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INTERPOLATION INEQUALITIES BETWEEN LORENTZ SPACE AND BMO: THE ENDPOINT CASE $(L^{1,\infty}, BMO)$

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Abstract. We prove interpolation inequalities by means of the Lorentz norm, BMO norm, and the fractional Sobolev norm. In particular, we obtain an interpolation inequality for $(L^{1,\infty}, BMO)$, that we call the endpoint case.

1. Introduction and statement of main results

The main purpose of this article is to study the interpolation inequalities between the Lorentz space $L^{p,\alpha}(\mathbb{R}^n)$ and the $BMO(\mathbb{R}^n)$ space, where $n \geq 1$. It is known that the interpolation inequalities play a crucial role in studying the boundedness of operators and in studying PDEs, see, e.g. [1, 2, 5, 6, 7, 8]. Thus, such an extension of the inequalities of this type is involved many purposes, for instance: the theory of Marcinkiewicz interpolation; the boundedness of the operators acting on Lorentz spaces (the Hardy-Littlewood maximal function, the Hilbert transform, and the Riesz transform); and the estimates in PDEs.

In this article, we want to prove an interpolation inequality between the Lorentz space $L^{q,\alpha}(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, for $q \geq 1$, and $\alpha > 0$. And we call the endpoint case when $q = 1$. Our result is as follows.

Theorem 1.1. *Let $1 \leq q < p$, and $0 < \alpha < \infty$. Let $f \in L^{q,\infty}(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$. Then*

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \lesssim \|f\|^{q/p}_{L^{q,\infty}(\mathbb{R}^n)} \|f\|^{1-\frac{q}{p}}_{BMO(\mathbb{R}^n)}. \tag{1.1}$$

This result extends the recent results in [2, 3]. As a consequence of Theorem 1.1, we obtain an interpolation inequality between $L^{q,\infty}$ and the critical Sobolev space $\dot{W}^{s,\frac{n}{s}}(\mathbb{R}^n)$ for $s \in (0, 1)$.

Corollary 1.2. *Let $1 \leq q < p$, and $\alpha > 0$. For any $0 < s < 1$, we have*

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \lesssim \|f\|^{q/p}_{L^{q,\infty}(\mathbb{R}^n)} \|f\|^{1-\frac{q}{p}}_{\dot{W}^{s,\frac{n}{s}}(\mathbb{R}^n)}. \tag{1.2}$$

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Note that (1.2) follows from (1.1) and the inclusion $\dot{W}^{s,\frac{n}{s}}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$.

Before proving Theorem 1.1, we recall the definitions of the Lorentz spaces, and BMO space. Given $q, \alpha > 0$, we set

$$\|g\|_{L^{q,\alpha}(\mathbb{R}^n)} := \begin{cases} \left(q\int_0^\infty (\lambda^q |\{x \in \mathbb{R}^n : |g(x)| > \lambda\}|)^{\alpha/q} \frac{d\lambda}{\lambda}\right)^{1/\alpha} & \text{if } \alpha < \infty, \\ \sup_{\lambda>0} \lambda\big(|\{x \in \mathbb{R}^n : |g(x)| > \lambda\}|\big)^{1/q} & \text{if } \alpha = \infty. \end{cases}$$

The Lorentz space is $L^{q,\alpha}(\mathbb{R}^n) = \{g : \mathbb{R}^n \to \mathbb{R} : \|g\|_{L^{q,\alpha}(\mathbb{R}^n)} < \infty\}$. Next, we define the sharp maximal function:

$$f^\sharp(x) = \sup_{R>0, x\in B_R} \frac{1}{|B_R|}\int_{B_R} |f(y) - (f)_{B_R}|dy,$$

with $(f)_\Omega = \frac{1}{|\Omega|}\int_\Omega f(x)dx$. Then, we have a result, the so called strong type (p,p) in $L^p(\mathbb{R}^n)$ as follows (see, e.g. [9]).

Theorem 1.3. *Let $p > 1$. Then*

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f^\sharp\|_{L^p(\mathbb{R}^n)}, \tag{1.3}$$

whenever the right hand side is well-defined.

After that, we denote by

$$BMO(\mathbb{R}^n) = \big\{f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{BMO(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} f^\sharp(x) < \infty\big\}.$$

Finally, we denote the homogeneous fractional Sobolev space by

$$\begin{aligned} &\dot{W}^{s,p}(\mathbb{R}^n) \\ &= \big\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\Big)^{1/p} < \infty\big\}, \end{aligned}$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of $\mathcal{S}(\mathbb{R}^n)$ (the Schwartz space). To end this part, we denote $A \lesssim B$ if $A \le cB$, where $c > 0$ is a constant.

2. Proof of Theorem 1.1

It suffices to show that (1.1) holds for $q = 1$. To start, we prove the following result.

Lemma 2.1. *Let $0 < q < p < r \le \infty$ and $\alpha > 0$. If $f \in L^{q,\infty}(\mathbb{R}^n) \cap L^{r,\infty}(\mathbb{R}^n)$, then $f \in L^{p,\alpha}(\mathbb{R}^n)$, and*

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \lesssim \|f\|^\theta_{L^{q,\infty}(\mathbb{R}^n)}\|f\|^{1-\theta}_{L^{r,\infty}(\mathbb{R}^n)}, \tag{2.1}$$

with $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$.

