Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 66, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

QUASI-NEUTRAL LIMIT OF THE FLOW OF A CHEMICALLY REACTING GASEOUS MIXTURE FOR IONIC DYNAMICS

YOUNG-SAM KWON

Communicated by Jesus Ildefonso Diaz

Abstract. In this article we consider the quasi-neutral limit of the compressible flow of a chemically reacting gaseous mixture in the periodic domain $\mathbb{T}^3$ with the well-prepared initial data. We prove that the weak solution of the compressible flow of a chemically reacting gaseous mixture converges to the strong solution of the compressible flow of reacting gaseous mixture as long as the latter exists.

1. Introduction

The flow of chemically reacting gaseous mixture arises in sciences and engineering and is associated with a variety of phenomena and processes: pollutant formation, biotechnology, fuel droplets in combustion, sprays, astrophysical plasma. Analyzing the physical regimes associated with various processes unfolds complex chemistry mechanisms and detailed transport phenomena. Many interesting problems in that context involve the behavior of solutions to the governing equations for multicomponent reactive flows as certain parameters vanish or become infinity. The objective of this work is to investigate quasi-neutral limit for such complex flows based on the relative entropy in the periodic domain.

As a physical model of fluids, we here consider the flow of chemically reacting gaseous mixture governed by Poisson equations in the periodic domain $\Omega = \mathbb{T}^3$ where $\mathbb{T}^3$ is the three dimensional periodic domain:

$$\partial_t \varrho_\epsilon + \operatorname{div}(\varrho_\epsilon \mathbf{u}_\epsilon) = 0, \tag{1.1}$$

$$\partial_t(\varrho_\epsilon \mathbf{u}_\epsilon) + \operatorname{div}(\varrho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \nabla \varrho_\epsilon^\gamma = \mu \Delta \mathbf{u}_\epsilon + (\mu + \nu)\nabla \operatorname{div} \mathbf{u}_\epsilon - \varrho_\epsilon \nabla G_\epsilon, \tag{1.2}$$

$$\partial_t(\varrho_\epsilon Y_\epsilon) + \operatorname{div}(\varrho_\epsilon Y_\epsilon \mathbf{u}_\epsilon) = d\Delta Y_\epsilon - k\varrho_\epsilon Y_\epsilon, \tag{1.3}$$

$$-\epsilon^2 \Delta G_\epsilon = \varrho_\epsilon - \exp G_\epsilon, \tag{1.4}$$

where $\mathbf{u}_\epsilon$ is the vector field, $\gamma > \frac{3}{2}$, ϱ_ϵ is the density, Y_ϵ is the reactant fraction, and G_ϵ is a potential function. Note that we assume that the viscosities μ, ν, d, k do not depend on ϵ because of the good regularity of density, velocity, and reactant fraction.

2010 *Mathematics Subject Classification.* 35L15, 35L53.
Key words and phrases. Reacting gaseous mixture; quasi-neutral limit.

Submitted March 23, 2018. Published May 13, 2019.

The existence of global weak solutions for the compressible flow of chemically reacting gaseous mixture (1.1)–(1.4) was proved by Donatelli, Trivisa, Marion, and Temmam [7, 9, 19]. Bresch, Desjardins, and Ducomet [1] studied the quasi-neutral limit for the isentropic compressible Navier-Stokes-Poission system for ions with capillary effect on $\mathbb{T}^3$. They established the existence of global weak solutions of the model and obtained that the weak solution the primitive model converges to the weak solutions of the compressible capillary Navier-Stokes equations in the torus $\mathbb{T}^3$. Later, Feireisl and Zhang [11] studied the quasi-neutral limit of the compressible Navier-Stokes-Poisson system for ions with an additional damping term in a bounded domain of $\mathbb{R}^3$ in the framework of weak solutions. For the ionic Euler-Poisson system, the quasi-neutral limit was studied, for example, in [3, 13, 12, 20].

If we replace the Poisson equation (1.4) by

$$-\epsilon^2 \Delta V = \varrho - D(x), \tag{1.5}$$

where $D(x)$ is a given function, we obtain the corresponding model for electrons and a few results on the the quasi-neutral limit are available [14, 2, 5, 17, 18]. Ju, Li, and Li [14] studied the quasi-neutral limit for local strong solutions to the Navier-Stokes-Fourier-Poisson system on $\mathbb{T}^3$. Chen, Donatelli, and Marcati [2] studied the quasi-neutral limit of a hydrodynamic model for charge-carrier transport in the framework of weak solutions. In [17], Ju and Li studied the combined quasi-neutral and zero-electron-mass limit of the Navier-Stokes-Fourier-Poisson system in the torus $\mathbb{T}^3$ and showed the limit is the the incompressible Navier-Stokes equations. Donatelli and Marcati [5] gave some descriptions on the quasineutral limit for the full Navier-Stokes-Poisson system in $\mathbb{R}^3$. Very recently, Li, Ju, and Xu [18] improved the result in [17] to allow the temperature have a large variation. Also, they are many results on quasi-neutral limit to the electric Euler-Poisson and Navier-Stokes-Poisson system, among others, we mention [15, 16, 4, 22, 21, 6].

