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EXISTENCE OF SOLUTIONS FOR KIRCHHOFF-TYPE PROBLEMS VIA THE METHOD OF LOWER AND UPPER SOLUTIONS

BAOQIANG YAN, DONAL O'REGAN, RAVI P. AGARWAL

Abstract. This article considers elliptic problems of Kirchhoff-type. We give some new definitions of lower and upper solutions for the problem and establish the method of lower and upper solutions when the upper and lower solutions are well ordered, i.e., the lower solution is less than the upper one, and we also consider the case when the upper and lower solutions have opposite ordering. In addition we use the relation between the topological degree and strict upper and lower solutions in both cases and using this we obtain multiplicity results for nonlinear Kirchhoff-type elliptic problems.

1. Introduction

In this article, we consider the nonlocal elliptic problem

$$\begin{gathered} -a\Big(\int_\Omega |\nabla u(x)|^2dx\Big)\Delta u(x) = f(x,u(x),\nabla u(x)) - g(x,u(x),\nabla u(x)), \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain and $a \in C([0,+\infty),(0,+\infty))$ with

$$a(t) \text{ nondecreasing on } [0,+\infty) \text{ and } a(t) \geq a(0) > 0,\ \forall t \in [0,+\infty). \tag{1.2}$$

Problem (1.1) is a generalization of the classical stationary Kirchhoff equation

$$\begin{gathered} -a\Big(\int_\Omega |\nabla u(x)|^2dx\Big)\Delta u(x) = f(x,u(x)), \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

which was proposed by Kirchhoff as a generalization of the well-known d'Alembert's equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2 u}{\partial x^2} = g(x,u)$$

for free vibrations of elastic strings; see [26]. Problem (1.2) received attention after the paper by Lions [32], where an abstract framework to the problem was proposed and variational methods were applied to establish existence and multiplicity of solutions for problem (1.2); see [3, 7, 14, 22, 23, 25, 28, 29, 30, 31, 33, 34, 35, 38] and the references therein.

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Note the nonlinearity $f-g$ in (1.1) contains the gradient term ∇u, which makes problem (1.1) nonvariational. Thus the variational method cannot be used in a direct way. It is possible to use other tools to discuss such problems. For example, Huy and Quan [24] discussed the problem

$$-M\Big(x,\int_\Omega |\nabla u(x)|^2dx\Big)\Delta_p u(x)=\lambda f(x,u(x),\nabla u(x))-g(x,u(x),\nabla u(x)),\quad \text{in } \Omega,$$
$$u=0,\quad \text{on } \partial\Omega,$$

where $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, λ is a real parameter and $M:\Omega\times\mathbb{R}^+\to\mathbb{R}^+$, f, $g:\Omega\times\mathbb{R}^+\times\mathbb{R}^N\to\mathbb{R}^+$ are suitable functions and using the fixed point index, they established existence results for both nondegenerate and degenerate cases of the function M. The method of lower and upper solutions is an important tool to establish existence of solutions for nonlinear boundary value problems; see [4, 5, 6, 13, 15, 19, 21, 27, 36, 37, 42]. It would be natural to try to define lower and upper solutions for Kirchhoff-type problems and consider a corresponding comparison principle. Unfortunately, in [16], [18], the authors showed that the Kirchhoff equation does not enjoy the usual comparison principles (both weak or strong) and the lower and upper solutions method has problems (and errors in the literature). Therefore it is of interest to obtain the existence of solutions for the Kirchhoff-type equation via a method of lower and upper solutions.

When $f\geq 0$ and $g=0$, Alves and Corrêa [2] established an existence result for (1.1) via the lower and upper approach, by using the theory of pseudomonotone operators. In [41], when $f-g$ is independent of ∇u, the authors established a theorem on lower and upper solutions and obtained some existence theorems for some special nonlinearities $f-g$. Also some interesting results for the Kirchhoff equation by the method of lower and upper solution can be found in [1, 8, 9, 10, 11, 12, 17, 39, 41] when $f-g$ is independent of ∇u. Usually in the literature authors assume that the lower and upper solutions are well ordered, i.e., the lower solution is less than upper one.

There are two main objectives in this article: (1) from some ideas in [13, 36, 37], we obtain relations between the topological degree and strict upper and lower solutions which are well ordered or opposite-ordered; (2) we obtain existence and multiplicity results for the nonlinear elliptic problem of Kirchhoff-type (1.1).

The article is organized as follows. In Section 2, we list some conditions and give the definitions of lower and upper solutions. Section 3 is devoted to proving a result on lower and upper solutions using topological degree with well-ordered lower and upper solutions. In Section 4, we use topological degree in the case of opposite-ordered lower and upper solutions. In Section 5, we discuss the multiplicity of solutions of problem (1.1). In section 6, two examples are listed to illustrate the applications of our theory.

2. Preliminaries

In this section, we list some conditions and give the definitions of lower and upper solutions for problem (1.1). Throughout this paper, we suppose the following conditions are satisfied:

(H1) f and $g\in C(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ with

$$f(x,u,z)\geq 0,\quad g(x,u,z)\geq 0,\forall (x,u,z)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N; \tag{2.1}$$

(H2) there exists $M > 0$ such that
$$|f(x,u,z) - g(x,u,z)| \le M, \quad \forall (x,u,z) \in [0,1] \times \mathbb{R} \times \mathbb{R}^N. \tag{2.2}$$

For $f - g$ unbounded we can use the method of a priori estimates and replace condition (2.2) by conditions of growth or sign types.

Suppose $G(x,y)$ is the Green's function for $-\Delta u(x) = h$ and $u|_{\partial\Omega} = 0$ and set
$$\begin{gathered} H(x) := \frac{M}{a_0} \int_\Omega |\nabla_x G(x,y)| dy, \quad x \in \overline{\Omega}, \\ a_0 = a(0), \quad b_0 := a\Big(\int_\Omega H^2(x) dx \Big). \end{gathered} \tag{2.3}$$

Let
$$C^1(\overline{\Omega}) = \{u : \overline{\Omega} \to \mathbb{R} : \ u(x) \text{ is continuously differentiable on } \overline{\Omega}\}$$
with the norm $\|u\| = \max\{\|u\|_0, \|\nabla u\|_0\}$, where $\|u\|_0 = \max_{x\in\overline{\Omega}} |u(x)|$ and $\|\nabla u\|_0 = \max_{x\in\overline{\Omega}} \sqrt{\sum_{i=1}^N (\frac{\partial u}{\partial x_i})^2}$. Note that $C^1(\overline{\Omega})$ is a Banach space.

Definition 2.1. A function $\alpha \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a lower solution of (1.1) if
$$\begin{gathered} -\Delta\alpha(x) \le \frac{1}{b_0} f(x, \alpha(x), \nabla\alpha(x)) - \frac{1}{a_0} g(x, \alpha(x), \nabla\alpha(x)), \text{ in } \Omega, \\ \alpha\big|_{\partial\Omega} \le 0, \end{gathered} \tag{2.4}$$
where a_0 and b_0 are defined by (2.3).

