmas 3.3 and 3.5 respectively, then there exists a $\zeta \geq 0$ such that*

$$(\bar{u}, \bar{v})(\xi) \geq (\underline{u}, \underline{v})(\xi - \zeta), \quad \xi \in \mathbb{R}.$$

Proof. According to Lemma 3.4, $\underline{u}$ has the following asymptotic behaviors

$$\underline{u}(\xi) = \underline{A} e^{\lambda_1(c)\xi} + o(\underline{A} e^{\lambda_1(c)\xi}), \quad \xi \to -\infty.$$

Since (3.10) is shifting invariant, we have for any fixed $\zeta \geq 0$,

$$\underline{u}(\xi - \zeta) = \underline{A} e^{-\lambda_1(c)\zeta} e^{\lambda_1(c)\xi} + o(\underline{A} e^{\lambda_1(c)\xi}), \quad \xi \to -\infty.$$

It follows that for a sufficiently large $\zeta \geq 0$,

$$\underline{A} e^{-\lambda_1(c)\zeta} < 1.$$

The boundary conditions of $(\bar{u}, \bar{v})(\xi)$ imply the existence of a large number $N > 0$ and $\zeta_0 \geq 0$ such that

$$(\bar{u}, \bar{v})(\xi) > (\underline{u}, \underline{v})(\xi - \zeta_0), \quad \xi \in (-\infty, -N] \cup [N, +\infty).$$

Since system (3.4) is monotone and the upper and lower solutions are monotonically increasing, then by Proposition 3.6 we have

$$(\bar{u}, \bar{v})(\xi) > (\underline{u}, \underline{v})(\xi - \zeta_0), \quad \xi \in (-N, N).$$

Hence after a shifting of the lower solution, we have the orderness of the upper and lower solutions. □

We will still denote the shifted lower solution as $(\underline{u}, \underline{v})(\xi)$, $\xi \in \mathbb{R}$.

Theorem 3.9. *Assume condition (3.2). Then for each $c \geq c^*$, system (3.4) has a unique traveling wave solution. The solution is strictly monotonically increasing on $\mathbb{R}$. There is no monotone traveling wave solution for $0 < c < c^*$, and c^* is the minimal wave speed.*

For $c = c^*$, the traveling wave has the following asymptotic behaviors:

$$\begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} A_{c^*} \xi e^{\lambda_1(c^*)\xi} \\ B_{c^*} \xi e^{\lambda_1(c^*)\xi} \end{pmatrix} + o \begin{pmatrix} \xi e^{\lambda_1(c^*)\xi} \\ \xi e^{\lambda_1(c^*)\xi} \end{pmatrix} \quad \text{as } \xi \to -\infty,$$

and

$$\begin{aligned} \begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} &= \begin{pmatrix} 1 - \bar{A}_{c^*} e^{-\frac{b}{c^*}\xi} - \bar{B}_{c^*} e^{-\bar{\lambda}_2(c^*)\xi} \\ 1 - \hat{A}_{c^*} e^{-\frac{b}{c^*}\xi} \end{pmatrix} \\ &\quad + o \begin{pmatrix} \bar{A}_{c^*} e^{-\frac{b}{c^*}\xi} + \bar{B}_{c^*} e^{-\bar{\lambda}_2(c^*)\xi}) \\ \hat{A}_{c^*} e^{-\frac{b}{c^*}\xi} \end{pmatrix} \quad \text{as } \xi \to +\infty. \end{aligned}$$

