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DECAY RATE OF STRONG SOLUTIONS TO COMPRESSIBLE NAVIER-STOKES-POISSON EQUATIONS WITH EXTERNAL FORCE

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Abstract. In this article, we consider the three dimensional compressible Navier-Stokes-Poisson equations with the effect of external potential force. First, the stationary solution is established by solving a nonlinear elliptic system. Next, we show global well-posedness of the strong solutions for the initial value problem to the three dimensional compressible Navier-Stokes-Poisson equations when the initial data are close to the stationary solution in $H^2(\mathbb{R}^3)$. Moreover, if the $L^1(\mathbb{R}^3)$-norm of initial perturbation is finite, we prove the optimal $L^p(\mathbb{R}^3)$ $(2 \le p \le 6)$ decay rates for such strong solution and $L^2(\mathbb{R}^3)$ decay rate of its first-order spatial derivatives via a low frequency and high frequency decomposition.

1. Introduction

This study is concerned with the initial value problem of the isentropic Navier-Stokes-Poisson equations

$$
\begin{gathered}
\partial_t\rho + \nabla\cdot(\rho u) = 0,\\
\rho[\partial_t u + (u\cdot\nabla)u] + \nabla P(\rho) = \rho\nabla\phi + \mu\Delta u + (\mu+\nu)\nabla(\nabla\cdot u) + \rho F,\\
\Delta\phi = \rho - \bar\rho,\ \lim_{|x|\to\infty}\phi(x,t) = 0,\\
(\rho,u)(x,0) = (\rho_0,u_0)(x).
\end{gathered}
\tag{1.1}
$$

Here the time variable is $t \ge 0$, and the spatial coordinate is $x \in \mathbb{R}^3$. The unknown functions are the density $\rho > 0$, the velocity u, and the electrostatic potential ϕ. $\bar\rho > 0$ stands for the constant background doping profile. The constants μ and ν are the viscosity coefficients satisfying $\mu > 0$ and $2\mu + 3\nu \ge 0$. $F(x) = (F_1(x), F_2(x), F_3(x))$ is a given external force. $P = P(\rho)$ is the pressure. In this paper, we always assume that $P = P(\rho)$ is a C^2-function in the neighborhood of $\bar\rho$ and satisfies $P'(\rho) > 0$ for $\rho > 0$. The typical examples are $P(\rho) = A\rho^\gamma$ corresponding to polytropic $(\gamma > 1)$ and isothermal fluid $(\gamma = 1)$. The Navier-Stokes-Poisson system is used to describe the motion of a compressible viscous isotropic Newtonian fluid in semiconductor devices [5, 12] or in plasmas [12, 21].

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Recently, there has has been a lot of research devoted to proving the global existence, uniqueness, quasineutral limit, zero-eletron-mass limit and time decay rates of solutions to the compressible Navier-Stokes-Poisson equations, cf. [2, 3, 4, 6, 8, 10, 11, 13, 14, 16, 23, 24, 25, 27, 29, 30, 31, 32] and references therein. We only mention some results about time decay rates of solutions to the compressible Navier-Stokes-Poisson equations. Li, Matsumura and Zhang [16], and Li and Zhang [17] studied global existence and the optimal decay estimate of classical solutions for the initial value problem to the isentropic compressible Navier-Stokes-Poisson system in $\mathbb{R}^3$. Li and Zhang [18] obtained the decay rates of more derivatives of solutions when the initial perturbation also is in the $H^{-s}(\mathbb{R}^3)$ (negative Sobolev norms) with $0 \le s < 3/2$. Wang and Wu [29] investigated the initial value problem for the for the isentropic compressible Navier-Stokes-Poisson system in $\mathbb{R}^n (n \ge 3)$ and obtained the pointwise estimates of the solution by a detailed analysis of the Green's function to the corresponding linearized equations. Wang and Wang [30] considered the initial value problem for the isentropic compressible Navier-Stokes-Poisson equations in three and higher dimensions and established new decay estimate of classical solutions. The decay rates of the solutions for non-isentropic compressible Navier-Stokes-Poisson equations also are discussed in [23, 24, 31]. It is worth noticing that all above results are showed for the compressible Navier-Stokes-Poisson equations without any external force. Recently, Zhao and Li [33] showed the global existence and the optimal L^2-decay rate of smooth solutions for the non-isentropic compressible Navier-Stokes-Poisson equations with the potential external force. However, all the previous decay rates were proved for the solutions in $H^3(\mathbb{R}^3)$ or more regular solutions. In this paper, we discuss the global existence and the optimal L^2-decay rate of strong solutions for compressible Navier-Stokes-Poisson system with external force (1.1) in $H^2(\mathbb{R}^3)$.

In this article, we consider the potential force, for simplicity, $F = -\nabla\psi(x)$. Then problem (1.1) can be rewritten as

$$
\begin{gathered}
\partial_t \rho + \nabla \cdot (\rho u) = 0, \\
\rho[\partial_t u + (u \cdot \nabla) u] + \nabla P(\rho) = \rho \nabla \phi + \mu \Delta u + (\mu + \nu) \nabla (\nabla \cdot u) - \rho \nabla \psi, \\
\Delta \phi = \rho - \bar{\rho}, \lim_{|x| \to \infty} \phi(x, t) = 0, \\
(\rho, u)(x, 0) = (\rho_0, u_0)(x).
\end{gathered}
\tag{1.2}
$$

We assume that the initial data satisfy

$$
(\rho_0, u_0)(x) \to (\bar{\rho}, 0) \quad \text{as } |x| \to \infty.
$$

The main purpose of this article is to show the global existence and decay rate of strong solutions for (1.2) for the initial data around stationary solutions. We first study the stationary problem

$$
\begin{gathered}
\nabla \cdot (\tilde{\rho} \tilde{u}) = 0, \\
\tilde{\rho} (\tilde{u} \cdot \nabla) \tilde{u} + \nabla P(\tilde{\rho}) = \tilde{\rho} \nabla \tilde{\phi} - \tilde{\rho} \nabla \psi + \mu \Delta \tilde{u} + (\mu + \nu) \nabla (\nabla \cdot \tilde{u}), \\
\Delta \tilde{\phi} = \tilde{\rho} - \bar{\rho}, \\
\tilde{\rho} \to \bar{\rho}, \tilde{u} \to 0 \quad \text{as } |x| \to \infty.
\end{gathered}
\tag{1.3}
$$

Before we state our main results, we introduce the following notation which is used in whole paper. $C > 0$ denotes the generic positive constant independent of time. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} (|\alpha| =$

$\sum_{i=1}^{3}\alpha_i)$, $\nabla=(\partial_1,\partial_2,\partial_3)$ with $\partial_i=\partial_{x_i}(i=1,2,3)$ and for any integer $l\geq 1$, $\nabla^l f$ denotes all of lth derivatives of f. For multi-indices $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2,\beta_3)$, $C_\beta^\alpha=\frac{\alpha!}{\beta!(\alpha-\beta)!}$ with $\beta\leq\alpha$ which means $\beta_i\leq\alpha_i$ for all $1\leq i\leq 3$. Let $L^p(1\leq p\leq\infty)$ denote the usual L^p-Lebesgue space on $\mathbb{R}^n$ with norm $\|\cdot\|_{L^p}$. In particular, $\|\cdot\|$ denotes the standard L^2 norm of the functions. For non-negative integer k, we denote by $W^{k,p}(1\leq p\leq\infty)$ the usual L^p-Sobolev space of order k whose norm is denoted by $\|\cdot\|_{W^{k,p}}=(\sum_{l=0}^{k}\|D^l\cdot\|^p)^{1/p}$. When $p=2$, we define $H^k=W^{k,2}$ with the norm $\|\cdot\|_k=(\sum_{l=0}^{k}\|D^l\cdot\|^2)^{1/2}$. Moreover, $C^k([0,T];H^l(\mathbb{R}^3))(k,l\geq 0)$ denotes the space of the k-times continuously differentiable functions on the interval $[0,T]$ with values in $H^l(\mathbb{R}^3)$. Finally, for a function f, we denote its Fourier transform by $\mathcal{F}[f]=\hat{f}$:

$$\mathcal{F}[f](\xi)=\hat{f}(\xi)=\int_{\mathbb{R}^n}f(x)e^{-\sqrt{-1}x\cdot\xi}dx.$$

The inverse of $\mathcal{F}$ is denoted by $\mathcal{F}^{-1}[f]=\check{f}$,

$$\mathcal{F}^{-1}[f](\xi)=\check{f}(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}f(\xi)e^{\sqrt{-1}\xi\cdot x}d\xi.$$

The following is our first main result on the existence and uniqueness of the stationary solutions.