Definition 2.2. A function $\beta \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is an upper solution of (1.1) if
$$\begin{gathered} -\Delta\beta(x) \ge \frac{1}{a_0} f(x, \beta(x), \nabla\beta(x)) - \frac{1}{b_0} g(x, \beta(x), \nabla\beta(x)), \quad \text{in } \Omega, \\ \beta\big|_{\partial\Omega} \ge 0, \end{gathered} \tag{2.5}$$
where a_0 and b_0 are defined by (2.3).

From the ideas in [13], we give the following definition.

Definition 2.3. Let $u, v \in C^1(\overline{\Omega})$. We say that $u \prec v$ if $u(x) < v(x)$ on Ω and for $x \in \partial\Omega$, either $u(x) < v(x)$ or $u(x) = v(x)$ and $\frac{\partial u}{\partial n} > \frac{\partial v}{\partial n}$.

Remark 2.4. The set $S = \{u \in C^1(\overline{\Omega}) : \alpha \prec u \prec \beta\}$ is open if $\alpha \prec \beta$.

We say that an open set $S \subseteq C^1(\overline{\Omega})$ is admissible for the degree (for the compact map A) if the compact operator A has no fixed point on its boundary ∂S and the set of fixed points of A in S is bounded. In this case, we define
$$\deg(I - A, S, 0) = \deg(I - A, S \cap B(0,R), 0),$$
where R is such that every fixed point u of A in S satisfies $\|u\| < R$. From the excision property this degree does not depend on R.

Suppose that (H1) and (H2) hold. Then the operators
$$\tilde{a} : C^1(\overline{\Omega}) \to (0, +\infty) : u \mapsto a\Big(\int_\Omega ||\nabla u(x)| - (|\nabla u(x)| - H(x))^+|^2 dx \Big)$$
and
$$N : C^1(\overline{\Omega}) \to C(\overline{\Omega}) : u \mapsto \frac{1}{\tilde{a}(u)} [f(x, u(x), \nabla u(x)) - g(x, u(x), \nabla u(x))]$$

are well-defined, continuous, and map bounded sets of $C^1(\overline{\Omega})$ to bounded sets in $C(\overline{\Omega})$; here $(|\nabla u(x)| - H(x))^+ = \max\{0, |\nabla u(x)| - H(x)\}$. Then, for fixed $\lambda_0 > 0$, the operator $A : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$

$$Au = (-\Delta + \lambda_0)^{-1}(Nu + \lambda_0 u) \tag{2.6}$$

is completely continuous.

If u is a fixed point of A defined by (2.6), then

$$u(x) = \frac{1}{\tilde{a}(u)} \int_\Omega G(x,y)[f(y,u(y),\nabla u(y)) - g(y,u(y),\nabla u(y))]dy$$

and then

$$\begin{aligned} |\nabla u(x)| &= \Big|\frac{1}{\tilde{a}(u)} \int_\Omega \nabla_x G(x,y)[f(y,u(y),\nabla u(y)) - g(y,u(y),\nabla u(y))]dy\Big| \\ &\le \frac{M}{a_0} \int_\Omega |\nabla_x G(x,y)|dy, \end{aligned}$$

which implies

$$\tilde{a}(u) = a\Big(\int_\Omega ||\nabla u(x)| - (|\nabla u(x)| - H(x))^+|^2 dx\Big) = a\Big(\int_\Omega |\nabla u(x)|^2 dx\Big)$$

Therefore,

$$u(x) = \frac{1}{a\big(\int_\Omega |\nabla u(x)|^2 ds\big)} \int_\Omega G(x,y)[f(y,u(y),\nabla u(y)) - g(y,u(y),\nabla u(y))]dy,$$

i.e., $u(x)$ satisfies (1.1). Consequently, the existence of solutions of (1.1) is equivalent to the existence of fixed points of the operator A defined in (2.6).

Definition 2.5. A lower solution α of (1.1) is said to be strict if every solution u of (1.1) such that $\alpha \le u$ on Ω satisfies $\alpha \prec u$.

In the same way a strict upper solution β of (1.1) is an upper solution such that every solution $u \le \beta$ is such that $u \prec \beta$.

3. Existence of solutions to (1.1) with well ordered lower and upper solutions

In this section, we assume that (1.1) has a pair of well ordered lower and upper solutions.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^N (N \ge 1)$ be a smooth bounded domain. Suppose that conditions (H1) and (H2) hold. Assume α and β are the lower solution and upper solution of (1.1) respectively such that*

$$\alpha(x) \le \beta(x). \tag{3.1}$$

Then problem (1.1) has at least one solution u such that for all $x \in \overline{\Omega}$,

$$\alpha(x) \le u(x) \le \beta(x).$$

If moreover $\alpha(x)$ and $\beta(x)$ are strict and satisfy $\alpha \prec \beta$, then

$$S = \{u \in C^1(\overline{\Omega}) : \ \alpha \prec u \prec \beta\}$$

is admissible for the degree (for the map A) and $\deg(I - A, S, 0) = 1$.

Proof. Let

$$\overline{f}(x,u,z)=\begin{cases} f(x,\alpha(x),z), & \text{if } u<\alpha(x);\\ f(x,u,z), & \text{if } \alpha(x)\le u\le\beta(x);\\ f(x,\beta(x),z), & \text{if } u>\beta(x)\end{cases}$$

and

$$\overline{g}(x,u,z)=\begin{cases} g(x,\alpha(x),z), & \text{if } u<\alpha(x);\\ g(x,u,z), & \text{if } \alpha(x)\le u\le\beta(x);\\ g(x,\beta(x),z), & \text{if } u>\beta(x).\end{cases}$$

We will study the modified problem

$$\begin{gathered} -\Delta u+\lambda_0 u=\frac{1}{\tilde a(u)}[\overline{f}(x,u,\nabla u(x))-\overline{g}(x,u,\nabla u(x))]+\lambda_0\gamma(x,u(x)),\quad x\in\Omega,\\ u\big|_{\partial\Omega}=0, \end{gathered}\tag{3.2}$$

where $\gamma(x,u)=\max\{\alpha(x),\min\{u,\beta(x)\}\}$ and λ_0 is as in (2.6).