Motivated by the results in [1, 11, 14, 17, 18], in this article we want to study the quasi-neutral limit to the system (1.1)–(1.4). Formally, letting ϵ tend to 0 in (1.4), we obtain $\varrho = \exp(G)$. Thus, the term $\varrho\nabla G$ in (1.2) turns to $\nabla\varrho$. Hence we can expect that, as ϵ tend to 0, the limiting system is the compressible Navier-Stokes system

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{1.6}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \Pi(\varrho) = \mu \Delta \mathbf{u} + (\mu + \nu) \nabla \operatorname{div} \mathbf{u}, \tag{1.7}$$

$$\partial_t(\varrho Y) + \operatorname{div}(\varrho Y \mathbf{u}) = d\Delta Y - k\varrho Y, \tag{1.8}$$

where $\Pi(\varrho) = \varrho^\gamma + \varrho$.

The purpose of this article is to give a rigorous proof of the above formal process for the well-prepared initial data case.

The remainder of this article is arranged as follows. In section 2 we define the weak solutions to the primitive system (1.1)–(1.4) and state our main results. In section 3 we give the proof of it.

2. Main results

We now introduce the notion of weak solution of the system (1.1)-1.4.

Definition 2.1. We say that a quantity $\{\varrho, \mathbf{u}, Y\}$ is a weak solution of the flow of a gaseous mixture (1.1)-1.3 supplemented with the initial data $\{\varrho_0, \mathbf{u}_0, Y_0\}$ provided that the following hold.

- $\varrho \in L^\infty(0,T;(L^\gamma + L^2)(\Omega))$, the velocity field $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, $\varrho|\mathbf{u}|^2 \in L^\infty(0,T;L^1(\Omega))$.
- The density $\varrho \in C_{\rm weak}([0,T];L^1(\Omega))$ and the equation of continuity (1.1) holds as a family of integral identities

$$\int_\Omega \Big[\varrho(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_0\varphi(0,\cdot)\Big]\,\mathrm{d}x = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{2.1}$$

for any $\tau \in [0,T]$ and any test function $\varphi \in C_c^\infty([0,T]\times\Omega)$.

- The balance of momentum holds in distributional sense, namely

$$\begin{aligned}
&\int_\Omega \varrho\mathbf{u}\cdot\vec{\varphi}(\tau,\cdot)\,\mathrm{d}x - \int_\Omega (\varrho\mathbf{u})_0\cdot\vec{\varphi}(0,\cdot)\,\mathrm{d}x \\
&= \int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\vec{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\vec{\varphi} + \varrho^\gamma\ \operatorname{div}\vec{\varphi} \\
&\quad - \mu\nabla\mathbf{u} : \nabla\vec{\varphi} - (\mu+\nu)\operatorname{div}\mathbf{u}\operatorname{div}\vec{\varphi} - \varrho\nabla G\cdot\vec{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{2.2}$$

for any test function $\vec{\varphi} \in \mathcal{D}([0,T);\mathbb{R}^3))$.

- The total energy of the system holds,

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\gamma-1}\varrho^\gamma + \frac{\epsilon^2}{2}|\nabla G|^2 + (G-1)\exp G\Big)(t,\cdot)\,\mathrm{d}x \\
&+ \int_0^\tau \int_\Omega \mu|\nabla\mathbf{u}|^2 + (\mu+\nu)(\operatorname{div}\mathbf{u})^2\,\mathrm{d}x\,\mathrm{d}t \le E_{0,\epsilon}
\end{aligned} \tag{2.3}$$

holds for a.e. $\tau \in (0,T)$ where

$$E_{0,\epsilon} = \int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{\gamma-1}\varrho_0^\gamma + \frac{\epsilon^2}{2}|\nabla G_0|^2 + (G-1)\exp G\Big)\,\mathrm{d}x\,.$$