Theorem 1.1. *There exists $\epsilon_1>0$ such that if $\|\Delta\psi\|_2+\sum_{k=0}^{1}\|(1+|x|)\nabla^k\Delta\psi\|\leq\epsilon_1$, the problem* (1.3) *has a unique solution $(\tilde{\rho},\tilde{u},\tilde{\phi})(x)$ satisfying*

$$\tilde{\rho}-\bar{\rho}\in H^4(\mathbb{R}^3),\quad \tilde{u}=0,\quad \nabla\tilde{\phi}\in H^3(\mathbb{R}^3),\quad \tilde{\phi}\in L^6(\mathbb{R}^3),$$

and

$$\frac{1}{2}\bar{\rho}\leq\tilde{\rho}(x)\leq 2\bar{\rho}. \tag{1.4}$$

Moreover, there exists a constant C such that

$$\|\tilde{\rho}-\bar{\rho}\|_4+\|\nabla\tilde{\phi}\|_3+\|\tilde{\phi}\|_{L^6}\leq C\epsilon_1, \tag{1.5}$$

$$\|(1+|x|)(\tilde{\rho}-\bar{\rho})\|_3+\|(1+|x|)\nabla\tilde{\phi}\|_2\leq C\epsilon_1. \tag{1.6}$$

Next, the global existence and optimal decay rate of strong solutions for (1.2) in $H^2(\mathbb{R}^3)$ space are stated as follows.

Theorem 1.2. *Let $(\rho_0-\bar{\rho},u_0)(x)\in H^2(\mathbb{R}^3)$, there exists $0<\delta_0<\epsilon_1$ such that if*

$$\|(\rho_0-\bar{\rho},u_0)\|_2+\|\Delta\psi\|_2+\sum_{k=0}^{1}\|(1+|x|)\nabla^k\Delta\psi\|\leq\delta_0, \tag{1.7}$$

then the initial value problem (1.2) *admits a unique solution $(\rho,u,\phi)(t,x)$ globally in time, which satisfies*

$$\begin{gathered}\rho-\tilde{\rho}\in C^0[0,\infty;H^2(\mathbb{R}^3))\cap C^1(0,\infty;H^1(\mathbb{R}^3)),\\ u\in C^0[0,\infty;H^2(\mathbb{R}^3))\cap C^1(0,\infty;L^2(\mathbb{R}^3)),\\ \phi-\tilde{\phi}\in L^6(0,\infty;\mathbb{R}^3),\nabla(\phi-\tilde{\phi})\in C^0[0,\infty;H^2(\mathbb{R}^3))\cap C^1(0,\infty;H^1(\mathbb{R}^3)).\end{gathered}$$

Moreover, if the initial data $(\rho_0-\bar{\rho},u_0)(x)\in L^1(\mathbb{R}^3)$ with

$$\|(\rho_0-\bar{\rho},u_0)\|_{L^1}<+\infty. \tag{1.8}$$

then the solution $(\rho, u, \phi)(x,t)$ enjoys the following decay-in-time estimates:

$$\|\nabla(\rho-\tilde{\rho}, u, \nabla\phi-\nabla\tilde{\phi})(t)\|_1 \le C(1+t)^{-3/4}, \tag{1.9}$$

and for $2 \le p \le 6$,

$$\|(\rho-\tilde{\rho})(t)\|_{L^p} \le C(1+t)^{-3/4}, \tag{1.10}$$

$$\|(u, \nabla\phi-\nabla\tilde{\phi})(t)\|_{L^p} \le C(1+t)^{-(1-\frac{3}{2p})}. \tag{1.11}$$

Remark 1.3. It should be noted that given the same kind of initial data, the velocity of global solution of compressible Navier-Stokes equations decays with an optimal rate $(1+t)^{-3/4}$ in $L^2(\mathbb{R}^3)$-norm, see [20, 26]. While the optimal decay rate in Theorem 1.2 implies that the momentum of the compressible Navier-Stokes-Poisson equations decays at the slower rate $(1+t)^{-1/4}$ in $L^2(\mathbb{R}^3)$-norm. This is caused by the coupling of the electric field and velocity field through Poisson equation, which also destroys the usual ascoustic wave propagation for the classical compressible viscous flow, see [16, 31]. Moreover, compared with the previous results about the compressible Navier-Stokes equations in [20, 26], we here assume only the smallness of gradient of ψ which is the force rather than the potential ψ itself.

The idea of the proof is outlined as follows. First, we show the existence and uniqueness of the stationary solution by the iteration method. The construction of the solutions themselves also gives the weighted energy estimate for the stationary solutions. Next, combining the local existence and global a-priori estimates which are derived by elaborate energy method, we apply the continuity argument to establish global existence of strong solutions for the nonlinear problem as that in [15, 19]. Finally, in order to establish the decay rates of the strong solution, we use the low-frequency and high-frequency decomposition of the solution to (1.2), which is utilized to obtain the optimal convergence rates for the strong solutions to the compressible Navier-Stokes equations with potential force in [20, 26]. More precisely, motivated by [7, 9], we show a Lyapunov-type energy inequality of all derivatives. These derivatives of solutions can be controlled only by the low frequency part of the perturbed density and the first order derivative of the velocity, which is different from that of [7, 9, 20, 26]. Next, from the spectral analysis on the corresponding linearized Navier-Stokes-Poisson equations, we can obtain the decay rate of the low frequency solution for the perturbed density and the first order derivative of the velocity. However, in order to face with less decay rates of $G_L^{11}(t)$ and $G_L^{21}(t)$, which are the part of the solution semigroup to the linearized equations, we note that $f_{11}+f_{12} = \nabla\cdot(nu + (\tilde{\rho}-\bar{\rho})u)$ and utilize the property of convolution product to obtain the decay estimate of $\|n_L\|$ and $\|\nabla u_L\|$. The derivation of the decay rates is different from that of the compressible Navier-Stokes equations with external force in $H^2(\mathbb{R}^3)$ in [20, 26].

The rest of this article is organized as follows. In Section 2, we present the unique existence of the stationary solution. Then, we reformulate the original problem in terms of the perturbed variables and give some important inequalities in Section. The global existence of strong solutions for the initial value problem (1.2) by energy methods is proved in Section 4. Finally, in Section 5, we prove the optimal $L^p(\mathbb{R}^3)$ $(2 \le p \le 6)$ decay rates for such strong solution, and $L^2(\mathbb{R}^3)$ decay rate of its first-order spatial derivatives.

2. Stationary solution

In this section, we mainly consider well-posedness and qualitative behavior of solutions for stationary problem (1.2). Since we focus on a small neighborhood of $(\bar{\rho}, 0, 0)$ in $H^2 \times H^2 \times H^2$ by Soblev's inequality, we may suppose $|\tilde{\rho} - \bar{\rho}|, |\tilde{u}| < \frac{1}{2}\bar{\rho}$. First, we manipulate (1.3) as follows:

$$\int_{\mathbb{R}^3} ((1.3)_1 \times \int_{\bar{\rho}}^{\tilde{\rho}} \frac{P_\rho(\eta)}{\eta} d\eta) dx + \int_{\mathbb{R}^3} (1.3)_2 \times \tilde{\rho}\tilde{u} dx = 0.$$

It follows from integration by parts and the mean value theorem that

$$\|\nabla \tilde{u}\|^2 \le C(\|\nabla \tilde{\rho}\| + \|\tilde{u}\|_1)\|\nabla \tilde{u}\|^2.$$

Therefore, if $\|\nabla \tilde{\rho}\|$ and $\|\tilde{u}\|_1$ are small enough, we conclude $\tilde{u} = 0$. Thus the stationary equation (1.3) is reduced to

$$\begin{gathered} \nabla P(\tilde{\rho}) - \tilde{\rho}\nabla\tilde{\phi} + \tilde{\rho}\nabla\psi = 0, \\ \Delta\tilde{\phi} = \tilde{\rho} - \bar{\rho}, \\ \tilde{\rho} \to \bar{\rho},\ \tilde{\phi} \to 0 \quad \text{as } |x| \to \infty. \end{gathered} \tag{2.1}$$

Let $h(s) = \int_0^s \frac{P'(\rho)}{\rho} d\rho$. Taking divergence of the first equation in (2.1) yields

$$\begin{gathered} \Delta h(\tilde{\rho}) = \tilde{\rho} - \bar{\rho} - \Delta\psi, \\ \Delta\tilde{\phi} = \tilde{\rho} - \bar{\rho}, \\ \tilde{\rho} \to \bar{\rho}, \tilde{\phi} \to 0 \quad \text{as } |x| \to \infty. \end{gathered} \tag{2.2}$$

Then, we have the following result about the existence of the stationary solution $(\tilde{\rho}, 0, \tilde{\phi})$.

Lemma 2.1. *Under the assumptions of Theorem 1.1, problem* (2.2) *has a unique solution* $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ *satisfying*

$$\tilde{\rho} - \bar{\rho} \in H^4(\mathbb{R}^3), \quad \tilde{u} = 0,\ \nabla\tilde{\phi} \in H^3(\mathbb{R}^3), \quad \tilde{\phi} \in L^6(\mathbb{R}^3),$$

and

$$\frac{1}{2}\bar{\rho} \le \tilde{\rho}(x) \le 2\bar{\rho}, \tag{2.3}$$

$$\|\tilde{\rho} - \bar{\rho}\|_4 + \|\nabla\tilde{\phi}\|_3 + \|\tilde{\phi}\|_{L^6} \le C\epsilon_1, \tag{2.4}$$

$$\|(1+|x|)(\tilde{\rho} - \bar{\rho})\|_3 + \|(1+|x|)\nabla\tilde{\phi}\|_2 \le C\epsilon_1. \tag{2.5}$$

Since the proof of Lemma 2.1 is similar to that in [33], we omit it.