Step 1. Every solution u of (3.2) satisfies $\alpha(x)\le u(x)\le\beta(x)$, $x\in\overline{\Omega}$.
Clearly, $||\nabla u(x)|-(|\nabla u(x)|-H(x))^+|^2\le H(x)^2$, which together with the monotonicity of $a(t)$ implies

$$a_0\le\tilde a(u)\le a\Big(\int_\Omega H(x)^2dx\Big)=b_0.$$

(a) We prove that $\alpha(x)\le u(x)$ on $\overline{\Omega}$. By contradiction, if $\alpha(x)\not\le u(x)$ on $\overline{\Omega}$, we have $\max_{x\in\overline{\Omega}}(\alpha(x)-u(x))=M>0$. Note that $\alpha(x)-u(x)\not\equiv M$ on $\overline{\Omega}$ ($\alpha(x)-u(x)\le 0$, $x\in\partial\Omega$). If $x_0\in\Omega$ is such that $\alpha(x_0)-u(x_0)=M$, it is easy to see that $\alpha(x_0)>u(x_0)$ and $\nabla u(x_0)=\nabla\alpha(x_0)$. It follows from (2.4) that

$$\begin{aligned} -\Delta(\alpha(x_0)-u(x_0)) &\le \frac{1}{b_0}f(x_0,\alpha(x_0),\nabla\alpha(x_0))-\frac{1}{a_0}g(x_0,\alpha(x_0),\nabla\alpha(x_0))\\ &\quad-\frac{1}{\tilde a(u)}[\overline{f}(x_0,u(x_0),\nabla u(x_0))-\overline{g}(x_0,u(x_0),\nabla u(x_0))]\\ &\quad+\lambda_0(u(x_0)-\gamma(x_0,u(x_0))\\ &=\frac{1}{b_0}f(x_0,\alpha(x_0),\nabla\alpha(x_0))-\frac{1}{\tilde a(u)}\overline{f}(x_0,u(x_0),\nabla u(x_0))\\ &\quad-\frac{1}{a_0}g(x_0,\alpha(x_0),\nabla\alpha(x_0))+\frac{1}{\tilde a(u)}\overline{g}(x_0,u(x_0),\nabla u(x_0))\\ &\quad+\lambda_0(u(x_0)-\alpha(x_0))\\ &\le\frac{1}{b_0}[f(x_0,\alpha(x_0),\nabla\alpha(x_0))-f(x_0,\alpha(x_0),\nabla\alpha(x_0)))\\ &\quad+\frac{1}{a_0}[g(x,\alpha(x_0),\nabla\alpha(x_0))-g(x_0,\alpha(x_0),\nabla\alpha(x_0))]\\ &\quad+\lambda_0(u(x_0)-\alpha(x_0))<0. \end{aligned}$$

This is a contradiction because x_0 is a maximum point.

(b) Now we prove that $\beta(x)\ge u(x)$ on $\overline{\Omega}$. By contradiction, assume that $\min_{x\in\overline{\Omega}}(\beta(x)-u(x))=-m<0$. Note that $\beta(x)-u(x)\not\equiv -m$ on $\overline{\Omega}$ ($\beta(x)-u(x)\ge 0$,

$x \in \partial\Omega$). If $x_0 \in \Omega$ is such that $\beta(x_0) - u(x_0) = -m < 0$, it is easy to see that $\beta(x_0) < u(x_0)$ and $\nabla u(x_0) = \nabla\beta(x_0)$. It follows from (2.5) that

$$\begin{aligned}
-\Delta(\beta(x_0) - u(x_0)) &\geq \frac{1}{a_0} f(x_0, \beta(x_0), \nabla\beta(x_0)) - \frac{1}{b_0} g(x_0, \beta(x_0), \nabla\beta(x_0)) \\
&\quad - \frac{1}{\tilde{a}(u)} [\overline{f}(x_0, u(x_0), \nabla u(x_0)) - \overline{g}(x_0, u(x_0), \nabla u(x_0))] \\
&\quad + \lambda_0(u(x_0) - \gamma(x_0, u(x_0))) \\
&= \frac{1}{a_0} f(x_0, \beta(x_0), \nabla\beta(x_0)) - \frac{1}{\tilde{a}(u)} \overline{f}(x_0, u(x_0), \nabla u(x_0)) \\
&\quad - \frac{1}{b_0} g(x_0, \beta(x_0), \nabla\beta(x_0)) + \frac{1}{\tilde{a}(u)} \overline{g}(x_0, u(x_0), \nabla u(x_0)) \\
&\quad + \lambda_0(u(x_0) - \beta(x_0)) \\
&\geq \frac{1}{a_0} [f(x_0, \beta(x_0), \nabla\beta(x_0)) - f(x_0, \beta(x_0), \nabla\beta(x_0))] \\
&\quad + \frac{1}{b_0} [g(x, \beta(x_0), \nabla u(x_0)) - g(x_0, \beta(x_0), \nabla\beta(x_0))] \\
&\quad + \lambda_0(u(x_0) - \beta(x_0)) > 0.
\end{aligned}$$

This is a contradiction because x_0 is a minimum point.

Combining (a) and (b), we have $\alpha(x) \leq u(x) \leq \beta(x)$, $\quad x \in \overline{\Omega}$.

Step 2. Every solution of (3.2) is a solution of (1.1).
Every solution of (3.2) satisfies $\alpha(x) \leq u(x) \leq \beta(x)$, $x \in \overline{\Omega}$. From the definitions of $\overline{f}$ and $\overline{g}$, we have

$$\begin{gathered}
\overline{f}(x, u(x), \nabla u(x)) = f(x, u(x), \nabla u(x)), \\
\overline{g}(x, u(x), \nabla u(x)) = g(x, u(x), \nabla u(x)), \\
|\nabla u(x)| \leq \frac{M}{a_0} \int_\Omega |\nabla_x G(x, y)| dy = H(x), \quad x \in \Omega
\end{gathered}$$

and so

$$\tilde{a}(u) = a\Big(\int_\Omega ||\nabla u(x)| - (|\nabla u(x)| - H(x))^+|^2 dx\Big) = a\Big(\int_\Omega |\nabla u(x)|^2 dx\Big).$$

Thus, u is a solution of (1.1).

Step 3. Equation (3.2) has at least one solution.
From (2.2) and the construction of $\overline{f}$ and $\overline{g}$, for every $u \in C^1(\overline{\Omega})$, we have

$$\begin{aligned}
&\Big|\frac{1}{\tilde{a}(u)} [\overline{f}(x, u(x), \nabla u(x)) - \overline{g}(x, u(x), \nabla u(x))] + \lambda_0 \gamma(x, u(x))\Big| \\
&\leq \frac{1}{a_0} M + \lambda_0(\|\alpha\|_0 + \|\beta\|_0), \quad \forall x \in \Omega.
\end{aligned}$$

Define operators

$$\overline{N} : C^1(\overline{\Omega}) \to C(\overline{\Omega}) : u \mapsto \frac{1}{\tilde{a}(u)} [\overline{f}(x, u(x), \nabla u(x)) - \overline{g}(x, u(x), \nabla u(x))] + \lambda_0 \gamma(x, u(x))$$

and $\overline{A} : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ by

$$\overline{A}u = (-\Delta + \lambda_0)^{-1}(\overline{N}u).$$

Note $\overline{A}$ is completely continuous and there exists a $K_0 > 0$ big enough such that for all $v \in \overline{A}(C^1(\overline{\Omega}))$, we have
$$\|v\| \le K_0. \tag{3.3}$$
Let $K = K_0 + 1$ and
$$S_1 = \{u \in C^1(\overline{\Omega}) :\ \|u(x)\|_0 < K\}.$$
Now S_1 is a open set in $C^1(\overline{\Omega})$ and (3.3) implies
$$\overline{A}(\overline{S}_1) \subseteq S_1,$$
so $\deg(I - \overline{A}, S_1, 0) = 1$. Therefore there exists a $u \in S_1$ such that $u = \overline{A}u$.