Proof. We rewrite

$$\|f\|^\alpha_{L^{p,\alpha}(\mathbb{R}^n)} = p\int_0^{\lambda_0} \lambda^\alpha\,|\{|f| > \lambda\}|^{\alpha/p}\,\frac{d\lambda}{\lambda} + p\int_{\lambda_0}^\infty \lambda^\alpha\,|\{|f| > \lambda\}|^{\alpha/p}\,\frac{d\lambda}{\lambda}. \tag{2.2}$$

Since $f \in L^{q,\infty}(\mathbb{R}^n) \cap L^{r,\infty}(\mathbb{R}^n)$, we have

$$\begin{aligned} \int_0^{\lambda_0} \lambda^\alpha |\{|f| > \lambda\}|^{\alpha/p}\frac{d\lambda}{\lambda} &\le \int_0^{\lambda_0} \lambda^\alpha\Big(\frac{\|f\|^q_{L^{q,\infty}(\mathbb{R}^n)}}{\lambda^q}\Big)^{\alpha/p}\frac{d\lambda}{\lambda} \\ &= \frac{\|f\|^{\alpha q/p}_{L^{q,\infty}(\mathbb{R}^n)}}{\alpha(1-q/p)}\lambda_0^{\alpha(1-q/p)}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\int_{\lambda_0}^{\infty} \lambda^{\alpha} |\{|f| > \lambda\}|^{\alpha/p} \frac{d\lambda}{\lambda} &\le \int_{\lambda_0}^{\infty} \lambda^{\alpha} \Big(\frac{\|f\|^r_{L^{r,\infty}(\mathbb{R}^n)}}{\lambda^r} \Big)^{\alpha/p} \frac{d\lambda}{\lambda} \\ &= \frac{\|f\|^{\alpha r/p}_{L^{r,\infty}(\mathbb{R}^n)}}{\alpha(r/p-1)} \lambda_0^{\alpha(1-r/p)}.\end{aligned} \tag{2.4}$$

By (2.2), (2.3) and (2.4), we obtain

$$\|f\|^{\alpha}_{L^{p,\alpha}(\mathbb{R}^n)} \le p \Big(\frac{\|f\|^{\alpha q/p}_{L^{q,\infty}(\mathbb{R}^n)}}{\alpha(1-q/p)} \lambda_0^{\alpha(1-q/p)} + \frac{\|f\|^{\alpha r/p}_{L^{r,\infty}(\mathbb{R}^n)}}{\alpha(r/p-1)} \lambda_0^{\alpha(1-r/p)} \Big).$$

Now, we take

$$\lambda_0^{r-q} = \frac{\|f\|^r_{L^{r,\infty}(\mathbb{R}^n)}}{\|f\|^q_{L^{q,\infty}(\mathbb{R}^n)}},$$

so the proof is complete. □

Thanks to Lemma 2.1, we have for any $r > p$

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \lesssim \|f\|^{\theta}_{L^{1,\infty}(\mathbb{R}^n)} \|f\|^{1-\theta}_{L^{r,\infty}(\mathbb{R}^n)}, \tag{2.5}$$

where $\frac{1}{p} = \theta + \frac{1-\theta}{r}$.

Since $r > p > 1$, and by (1.3), we obtain

$$\begin{aligned}\|f\|^r_{L^{r,\infty}(\mathbb{R}^n)} &\le \|f\|^r_{L^r(\mathbb{R}^n)} \lesssim \|f^{\sharp}\|^r_{L^r(\mathbb{R}^n)} \\ &\lesssim \|f\|^{r-p}_{BMO(\mathbb{R}^n)} \|f^{\sharp}\|^p_{L^p(\mathbb{R}^n)} \\ &\lesssim \|f\|^{r-p}_{BMO(\mathbb{R}^n)} \|f\|^p_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6) yields

$$\begin{aligned}\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} &\lesssim \|f\|^{\theta}_{L^{1,\infty}(\mathbb{R}^n)} \Big(\|f\|^{1-\frac{p}{r}}_{BMO(\mathbb{R}^n)} \|f\|^{\frac{p}{r}}_{L^p(\mathbb{R}^n)} \Big)^{1-\theta} \\ &\lesssim \|f\|^{\theta}_{L^{1,\infty}(\mathbb{R}^n)} \Big(\|f\|^{1-\frac{p}{r}}_{BMO(\mathbb{R}^n)} \|f\|^{\frac{p}{r}}_{L^{p,\alpha}(\mathbb{R}^n)} \Big)^{1-\theta}.\end{aligned}$$

Then

$$\begin{gathered}\|f\|^{1-\frac{p}{r}(1-\theta)}_{L^{p,\alpha}(\mathbb{R}^n)} \lesssim \|f\|^{\theta}_{L^{1,\infty}(\mathbb{R}^n)} \|f\|^{(1-\frac{p}{r})(1-\theta)}_{BMO(\mathbb{R}^n)}, \\ \|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \lesssim \|f\|^{\frac{1}{p}}_{L^{1,\infty}(\mathbb{R}^n)} \|f\|^{1-\frac{1}{p}}_{BMO(\mathbb{R}^n)}.\end{gathered}$$

Thus, the proof is complete.

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