- The reactant mass fraction Y is a bounded measurable function on $(0,T)\times\Omega$,

$$0 \le Y(t,x) \le 1 \text{ for a.e. } t\in(0,T),\ x\in\Omega, \tag{2.4}$$

and the integral identity

$$\begin{aligned}
&-\int_0^T \int_\Omega \Big(\varrho Y\partial_t\varphi + \varrho Y\mathbf{u}\cdot\nabla_x\varphi - d\nabla_x Y\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \kappa\int_0^T \int_\Omega \varrho Y\varphi\,\mathrm{d}x\ \mathrm{d}t + \int_\Omega \varrho_0 Y_0\varphi(0,\cdot)\,\mathrm{d}x,
\end{aligned} \tag{2.5}$$

to be satisfied for any test function $\varphi \in \mathcal{D}([0,T)\times\Omega)$, together with

$$\begin{aligned}
&-\int_0^T \partial_t\psi \int_\Omega \varrho B(\mathbf{Y})\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \psi\int_\Omega d\underline{G}|\nabla_x Y|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_0^T \psi\int_\Omega \kappa\varrho\frac{\partial B(\mathbf{Y})}{\partial Y}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \varrho_0 B(\mathbf{Y}_0)\psi(0)\,\mathrm{d}x
\end{aligned} \tag{2.6}$$

for any $\psi \in \mathcal{D}[0,T)$, $\psi \ge 0$, and any convex $B \in C^2(\mathbb{R}^3)$,

$$\underline{B} = \inf_{Y\in\mathbb{R}} B''(Y)\,.$$

• The chemical energy inequality

$$\frac{1}{2}\int_\Omega \varrho Y^2 \,\mathrm{d}x + \kappa\int_0^T\int_\Omega \varrho Y^2 \,\mathrm{d}x\,\mathrm{d}t + d\int_0^T\int_\Omega |\nabla Y|^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}\int_\Omega \varrho_0 Y_0^2 \,\mathrm{d}x \tag{2.7}$$

is satisfied in $\mathcal{D}'(0,T)$.

• Equation (1.4) holds in $\mathcal{D}'((0,\infty)\times\Omega)$.

Remark 2.2. The existence of weak solutions in $(0,T)\times\Omega$ to the flow of a chemical reacting gaseous mixture (1.1)–(1.4) can be established by slightly modifying the arguments in [7, 9]. Since we are mainly interested int the quasi-neutral limit, we omit the details on existence theory here.

Before stating our main results, we recall the local existence of smooth solutions to the problem (1.6)–(1.8). Since the system (1.6)–(1.8) is a parabolic-hyperbolic one, the results in [23] imply that

Proposition 2.3 ([23]). *Let $s > 7/2$ be an integer and assume that the initial data $(\varrho_0, Y_0, \mathbf{u}_0)$ satisfy*

$$\varrho_0, Y_0, \mathbf{u}_0 \in H^{s+2}(\Omega), \quad 0 < \bar\rho \le \rho_0(x), \tag{2.8}$$

for a positive constant $\bar\rho$. Then there exist positive constants T_* (the maximal time interval, $0 < T_* \le +\infty$), and $\hat\rho$, such that (1.6)–(1.8) with initial data $(\varrho, Y, \mathbf{u})|_{t=0} = (\varrho_0, Y_0, \mathbf{u}_0)$ has a unique classical solution $(\rho, Y, \mathbf{u})$ satisfying

$$\rho \in C^l([0,T_*), H^{s+2-l}(\Omega)), \quad \mathbf{u}, Y \in C^l([0,T_*), H^{s+2-2l}(\Omega)),\ l = 0,1;$$
$$0 < \hat\rho \le \rho(x,t).$$

Now we state our main results.

Theorem 2.4. *Let$(\varrho_\epsilon, \mathbf{u}_\epsilon, Y_\epsilon, G_\epsilon)$ the global weak solution of (1.1)–(1.4) with the initial data $(\varrho_{0,\epsilon}, \mathbf{u}_{0,\epsilon}, Y_{0,\epsilon}, G_{0,\epsilon})$. Assume that $(\varrho_{0,\epsilon}, \mathbf{u}_{0,\epsilon}, Y_{0,\epsilon}, G_{0,\epsilon})$ satisfy*

$$\int_\Omega |\varrho_{0,\epsilon} - \varrho_0|^2 \,\mathrm{d}x + \int_\Omega |Y_{0,\epsilon} - Y_0|^2 \,\mathrm{d}x + \int_\Omega |\exp(G_{0,\epsilon}) - \varrho_0|^2 \,\mathrm{d}x \le C\epsilon, \tag{2.9}$$

$$\|\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon} - \sqrt{\varrho_0}\mathbf{u}_{0,\epsilon}\|^2_{L^2(\Omega)} \le C\epsilon, \tag{2.10}$$

$$\epsilon\|\nabla G_{0,\epsilon}\|^2_{L^2(\Omega)} \le C, \tag{2.11}$$

where $(\varrho_0, Y_0, \mathbf{u}_0)$ satisfy the conditions (2.8). Then, for $0 < T < T^*$ (defined in Proposition 2.3), one has

$$\|\varrho_\epsilon - \varrho\|_{L^\infty(0,T;(L^2+L^\gamma))(\Omega)} \le C\sqrt{\epsilon}, \tag{2.12}$$

$$\|Y_\epsilon - Y\|_{L^\infty(0,T;L^2(\Omega))} \le C\sqrt{\epsilon}, \tag{2.13}$$

$$\|\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon - \sqrt{\varrho}\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))} \le C\sqrt{\epsilon}, \tag{2.14}$$

$$\|\sqrt{\exp(V_\epsilon)} - \sqrt{\varrho}\|_{L^\infty(0,T;L^2(\Omega))} \le C\sqrt{\epsilon}. \tag{2.15}$$