For $c > c^*$,

$$\begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} A_c e^{\lambda_1(c)\xi} \\ B_c e^{\lambda_1(c)\xi} \end{pmatrix} + o(e^{\lambda_1(c)\xi}) \quad \text{as } \xi \to -\infty,$$

and

$$\begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} 1 - \bar{A}_c e^{-\frac{b}{c}\xi} - \bar{B}_c e^{-\bar{\lambda}_2(c)\xi} \\ 1 - \hat{A}_c e^{-\frac{b}{c}\xi} \end{pmatrix}$$

$$+ o\begin{pmatrix} \bar{A}_c e^{-\frac{b}{c}\xi} + \bar{B}_c e^{-\bar{\lambda}_2(c)\xi}) \\ \hat{A}_c e^{-\frac{b}{c}\xi} \end{pmatrix} \quad \text{as } \xi \to +\infty,$$

where $\bar{\lambda}_2$ is the smaller one of the real roots of equation $\int_{\mathbb{R}} e^{\lambda\xi} J(\xi)d\xi - 1 - c\lambda - 1 = 0$ and $-c\lambda + b = 0$. A_{c^*}, B_{c^*}, $\bar{A}_{c^*}$, $\bar{B}_{c^*}$, $\hat{A}_{c^*}$, and A_c, B_c, $\bar{A}_c$, $\bar{B}_c$, $\hat{A}_c$ are real numbers and at least one of $\bar{A}_c$, $\bar{B}_c$ (or hatA_c) is non-zero, the sign of $\bar{A}_c$ (or $\hat{A}_c$) and $\bar{B}_c$ should be chosen such that the above equations are well defined.

Proof. The proof is divided into five steps.

Step 1. The existence of the traveling wave solution for system (3.4) comes from Theorem 2.9 and Lemma 3.3 through Lemmas 3.5 and 3.8. Since the traveling wave solution is a fixed point in the profile set, it is monotone. In addition, the component u of the traveling wave solution is strictly monotonically increasing because of the maximum principle.

For the component v, from the construction of upper and lower solutions, we have $v < 1$ for $\xi < 0$. Therefore, there exists a positive constant M such that $v' = \frac{b}{c}u(1-v) > 0$ for $\xi \le -M$. Thus, v is strictly monotonically increasing for $\xi \in (-\infty, -M]$. Let $(w_1(\xi), w_2(\xi)) = (u'(\xi), v'(\xi))$. Then we have

$$-cw_2' + b(1-v)w_1 - buw_2 = 0.$$

We rewrite it as

$$cw_2' + buw_2 = b(1-v)w_1 \ge 0.$$

Then we have

$$(e^{\frac{b}{c}\int_{-M}^{\xi} u(s)ds} w_2(\xi))' \ge 0, \quad \xi \in [-M, +\infty)$$

which means that $e^{\frac{b}{c}\int_{-M}^{\xi} u(s)ds} w_2(\xi)$ is increasing on $[-M, +\infty)$. This together with $w_2(-M) > 0$ implies that $w_2(\xi) > 0$ on $(-M, +\infty)$. Therefore v is strictly monotonically increasing for $\xi \in [-M, +\infty)$. Hence, v is also strictly monotonically increasing for $\xi \in (-\infty, +\infty)$.

Step 2. We derive the asymptotics of the traveling wave solutions at infinities. Comparing the decay rates of the upper and lower solutions at $-\infty$, we have the asymptotics of the u component at $-\infty$,

$$u(\xi) = A_c e^{\lambda_1(c)\xi} + o(e^{\lambda_1(c)\xi}).$$

To derive the asymptotic of the v component at $-\infty$, we investigate the derivative $(w_1(\xi), w_2(\xi))$ of the traveling wave solution $(u(\xi), v(\xi))$, which satisfies the system

$$\begin{gathered} J * w_1 - w_1 - cw_1' + w_1(1 - r - 2u + rv) + rw_1w_2 = 0, \\ -cw_2' + b(1-v)w_1 - buw_2 = 0, \\ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(-\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(+\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{gathered} \tag{3.16}$$

The limit system of (3.16) at $-\infty$ is

$$\begin{gathered} J * w_1^- - w_1^- - c(w_1^-)' + w_1^-(1-r) = 0, \\ -c(w_2^-)' + bw_1^- = 0. \end{gathered} \tag{3.17}$$