3. Reformulation of original problem

In this section, we reformulate problem (1.2). Let $(\rho, u, \phi) = (n + \tilde{\rho}, u, \Phi + \tilde{\phi})$. Then (1.2) is equivalent to

$$\begin{aligned} &\partial_t n + \nabla \cdot ((n + \tilde{\rho})u) = 0, \\ &\partial_t u + (u \cdot \nabla)u - \frac{1}{n + \tilde{\rho}}[\mu\Delta u + (\mu + \nu)\nabla(\nabla \cdot u)] + \frac{P'(n + \tilde{\rho})}{n + \tilde{\rho}}\nabla(n + \tilde{\rho}) \\ &= \nabla\Phi + \frac{P'(\tilde{\rho})}{\tilde{\rho}}\nabla\tilde{\rho}, \end{aligned}$$

$$\Delta\Phi = n, \lim_{|x|\to\infty} \Phi(x,t) = 0,$$

which together with $\mu_1 = \mu/\bar{\rho}$ and $\mu_2 = (\mu+\nu)/\bar{\rho}$, yield

$$\begin{gathered}
\partial_t n + \bar{\rho}\nabla\cdot u = f_{11} + f_{12},\\
\partial_t u - \mu_1\Delta u - \mu_2\nabla(\nabla\cdot u) + \frac{P'(\bar{\rho})}{\bar{\rho}}\nabla n = \nabla\Phi + f_{21} + f_{22},\\
\Delta\Phi = n, \lim_{|x|\to\infty} \Phi(x,t) = 0,
\end{gathered} \tag{3.1}$$

with the initial data

$$(n,u)(x,0) = (n_0,u_0)(x) = (\rho_0 - \tilde{\rho}, u_0)(x). \tag{3.2}$$

Here

$$f_{11} = -\nabla\cdot((\tilde{\rho}-\bar{\rho})u), \tag{3.3}$$

$$f_{12} = -\nabla\cdot(nu), \tag{3.4}$$

$$\begin{aligned}
f_{21} = &-\Big(\frac{P'(n+\tilde{\rho})}{n+\tilde{\rho}} - \frac{P'(\tilde{\rho})}{\tilde{\rho}}\Big)\nabla\tilde{\rho} - \Big(\frac{P'(\tilde{\rho})}{\tilde{\rho}} - \frac{P'(\bar{\rho})}{\bar{\rho}}\Big)\nabla n\\
&+ (\tilde{\rho}-\bar{\rho})(\frac{\mu_1}{\tilde{\rho}}\Delta u + \frac{\mu_2}{\tilde{\rho}}\nabla\operatorname{div} u),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
f_{21} = &-(u\cdot\nabla)u - \Big(\frac{P'(n+\tilde{\rho})}{n+\tilde{\rho}} - \frac{P'(\tilde{\rho})}{\tilde{\rho}}\Big)\nabla n\\
&+ \Big(\frac{1}{n+\tilde{\rho}} - \frac{1}{\tilde{\rho}}\Big)(\mu\Delta u + (\mu+\nu)\nabla(\nabla\cdot u)),
\end{aligned} \tag{3.6}$$

In what follows, we consider the global existence and time decay rates of the solution $(n,u,\Phi)(t,x)$ to the steady state $(\tilde{\rho},0,\tilde{\phi})(x)$; that is, the existence and decay rates of the perturbed solution $(n,u,\Phi)(x,t)$ to the problem (3.1)-(3.2).

To close this section, some inequalities are listed as follows which will be used in the subsequent. One can found them in [1, 16, 17, 22].

Lemma 3.1 (see [1, 22])**.** *(i) If $u(x)\in H^1(\mathbb{R}^3)$, then the following inequalities hold:*

$$\begin{gathered}
\|\frac{u}{|x|}\| \le C\|\nabla u\|,\\
\|u\|_{L^6} \le C\|\nabla u\|,\\
\|u\|_{L^3} \le C(\|u\| + \|u\|_{L^6}) \le C\|u\|_1.
\end{gathered}$$

(ii) Assume $u(x)\in H^2(\mathbb{R}^3)$, then

$$\|u\|_{L^\infty} \le C\|\nabla u\|_1,$$

Lemma 3.2 (see [17, 18, 26, 28])**.** *Let $r_1, r_2 > 0$, then*

$$\int_0^t (1+t-\tau)^{-r_1}(1+\tau)^{-r_2}d\tau \le C(r_1,r_2)(1+t)^{-\min\{r_1,r_2,r_1+r_2-1-\eta\}}, \tag{3.7}$$

for an arbitrarily small $\eta > 0$.

4. Existence of a global solution

In this section, we establish the existence of global solutions to the problem (3.1)-(3.2) in the H^2-framework by using the energy method. First, we have the following local-in-time existence of solutions of (3.1)-(3.2).

Lemma 4.1. *If $(n_0, u_0)(x) \in H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$, there exists a positive constant T such that the initial value problem (3.1)-(3.2) has a local solution $(n, u, \Phi)(x, t)$, which satisfies*

$$
\begin{gathered}
n \in C^0([0,T], H^2(\mathbb{R}^3)) \cap C^1([0,T], H^1(\mathbb{R}^3)) \cap L^2([0,T], H^2(\mathbb{R}^3)),\\
u \in C^0([0,T], H^2(\mathbb{R}^3)) \cap C^1([0,T], L^2(\mathbb{R}^3)),\ \ \nabla u \in L^2([0,T], H^2(\mathbb{R}^3)),\\
\Phi \in L^6([0,T]; \mathbb{R}^3), \nabla\Phi \in C^0([0,T], H^2(\mathbb{R}^3)) \cap C^1([0,T], H^1(\mathbb{R}^3)),
\end{gathered}
$$

and

$$
\|(n,u)(\cdot,t)\|_2^2 + \|\nabla\Phi(\cdot,t)\|_2^2 + \int_0^t \|(n,\nabla u)(\cdot,s)\|_2^2 ds \le C.
$$

Proof. The proof can be done by using the framework in [3, 4, 15, 19, 32], which is based on standard iteration arguments and the contraction map theorems. The key point is that the electric field $\nabla\Phi$ can be expressed by $(3.1)_1$ and the Riesz potential as a nonlocal term

$$
\nabla\Phi = \nabla\Phi_0 + \nabla(-\Delta)^{-1}\operatorname{div}\int_0^t (n+\tilde\rho)u\,ds, \quad \text{where } \Phi_0 = \Delta^{-1}n_0.
$$

Note that

$$
\|\nabla\Delta^{-1}\operatorname{div}\int_0^t ((n+\tilde\rho)u)\,ds\|_k \le C\int_0^t \|(n+\tilde\rho)u\|_k\,ds,\ \ k\ge 0.
$$

Then the remaining part to obtain local existence is almost the same to that in [3, 4, 15, 19, 32] and the better regularity for Φ than n comes from the estimate for the Poisson equation. Here we omit the details. This completes the proof. $\square$

By the standard continuity argument (see [15, 19]), the global existence of solutions to the initial value problem (3.1)-(3.2) will be obtained from the combination of the local existence result with the following a priori estimates.

Proposition 4.2. *For $T > 0$, let $(n, u, \Phi)(x, t)$ be a solution of (3.1)-(3.2) in $[0, T]$ and introduce $E(T) = \sup_{0\le t\le T} \|(n, u)(\cdot, t)\|_2$. Then there exists $\delta > 0$ such that if*

$$
E(T) + \epsilon_1 \le \delta, \tag{4.1}
$$

then the following a-priori estimate holds

$$
\|(n,u,\nabla\Phi)(\cdot,t)\|_2^2 + \int_0^t \|(n,\nabla u,\nabla^2\Phi)(\cdot,s)\|_2^2 ds \le C\|(n_0,u_0)\|_2^2, \tag{4.2}
$$

for any $t \in [0, T]$, where C is a positive constant independent of t.

In the following, we focus on the proof of Proposition 4.2. First of all, by (4.1) and the Sobolev's inequality, we have

$$
\|n\|_{L^\infty} \le C\delta,
$$

which together with (**??**) yields

$$
\frac{1}{4}\bar\rho \le n + \tilde\rho \le 4\bar\rho. \tag{4.3}
$$

Before proving Proposition 4.2, we need Lemmas 4.3, 4.4 and 4.5 for the sake of clarity.