Here $(\varrho, Y, \mathbf{u})$ is the solution to the system (1.6)–(1.8) constructed in Proposition 2.3.

3. Proof of Theorem 2.4

In this section we give a rigorous proof of Theorem 2.4 by applying and modifying the relative entropy method. The main difficulty here is that the target system is the flow of a chemical reacting gaseous mixture. Thus, we need to pay more an attention on construct the modified energy inequality and deal with the remainders.

Step I. Let us set

$$h(\varrho_\epsilon)=\frac{1}{\gamma-1}(\varrho_\epsilon^\gamma-\varrho^\gamma-\gamma\varrho^{\gamma-1}(\varrho_\epsilon-\varrho)),$$

and define the relative entropy:

$$\begin{aligned}\mathcal{E}_\epsilon(\tau)=\int_\Omega\Big(&\frac{1}{2}\varrho_\epsilon|\mathbf{u}_\epsilon-\mathbf{u}|^2+h(\varrho_\epsilon)+\frac{1}{2}\varrho_\epsilon|Y_\epsilon-Y|^2+\frac{1}{2}\epsilon^2|\nabla G_\epsilon|^2\\&+m_\epsilon\ln\Big(\frac{m_\epsilon}{\varrho}\Big)-m_\epsilon+\varrho\Big)\,\mathrm{d}x,\end{aligned}\tag{3.1}$$

with $m_\epsilon=\exp G_\epsilon$ where $(\varrho_\epsilon,Y_\epsilon,\mathbf{u}_\epsilon,G_\epsilon)$ is a solution of (1.1)-1.4 and $(\varrho,Y,\mathbf{u})$ is a solution of (1.6)-1.8.

We remark that in (3.1), we have used

$$\int_\Omega(G_\epsilon-1)\exp G_\epsilon\,\mathrm{d}x=\int_\Omega(m_\epsilon\ln m_\epsilon-m_\epsilon)\,\mathrm{d}x$$

and

$$\begin{aligned}\int_\Omega m_\epsilon\ln(1/\varrho)\,\mathrm{d}x&=\int_\Omega m_{0,\epsilon}\ln(1/\varrho_0)\,\mathrm{d}x-\int_0^\tau\int_\Omega m_\epsilon\partial_t\ln\varrho\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\epsilon\int_0^\tau\int_\Omega\partial_t\Delta G_\epsilon\ln(1/\varrho)\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega\varrho_\epsilon\mathbf{u}_\epsilon\cdot\nabla\ln\varrho\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{3.2}$$

where we have used (1.3).

Let us observe that

$$\begin{aligned}&\int_0^\tau\int_\Omega\varrho_\epsilon(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\nabla\ln\varrho\,\mathrm{d}x\,\mathrm{d}t\\&=\int_0^\tau\int_\Omega m_\epsilon\mathbf{u}\cdot\nabla\ln\varrho-\epsilon\int_0^\tau\int_\Omega\Delta G_\epsilon\mathbf{u}\cdot\nabla\ln\varrho\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^\tau\int_\Omega\varrho_\epsilon\mathbf{u}_\epsilon\cdot\nabla\ln\varrho\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.3}$$

Using (3.3), (3.2) we have

$$\begin{aligned}&\int_\Omega m_\epsilon\ln(1/\varrho)\,\mathrm{d}x\\&=\int_\Omega m_{0,\epsilon}\ln(1/\varrho_0)\,\mathrm{d}x-\int_0^\tau\int_\Omega m_\epsilon\partial_t\ln\varrho\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\epsilon\int_0^\tau\int_\Omega\partial_t\Delta G_\epsilon\ln(1/\varrho)\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega\varrho_\epsilon(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\nabla\ln\varrho\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\int_0^\tau\int_\Omega m_\epsilon\mathbf{u}\cdot\nabla\ln\varrho+\epsilon\int_0^\tau\int_\Omega\Delta G_\epsilon\mathbf{u}\cdot\nabla\ln\varrho\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.4}$$