Since $w_1(\xi)$ is a derivative of $u(\xi)$, we have $w_1^-(\xi) \sim e^{\lambda_1(c)\xi}$ as $\xi \to -\infty$. By integrating the second equation of (3.17), we have

$$w_2^-(\xi) = \bar{d}e^{\lambda_1(c)\xi} + o(e^{\lambda_1(c)\xi}). \tag{3.18}$$

From the second equation of (3.17), we see $\bar{d} > 0$. Then the asymptotic behavior of $v(\xi)$ is obtained by integrating $w_2^-(\xi)$ from $-\infty$ to ξ. So the traveling wave solution has the following asymptotic behaviors:

$$\begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} A_c e^{\lambda_1(c)\xi} \\ B_c e^{\lambda_1(c)\xi} \end{pmatrix} + o \begin{pmatrix} e^{\lambda_1(c)\xi} \\ e^{\lambda_1(c)\xi} \end{pmatrix} \quad \text{as } \xi \to -\infty.$$

Next, we derive the asymptotics of the traveling waves at $+\infty$. By introducing transformations: $\hat{u} = 1 - u$ and $\hat{v} = 1 - v$, system (3.4) is changed into

$$\begin{gathered} J * \hat{u} - \hat{u} - c(\hat{u})' + (1 - \hat{u})(r\hat{v} - \hat{u}) = 0, \\ c(\hat{v})' + b(1 - \hat{u})\hat{v} = 0 \\ (\hat{u}, \hat{v})(-\infty) = (1, 1), \ (\hat{u}, \hat{v})(+\infty) = (0, 0). \end{gathered} \tag{3.19}$$

Further, by the transformation $\bar{\xi} = -\xi$, system (3.19) is changed into

$$\begin{gathered} J * \hat{u} - \hat{u} + c(\hat{u})' + (1 - \hat{u})(r\hat{v} - \hat{u}) = 0, \\ -c(\hat{v})' + b(1 - \hat{u})\hat{v} = 0, \\ (\hat{u}, \hat{v})(-\infty) = (0, 0), \quad (\hat{u}, \hat{v})(+\infty) = (1, 1), \end{gathered} \tag{3.20}$$

where the derivative is taken with respect to $\bar{\xi}$. Then we need to study the asymptotics of $(\hat{u}, \hat{v})$ at $-\infty$ for the system (3.20), which can be rewritten as

$$\begin{gathered} J * \hat{u} - \hat{u} + c(\hat{u})' - \hat{u} + r\hat{v} = -\hat{u} + r\hat{v} - (1 - \hat{u})(r\hat{v} - \hat{u}) \doteq R_1(\hat{u}, \hat{v}), \\ -c(\hat{v})' + b\hat{v} = b\hat{v} - b(1 - \hat{u})\hat{v} \doteq R_2(\hat{u}, \hat{v}). \end{gathered} \tag{3.21}$$

Similar to the proof of [14, Lemma 10], we can show that for the solutions of (3.20) there exists a positive constant γ such that

$$\begin{pmatrix} \hat{u}(\bar{\xi}) \\ \hat{v}(\bar{\xi}) \end{pmatrix} = \begin{pmatrix} O(e^{\gamma\bar{\xi}}) \\ O(e^{\gamma\bar{\xi}}) \end{pmatrix} \quad \text{as } \bar{\xi} \to -\infty.$$

For λ's such that $-\gamma < \operatorname{Re}\lambda < 0$, the two side Laplace transform of $\hat{u}$ and $\hat{v}$ are well defined. Let

$$(U(\lambda), V(\lambda)) = \Big(\int_{\mathbb{R}} e^{-\lambda\bar{\xi}} \hat{u}(\bar{\xi}) d\bar{\xi}, \ \int_{\mathbb{R}} e^{-\lambda\bar{\xi}} \hat{v}(\bar{\xi}) d\bar{\xi} \Big),$$

then system (3.21) can be written as

$$\begin{pmatrix} \int_{\mathbb{R}} e^{\lambda\bar{\xi}} J(\bar{\xi}) d\bar{\xi} - 1 + c\lambda - 1 & r \\ 0 & -c\lambda + b \end{pmatrix} \begin{pmatrix} U(\lambda) \\ V(\lambda) \end{pmatrix} = \int_{\mathbb{R}} e^{-\lambda\bar{\xi}} \begin{pmatrix} R_1(\hat{u}(\bar{\xi}), \hat{v}(\bar{\xi})) \\ R_2(\hat{u}(\bar{\xi}), \hat{v}(\bar{\xi})) \end{pmatrix} d\bar{\xi}$$