Lemma 4.3. *Under the a priori assumption* (4.1)*, we have*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{P'(\bar\rho)}{\bar\rho}n^2+\bar\rho u^2+|\nabla\Phi|^2\Big)\,dx+C\|\nabla u\|^2\le C\delta\|(n,\nabla n,\nabla u,\nabla^2u)\|^2. \tag{4.4}$$

Proof. Multiplying $(3.1)_1$ and $(3.1)_2$ by $\frac{P'(\bar\rho)}{\bar\rho}n$ and $\bar\rho u$, respectively, integrating by parts over $\mathbb{R}^3$, and summing the resultant equalities up, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{P'(\bar\rho)}{\bar\rho}n^2+\bar\rho u^2\Big)dx+\mu\int_{\mathbb{R}^3}(\nabla u)^2dx\\
&+(\mu+\nu)\int_{\mathbb{R}^3}(\nabla\cdot u)^2dx-\int_{\mathbb{R}^3}\bar\rho\nabla\Phi u\,dx\\
&=\int_{\mathbb{R}^3}\frac{P'(\bar\rho)}{\bar\rho}(f_{11}+f_{12})n\,dx+\int_{\mathbb{R}^3}\bar\rho(f_{21}+f_{22})u\,dx.
\end{aligned}\tag{4.5}$$

Moreover, it follows from integration by parts and $(3.1)_{1,3}$ that

$$\begin{aligned}
-\int_{\mathbb{R}^3}\bar\rho\nabla\Phi u\,dx&=\int_{\mathbb{R}^3}\bar\rho\Phi\nabla\cdot udx=-\int_{\mathbb{R}^3}\Phi(n_t-\nabla\cdot(nu)-\nabla\cdot(\tilde\rho-\bar\rho)u))dx\\
&=-\int_{\mathbb{R}^3}\Phi(\Delta\Phi_t-\nabla\cdot(nu)-\nabla\cdot(\tilde\rho-\bar\rho)u))dx\\
&=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\nabla\Phi)^2dx-\int_{\mathbb{R}^3}\nabla\Phi(nu+(\tilde\rho-\bar\rho)u)dx,
\end{aligned}$$

which together with (4.4) imply

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{P'(\bar\rho)}{\bar\rho}n^2+\bar\rho u^2+|\nabla\Phi|^2\Big)dx\\
&+\mu\int_{\mathbb{R}^3}(\nabla u)^2dx+(\mu+\nu)\int_{\mathbb{R}^3}(\nabla\cdot u)^2dx\\
&=\int_{\mathbb{R}^3}\nabla\Phi(nu+(\tilde\rho-\bar\rho)u)dx+\int_{\mathbb{R}^3}\frac{P'(\bar\rho)}{\bar\rho}(f_{11}+f_{12})n\,dx\\
&\quad+\int_{\mathbb{R}^3}\bar\rho(f_{21}+f_{22})u\,dx.
\end{aligned}\tag{4.6}$$

Next, we estimate the integral terms on the right-hand side of (4.5). First, using Hölder's inequality, Young's inequality, (4.1), Lemma 2.1 and Lemma 3.1, we have

$$\begin{aligned}
&-\int_{\mathbb{R}^3}\nabla\Phi(nu+(\tilde\rho-\bar\rho)u)dx\\
&\le\|\nabla\Phi\|_{L^6}(\|n\|_{L^3}\|u\|+\|\frac{u}{1+|x|}\|\|(1+|x|)(\tilde\rho-\bar\rho)\|_{L^3})\\
&\le C\delta\|(n,\nabla n,\nabla u)\|^2,
\end{aligned}\tag{4.7}$$

here we used the inequality

$$\|\nabla^k\nabla\phi\|\le C\|\nabla^{k-1}n\|,\quad k\ge1,\tag{4.8}$$

which is derived by L^2 estimate for the Poisson equation $(3.1)_3$.

Since

$$f_{11}\sim\partial_i(\tilde\rho-\bar\rho)u_i+(\tilde\rho-\bar\rho)\partial_iu_i,\tag{4.9}$$

$$f_{12} \sim \partial_i n u_i + n \partial_i u_i, \tag{4.10}$$

it follows from Hölder's inequality, Young's inequality, (4.1), Lemma 2.1 and 3.1, that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{P'(\bar\rho)}{\bar\rho} f_{11} n\,dx \\
&\le C\|u\|_{L^6}\|\frac{n}{1+|x|}\|\|(1+|x|)\nabla(\tilde\rho-\bar\rho)\|_{L^3} + \|n\|_{L^6}\|(\tilde\rho-\bar\rho)\|_{L^3}\|\nabla\cdot u\| \\
&\le C\delta\|(\nabla n, \nabla u, \nabla\cdot u)\|^2,
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{P'(\bar\rho)}{\bar\rho} f_{12} n\,dx &\le C\|n\|_{L^3}(\|u\|_{L^6}\|\nabla n\| + \|n\|_{L^6}\|\nabla\cdot u\|) \\
&\le C\delta\|(\nabla n, \nabla u, \nabla\cdot u)\|^2.
\end{aligned} \tag{4.12}$$

Similarly, because

$$f_{21} \sim (\tilde\rho-\bar\rho)\partial_i\partial_i u_j + (\tilde\rho-\bar\rho)\partial_j\partial_i u_i + (\tilde\rho-\bar\rho)\partial_j n + n\partial_j(\tilde\rho-\bar\rho), \tag{4.13}$$

$$f_{22} \sim u_i\partial_i u_j + n\partial_i\partial_i u_j + n\partial_j\partial_i u_i + n\partial_j n, \tag{4.14}$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \bar\rho f_{21} u\,dx \\
&\le C\|u\|_{L^6}\Big(\|\Delta u\|\|\tilde\rho-\bar\rho\|_{L^3} + \|\nabla\nabla\cdot u\|\|\tilde\rho-\bar\rho\|_{L^3} + \|\nabla n\|\|\tilde\rho-\bar\rho\|_{L^3} \\
&\quad + \|\frac{n}{1+|x|}\|\|(1+|x|)\nabla(\tilde\rho-\bar\rho)\|_{L^3}\Big) \\
&\le C\delta\|(\nabla n, \nabla u, \nabla^2 u)\|^2,
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^3} \bar\rho f_{22} u\,dx \\
&\le C\|u\|_{L^6}(\|\nabla u\|\|u\|_{L^3} + \|\Delta u\|\|n\|_{L^3} + \|\nabla\nabla\cdot u\|\|n\|_{L^3} + \|\nabla n\|\|n\|_{L^3}) \\
&\le C\delta\|(\nabla n, \nabla u, \nabla^2 u)\|^2.
\end{aligned} \tag{4.16}$$

Therefore, putting (**??**) and (**??**)–(**??**) into (4.5) yields (4.3). This completes the proof. □

Lemma 4.4. *Under the a priori assumption* (4.1), *it holds that*

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_{\mathbb{R}^3}(\frac{P'(\bar\rho)}{\bar\rho}(\partial_x^\alpha n)^2 + \bar\rho(\partial_x^\alpha u)^2 + (\partial_x^\alpha\nabla\Phi)^2)dx + C\int_{\mathbb{R}^3}(\nabla\partial_x^\alpha u)^2 dx \\
&\le C\delta(\|\nabla n\|_1^2 + \|\nabla u\|_2^2), \quad 1 \le |\alpha| \le 2.
\end{aligned} \tag{4.17}$$

Proof. For each multi-index α with $|\alpha| = k$ $(k = 1, 2)$, multiplying ∂_x^α $(3.1)_1$ and ∂_x^α $(3.1)_2$ by $\frac{P'(\bar\rho)}{\bar\rho}\partial_x^\alpha n$ and $\bar\rho\partial_x^\alpha u$, respectively, and integrating over $\mathbb{R}^3$ by parts, and noting that

$$\begin{aligned}
-\int_{\mathbb{R}^3} \bar\rho\nabla\partial_x^\alpha\Phi\partial_x^\alpha u\,dx &= \int_{\mathbb{R}^3} \bar\rho\partial_x^\alpha\Phi\partial_x^\alpha \operatorname{div} u\,dx \\
&= -\int_{\mathbb{R}^3} \partial_x^\alpha\Phi(n_t - \partial_x^\alpha(f_{11}+f_{12}))dx
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^3} \partial_x^\alpha \Phi(\Delta \partial_x^\alpha \Phi_t - \partial_x^\alpha (f_{11}+f_{12}))dx \\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} (\nabla \partial_x^\alpha \Phi)^2 dx + \int_{\mathbb{R}^3} \partial_x^\alpha \Phi \partial_x^\alpha (f_{11}+f_{12})dx,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \Big(\frac{P'(\bar\rho)}{\bar\rho}(\partial_x^\alpha n)^2 + \bar\rho(\partial_x^\alpha u)^2 + (\partial_x^\alpha \nabla\Phi)^2\Big)dx \\
&+ \mu \int_{\mathbb{R}^3} (\nabla \partial_x^\alpha u)^2 dx + (\mu+\nu)\int_{\mathbb{R}^3} (\operatorname{div} \partial_x^\alpha u)^2 dx \\
&= \int_{\mathbb{R}^3} \frac{P'(\bar\rho)}{\bar\rho}\partial_x^\alpha f_{11}\partial_x^\alpha n\, dx + \int_{\mathbb{R}^3} \frac{P'(\bar\rho)}{\bar\rho}\partial_x^\alpha f_{12}\partial_x^\alpha n\, dx + \int_{\mathbb{R}^3} \bar\rho \partial_x^\alpha f_{21}\partial_x^\alpha u\, dx \\
&\quad + \int_{\mathbb{R}^3} \bar\rho\partial_x^\alpha f_{22}\partial_x^\alpha u\, dx - \int_{\mathbb{R}^3} \partial_x^\alpha \Phi \partial_x^\alpha (f_{11}+f_{12})dx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{4.18}
$$