Now Step 1 and Step 2 yield
$$\begin{gathered}\alpha(x) \le u(x) \le \beta(x),\\ \tilde{a}(u) = a\Big(\int_\Omega |\nabla u(x)|^2 dx\Big), \quad x \in \Omega;\end{gathered}$$
so $u(x)$ is a solution to (1.1).

Step 4. If $\alpha(x)$ and $\beta(x)$ are strict lower solution and upper solution, we show $\deg(I - A, S, 0) = 1$.

Since $\alpha(x)$ and $\beta(x)$ are strict lower solution and upper solution, A has no fixed point on ∂S and so $\deg(I - A, S, 0)$ is well defined. On the other hand, by step 1, if $u \in S_1$ is a fixed point in $\overline{A}$, we have
$$\alpha(x) \le u(x) \le \beta(x),$$
and $u(x)$ is solution of (1.1). Since $\alpha(x)$ and $\beta(x)$ are strict lower solution and upper solution, one has $\alpha \prec u \prec \beta$. Therefore, $\overline{A}$ has no fixed point in $S_1 - S$, which implies that
$$\deg(I - \overline{A}, S_1, 0) = \deg(I - \overline{A}, S, 0) = 1.$$

Set $H(t,u) := u - (tAu + (1-t)\overline{A}u)$, $(t,u) \in [0,1] \times \overline{S}$. We show that $H(t,u) \ne 0$, for all $(t,u) \in [0,1] \times \partial S$. Suppose there is a $(t_0, u_0) \in [0,1] \times \partial S$ such that $H(t_0, u_0) = 0$. Since $u_0 \in \partial S$, we have
$$\alpha(x) \le u_0(x) \le \beta(x)$$
and then $\gamma(u_0(x)) = u_0(x)$ for all $x \in \Omega$, and
$$\begin{gathered}\begin{aligned}-\Delta u_0(x) &= t_0 \frac{1}{\tilde{a}(u_0)}[f(x, u_0(x), \nabla u_0(x)) - g(x, u_0(x), \nabla u_0(x))]\\ &\quad + (1-t_0)\frac{1}{\tilde{a}(u_0)}[\overline{f}(x, u_0(x), \nabla u_0(x)) - \overline{g}(x, u_0(x), \nabla u_0(x))], \quad x \in \Omega,\end{aligned}\\ u_0\big|_{\partial\Omega} = 0.\end{gathered}$$
Hence,
$$\begin{aligned}|\nabla u_0(x)| &\le \frac{t_0}{a_0}\int_\Omega |\nabla_x G(x,y)| f(y, u_0(y), \nabla u_0(y)) - g(y, u_0(y), \nabla u_0(y))| dy\\ &\quad + \frac{1-t_0}{a_0}\int_\Omega |\nabla_x G(x,y)|\overline{f}(y, u_0(y), \nabla u_0(y)) - \overline{g}(y, u_0(y), \nabla u_0(y))| dy\\ &\le \frac{M}{a_0}\int_\Omega |\nabla_x G(x,y)| dy,\end{aligned}$$

which implies

$$\tilde{a}(u_0) = a\Big(\int_\Omega |\nabla u_0(x)|^2 dx\Big), \quad x \in \Omega.$$

Then $u_0(x)$ satisfies

$$-a\Big(\int_\Omega |\nabla u_0(x)|^2 dx\Big)\Delta u_0(x) = f(x, u_0(x), \nabla u_0(x)) - g(x, u_0(x), \nabla u_0(x)), \quad x \in \Omega,$$
$$u_0 = 0, \quad \text{on } \partial\Omega.$$

Since α and β are strict lower and upper solutions, one has $\alpha \prec u_0 \prec \beta$ which contradicts $u_0 \in \partial S$. By the homotopy of topological degree we have

$$\deg(I - A, S, 0) = \deg(I - \overline{A}, S, 0) = 1.$$

The proof is complete. □

4. Existence of solutions to (1.1) with opposite-ordered upper and lower solutions

In this section, we suppose that (1.1) has upper and lower solutions with opposite order.

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain. Suppose that the conditions (H1) and (H2) hold and α, β are lower solution and upper solution of (1.1) respectively. If*

$$\alpha, \beta \text{ are strict and satisfy } \beta \prec \alpha, \tag{4.1}$$

then $\deg(I - A, S_2, 0) = -1$, where

$$S_2 = \{u \in C^1(\overline{\Omega}) : \|u\|_0 < B, \exists x_u \in \overline{\Omega} \text{ such that } \beta(x_u) < u(x_u) < \alpha(x_u)\}$$

with $B > \|\alpha\|_0 + \|\beta\|_0 + \frac{2M}{a_0}\int_\Omega |G(x, y)|dy + 1$, i.e. problem (1.1) has at least one solution in S_2.

Proof. Let

$$\overline{f}(x, u, z) = \begin{cases} f(x, u, z) - M, & \text{if } u \geq B + 1;\\ f(x, u, z) - (y - B)M, & \text{if } B < u < B + 1;\\ f(x, u, z), & \text{if } u \leq B \end{cases}$$

and

$$\overline{g}(x, u, z) = \begin{cases} g(x, u, z) - M, & \text{if } u \leq -B - 1;\\ g(x, u, z) - (y + B)M, & \text{if } -B < u < -B - 1;\\ g(x, u, z), & \text{if } u \geq -B. \end{cases}$$

From the construction of $\overline{f}$ and $\overline{g}$, we have

$$|\overline{f}(x, u, z) - \overline{g}(x, u, z)| \leq 2M, \quad \forall (x, u, z) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N.$$

Set

$$\overline{a}(t) := \begin{cases} a(t), & t \in [0, t_0];\\ a(t_0), & t > t_0, \end{cases}$$

where $t_0 = \int_\Omega H^2(x)dx$; here $H(x)$ is defined as in (2.3).

Now we consider the modified problem

$$\begin{gathered}-\Delta u=\frac{\overline{f}(x,u,\nabla u(x))-\overline{g}(x,u,\nabla u(x))}{\overline{a}(\int_\Omega|\nabla u(x)|^2dx)},\quad x\in\Omega,\\ u\big|_{\partial\Omega}=0.\end{gathered}\tag{4.2}$$

Let $\overline{\alpha}(x)=-B-1$ and $\overline{\beta}(x)=B+1$. Clearly, $\overline{\alpha}\prec\beta,\quad \alpha\prec\overline{\beta}$.