or equivalently,

$$\begin{aligned} &\begin{pmatrix} U(\lambda) \\ V(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} \int_{\mathbb{R}} e^{\lambda\bar{\xi}} J(\bar{\xi}) d\bar{\xi} - 1 + c\lambda - 1 & r \\ 0 & -c\lambda + b \end{pmatrix}^{-1} \int_{\mathbb{R}} e^{-\lambda\bar{\xi}} \begin{pmatrix} R_1(\hat{u}(\bar{\xi}), \hat{v}(\bar{\xi})) \\ R_2(\hat{u}(\bar{\xi}), \hat{v}(\bar{\xi})) \end{pmatrix} d\bar{\xi} \\ &= \begin{pmatrix} \frac{1}{\int_{\mathbb{R}} e^{\lambda\bar{\xi}} J(\bar{\xi}) d\bar{\xi} - 1 + c\lambda - 1} & \frac{-r}{[\int_{\mathbb{R}} e^{\lambda\bar{\xi}} J(\bar{\xi}) d\bar{\xi} - 1 + c\lambda - 1][-c\lambda + b]} \\ 0 & \frac{1}{-c\lambda + b} \end{pmatrix} \\ &\times \int_{\mathbb{R}} e^{-\lambda\bar{\xi}} \begin{pmatrix} R_1(\hat{u}(\bar{\xi}), \hat{v}(\bar{\xi})) \\ R_2(\hat{u}(\bar{\xi}), \hat{v}(\bar{\xi})) \end{pmatrix} d\bar{\xi} \end{aligned}$$

$$\doteq \tilde{M}\int_{\mathbb{R}} e^{-\lambda\bar{\xi}}\begin{pmatrix} R_1(\hat{u}(\bar{\xi}),\hat{v}(\bar{\xi}))\\ R_2(\hat{u}(\bar{\xi}),\hat{v}(\bar{\xi}))\end{pmatrix} d\bar{\xi}.$$

Further, system (3.21) can be rewritten as

$$\begin{pmatrix}\int_0^{+\infty}\hat{u}(\bar{\xi})e^{\lambda\bar{\xi}}d\bar{\xi}\\ \int_0^{+\infty}\hat{v}(\bar{\xi})e^{\lambda\bar{\xi}}d\bar{\xi}\end{pmatrix} = \tilde{M}\int_{\mathbb{R}} e^{-\lambda\bar{\xi}}\begin{pmatrix} R_1(\hat{u}(\bar{\xi}),\hat{v}(\bar{\xi}))\\ R_2(\hat{u}(\bar{\xi}),\hat{v}(\bar{\xi}))\end{pmatrix} d\bar{\xi} - \begin{pmatrix}\int_0^{+\infty}\hat{u}(\bar{\xi})e^{-\lambda\bar{\xi}}d\bar{\xi}\\ \int_0^{+\infty}\hat{v}(\bar{\xi})e^{-\lambda\bar{\xi}}d\bar{\xi}\end{pmatrix}.$$

Since $\hat{u}$ and $\hat{v}$ are monotonically decreasing, by Ikehara's Tauberian Theorem [5] the asymptotics of $\hat{v}$ and $\hat{u}$ are

$$\hat{u}(\bar{\xi}) \sim e^{\bar{\lambda}_2\bar{\xi}}, \quad \hat{v}(\bar{\xi}) \sim e^{\frac{b}{c}\bar{\xi}}, \quad \bar{\xi}\to-\infty,$$

with $\bar{\lambda}_2 > 0$ being the smaller one of the real roots of equations $\int_{\mathbb{R}} e^{\lambda\xi}J(\bar{\xi})d\bar{\xi} - 1 + c\lambda - 1 = 0$ and $-c\lambda + b = 0$. Therefore, the asymptotics of the traveling wave $(\hat{u},\ \hat{v})$ at $-\infty$ are