Taking $\frac{1}{2}|\mathbf{u}|^2$ and $p'(\varrho)$ with $p(\varrho)=\frac{1}{\gamma-1}\varrho^\gamma$ as a test function in (2.1), we obtain

$$\int_\Omega \frac{1}{2}\varrho_\epsilon|\mathbf{u}|^2\,\mathrm{d}x=\int_\Omega \frac{1}{2}\varrho_{0,\epsilon}|\mathbf{u}_0|^2\,\mathrm{d}x+\int_0^\tau\int_\Omega \Big(\varrho_\epsilon\mathbf{u}\cdot\partial_t\mathbf{u}+\varrho_\epsilon\mathbf{u}_\epsilon\cdot\nabla\mathbf{u}\cdot\mathbf{u}\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{3.5}$$

and

$$\int_\Omega \frac{1}{2}\varrho_\epsilon p'(\varrho)\,\mathrm{d}x=\int_\Omega \frac{1}{2}\varrho_{0,\epsilon}p'(\varrho_0)\,\mathrm{d}x+\int_0^\tau\int_\Omega \Big(\varrho_\epsilon\partial_t p'(\varrho)+\varrho_\epsilon\mathbf{u}_\epsilon\cdot\nabla p'(\varrho)\Big)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.6}$$

We choose $\mathbf{u}$ as a test function to the moment equation (1.2); it provides

$$\begin{aligned}
&-\int_\Omega(\varrho_\epsilon\mathbf{u}_\epsilon\cdot\mathbf{u})(\tau)\,\mathrm{d}x\\
&=-\int_\Omega(\varrho_{0,\epsilon}\mathbf{u}_{0,\epsilon})\cdot\mathbf{u}_0\,\mathrm{d}x-\int_0^\tau\int_\Omega\varrho_\epsilon\mathbf{u}_\epsilon\cdot\partial_t\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega\Big(\varrho_\epsilon\mathbf{u}_\epsilon\otimes\mathbf{u}_\epsilon:\nabla\mathbf{u}+\varrho_\epsilon^\gamma\operatorname{div}\mathbf{u}-\mu\nabla\mathbf{u}_\epsilon:\nabla\mathbf{u}-(\mu+\nu)\operatorname{div}\mathbf{u}_\epsilon\operatorname{div}\mathbf{u}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega\Big(m_\epsilon\operatorname{div}\mathbf{u}-\epsilon^2\mathrm{D}\mathbf{u}:(\nabla G_\epsilon\otimes\nabla G_\epsilon)-\frac{\epsilon^2}{2}|\nabla G_\epsilon|^2\operatorname{div}\mathbf{u}\Big)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{3.7}$$

where $\mathrm{D}\mathbf{u}=\frac{1}{2}(\nabla\mathbf{u}+\nabla^T\mathbf{u})$ while the equation (1.4) together with using the integration by parts provides

$$\begin{aligned}
&-\int_0^\tau\int_\Omega\varrho_\epsilon\nabla G_\epsilon\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\\
&=-\int_0^\tau\int_\Omega(m_\epsilon-\epsilon^2\Delta G_\epsilon)\nabla G_\epsilon\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^\tau\int_\Omega m_\epsilon\operatorname{div}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t-\epsilon^2\int_0^\tau\int_\Omega\Big(\mathrm{D}\mathbf{u}:(\nabla G_\epsilon\otimes\nabla G_\epsilon)+\frac{1}{2}|\nabla G_\epsilon|^2\operatorname{div}\mathbf{u}\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We also obtain

$$p'(\varrho)\varrho-p(\varrho)=\varrho^\gamma\,.$$

Thus, we deduce, after adding (2.3), (2.7) (3.2), (3.3), (3.5), (3.6), and (3.7), the inequality