Utilizing (4.6), Hölder's inequality, Young's inequality, Lemmas 2.1 and 3.1, we have the estimate

$$
\begin{aligned}
I_1 &= \frac{P'(\bar\rho)}{\bar\rho}\int_{\mathbb{R}^3} \partial_x^\alpha n \partial_x^\alpha(\partial_i(\tilde\rho-\bar\rho)u_i + (\tilde\rho-\bar\rho)\partial_i u_i)dx \\
&\le C\sum_{|\beta|\le|\alpha|}\int_{\mathbb{R}^3} |\partial_x^\alpha n \partial_x^\beta \partial_i(\tilde\rho-\bar\rho)\partial_x^{\alpha-\beta}u_i|dx \\
&\quad + C\Big(\sum_{|\beta|=0} + \sum_{1\le|\beta|\le|\alpha|}\Big)\int_{\mathbb{R}^3} |\partial_x^\alpha n \partial_x^\beta(\tilde\rho-\bar\rho)\partial_x^{\alpha-\beta}\partial_i u_i|dx \\
&\le C\sum_{|\beta|\le|\alpha|}\|\partial_x^\alpha n\|\|\partial_x^\beta\partial_i(\tilde\rho-\bar\rho)\|_{L^3}\|\partial_x^{\alpha-\beta}u_i\|_{L^6} + C\|\partial_x^\alpha n\|\|\tilde\rho-\bar\rho\|_{L^\infty}\|\partial_x^\alpha\partial_i u_i\| \\
&\quad + C\sum_{1\le|\beta|\le|\alpha|}\|\partial_x^\alpha n\|\partial_x^\beta(\tilde\rho-\bar\rho)\|_{L^3}\|\partial_x^{\alpha-\beta}\partial_i u_i\|_{L^6} \\
&\le C\delta(\|\partial_x^\alpha n\|^2 + \|\nabla u\|_2^2).
\end{aligned}
$$

Next, by (4.7), and Lemma 3.1, (4.1), Hölder's inequality, Young's inequality, and integration by parts, it holds that

$$
\begin{aligned}
I_2 &= \frac{P'(\bar\rho)}{\bar\rho}\Big(\int_{\mathbb{R}^3} \partial_x^\alpha n\partial_x^\alpha\partial_i n u_i dx + \sum_{\beta\le\alpha,|\beta|\le|\alpha|-1} C_\beta^\alpha \int_{\mathbb{R}^3} \partial_x^\alpha n \partial_x^\beta \partial_i n \partial_x^{\alpha-\beta}u_i dx \\
&\quad + \int_{\mathbb{R}^3} \partial_x^\alpha n\partial_x^\alpha \partial_i u_i n\, dx + \sum_{\beta\le\alpha,|\beta|\le|\alpha|-1} C_\beta^\alpha\int_{\mathbb{R}^3} \partial_x^\alpha n\partial_x^\beta\partial_i u_i\partial_x^{\alpha-\beta} n\, dx\Big) \\
&= \frac{P'(\bar\rho)}{\bar\rho}\Big(-\frac{1}{2}\int_{\mathbb{R}^3} \partial_i u_i(\partial_x^\alpha n)^2 dx + \sum_{\beta\le\alpha,|\beta|\le|\alpha|-1} C_\beta^\alpha\int_{\mathbb{R}^3} \partial_x^\alpha n\partial_x^\beta\partial_i n\partial_x^{\alpha-\beta}u_i dx \\
&\quad + \int_{\mathbb{R}^3} n\partial_x^\alpha n\partial_x^\alpha\partial_i u_i dx + \sum_{\beta\le\alpha,|\beta|\le|\alpha|-1} C_\beta^\alpha\int_{\mathbb{R}^3} \partial_x^\alpha n\partial_x^\beta\partial_i u_i\partial_x^{\alpha-\beta} n\, dx\Big) \\
&\le C\|\partial_i u_i\|_{L^\infty}\|\partial_x^\alpha n\|^2 + C\Big(\sum_{|\beta|=0} + \sum_{1\le|\beta|\le|\alpha|-1}\Big)\int_{\mathbb{R}^3} |\partial_x^\alpha n\partial_x^\beta\partial_i n\partial_x^{\alpha-\beta}u_i|dx
\end{aligned}
$$

$$
\begin{aligned}
&\quad + C\Big(\sum_{|\beta|=0} + \sum_{1\le|\beta|\le|\alpha|-1}\Big)\int_{\mathbb{R}^3} |\partial_x^\alpha n \partial_x^\beta \partial_i u_i \partial_x^{\alpha-\beta} n| dx + C\|n\|_{L^\infty}\|\partial_x^\alpha \partial_i u\|\|\partial_x^\alpha n\| \\
&\le C\|\partial_i u_i\|_{L^\infty}\|\partial_x^\alpha n\|^2 + C\|\partial_i n\|_{L^3}\|\partial_x^\alpha u_i\|_{L^6}\|\partial_x^\alpha n\| + C\|n\|_{L^\infty}\|\partial_x^\alpha \partial_i u\|\|\partial_x^\alpha n\| \\
&\quad + \delta\|\partial_x^\alpha n\|^2 + \frac{C}{\delta}\sum_{1\le|\beta|\le|\alpha|-1}(\|\partial_x^\beta \partial_i n\|^2\|\partial_x^{\alpha-\beta} u_i\|^2 + \|\partial_x^\beta \partial_i u_i\|_1^2\|\partial_x^{\alpha-\beta}\nabla n\|^2) \\
&\le C\delta(\|\partial_x^\alpha n\|^2 + \|\nabla u\|_2^2),
\end{aligned}
$$

where we note that the terms including the sum of β with $1 \le |\beta| \le |\alpha| - 1$ will be vanished if $|\alpha| = 1$.

From (**??**) and (**??**), we have

$$
\begin{gathered}
I_3 \sim C\int_{\mathbb{R}^3} \partial_x^\alpha u \partial_x^\alpha((\tilde\rho - \bar\rho)\partial_i\partial_i u_j + (\tilde\rho - \bar\rho)\partial_j\partial_i u_i + (\tilde\rho - \bar\rho)\partial_j n + n\partial_j(\tilde\rho - \bar\rho))dx, \\
I_4 \sim C\int_{\mathbb{R}^3} \partial_x^\alpha u \partial_x^\alpha(u_i\partial_i u_j + n\partial_i\partial_i u_j + n\partial_j\partial_i u_i + n\partial_j n)dx.
\end{gathered}
$$

Then as for the estimates of I_1 and I_2, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} |\partial_x^\alpha u \partial_x^\alpha(n\partial_j(\tilde\rho - \bar\rho))| dx &= \Big(\sum_{|\beta|=0} + \sum_{1\le|\beta|\le|\alpha|}\Big)\int_{\mathbb{R}^3} |\partial_x^\alpha u \partial_x^\beta n \partial_x^{\alpha-\beta}\partial_j(\tilde\rho - \bar\rho)| dx \\
&\le C\|\frac{n}{1+|x|}\|\|\partial_x^\alpha u\|_{L^6}\|(1+|x|)\partial_x^\alpha \partial_j(\tilde\rho - \bar\rho))\|_{L^3} \\
&\quad + C\sum_{1\le|\beta|\le|\alpha|-1}\|\partial_x^\beta n\|_{L^3}\|\partial_x^\alpha u\|_{L^6}\|\partial_x^{\alpha-\beta}\partial_j(\tilde\rho - \bar\rho)\| \\
&\le C\delta(\|\partial_x^\alpha \nabla u\|^2 + \|\nabla n\|_1^2),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \partial_x^\alpha u \partial_x^\alpha(n\partial_i\partial_i u_j)dx \\
&= \Big(\sum_{|\beta|=0} + \sum_{1\le|\beta|\le|\alpha|}\Big)\int_{\mathbb{R}^3} \partial_x^\alpha u \partial_x^\beta n \partial_x^{\alpha-\beta}\partial_i\partial_i u_j dx \\
&\le \int_{\mathbb{R}^3} n(\partial_x^\alpha \partial_i u)^2 dx + \int_{\mathbb{R}^3} |\partial_x^\alpha u \partial_i n \partial_x^\alpha \partial_i u_j| dx \\
&\quad + \int_{\mathbb{R}^3} |\partial_x^\alpha u \partial_x^\alpha n \partial_i\partial_i u_j| dx + \sum_{1\le|\beta|\le|\alpha|-1}\int_{\mathbb{R}^3} |\partial_x^\alpha u \partial_x^\beta n \partial_x^{\alpha-\beta}\partial_i\partial_i u_j| dx \\
&\le C\|n\|_{L^\infty}\|\partial_x^\alpha \partial_i u\|^2 + C\|\partial_i n\|_{L^3}\|\partial_x^\alpha u\|_{L^6}\|\partial_x^\alpha \partial_i u_j\| \\
&\quad + C\sum_{1\le|\beta|\le|\alpha|-1}\|\partial_x^\beta n\|_{L^3}\|\partial_x^\alpha u\|_{L^6}\|\partial_x^{\alpha-\beta}\partial_i\partial_i u_j\| + \|\partial_x^\alpha n\|\|\partial_x^\alpha u\|_{L^6}\|\partial_i\partial_i u_j\|_{L^3} \\
&\le C\delta(\|\partial_x^\alpha \nabla u\|^2 + \|\nabla^2 u\|_1^2).
\end{aligned}
$$