$$\begin{pmatrix}\hat{u}(\bar{\xi})\\ \hat{v}(\bar{\xi})\end{pmatrix} = \begin{pmatrix}\bar{A}_c e^{\frac{b}{c}\bar{\xi}} + \bar{B}_c e^{\bar{\lambda}_2(c)\bar{\xi}}\\ \hat{A}_c e^{\frac{b}{c}\bar{\xi}}\end{pmatrix} + o\begin{pmatrix}\bar{A}_c e^{\frac{b}{c}\bar{\xi}} + \bar{B}_c e^{\bar{\lambda}_2(c)\bar{\xi}})\\ \hat{A}_c e^{\frac{b}{c}\bar{\xi}}\end{pmatrix}, \quad \bar{\xi}\to-\infty,$$

where $\bar{\lambda}_2$ is the smaller one of the real roots of equations $\int_{\mathbb{R}} e^{\lambda\xi}J(\xi)d\xi - 1 + c\lambda - 1 = 0$ and $-c\lambda + b = 0$, and $\bar{A}_c$, $\bar{B}_c$ $(\hat{A}_c)$ are real number and at least one of $\bar{A}_c$, $\bar{B}_c$ $(\hat{A}_c)$ is non-zero, the sign of $\bar{A}_c$, $(\hat{A}_c)$ and $\bar{B}_c$ should be chosen such that the above equations are well defined.

On changing back to u and v, we have the estimates:

$$\begin{pmatrix}u(\xi)\\ v(\xi)\end{pmatrix} = \begin{pmatrix}1 - \bar{A}_c e^{-\frac{b}{c}\xi} - \bar{B}_c e^{-\bar{\lambda}_2(c)\xi}\\ 1 - \hat{A}_c e^{-\frac{b}{c}\xi}\end{pmatrix} + o\begin{pmatrix}\bar{A}_c e^{-\frac{b}{c}\xi} + \bar{B}_c e^{-\bar{\lambda}_2(c)\xi})\\ \hat{A}_c e^{-\frac{b}{c}\xi}\end{pmatrix}, \quad \xi\to+\infty,$$

where $\bar{\lambda}_2$ and $\bar{A}_c$, $\bar{B}_c$, $\hat{A}_c$ are the same as above.

Step 3. We show that for $c = c^*$, system (3.4) also has a unique monotone traveling wave solution. Let $c > c^*$ be a wave speed and (u, v) be the corresponding wave. We can normalize u such that $u(0) = 1/2$ and the corresponding $v(0) = \tilde{v}$ due to the shifting invariance of system (3.4). Choose $c_n > c^*$ such that $\lim_{n\to\infty} c_n = c^*$ and let (u_n, v_n) be the corresponding wave satisfying the above normalization.

> *claim* the functions $\{(u_n, v_n)\}$, $\{J * u_n\}$ and $\{(u_n', v_n')\}$ for $n = 1, 2, \dots$ are uniformly bounded and equi-continuous.

In fact, since (u_n, v_n) is a wave solution for the speed c_n, we have $0 \le u_n(\xi), v_n(\xi) \le 1$, $\xi \in \mathbb{R}$ and $n \ge 1$ which means that (u_n, v_n) are uniformly bounded. The continuity of (F_1, F_2) on the interval $[(0,0),(1,1)]$ implies that

$$c_n|v_n'| \le \max_{0\le u_n, v_n\le 1} F_2.$$

This shows that v_n' is uniformly bounded and the equi-continuity for v_n follows easily from the mean value theorems.

We can similarly show that $J * u_n$ and u_n' are uniformly bounded on $\mathbb{R}$ and the equi-continuity of u_n.