First, if u is a solution of (4.2), we have

$$\begin{aligned}|u(x)|&=\frac{|\int_\Omega G(x,y)(\overline{f}(y,u(y),\nabla u(y))-\overline{g}(y,u(y),\nabla u(y)))dy|}{\overline{a}(\int_\Omega|\nabla u(x)|^2dx)}\\ &\le\frac{2M}{a_0}\int_\Omega|G(x,y)|dy<B,\end{aligned}$$

which implies that

$$\text{if } u \text{ is a solution of (4.2), then } \|u\|_0<B\tag{4.3}$$

and so $\overline{\alpha}(x)<u(x)$ and $u(x)<\overline{\beta}(x)$ for all $x\in\overline{\Omega}$. Clearly,

$$-\Delta\beta(x)\equiv 0,\quad -\Delta\alpha(x)\equiv 0\tag{4.4}$$

and for all x in Ω,

$$\begin{gathered}\frac{1}{a_0}\overline{f}(x,\beta(x),\nabla\beta(x))-\frac{1}{b_0}\overline{g}(x,\beta(x),\nabla\beta(x))<0,\\ \frac{1}{b_0}\overline{f}(x,\alpha(x),\nabla\alpha(x))-\frac{1}{a_0}\overline{g}(x,\alpha,\nabla\alpha(x))>0.\end{gathered}\tag{4.5}$$

Combining (4.4) and (4.5), we have

$$\begin{gathered}-\Delta\overline{\alpha}(x)\le\frac{1}{b_0}\overline{f}(x,\overline{\alpha}(x),\nabla\overline{\alpha}(x))-\frac{1}{a_0}\overline{g}(x,\overline{\alpha}(x),\nabla\overline{\alpha}(x)),\quad x\in\Omega,\\ \overline{\alpha}\big|_{\partial\Omega}<0,\end{gathered}$$

and

$$\begin{gathered}-\Delta\overline{\beta}(x)\ge\frac{1}{a_0}\overline{f}(x,\overline{\beta}(x),\nabla\overline{\beta}(x))-\frac{1}{b_0}\overline{g}(x,\overline{\beta}(x),\nabla\overline{\beta}(x)),\quad x\in\Omega,\\ \overline{\beta}\big|_{\partial\Omega}>0,\end{gathered}$$

which guarantees that $\overline{\alpha}(x)$ and $\overline{\beta}(x)$ are lower solution and upper solution of problem (4.2). From the construction of $\overline{f}$ and $\overline{g}$, it is easy to see that α and β are strict upper solution and lower solution of problem (4.2) also.

From the boundedness of $\overline{f}$ and $\overline{g}$, the operator

$$N^*:C^1(\overline{\Omega})\to C(\overline{\Omega}):u\mapsto\frac{\overline{f}(x,u(x),\nabla u(x))-\overline{g}(x,u(x),\nabla u(x))}{\overline{a}\big(\int_\Omega|\nabla u(x)|^2dx\big)}$$

is well-defined, continuous, and maps $\|\cdot\|_0$-bounded sets of $C^1(\overline{\Omega})$ to bounded sets in $C(\overline{\Omega})$. Then, for fixed $\lambda_0>0$ in (2.6), the operator $A^*:C^1(\overline{\Omega})\to C^1(\overline{\Omega})$

$$A^*u=(-\Delta+\lambda_0)^{-1}(N^*u+\lambda_0u)$$

is completely continuous and a fixed point of A^* is a solution of (4.2).

Let

$$\begin{gathered}S_3=\{u\in C^1(\overline{\Omega}):\ \overline{\alpha}\prec u\prec\overline{\beta}\},\\ S_4=\{u\in S_3:\ \alpha\prec u\},\quad S_5=\{u\in S_3:\ u\prec\beta\}.\end{gathered}$$

From the proof of Theorem 3.1, we have

$$\deg(I-A^*,S_3,0)=1,\quad \deg(I-A^*,S_4,0)=1,\quad \deg(I-A^*,S_5,0)=1.$$

The additivity of topological degree implies that

$$\begin{aligned}&\deg(I-A^*,S_3-\overline{S_4\cup S_5},0)\\ &=\deg(I-A^*,S_3,0)-\deg(I-A^*,S_4,0)-\deg(I-A^*,S_5,0)=-1.\end{aligned}$$

It is easy to see that

$$S_3-\overline{S_4\cup S_5}=\{u\in S_3:\exists x_u\in\overline{\Omega}:\beta(x_u)<u(x_u)<\alpha(x_u)\}.$$

From (4.3), A^* has no fixed point in $S_3-\overline{S_4\cup S_5}-\overline{S}_2$. The excision property of topological degree guarantees that

$$\deg(I-A^*,S_3-\overline{S_4\cup S_5},0)=\deg(I-A^*,S_2,0)=-1.$$

From the definitions of A and A^*, we have $A^*u=Au$ for all $u\in\overline{S_2}$. Then $\deg(I-A,S_2,0)=-1$, which guarantees that problem (1.1) has at least one solution in S_2. The proof is complete. □

5. Multiplicity results

In this section, using Theorems 3.1 and 4.1, we obtain several multiplicity results for problem (1.1).

Theorem 5.1. *Let* (H1) *and* (H2) *hold and let α_1, α_2, α_3 be strict lower, upper, and lower solutions of* (1.1) *with*

$$\alpha_1\prec\alpha_2\prec\alpha_3. \tag{5.1}$$

Then (1.1) *has at least two different solutions u, v satisfying $\alpha_1\prec u\prec\alpha_2$ and*

$$\alpha_2(x_v)<v(x_v)<\alpha_3(x_v),\quad \text{for a } x_v\in\Omega.$$

Proof. Let

$$S_1=\{u\in C^1(\overline{\Omega}):\ \alpha_1\prec u\prec\alpha_2\},$$
$$S_2=\{u\in C^1(\overline{\Omega}):\ \|u\|_0<B,\exists x_u \text{ such that } \alpha_2(x_u)<u(x_u)<\alpha_3(x_u)\},$$

where

$$B>\|\alpha_1\|_0+\|\alpha_3\|_0+\frac{2M}{a_0}\int_\Omega|G(x,y)|dy.$$

Now (5.1), Theorems 3.1 and 4.1 guarantee that

$$\deg(I-A,S_1,0)=1,\quad \deg(I-A,S_2,0)=-1,$$

which implies that (1.1) has at least two solution u and v with $\alpha_1\prec u\prec\alpha_2$

$$\alpha_2(x_v)<v(x_v)<\alpha_3(x_v),\ \text{for a } x_v\in\Omega.$$

The proof is complete. □

Theorem 5.2. *Let conditions* (H1) *and* (H2) *hold and let α_1, α_2, α_3 be strict upper, lower, upper solutions of* (1.1) *with*

$$\alpha_1\prec\alpha_2\prec\alpha_3. \tag{5.2}$$

Then (1.1) *has at least two different solutions u, v satisfying $\alpha_2\prec v\prec\alpha_3$ and*

$$\alpha_1(x_u)<u(x_n)<\alpha_2(x_u),\quad \text{for an } x_u\in\Omega.$$

From condition (5.2) and Theorems 3.1 and 4.1, the argument of the proof for Theorem 5.2 is the same as that in Theorem 5.1 so we omit the proof.