The other terms in I_3 and I_4 can be estimated similarly. Thus,

$$
I_3 + I_4 \le C\delta(\|\partial_x^\alpha u\|^2 + \|\partial_x^\alpha \nabla u\|^2 + \|\nabla n\|_1^2 + \|\nabla u\|_2^2).
$$

Finally, let us consider I_5. Indeed, utilizing Lemmas 2.1 and 3.1, (4.1), (**??**), Hölder's inequality, Young's inequality, and integration by parts, we have

$$
\begin{aligned}
I_5 &= -\int_{\mathbb{R}^3} \partial_x^\alpha \Phi \partial_x^\alpha (\nabla\cdot(nu) + \nabla\cdot((\tilde\rho-\bar\rho)u))dx \\
&= \int_{\mathbb{R}^3} \nabla\partial_x^\alpha \Phi \partial_x^\alpha (nu + (\tilde\rho-\bar\rho)u)dx \\
&= \Big(\sum_{|\beta|=0} + \sum_{1\le|\beta|\le|\alpha|}\Big) C_\beta^\alpha \int_{\mathbb{R}^3} \nabla\partial_x^\alpha\Phi[\partial_x^\beta(\tilde\rho-\bar\rho)\partial_x^{\alpha-\beta}u + \partial_x^\beta n\partial_x^{\alpha-\beta}u)]dx \\
&\le C\|\nabla\partial_x^\alpha\Phi\|_{L^6}\|\partial_x^\alpha u\|_{L^3}\|\tilde\rho-\bar\rho\| + C\|\nabla\partial_x^\alpha\Phi\|_{L^6}\|\partial_x^\alpha u\|_{L^3}\|n\| \\
&\quad + C\|\nabla\partial_x^\alpha\Phi\|_{L^6}\|\frac{u}{1+|x|}\|\|(1+|x|)\partial_x^\alpha(\tilde\rho-\bar\rho)\|_{L^3} + C\|\nabla\partial_x^\alpha\Phi\|_{L^6}\|u\|\|\partial_x^\alpha n\|_{L^3} \\
&\quad + C\sum_{1\le|\beta|\le|\alpha|-1} [\|\nabla\partial_x^\alpha\Phi\|_{L^6}\|\partial_x^{\alpha-\beta}u\|_{L^3}\|\partial_x^\beta(\tilde\rho-\bar\rho)\| \\
&\quad + \|\nabla\partial_x^\alpha\Phi\|_{L^6}\|\partial_x^{\alpha-\beta}u\|_{L^3}\|\partial_x^\beta n\|] \\
&\le C\delta(\|\partial_x^\alpha u\|^2 + \|\partial_x^\alpha\nabla u\|^2 + \|\nabla n\|_1^2 + \|\nabla u\|_1^2).
\end{aligned}
$$

Therefore, insertion the estimates of I_i $(i = 1,2,3,4,5)$ into (4.9) implies (4.8). This completes the proof. □

Finally, let us focus on the integral estimate for the deviation of density.

Lemma 4.5. *Under the a priori assumption* (4.1)*, we have*

$$
\frac{d}{dt}\int_{\mathbb{R}^3} u\cdot\nabla n dx + C\|(n,\nabla n)\|^2 \le C\|(\nabla u,\nabla^2 u)\|^2, \tag{4.19}
$$

$$
\frac{d}{dt}\int_{\mathbb{R}^3} \nabla u\cdot\nabla^2 n\, dx + C\|(\nabla n,\nabla^2 n)\|^2 \le C\|(\nabla u,\nabla^2 u,\nabla^3 u)\|^2. \tag{4.20}
$$

Proof. First, taking inner product of $(3.1)_2$ and ∇n over $\mathbb{R}^3$, and using $(3.1)_1$ and integration by parts, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} u\nabla n\, dx + \int_{\mathbb{R}^3}\frac{P'(\bar\rho)}{\bar\rho}(\nabla n)^2 dx - \int_{\mathbb{R}^3}\nabla\Phi\nabla n\, dx \\
&= \int_{\mathbb{R}^3} u\nabla n_t dx + \int_{\mathbb{R}^3}(f_{21}+f_{22})\nabla n\, dx + \int_{\mathbb{R}^3}(\mu_1\Delta u + \mu_2\nabla\operatorname{div} u)\nabla n\, dx \\
&= -\int_{\mathbb{R}^3}\nabla u(f_{11}+f_{12}-\bar\rho\operatorname{div} u)dx + \int_{\mathbb{R}^3}(f_{21}+f_{22})\nabla n\, dx \\
&\quad + \int_{\mathbb{R}^3}[\mu_1\Delta u + \mu_2\nabla\operatorname{div} u]\nabla n\, dx.
\end{aligned} \tag{4.21}
$$

First, it is easy to obtain

$$
-\int_{\mathbb{R}^3}\nabla\Phi\cdot\nabla n\, dx = \int_{\mathbb{R}^3}\Delta\Phi n\, dx = \int_{\mathbb{R}^3} n^2 dx. \tag{4.22}
$$

Moreover, from Young's inequality, one obtains

$$
\int_{\mathbb{R}^3}\bar\rho\nabla u\operatorname{div} u dx + \int_{\mathbb{R}^3}(\mu_1\Delta u + \mu_2\nabla\operatorname{div} u)\nabla n\, dx \le C\|\nabla u\|_1^2 + \frac{P'(\bar\rho)}{4\bar\rho}\|\nabla n\|^2. \tag{4.23}
$$

As for (**??**)-(**??**) and (**??**)-(**??**), we have

$$-\int_{\mathbb{R}^3}(f_{11}+f_{12})\nabla u\,dx \le C\delta(\|\nabla n\|^2+\|\nabla u\|^2), \tag{4.24}$$

$$\int_{\mathbb{R}^3}(f_{21}+f_{22})\nabla n dx \le C\delta(\|\nabla n\|^2+\|\nabla u\|_1^2). \tag{4.25}$$

Hence from (4.12) and (4.13)-(4.16), we have (4.10).

Performing the similar computations for $\int_{\mathbb{R}^3}\partial_x^\beta(3.1)_2\nabla\partial_x^\beta n\,dx$ for $|\beta|=1$ leads to (4.11). This completes the proof. □

Proof of Proposition 4.2. Combining (4.3), (4.8) with $1\le|\alpha|\le 2$, (4.10) and (4.11) with $|\beta|=1$, and using Gronwall's inequality, we immediately have

$$\|(n,u,\nabla\Phi)(\cdot,t)\|_2^2+\int_0^t\|(n,\nabla u)(\cdot,s)\|_2^2 ds\le C\|(n_0,u_0)\|_2^2,$$

which together with (**??**) yields (4.2). This completes the proof. □

5. Decay rate

In this section, we get decay rates of solutions to the problem (3.1), (3.2). To begin, we set $U=(n,u)^t$, $U_0=(n_0,u_0)^t$, $Q=(f_{11}+f_{12},f_{21}+f_{22})^t$ and $G(t)$ as the solution semigroup defined by $G(t)=e^{-tA}(t\ge 0)=(G^{ij}(t))_{2\times 2}$, with A being a matrix-valued differential operator given by

$$A=\begin{pmatrix}0 & \bar\rho\,\mathrm{div}\\ \frac{P'(\bar\rho)}{\bar\rho}\nabla-\nabla\Delta^{-1} & -\mu_1\Delta-\mu_2\nabla\,\mathrm{div}\end{pmatrix}.$$

For a function $f(x,t)$, we have $G(t)*f=\mathcal{F}^{-1}(e^{-t\hat A(\xi)}\hat f(\xi,t))$. Then, we rewrite the solution of (3.1)-(3.2) as

$$U(t)=G(t)*U_0+\int_0^t G(t-s)*Q(U(s))\,ds, \tag{5.1}$$

and

$$\begin{aligned}\nabla\Phi=\mathcal{E}^1(t)*n_0+\mathcal{E}^2(t)*u_0+\int_0^t(\mathcal{E}^1(t-s)*(f_{11}+f_{12})(U(s))\\ +\mathcal{E}^2(t-s)*(f_{21}+f_{22})(U(s)))ds,\end{aligned} \tag{5.2}$$

where $\mathcal{E}^1$ and $\mathcal{E}^2$ be the respective inverse Fourier transform of the following $\hat{\mathcal{E}}^1$ and $\hat{\mathcal{E}}^2$

$$\hat{\mathcal{E}}^1=\frac{i\xi}{|\xi|^2}\otimes\hat G_{11},\quad \hat{\mathcal{E}}^2(t)=\frac{i\xi}{|\xi|^2}\otimes\hat G_{12}.$$