$$\begin{aligned}
&\int_\Omega\Big(\frac{1}{2}\varrho_\epsilon|\mathbf{u}_\epsilon-\mathbf{u}|^2+p(\varrho_\epsilon)-\varrho_\epsilon p'(\varrho)+\frac{1}{2}\varrho_\epsilon|Y_\epsilon-Y|^2+\frac{1}{2}\epsilon^2|\nabla G_\epsilon|^2\\
&\quad+m_\epsilon\ln\Big(\frac{m_\epsilon}{\varrho}\Big)-m_\epsilon+\varrho\Big)\,\mathrm{d}x\\
&\quad+\int_0^\tau\int_\Omega\Big(\mu|\nabla\mathbf{u}_\epsilon-\nabla\mathbf{u}|^2+(\mu+\nu)|\operatorname{div}\mathbf{u}_\epsilon-\operatorname{div}\mathbf{u}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\le\int_\Omega\Big(\frac{1}{2}\varrho_{0,\epsilon}|\mathbf{u}_{0,\epsilon}-\mathbf{u}_0|^2+p(\varrho_{0,\epsilon})-\varrho_{0,\epsilon}p'(\varrho_0)+\frac{1}{2}\epsilon^2|\nabla G_{0,\epsilon}|^2\\
&\quad+m_{0,\epsilon}\ln\Big(\frac{m_{0,\epsilon}}{\varrho_0}\Big)-m_{0,\epsilon}+\varrho_0\Big)\,\mathrm{d}x\\
&\quad+\int_0^\tau\int_\Omega\Big(\varrho_\epsilon(\partial_t\mathbf{u}+\mathbf{u}_\epsilon\cdot\nabla\mathbf{u})\cdot(\mathbf{u}-\mathbf{u}_\epsilon)\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\mu\int_0^\tau\int_\Omega\nabla\mathbf{u}:\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\,\mathrm{d}x\,\mathrm{d}t+(\mu+\nu)\int_0^\tau\int_\Omega\operatorname{div}\mathbf{u}(\operatorname{div}\mathbf{u}-\operatorname{div}\mathbf{u}_\epsilon)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^\tau \int_\Omega (\varrho_\epsilon \partial_t p(\varrho) + \varrho_\epsilon \mathbf{u}_\epsilon \cdot \nabla p'(\varrho)) \,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_\Omega \varrho_\epsilon^\gamma \operatorname{div} \mathbf{u} \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau \int_\Omega m_\epsilon \partial_t \ln \varrho \,\mathrm{d}x\,\mathrm{d}t + \epsilon \int_0^\tau \int_\Omega \partial_t \Delta G_\epsilon \ln(1/\varrho) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \varrho_\epsilon (\mathbf{u} - \mathbf{u}_\epsilon) \cdot \nabla \ln \varrho \,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_\Omega m_\epsilon \mathbf{u} \cdot \nabla \ln \varrho \,\mathrm{d}x\,\mathrm{d}t \\
&+ \epsilon \int_0^\tau \int_\Omega \Delta G_\epsilon \mathbf{u} \cdot \nabla \ln \varrho \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \Big(m_\epsilon \operatorname{div} \mathbf{u} - \epsilon^2 \mathrm{D}\mathbf{u} : (\nabla G_\epsilon \otimes \nabla G_\epsilon) - \frac{\epsilon^2}{2} |\nabla G_\epsilon|^2 \operatorname{div} \mathbf{u} \Big) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{2} \int_\Omega \varrho_\epsilon Y^2 \,\mathrm{d}x + \frac{1}{2} \int_\Omega \varrho_{0,\epsilon} Y_{0,\epsilon}^2 \,\mathrm{d}x - \int_\Omega \varrho_\epsilon Y_\epsilon Y \,\mathrm{d}x \\
&- d \int_0^\tau \int_\Omega |\nabla Y_\epsilon|^2 \,\mathrm{d}x\,\mathrm{d}t - k \int_0^\tau \int_\Omega \varrho_\epsilon Y_\epsilon \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.8}
$$

Note that

$$
\begin{aligned}
\int_\Omega \varrho^\gamma \,\mathrm{d}x - \int_\Omega \varrho_0^\gamma \,\mathrm{d}x &= \int_0^\tau \int_\Omega \partial_t \varrho^\gamma \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_\Omega \Big(\varrho \partial_t p'(\varrho) + \varrho \nabla p'(\varrho) \cdot \mathbf{u} + \varrho^\gamma \operatorname{div} \mathbf{u} \Big) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.9}
$$

Using (3.9), the relative entropy in (3.8) can be written as

$$
\Big[\mathcal{E}_\epsilon(t) \Big]_0^\tau + \int_0^\tau \int_\Omega \Big(\mu |\nabla \mathbf{u}_\epsilon - \nabla \mathbf{u}|^2 + (\mu + \nu) |\operatorname{div} \mathbf{u}_\epsilon - \operatorname{div} \mathbf{u}|^2 \Big) \,\mathrm{d}x\,\mathrm{d}t \le \sum_{j=1}^8 A_\epsilon^j, \tag{3.10}
$$