Theorem 5.3. *Let conditions* (H1) *and* (H2) *hold and let α_1, α_2, α_3, α_4 be strict lower, upper, lower and upper solutions of* (1.1) *with*

$$\alpha_1 \prec \alpha_2 \prec \alpha_3 \prec \alpha_4. \tag{5.3}$$

Then (1.1) *has at least three different solutions u, v, w satisfying*

$$\alpha_1 \prec u \prec \alpha_2, \quad \alpha_3 \prec v \prec \alpha_4,$$
$$\exists x_w \text{ such that } \alpha_2(x_w) < w(x_w) < \alpha_3(x_w).$$

Proof. Let

$$\begin{aligned} S &= \{u \in C^1(\overline{\Omega}) : \ \alpha_1 \prec u \prec \alpha_4\}, \\ S_1 &= \{u \in C^1(\overline{\Omega}) : \ \alpha_1 \prec u \prec \alpha_2\}, \\ S_2 &= \{u \in C^1(\overline{\Omega}) : \ \alpha_3 \prec u \prec \alpha_4\}. \end{aligned}$$

Now (5.3) and Theorem 3.1 guarantee that

$$\deg(I - A, S, 0) = 1, \quad \deg(I - A, S_1, 0) = 1, \quad \deg(I - A, S, 0) = 1. \tag{5.4}$$

From the additivity of topological degree, we have

$$\deg(I - A, S, 0) = \deg(I - A, S_1, 0) + deg(I - A, S_2, 0) + \deg(I - A, I - (\overline{S_1 \cup S_2}), 0),$$

which together (5.4) implies that

$$\deg(I - A, I - \overline{S_1 \cup S_2}, 0) = -1. \tag{5.5}$$

From (5.4)-(5.5), (1.1) has at least three different solutions u, v, w satisfying $\alpha_1 \prec u \prec \alpha_2$, $\alpha_3 \prec v \prec \alpha_4$ and $\exists x_w$such that $\alpha_2(x_w) < w(x_w) < \alpha_3(x_w)$. The proof is complete. □

6. Examples

In this section, we give two examples to illustrate the theory.

Example 6.1. We consider the problem

$$\begin{gathered} -\Big(1 + \int_\Omega |\nabla u|^2 dx\Big)\Delta u = \sin^2 u + 1 + \frac{1}{2}\cos^4(|\nabla u|^2) - \sin^4 u, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{6.1}$$

where Ω is a bounded open domain with smooth boundary. We conclude that (6.1) has at least one positive solution.

Proof. The proof is divided into three steps.

Step 1. We show that (1.2), (H1) and (H2) hold.

Set $a(t) := 1 + t$ for $t \in [0, +\infty)$; $f(x, u, z) := \sin^2 u + 1 + \frac{1}{2}\cos^4(|z|^2)$, $g(x, u, z) := \sin^4 u$ for $(x, u, z) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. We have

(1) f and $g \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ with

$$f(x, u, z) \geq 0, \quad g(x, u, z) \geq 0, \quad \forall (x, u, z) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N;$$

(2) with $M = \frac{7}{2} > 0$,

$$|f(x, u, z) - g(x, u, z)| \leq M, \forall (x, u, z) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N;$$

(3) $a(t) = 1 + t \geq 1 = a_0 > 0$ for $t \geq 0$.

Step 2. We construct a strict upper solution as in Definitions 2.2 and 2.5. Set $\beta(x) := 8e(x)$, where $e(x)$ is the unique positive solution of the problem

$$\begin{gathered} -\Delta u = 1, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{6.2}$$

Suppose $G(x,y)$ is the Green's function for $-\Delta u(x) = h$ and $u|_{\partial\Omega} = 0$ and set

$$\begin{gathered} H(x) := \frac{M}{a_0}\int_\Omega |\nabla_x G(x,y)|dy = \frac{7}{2}\int_\Omega |\nabla_x G(x,y)|dy, \quad x \in \overline{\Omega}, \\ b_0 := 1 + \Big(\int_\Omega H^2(x)dx\Big). \end{gathered}$$

From

$$\frac{1}{a_0}f(x,\beta(x),\nabla\beta(x)) - \frac{1}{b_0}g(x,\beta(x),\nabla\beta(x)) \le \frac{5}{2} < 8, \quad x \in \Omega$$

and

$$-\Delta\beta(x) = 8, \quad x \in \Omega, \tag{6.3}$$

with (6.2) we have

$$\begin{gathered} -\Delta\beta(x) > \frac{1}{a_0}f(x,\beta(x),\nabla\beta(x)) - \frac{1}{b_0}g(x,\beta(x),\nabla\beta(x)), \quad x \in \Omega, \\ \beta\big|_{\partial\Omega} = 0, \end{gathered} \tag{6.4}$$

which implies that β is an upper solution of problem (6.1) as in Definition 2.2.

Now we claim that β works for Definition 2.5, that is, if $u(x)$ is a solution of problem (6.1) with $u \le \beta$, we have $u \prec \beta$. In fact, since u is a solution of problem (6.1), we have

$$\begin{aligned} -\Delta u &= \frac{1}{1+\int_\Omega |\nabla u|^2dx}f(x,u(x),\nabla(x)) - \frac{1}{1+\int_\Omega |\nabla u|^2dx}g(x,u(x),\nabla u(x)) \\ &\le \frac{5}{2} < 8, \quad x \in \Omega, \end{aligned}$$

which together with (6.3) yields

$$-\Delta(\beta(x)-u(x)) > 0, \quad x \in \Omega; \quad \beta(x)-u(x) = 0, \quad x \in \partial\Omega.$$

The strong maximum theorem guarantees that

$$\beta(x) - u(x) > 0, \quad \forall x \in \Omega.$$

Now, for given $x \in \partial\Omega$, one has

$$\begin{gathered} \beta(x) - u(x) = 0, \\ \beta(y) - u(y) > 0 \quad \forall y \in \Omega, \\ -\Delta(\beta - u)(y) > 0, \quad y \in \Omega. \end{gathered}$$

The Hopf Lemma guarantees that

$$\frac{\partial(\beta-u)}{\partial n}\Big|_x < 0, \quad \text{i.e. } \frac{\partial\beta(x)}{\partial n} < \frac{\partial u(x)}{\partial n}, \quad \forall x \in \partial\Omega. \tag{6.5}$$

Consequently, $u \prec \beta$, which together with (6.4) implies that β is a strict upper solution of problem (6.1).

Step 3. We construct a strict lower solution as in Definitions 2.1 and 2.5. Suppose that $\varphi_1(x)$, with $\|\varphi_1\| = 1$, is the eigenfunction corresponding to the principle eigenvalue λ_1 of the problem

$$\begin{gathered} -\Delta u = \lambda u, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega. \end{gathered}$$

Let $\varepsilon' < 1/\lambda_1$. Then

$$-\Delta(\beta(x) - \varepsilon'\varphi_1(x)) = 8 - \varepsilon'\lambda_1 > 0, \quad x \in \Omega; \quad \beta(x) - \varepsilon'\varphi_1(x) = 0, \quad x \in \partial\Omega.$$

The strong maximum theorem implies that

$$\beta(x) - \varepsilon'\varphi_1(x) > 0, \quad \forall x \in \Omega.$$

A similar argument to that in proving (6.5) shows that

$$\frac{\partial\beta(x)}{\partial n} < \frac{\partial\varepsilon'\varphi_1(x)}{\partial n}, \quad \forall x \in \partial\Omega.$$

Hence, $\varepsilon'\varphi_1 \prec \beta$.