Moreover, let $\hat\chi$ be a cutoff function defined by

$$\hat\chi(\xi)=\begin{cases}1, & \text{for } |\xi|<r\\ 0, & \text{for } |\xi|\ge r\end{cases}, \tag{5.3}$$

Here $r>0$ is some fixed constant. Now, based on the Fourier transform and (5.3), we can define the low frequency and high frequency decomposition $(f_L(x),f_H(x))$ for a function $f(x)$ as follows

$$f_L:=\mathcal{F}^{-1}(\hat\chi\hat f),\quad f_H=f-f_L. \tag{5.4}$$

Using the definitions (5.3) and (5.4) and the Plancherel's theorem, we can obtain directly the following estimates

$$\|\nabla^k f\| \le \|\nabla^k f_L\| + \|\nabla^k f_H\|, \quad \|\nabla^k f_L\| \le \|f\|, \quad k \ge 0, \tag{5.5}$$

$$C\|f_H\| \le C\|\nabla f_H\|, \quad C\|\nabla^k f_H\| \le \|\nabla^k f\|, \quad k \ge 1, \tag{5.6}$$

Then, using these definitions, (5.1) and (5.2), we have

$$U_L(t) = G_L(t) * U_0 + \int_0^t G_L(t-s) * Q(U(s))\, ds, \tag{5.7}$$

and

$$\begin{aligned}\nabla\Phi_L = \mathcal{E}_L^1(t) * n_0 + \mathcal{E}_L^2(t) * u_0 + \int_0^t (\mathcal{E}_L^1(t-s) * (f_{11}+f_{12})(U(s)) \\ + \mathcal{E}_L^2(t-s) * (f_{21}+f_{22})(U(s)))ds.\end{aligned} \tag{5.8}$$

Next we first give the decay rates of the low frequency solution, namely, $n_L(t)$ and ∇u_L. For this, we need the L^2-type of the time decay estimates on the low-frequency part of the semigroup $G(t), \mathcal{E}^1$ and $\mathcal{E}^2$. From the results in [16, 17], one has the following decay estimates.

Lemma 5.1. *Let $k \ge 0$ be an integer. Then, for any $t \ge 0$, we have*

$$\begin{aligned}
\|\partial_x^k G_L^{11} * U_0\| \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U_0\|_{L^1}, \quad \|\partial_x^k G_L^{12} * U_0\| \le C(1+t)^{-\frac{5}{4}-\frac{k}{2}}\|U_0\|_{L^1},\\
\|\partial_x^k G_L^{21} * U_0\| \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{L^1}, \quad \|\partial_x^k G_L^{22} * U_0\| \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U_0\|_{L^1},\\
\|\partial_x^k \mathcal{E}_L^{1} * U_0\| \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{L^1}, \quad \|\partial_x^k \mathcal{E}_L^{2} * U_0\| \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U_0\|_{L^1}.
\end{aligned}$$

Now we can estimate the time-decay rate for $\|n_L\|$ and $\|\nabla u_L\|$.

Lemma 5.2. *Let $K_0 = \|U_0\|_{L^1}$. Then we have*

$$\|n_L(t)\| \le CK_0(1+t)^{-3/4} + C\delta\int_0^t (1+t-s)^{-5/4}\|(n, \nabla n, \nabla u, \nabla^2 u)(s)\|ds, \tag{5.9}$$

$$\begin{aligned}&\|\nabla u_L(t)\|\\ &\le CK_0(1+t)^{-3/4} + C\delta\int_0^t (1+t-s)^{-5/4}\|(n, \nabla n, \nabla u, \nabla^2 u)(s)\|ds.\end{aligned} \tag{5.10}$$

Proof. Applying (5.7), we have

$$\begin{aligned}
n_L(t) &= G_L^{11}(t) * n_0 + G_L^{12}(t) * u_0 + \int_0^t G_L^{11}(t-s) * (f_{11}+f_{12})(U(s))ds\\
&\quad + \int_0^t G_L^{12}(t-s) * (f_{21}+f_{22})(U(s))ds\\
&= G_L^{11}(t) * n_0 + G_L^{12}(t) * u_0 - \int_0^t \nabla G_L^{11}(t-s) * (nu + (\tilde\rho - \bar\rho)u)(s)ds\\
&\quad + \int_0^t G_L^{12}(t-s) * (f_{21}+f_{22})(U(s))ds.
\end{aligned}$$

Further, using Lemma 5.1, one obtains

$$\begin{aligned}
\|n_L(t)\| &\leq C(1+t)^{-3/4}\|n_0\|_{L^1} + C(1+t)^{-5/4}\|u_0\|_{L^1} \\
&\quad + C\int_0^t (1+t-s)^{-5/4}(\|nu(s)\|_{L^1} + \|(\tilde{\rho}-\bar{\rho})u(s)\|_{L^1})ds \\
&\quad + C\int_0^t (1+t-s)^{-5/4}(\|f_{21}(U)(s)\|_{L^1} + \|f_{22}(U)(s)\|_{L^1})ds \\
&\leq CK_0(1+t)^{-3/4} \\
&\quad + C\int_0^t (1+t-s)^{-5/4}(\|nu(s)\|_{L^1} + \|(\tilde{\rho}-\bar{\rho})u(s)\|_{L^1})ds \\
&\quad + C\int_0^t (1+t-s)^{-5/4}(\|f_{21}(U)(s)\|_{L^1} + \|f_{22}(U)(s)\|_{L^1})ds.
\end{aligned} \tag{5.11}$$

Similarly, using (5.7), we have

$$\begin{aligned}
&\nabla u_L(t) \\
&= \nabla G_L^{21}(t) * n_0 + \nabla G_L^{22}(t) * u_0 + \int_0^t \nabla G_L^{21}(t-s) * (f_{11}+f_{12})(U(s))ds \\
&\quad + \int_0^t \nabla G_L^{22}(t-s) * (f_{21}+f_{22})(U(s))ds \\
&= \nabla G_L^{21}(t) * n_0 + \nabla G_L^{22}(t) * u_0 - \int_0^t \nabla^2 G_L^{21}(t-s) * (nu + (\tilde{\rho}-\bar{\rho})u)(s)ds \\
&\quad + \int_0^t \nabla G_L^{22}(t-s) * f_2(s)ds
\end{aligned}$$

This and Lemma 5.1 lead to

$$\begin{aligned}
&\|\nabla u_L(t)\| \\
&\leq C(1+t)^{-3/4}\|n_0\|_{L^1} + C(1+t)^{-5/4}\|u_0\|_{L^1} \\
&\quad + C\int_0^t (1+t-s)^{-5/4}(\|nu(s)\|_{L^1} + \|(\tilde{\rho}-\bar{\rho})u(s)\|_{L^1})ds \\
&\quad + C\int_0^t (1+t-s)^{-5/4}(\|f_{21}(U)(s)\|_{L^1} + \|f_{22}(U)(s)\|_{L^1})ds \\
&\leq CK_0(1+t)^{-3/4} \\
&\quad + C\int_0^t (1+t-s)^{-5/4}(\|nu(s)\|_{L^1} + \|(\tilde{\rho}-\bar{\rho})u(s)\|_{L^1})ds \\
&\quad + C\int_0^t (1+t-s)^{-5/4}(\|f_{21}(U)(s)\|_{L^1} + \|f_{22}(U)(s)\|_{L^1})ds.
\end{aligned} \tag{5.12}$$

To obtain (5.9) and (5.10), we need only to control $\|nu\|_{L^1}, \|(\tilde{\rho}-\bar{\rho})u\|_{L^1}, \|f_{21}(U)\|_{L^1}$ and $\|f_{22}(U)\|_{L^1}$ by the L^2-norm of n and the derivatives of (n,u) at least first order. First, from (1.5), and using Hölder's inequality and Lemma 3.1, it is easy to have

$$\|(\tilde{\rho}-\bar{\rho})u\|_{L^1} \leq \|(1+|x|)(\tilde{\rho}-\bar{\rho})\|\|\frac{u}{1+|x|}\| \leq C\delta\|\nabla u\|. \tag{5.13}$$

Meanwhile, utilizing Hölder's inequality and (4.1)

$$\|nu(s)\|_{L^1} \leq C\delta\|n\|. \tag{5.14}$$

In the same way, we can show that

$$\|f_{21}(U)\|_{L^1} \le C\delta(\|\nabla n\| + \|\nabla^2 u\|)\|, \tag{5.15}$$

$$\|f_{22}(U)\|_{L^1} \le C\delta(\|\nabla n\| + \|\nabla u\| + \|\nabla^2 u\|)\|. \tag{5.16}$$