where

$$
\begin{aligned}
A_\epsilon^1 &= \int_0^\tau \int_\Omega \Big(\varrho_\epsilon \Big(\partial_t \mathbf{u} + \mathbf{u}_\epsilon \cdot \nabla \mathbf{u} + \nabla \ln \varrho \Big) \cdot (\mathbf{u} - \mathbf{u}_\epsilon) \Big) \,\mathrm{d}x\,\mathrm{d}t \\
A_\epsilon^2 &= \mu \int_0^\tau \int_\Omega \nabla \mathbf{u} : \nabla (\mathbf{u} - \mathbf{u}_\epsilon) \,\mathrm{d}x\,\mathrm{d}t + (\mu + \nu) \int_0^\tau \int_\Omega \operatorname{div} \mathbf{u} (\operatorname{div} \mathbf{u} - \operatorname{div} \mathbf{u}_\epsilon) \,\mathrm{d}x\,\mathrm{d}t \\
A_\epsilon^3 &= \int_0^\tau \int_\Omega \Big((\varrho - \varrho_\epsilon) \partial_t p'(\varrho) + \nabla p'(\varrho) \cdot (\varrho \mathbf{u} - \varrho_\epsilon \mathbf{u}_\epsilon) - \operatorname{div} \mathbf{u} (\varrho_\epsilon^\gamma - \varrho^\gamma) \Big) \,\mathrm{d}x\,\mathrm{d}t \\
A_\epsilon^4 &= - \int_0^\tau \int_\Omega m_\epsilon (\partial_t \ln \varrho + \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla \ln \varrho) \,\mathrm{d}x\,\mathrm{d}t \\
A_\epsilon^5 &= \epsilon \int_0^\tau \int_\Omega \partial_t \Delta G_\epsilon \ln(1/\varrho) \,\mathrm{d}x\,\mathrm{d}t + \epsilon \int_0^\tau \int_\Omega \Delta G_\epsilon \mathbf{u} \cdot \nabla \ln \varrho \,\mathrm{d}x\,\mathrm{d}t \\
A_\epsilon^6 &= \int_0^\tau \int_\Omega \Big(\epsilon^2 \mathrm{D}\mathbf{u} : (\nabla G_\epsilon \otimes \nabla G_\epsilon) + \frac{\epsilon^2}{2} |\nabla G_\epsilon|^2 \operatorname{div} \mathbf{u} \Big) \,\mathrm{d}x\,\mathrm{d}t \\
A_\epsilon^7 &= \frac{1}{2} \int_\Omega \varrho_\epsilon Y^2 \,\mathrm{d}x + \frac{1}{2} \int_\Omega \varrho_{0,\epsilon} Y_{0,\epsilon}^2 \,\mathrm{d}x - \int_\Omega \varrho_\epsilon Y_\epsilon Y \,\mathrm{d}x \\
A_\epsilon^8 &= -d \int_0^\tau \int_\Omega |\nabla Y_\epsilon|^2 \,\mathrm{d}x\,\mathrm{d}t - k \int_0^\tau \int_\Omega \varrho_\epsilon Y_\epsilon \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Step II. We introduce a result of the convex function h as follows:

$$h(\varrho_\epsilon) \geq \begin{cases} C(K)(|\varrho_\epsilon - \varrho|^2), & \text{if } \varrho_\epsilon \in K \\ C(K)(1+\varrho_\epsilon^\gamma), & \text{if } \varrho_\epsilon \in (0,\infty)\setminus K \end{cases} \tag{3.11}$$

for any compact subset $K \subset (0,\infty)$ and some $C(K) > 0$. The following notation will be used later:

$$[h]_{\rm ess} = h 1_{\varrho/2<\varrho_\epsilon<2\varrho}, \quad h = [h]_{\rm ess} + [h]_{\rm res}.$$

We first compute the residue and essential part which will be used later. From Holler's inequality, we obtain

$$\begin{aligned} \Big|\int_\Omega \big[\frac{\varrho_\epsilon - \varrho}{\varrho}\big]_{\rm ess} H(\mathbf{u})\cdot(\mathbf{u}_\epsilon - \mathbf{u})\,\mathrm{d}x\Big| &\leq \|[\varrho_\epsilon - \varrho]_{\rm ess}\|_{L^2(\Omega)} \|\frac{H(\mathbf{u})}{\varrho}\|_{L^3(\Omega)} \|\mathbf{u}_\epsilon - \mathbf{u}\|_{L^6(\Omega)} \\ &\leq C(\theta)\mathcal{E}_\epsilon(\tau) + \theta\|\mathbf{u}_\epsilon - \mathbf{u}\|^2_{L^6(\Omega)} \end{aligned} \tag{3.12}$$

where $H(\mathbf{u}) = \mu\Delta\mathbf{u} + (\mu+\nu)\nabla \operatorname{div}\mathbf{u}$. We also obtain

$$\begin{aligned} &\Big|\int_\Omega \big[\frac{\varrho_\epsilon - \varrho}{\varrho}\big]_{\rm res} H(\mathbf{u})\cdot(\mathbf{u}_\epsilon - \mathbf{u})\,\mathrm{d}x\Big| \\ &\leq \|[\varrho_\epsilon^{\gamma/2}]_{\rm res}\|_{L^2(\Omega)} \|\frac{H(\mathbf{u})}{\varrho}\|_{L^3(\Omega)} \|\sqrt{\varrho_\epsilon}(\mathbf{u}_\epsilon - \mathbf{u})\|_{L^2(\Omega)} \\ &\quad + \|[1]_{\rm res}\|_{L^2(\Omega)} \|H(\mathbf{u})\|_{L^3(\Omega)} \|\mathbf{u}_\epsilon - \mathbf{u}\|_{L^6(\Omega)} \\ &\leq C(\theta)\mathcal{E}_\epsilon(\tau) + \theta\|\mathbf{u}_\epsilon - \mathbf{u}\|^2_{L^6(\Omega)}. \end{aligned} \tag{3.13}$$