We next show the equi-continuity of $J * u_n$. For any two points $\xi_1, \xi_2 \in \mathbb{R}$,

$$\begin{aligned}&|J * u_n(\xi_1) - J * u_n(\xi_2)|\\ &= |\int_{\mathbb{R}} J(\xi_1 - y)u_n(y)dy - \int_{\mathbb{R}} J(\xi_2 - y)u_n(y)dy|\end{aligned}$$

$$
\begin{aligned}
&= |\int_{\mathbb{R}} J(y)u_n(\xi_1+y)dy - \int_{\mathbb{R}} J(y)u_n(\xi_2+y)dy| \\
&= |\int_{\mathbb{R}} J(y)(u_n(\xi_1+y) - u_n(\xi_2+y)dy| \\
&\le \int_{\mathbb{R}} J(y)|u_n(\xi_1+y) - u_n(\xi_2+y)|dy
\end{aligned}
$$

which together with the equi-continuity of u_n imply the equi-continuity of $J * u_n$ and hence that of u_n' follow from this. The claim is then proved.

By Arzela-Ascoli Thoerem and a diagonalization process, we can find a subsequence of $\{(u_n, v_n)\}$, which we still denote $\{(u_n, v_n)\}$ such that $\{(u_n, v_n)\}$, $\{J*u_n\}$, and $\{(u_n', v_n')\}$ converge uniformly on every bounded interval to functions (u^*, v^*), $J*u_n$, w^* and (X^*, Y^*). A standard argument [5] shows that $(u^*, v^*)' = (X^*, Y^*)$. Letting $n \to +\infty$, we have

$$
\begin{gathered}
J * u^* - u^* - c^*(u^*)' + u^*(1 - a_1 - u^* + a_1 v^*) = 0 \\
-c^*(v^*)' + bu^*(1 - v^*) = 0
\end{gathered}
$$

with the boundary conditions

$$
(u^*, v^*)(-\infty) = (0, 0), \ (u^*, v^*)(+\infty) = (1, 1),
$$

which shows the existence of traveling wave solution to system (3.4) for $c = c^*$.

Step 4. We next show that the traveling wave solution is unique up to a translation of the origin for $c \ge c^*$. If there are two wave solutions of (3.4) $(u, v)(\xi)$ and $(u_1, v_1)(\xi)$ with the same asymptotic properties as described by Step 2 at $-\infty$ and $+\infty$, then there exist a positive constant N and a point ζ such that

$$
\begin{pmatrix} u(\xi+\zeta) \\ v(\xi+\zeta) \end{pmatrix} > \begin{pmatrix} u_1(\xi) \\ v_1(\xi) \end{pmatrix}, \quad \xi \in (-\infty, -N] \cup [N, +\infty). \tag{3.22}
$$

In fact, since the traveling wave solution has the following asymptotic behaviors

$$
\begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} A_c e^{\lambda_1(c)\xi} \\ B_c e^{\lambda_1(c)\xi} \end{pmatrix} + o\begin{pmatrix} e^{\lambda_1(c)\xi} \\ e^{\lambda_1(c)\xi} \end{pmatrix} \quad \text{as } \xi \to -\infty,
$$

and

$$
\begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} 1 - \bar{A}_c e^{-\frac{b}{c}\xi} - \bar{B}_c e^{-\bar{\lambda}_2(c)\xi} \\ 1 - \hat{A}_c e^{-\frac{b}{c}\xi} \end{pmatrix} + o\begin{pmatrix} \bar{A}_c e^{-\frac{b}{c}\xi} + \bar{B}_c e^{-\bar{\lambda}_2(c)\xi}) \\ \hat{A}_c e^{-\frac{b}{c}\xi} \end{pmatrix} \quad \text{as } \xi \to +\infty,
$$

in addition if

$$
\begin{pmatrix} u_1(\xi) \\ v_1(\xi) \end{pmatrix} = \begin{pmatrix} A_c' e^{\lambda_1(c)\xi} \\ B_c' e^{\lambda_1(c)\xi} \end{pmatrix} + o(e^{\lambda_1(c)\xi}) \quad \text{as } \xi \to -\infty,
$$