Therefore, putting the above inequalities (**??**)-(**??**) into (**??**) and (**??**), we immediately have (5.9) and (5.10). This completes the proof. □

Next, from the a priori estimates obtained in Section 4 and the low frequency and high frequency decomposition (5.4), we derive a new Lyapunov-type inequality. From (4.8) and (4.11), we obtain

$$\frac{dD(t)}{dt} + \|\nabla^2 u\|_1^2 + \|(\nabla n, \nabla^2 n)\|^2 \le C\delta\|\nabla u\|^2, \tag{5.17}$$

here $D(t) = \|(\nabla n, \nabla u, \nabla^2\Phi)\|_1^2$. By using (5.5) and (5.6), and from (5.11), there exists a constant $C_1 > 0$ such that

$$\frac{dD(t)}{dt} + C_1\|(n_H, \nabla u_H)\| + \frac{1}{2}\|(\nabla n, \nabla^2 n, \nabla^2 u, \nabla^3 u)\|^2 \le C\delta(\|\nabla u_L\|^2 + \|\nabla u_H\|^2),$$

which together with the smallness of δ imply

$$\frac{dD(t)}{dt} + C_2\|(n_H, \nabla u_H)\| + \frac{1}{2}\|(\nabla n, \nabla^2 n, \nabla^2 u, \nabla^3 u)\|^2 \le C\delta\|\nabla u_L\|^2, \tag{5.18}$$

for some positive constant C_2. Further, adding $C_2(\|n_L\|^2 + \|\nabla u_L\|^2)$ to both sides of (5.12) and using (**??**), there exists some constants C_3 and C_4 such that

$$\frac{dD(t)}{dt} + C_3 D(t) \le C_4(\|n_L\|^2 + \|\nabla u_L\|^2), \quad t \ge 0. \tag{5.19}$$

To prove the decay estimates stated in Theorem 1.1, we first define

$$M(t) = \sup_{0\le s\le t}(1+s)^{3/2}(D(s), \|n(s)\|^2). \tag{5.20}$$

Notice that $M(t)$ is non-decreasing, and

$$\|n(s)\|, \|(\nabla n, \nabla u, \nabla^2\Phi)\|_1 \le C(1+t)^{-3/4}\sqrt{M(t)}, \quad 0 \le s \le t. \tag{5.21}$$

Then, from (5.9) and (5.10), by using Lemma 3.2 and (**??**), we obtain

$$\|n_L(t)\| \le CK_0(1+t)^{-3/4} + C\delta(1+t)^{-3/4}\sqrt{M(t)}, \tag{5.22}$$

$$\|\nabla u_L(t)\| \le CK_0(1+t)^{-3/4} + C\delta(1+t)^{-3/4}\sqrt{M(t)}. \tag{5.23}$$

By Gronwall's inequality and (**??**) and (**??**), from (5.13), we have

$$\begin{aligned} D(t) &\le D(0)e^{-C_3 t} + C_4\int_0^t e^{-C_3(t-s)}(\|n_L(s)\|^2 + \|\nabla u_L(s)\|^2)ds \\ &\le D(0)e^{-C_3 t} + C_4\int_0^t e^{-C_3(t-s)}(1+s)^{-\frac{3}{2}}(K_0^2 + \delta^2 M(s))ds \\ &\le C(1+t)^{-\frac{3}{2}}(D(0) + K_0^2 + \delta^2 M(t)). \end{aligned} \tag{5.24}$$

where $D(0)$ is equivalent to $\|(\nabla n_0, \nabla u_0)\|_1^2$.

In terms of $M(t)$, from (**??**) we have

$$M(t) \le C(D(0) + K_0^2) + C\delta^2 M(t).$$

Therefore, if $\delta > 0$ is small enough, then

$$M(t) \le C(D(0) + K_0^2) \le CK_0.$$

This in turn gives (1.9), and

$$\|n(t)\| \le C(1+t)^{-3/4}, \tag{5.25}$$

Next, we define the temporal energy functional $D_1(t) = \|(n, u, \nabla\Phi)\|_2^2$. Then from (4.2), we have

$$\frac{dD_1(t)}{dt} + C\|(n, \nabla u)\|_2^2 \le 0, \tag{5.26}$$

Similar as (5.12), from (5.15), there exists some constants C_5 and C_6 such that

$$\frac{dD_1(t)}{dt} + C_5 D_1(t) \le C_6(\|u_L\|^2 + \|\nabla\Phi_L\|^2),\ t \ge 0. \tag{5.27}$$

Again applying (5.7), we have

$$\begin{aligned} u_L(t) &= G_L^{21}(t) * n_0 + G_L^{22}(t) * u_0 + \int_0^t G_L^{21}(t-s) * (f_{11} + f_{12})(U(s))ds \\ &\quad + \int_0^t G_L^{22}(t-s) * (f_{21} + f_{22})(U(s))ds \\ &= G_L^{21}(t) * n_0 + G_L^{22}(t) * u_0 - \int_0^t \nabla G_L^{21}(t-s) * (nu + (\tilde{\rho} - \bar{\rho})u)(s)ds \\ &\quad + \int_0^t G_L^{22}(t-s) * (f_{21} + f_{22})(U(s))ds, \end{aligned}$$

which together with Lemma 5.1, (**??**)-(**??**) and (1.9) yield

$$\begin{aligned} \|u_L(t)\| &\le C(1+t)^{-1/4}\|n_0\|_{L^1} + C(1+t)^{-3/4}\|u_0\|_{L^1} \\ &\quad + C\int_0^t (1+t-s)^{-3/4}(\|nu(s)\|_{L^1} + \|(\tilde{\rho} - \bar{\rho})u(s)\|_{L^1})ds \\ &\quad + C\int_0^t (1+t-s)^{-3/4}(\|f_{21}(U)(s)\|_{L^1} + \|f_{22}(U)(s)\|_{L^1})ds \\ &\le CK_0(1+t)^{-1/4} \\ &\quad + C\int_0^t (1+t-s)^{-3/4}(\|nu(s)\|_{L^1} + \|(\tilde{\rho} - \bar{\rho})u(s)\|_{L^1})ds \\ &\quad + C\int_0^t (1+t-s)^{-3/4}(\|f_{21}(U)(s)\|_{L^1} + \|f_{22}(U)(s)\|_{L^1})ds \\ &\le CK_0(1+t)^{-1/4} + C\delta\int_0^t (1+t-s)^{-3/4}(1+s)^{-3/4}ds \\ &\le C(1+t)^{-1/4}. \end{aligned} \tag{5.28}$$

In the same way, from (5.8) and Lemma 5.1, we can show that

$$\|\nabla\Phi_L(t)\| \le C(1+t)^{-1/4}. \tag{5.29}$$

Furthermore, by the Gronwall inequality and (**??**), (**??**), from (5.16), we have

$$\begin{aligned} D_1(t) &\le D_1(0)e^{-C_5 t} + C_6\int_0^t e^{-C_5(t-s)}\|u_L(s)\|^2 ds \\ &\le D_1(0)e^{-C_5 t} + C_6(K_0^2 + \delta^2)\int_0^t e^{-C_5(t-s)}(1+s)^{-1/2}ds \\ &\le C(D_1(0) + K_0^2 + \delta^2)(1+t)^{-1/2}, \end{aligned}$$

which implies

$$\|u(t)\| \le C(1+t)^{-1/4}, \tag{5.30}$$

$$\|\nabla\Phi(t)\| \le C(1+t)^{-1/4}. \tag{5.31}$$

Here $D_1(0)$ is equivalent to $\|(n_0, u_0, \nabla\Phi_0)\|_1^2$.

Finally, let us focus on (1.10) and (1.11). First by Sobolev's inequality and (1.9), we have

$$\|(n, u, \nabla\Phi)(t)\|_{L^6} \le C\|(\nabla n, \nabla u, \nabla^2\Phi)(t)\| \le C(1+t)^{-3/4}. \tag{5.32}$$

Hence, by interpolation, it follows from (5.14), (**??**), (**??**) and (5.17) that for any $2 \le p \le 6$,

$$\|n(t)\|_{L^p} \le \|n(t)\|^\theta \|n(t)\|_{L^6}^{1-\theta} \le C\|n(t)\|^\theta \|\nabla n(t)\|^{1-\theta} \le C(1+t)^{-3/4},$$

and

$$\begin{aligned}\|(u, \nabla\phi)(t)\|_{L^p} &\le \|(u, \nabla\Phi)(t)\|^\theta \|(u, \nabla\Phi)(t)\|_{L^6}^{1-\theta}\\ &\le C\|(u, \nabla\Phi)(t)\|^\theta \|(\nabla u, \nabla^2\Phi)(t)\|^{1-\theta}\\ &\le C(1+t)^{-(1-\frac{3}{2p})},\end{aligned}$$

where $\theta = \frac{6-p}{2p}$. So we have (1.10) and (1.11). This finishes the proof of Theorem 1.2.

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