Let $0 < \varepsilon_0 < \varepsilon'$ be small enough such that

$$\begin{gathered} 2\varepsilon_0 < \min\big\{\frac{1}{2\lambda_1 b_0}, 1\big\}, \\ (2\varepsilon_0\varphi_1(x))^4 + \varepsilon_0\lambda_1\varphi_1(x) < \frac{1}{b_0}, \quad \forall x \in \Omega, \\ \varepsilon_0\varphi_1 \prec \beta. \end{gathered} \tag{6.6}$$

Set $\alpha(x) := \varepsilon_0\varphi_1(x)$. From (6.6), we have

$$\begin{aligned} &\frac{1}{b_0} f(x, \alpha(x), \nabla\alpha(x)) - \frac{1}{a_0} g(x, \alpha(x), \nabla\alpha(x)) \\ &\geq \frac{1}{b_0} - \sin^4\alpha(x) > \frac{1}{b_0} - (\frac{1}{2b_0})^4 > \frac{1}{2b_0}, \quad x \in \Omega \end{aligned}$$

and $-\Delta\alpha(x) = \varepsilon_0\lambda_1\varphi_1(x)$, $x \in \Omega$, which implies

$$\begin{gathered} -\Delta\alpha(x) < \frac{1}{b_0} f(x, \alpha(x), \nabla\alpha(x)) - \frac{1}{a_0} g(x, \alpha(x), \nabla\alpha(x)), \quad \text{in } \Omega, \\ \alpha\big|_{\partial\Omega} = 0, \end{gathered} \tag{6.7}$$

which guarantees that α is a lower solution of problem (6.1).

Now we claim that α works for Definition 2.5, that is, if $u(x)$ is a solution of problem (6.1) with $u \geq \alpha$, we have

$$\alpha \prec u. \tag{6.8}$$

We prove that $u(x) > \alpha(x)$ for all $x \in \Omega$. Arguing by contradiction, we suppose that there exists an $x' \in \Omega$ such that $u(x') = \alpha(x')$. Since $u(x) - \alpha(x)$ is continuous at x', there exists an $r > 0$ small enough such that $\alpha(x) \leq u(x) \leq \alpha(x) + \varepsilon_0\varphi_1(x)$ for all $x \in B(x', r) \subseteq \Omega$. From (6.6), we have

$$\begin{aligned} -\Delta u(x) &= \frac{1}{1 + \int_\Omega |\nabla u|^2 dx}(1 + \sin^2 u(x) + \cos^4(|\nabla u|^2)) - \frac{1}{1 + \int_\Omega |\nabla u|^2 dx} \sin^4 u(x) \\ &> \frac{1}{b_0} - (2\varepsilon_0\varphi_1(x))^4 \\ &> \varepsilon_0\varphi_1(x), \quad x \in B(x', r), \end{aligned}$$

and then

$$-\Delta(u(x)-\alpha(x))>0,\quad x\in\Omega,\quad (u(x)-\alpha(x))\big|_{x\in\partial B(x',r)}\geq 0.$$

The strong maximum theorem guarantees that $u(x)>\alpha(x)$ for all $x\in B(x',r)$. This contradicts $u(x')=\alpha(x')$. Thus, $\alpha(x)<u(x)$ for all $x\in\Omega$. A similar argument to that in proving (6.5) shows that

$$\frac{\partial(u-\alpha)}{\partial n}\Big|_x<0,\quad \text{i.e. } \frac{\partial u(x)}{\partial n}<\frac{\partial\alpha(x)}{\partial n},\quad \forall x\in\partial\Omega.$$

Hence, (6.8) holds.

From (6.7) and (6.8), α is a strict lower solution of problem (6.1). Consequently, Theorem 3.1 guarantees that problem (6.1) has at least one solution between α and β. The proof is complete. □

Note that the problem

$$\begin{gathered}-u''=1,\quad t\in(0,1),\\ u(0)=u(1)=0\end{gathered}\tag{6.9}$$

has a unique positive solution $e(t)=\frac{1}{2}t(1-t)$. From [20], the two-point boundary value problem

$$\begin{gathered}-u''=\frac{84\pi u^2}{2+\frac{\pi}{2}},\quad t\in(0,1),\\ u(0)=u(1)=0\end{gathered}\tag{6.10}$$

has a positive solution u_0. Let $T_0=1+\max_{t\in[0,1]}u_0(t)$ and

$$f(u)=\begin{cases}82\pi u^2, & |u|\leq T_0,\\ 82\pi T_0^2, & |u|\geq T_0.\end{cases}\tag{6.11}$$

Example 6.2. We consider the problem

$$\begin{gathered}-\Big(1+\arctan\Big(\int_\Omega|\nabla u|^2\Big)dx\Big)u''=f(u),\quad t\in(0,1),\\ u(0)=u(1)=0.\end{gathered}\tag{6.12}$$

We conclude that (6.12) has at least two positive solutions.

Proof. We present the existence of positive solutions of problem (6.12) in four steps.

Step 1. We show that (1.2), (H1) and (H2) hold.

(1) Let $f(t,u,z)=f(u)$ be defined in (6.11) and $g(t,u,z)=0$ for $t\in(0,1)\times\mathbb{R}\times\mathbb{R}$. Now

$$f(t,u,z)\geq 0,\quad g(t,u,z)=0,\forall(t,u,z)\in[0,1]\times\mathbb{R}\times\mathbb{R}.$$

(2) With $M=82\pi T_0^2>0$,

$$|f(x,u,z)|\leq M,\forall(t,u,z)\in[0,1]\times\mathbb{R}\times\mathbb{R}.$$

(3) $a(t)=1+\arctan t>a_0=1>0$ for $t\geq 0$.

Step 2. We construct a strict upper solution as in Definitions 2.2 and 2.5. Choose $M_1>82\pi T_0^2+1$ big enough such that $u_0\prec M_1e$, where u_0 is the positive solution of problem (6.10) and e is the unique positive solution of problem (6.9). Set $\beta(x):=M_1e(t)$.