and

$$
\begin{pmatrix} u_1(\xi) \\ v_1(\xi) \end{pmatrix} = \begin{pmatrix} 1 - \bar{A}_c' e^{-\frac{b}{c}\xi} - \bar{B}_c' e^{-\bar{\lambda}_2(c)\xi} \\ 1 - \hat{A}_c' e^{-\frac{b}{c}\xi} \end{pmatrix} + o\begin{pmatrix} \bar{A}_c' e^{-\frac{b}{c}\xi} + \bar{B}_c' e^{-\bar{\lambda}_2(c)\xi}) \\ \hat{A}_c' e^{-\frac{b}{c}\xi} \end{pmatrix}
$$

as $\xi \to +\infty$, Then for $\xi \to -\infty$, we have

$$
\begin{aligned}
\begin{pmatrix} u(\xi+\zeta) \\ v(\xi+\zeta) \end{pmatrix} &= \begin{pmatrix} A_c e^{\lambda_1(c)(\xi+\zeta)} \\ B_c e^{\lambda_1(c)(\xi+\zeta)} \end{pmatrix} + o\begin{pmatrix} e^{\lambda_1(c)(\xi+\zeta)} \\ e^{\lambda_1(c)(\xi+\zeta)} \end{pmatrix} \\
&= \begin{pmatrix} A_c e^{\lambda_1(c)\zeta} e^{\lambda_1(c)\xi} \\ B_c e^{\lambda_1(c)\zeta} e^{\lambda_1(c)\xi} \end{pmatrix} + o\begin{pmatrix} e^{\lambda_1(c)(\xi+\zeta)} \\ e^{\lambda_1(c)(\xi+\zeta)} \end{pmatrix},
\end{aligned}
$$

and for $\xi \to +\infty$, we have

$$
\begin{aligned}
&\begin{pmatrix} u(\xi+\zeta) \\ v(\xi+\zeta) \end{pmatrix} \\
&= \begin{pmatrix} 1-\bar{A}_c e^{-\frac{b}{c}(\xi+\zeta)} - \bar{B}_c e^{-\bar{\lambda}_2(c)(\xi+\zeta)} \\ 1-\hat{A}_c e^{-\frac{b}{c}(\xi+\zeta)} \end{pmatrix} + o\begin{pmatrix} \bar{A}_c e^{-\frac{b}{c}(\xi+\zeta)} + \bar{B}_c e^{-\bar{\lambda}_2(c)(\xi+\zeta)}) \\ \hat{A}_c e^{-\frac{b}{c}(\xi+\zeta)}) \end{pmatrix} \\
&= \begin{pmatrix} 1-\bar{A}_c e^{-\frac{b}{c}\zeta} e^{-\frac{b}{c}\xi} - \bar{B}_c e^{-\bar{\lambda}_2(c)\zeta} e^{-\bar{\lambda}_2(c)\xi} \\ 1-\hat{A}_c e^{-\frac{b}{c}\zeta} e^{-\frac{b}{c}\xi} \end{pmatrix} \\
&\quad + o\begin{pmatrix} \bar{A}_c e^{-\frac{b}{c}(\xi+\zeta)} + \bar{B}_c e^{-\bar{\lambda}_2(c)(\xi+\zeta)}) \\ \hat{A}_c e^{-\frac{b}{c}(\xi+\zeta)} \end{pmatrix}.
\end{aligned}
$$

Noting that there exists an $\hat{N} > 0$ such that for any $|\zeta| > \hat{N}$, we have $A_c e^{\bar{\lambda}_1(c)\zeta} > \bar{A}'_c$, $B_c e^{\bar{\lambda}_1(c)\zeta} > \bar{B}'_c$, and $\bar{A}_c e^{-\bar{\lambda}_1(c)\zeta} < \bar{A}'_c$, $\bar{B}_c e^{-\bar{\lambda}_2(c)\zeta} < \bar{B}'_c$, $\hat{A}_c e^{-\bar{\lambda}_1(c)\zeta} < \hat{A}'_c$, where $\bar{\lambda}_1(c) = b/c$. So, the inequality (3.22) holds. By Proposition 3.7, we have

$$
\begin{pmatrix} u(\xi+\zeta) \\ v(\xi+\zeta) \end{pmatrix} > \begin{pmatrix} u_1(\xi) \\ v_1(\xi) \end{pmatrix}, \quad \xi \in (-N,\ N). \tag{3.23}
$$

For ζ is decreasing, we have to deal with two cases. The first case is that there exists a $\zeta \in \mathbb{R}$ such that

$$
\begin{pmatrix} u(\xi+\zeta) \\ v(\xi+\zeta) \end{pmatrix} = \begin{pmatrix} u_1(\xi) \\ v_1(\xi) \end{pmatrix}, \quad \xi \in (-\infty,\ +\infty). \tag{3.24}
$$

We then have the uniqueness. The other case is that there is a first ζ and a first ξ, which are denoted by ζ_1 and ξ_1 separately, such that there is at least one component, for example, the component v, satisfying $v(\xi_1 + \zeta_1) = v_1(\xi_1)$. Let $\bar{W}(\xi) = (u(\xi), v(\xi)) - (u_1(\xi), v_1(\xi))$ then $w_i(\xi)$, $i = 1, 2$, satisfies

$$
\begin{gathered}
J * w_1 - w_1 - cw_1' + (1 - r - 2u + rv)w_1 \\
\le J * w_1 - w_1 - cw_1' + (1 - r - 2u + rv)w_1 + ruw_2 \le 0, \\
-cw_2' - buw_2 \le -cw_2' + b(1 - v)w_1 - buw_2 \le 0, \\
w_1(\xi) \ge 0, \quad w_2(\xi_1) = 0, \quad \xi \in (-\infty, +\infty).
\end{gathered}
$$

By the maximum principle for w_1 and the same argument as the one in Proposition 3.7 for w_2, this is impossible. Therefore,

$$
\begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} u_1(\xi) \\ v_1(\xi) \end{pmatrix}, \quad \xi \in (-\infty, +\infty).
$$

This proves the conclusion.

Step 5. We show that there is no monotone traveling wave solution for $0 < c < c^*$. Since the traveling wave solution has the following asymptotic behaviors at $-\infty$,

$$
\begin{pmatrix} u(\xi) \\ v(\xi) \end{pmatrix} = \begin{pmatrix} A_c e^{\lambda_1(c)\xi} \\ B_c e^{\lambda_1(c)\xi} \end{pmatrix} + o\begin{pmatrix} e^{\lambda_1(c)\xi} \\ e^{\lambda_1(c)\xi} \end{pmatrix} \quad \text{as } \xi \to -\infty,
$$

we can define the two-sided Laplace transform

$$
(U(\lambda), V(\lambda)) = \Big(\int_{\mathbb{R}} e^{-\lambda\bar{\xi}} \hat{u}(\bar{\xi}) d\bar{\xi},\ \int_{\mathbb{R}} e^{-\lambda\bar{\xi}} \hat{v}(\bar{\xi}) d\bar{\xi} \Big).
$$

Then the first equation of system (3.4) can be written as

$$[-c\lambda + \int_{\mathbb{R}} e^{\lambda y} J(y)dy - 1 + (1-r)]U(\lambda) = \int_{\mathbb{R}} e^{-\lambda x}[u - u(u - rv)]dx.$$

Since $-c\lambda + \int_{\mathbb{R}} e^{\lambda y} J(y)dy - 1 + (1-r) = 0$ has no real zeros for $0 < c < c^*$, $U(\lambda)$ is defined for all λ such that $\operatorname{Re}\lambda < 0$. (3) can be written as

$$\int_{\mathbb{R}} e^{-\lambda\xi}\Big[\Big(-c\lambda + \int_{\mathbb{R}} e^{\lambda y} J(y)dy - 1 + (1-r)\Big)u - u(u - rv)\Big]dx = 0.$$

Since $-c\lambda + \int_{\mathbb{R}} e^{\lambda y} J(y)dy - 1 + (1-r) \to +\infty$ as $\lambda \to -\infty$, we reach a contradiction. Therefore, there is no monotone traveling wave solution for $0 < c < c^*$, and c^* is the minimal wave speed. □

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