Suppose that $G(t,s)$ is the Green's function of problem (6.9). Set

$$H(t) := \frac{M}{a_0}\int_\Omega |G'_t(t,s)|ds = M\int_\Omega |G'_t(t,s)|ds, \quad t\in[0,1],$$
$$b_0 := 1+\arctan\Big(\int_0^1 H^2(t)dt\Big).$$

Obviously $b_0 < 1+\frac{\pi}{2}$. From

$$\frac{1}{a_0}f(t,\beta(t),\nabla\beta(t)) \le 82\pi T_0^2 < M_1, \quad t\in(0,1)$$

and

$$-\beta''(t) = M_1, \quad t\in(0,1), \tag{6.13}$$

we have

$$\begin{gathered} -\beta''(t) > \frac{1}{a_0}f(t,\beta(t),\beta'(t)), \quad t \text{ in } (0,1),\\ \beta(0)=\beta(1)=0. \end{gathered} \tag{6.14}$$

Now we claim that β works for Definition 2.5, that is, if $u(t)$ is a solution of problem (6.12) with $u\le\beta$, we show that

$$u \prec \beta. \tag{6.15}$$

Since

$$\frac{1}{1+\arctan\int_\Omega |u'|^2dt}f(t,u(t),u'(t)) \le 82\pi T_0^2 < M_1, \quad x\in\Omega,$$

which together with (6.13) yields

$$-(\beta(t)-u(t))'' > 0, \quad t\in(0,1); \quad \beta(0)=u(0)=\beta(1)=u(1)=0.$$

The strong maximum theorem guarantees that $\beta(t) > u(t)$ for all $t\in(0,1)$.

We now show that

$$\beta'(0) > u'(0), \quad \beta'(1) < \alpha'(1). \tag{6.16}$$

Since $\beta(t) > u(t)$ for $t\in(0,1)$, it is easy to see that $\beta'(0)\ge u'(0)$ and $\beta'(1)\le u'(1)$. If $\beta'(0)=u'(0)$, we have

$$(\beta(t)-u(t))' = \int_0^t (\beta(s)-u(s))''ds < 0, \quad \forall t\in(0,1),$$

which implies $\beta(t)-u(t)$ is decreasing on $[0,1]$. Since $\beta(0)-u(0)=0$, one has $\beta(t)-u(t)<0$ for all $t\in(0,1)$. This contradicts $\beta(t)-u(t)>0$ for all $t\in(0,1)$. Thus, $\beta'(0)>u'(0)$. From a similar argument we get $\beta'(1)<u'(1)$. Hence, (6.16) holds. Therefore, we have (6.15).

Now (6.14) and (6.15) imply that β is a strict upper solution of problem (6.12).

Step 3. We construct a strict lower solution as in Definitions 2.1 and 2.5.
Set $\alpha(t)=u_0(t)$, $t\in[0,1]$, where u_0 is the positive solution obtained in problem (6.10). Then

$$\begin{gathered} -\alpha''(t) = \frac{84\pi\alpha^2(t)}{2+\frac{\pi}{2}} < \frac{1}{b_0}f(t,\alpha(t),\alpha'(t)), \quad t \text{ in } (0,1),\\ \alpha(0)=\alpha(1)=0. \end{gathered} \tag{6.17}$$

Now we claim that α works for Definition 2.5, that is, if $u(t)$ is a solution of problem (6.12) with $u\ge\alpha$, we claim that

$$\alpha \prec u. \tag{6.18}$$

We only prove that $u(t) > \alpha(t)$ for all $t \in (0,1)$. Arguing by contradiction, we suppose that there exists an $t_0 \in (0,1)$ such that $u(t_0) = \alpha(t_0)$, $u'(t_0) = \alpha'(t_0)$. Since $u(t) - \alpha(t)$ is continuous at $t = t_0$, there exists a $\delta > 0$ such that $\alpha(t) \leq u(t) \leq \alpha(t) + 1 = u_0(t) + 1 \leq T_0$ for all $t \in [t_0, t_0 + \delta]$, where T_0 is defined in (6.11). Hence

$$\begin{aligned}
-(u(t) - \alpha(t))'' &= \frac{1}{1 + \arctan \int_0^1 |u'|^2 dt}(82\pi u^2) - \frac{82\pi\alpha^2(t)}{2 + \frac{\pi}{2}} \\
&> \frac{1}{2 + \frac{\pi}{2}}(82\pi u^2) - \frac{82\pi\alpha^2(t)}{2 + \frac{\pi}{2}} \\
&= \frac{1}{2 + \frac{\pi}{2}} 82\pi(u + \alpha)(u - \alpha) \\
&> 0, \quad t \in (t_0, t_0 + \delta).
\end{aligned}$$

Then

$$(u(t) - \alpha(t))' = \int_{t_0}^{t} (u(s) - \alpha(s))'' ds < 0, \quad t \in (t_0, t_0 + \delta),$$

which contradicts that t_0 is the minimum point of $u(t) - \alpha(t)$. Thus, $\alpha(t) < u(t)$ for all $t \in (0,1)$. A similar argument to that in proving (6.16) shows that $\alpha'(0) < u'(0)$ and $\alpha'(1) > u'(1)$. Hence, (6.18) holds. From (6.17) and (6.18), we have that $\alpha(t)$ is a strict lower solution of problem (6.12).

Step 4. We construct another strict upper solution of problem (6.12) as in Definitions 2.2 and 2.5.
Choose $\varepsilon_0 > 0$ small enough such that

$$\varepsilon_0 e \prec u_0, \quad 164\pi\varepsilon_0 < 1. \tag{6.19}$$

Set $\gamma(t) := \varepsilon_0 e(t)$, $t \in [0,1]$. Then

$$\begin{gathered}
-\gamma''(t) = \varepsilon_0, \\
\frac{1}{a_0} f(t, \gamma(t), \gamma'(t)) \leq 82\pi\varepsilon_0^2,
\end{gathered} \tag{6.20}$$

which together with (6.19) and (6.20) yields

$$\begin{gathered}
-\gamma(t)''(t) > \frac{1}{a_0} f(t, \gamma(t), \gamma'(t)), \quad t in \ (0,1), \\
\gamma(0) = \gamma(1) = 0.
\end{gathered} \tag{6.21}$$

Now we claim that γ works for Definition 2.5, that is, if $u(t)$ is a solution of problem (6.12) with $u \leq \gamma$, we have

$$u \prec \gamma. \tag{6.22}$$

From (6.19), we have

$$\frac{1}{1 + \arctan \int_\Omega |u'|^2 dt} f(t, u(t), u'(t)) \leq 82\pi\varepsilon_0^2 < \varepsilon_0, \quad t \in (0,1),$$

which together with (6.20) implies

$$-(\gamma(t) - u(t))'' > 0, \quad t \in (0,1); \quad \gamma(0) = u(0) = \gamma(1) = u(1) = 0.$$

The strong maximum guarantees that $\gamma(t) > u(t)$ for all $t \in (0,1)$.

A similar argument to that in proving (6.16) shows that

$$\gamma'(0) > u'(0), \quad \gamma'(1) < u'(1).$$

Hence, (6.22) holds.

Now (6.21) and (6.22) imply that $\gamma(t)$ is a strict upper solution of problem (6.12). Consequently, Theorem 5.2 guarantees that problem (6.12) has at least two positive solutions. The proof is complete. □

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Baoqiang Yan
School of Mathematical Sciences, Shandong Normal University, Jinan, 250014, China
Email address: `yanbqcn@aliyun.com`

Donal O'Regan
School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
Email address: `donal.oregan@nuigalway.ie`

Ravi P. Agarwal
Department of Mathematics, Texas A and M University-Kingsville, Texas 78363, USA
Email address: Ravi.Agarwal@tamuk.edu