We next control the velocity term in A^1_ϵ and the first term can be written as

$$\begin{aligned} &\int_0^\tau \int_\Omega \varrho_\epsilon(\partial_t \mathbf{u} + \mathbf{u}_\epsilon\cdot\nabla\mathbf{u} + \nabla\ln\varrho)(\mathbf{u} - \mathbf{u}_\epsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^\tau \int_\Omega \varrho_\epsilon(\mathbf{u} - \mathbf{u}_\epsilon)\otimes(\mathbf{u} - \mathbf{u}_\epsilon) : \nabla\mathbf{u}\ \mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega \varrho_\epsilon(\mathbf{u} - \mathbf{u}_\epsilon)\cdot(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\ln\varrho)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C\int_0^\tau \mathcal{E}_\epsilon(t)\,\mathrm{d}t + \int_0^\tau \int_\Omega \varrho_\epsilon(\mathbf{u} - \mathbf{u}_\epsilon)\cdot(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\ln\varrho)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.14}$$

Using (3.12) and (3.13), we rewrite (3.14) as

$$\begin{aligned} &\int_0^\tau \int_\Omega \varrho_\epsilon(\mathbf{u} - \mathbf{u}_\epsilon)\cdot(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\ln\varrho)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^\tau \int_\Omega \varrho_\epsilon(\mathbf{u}_\epsilon - \mathbf{u})\cdot\nabla p'(\varrho)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega \frac{\varrho_\epsilon - \varrho}{\varrho}(\mathbf{u} - \mathbf{u}_\epsilon)\cdot\Big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla\operatorname{div}\mathbf{u}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega (\mathbf{u} - \mathbf{u}_\epsilon)\cdot\Big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla\operatorname{div}\mathbf{u}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^\tau \int_\Omega \varrho_\epsilon(\mathbf{u}_\epsilon - \mathbf{u})\cdot\nabla p'(\varrho)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega \Big[\frac{\varrho_\epsilon - \varrho}{\varrho}(\mathbf{u} - \mathbf{u}_\epsilon)\cdot\Big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla\operatorname{div}\mathbf{u}\Big)\Big]_{\rm ess}\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\int_0^\tau\int_\Omega\Big[\frac{\varrho_\epsilon-\varrho}{\varrho}(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\Big(\mu\Delta\mathbf{u}+(\mu+\nu)\nabla\operatorname{div}\mathbf{u}\Big)\Big]_{\rm res}\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^\tau\int_\Omega(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\Big(\mu\Delta\mathbf{u}+(\mu+\nu)\nabla\operatorname{div}\mathbf{u}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\le C(\theta)\int_0^\tau\mathcal{E}_\epsilon(t)\,\mathrm{d}t+\theta\int_0^\tau\|\mathbf{u}_\epsilon-\mathbf{u}\|^2_{L^6(\Omega)}\,\mathrm{d}t\\
&+\int_0^\tau\int_\Omega\mu\nabla(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla\mathbf{u}+(\mu+\nu)\operatorname{div}(\mathbf{u}_\epsilon-\mathbf{u})\operatorname{div}\mathbf{u}+\varrho_\epsilon(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla p'(\varrho)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

where we have here used Höllder's inequality, integration by parts, and the property in (3.11). Thus, we obtain

$$
\begin{aligned}
&\int_0^\tau\int_\Omega\varrho_\epsilon(\partial_t\mathbf{u}+\mathbf{u}_\epsilon\cdot\nabla\mathbf{u}+\nabla\ln\varrho)(\mathbf{u}-\mathbf{u}_\epsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\le\int_0^\tau\int_\Omega\mu\nabla(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla\mathbf{u}+(\mu+\nu)\operatorname{div}(\mathbf{u}_\epsilon-\mathbf{u})\operatorname{div}\mathbf{u}+\varrho_\epsilon(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla p'(\varrho)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\theta\int_0^\tau\|\mathbf{u}_\epsilon-\mathbf{u}\|^2_{L^6(\Omega)}\,\mathrm{d}t+C(\theta)\int_0^\tau\mathcal{E}_\epsilon(t)\,\mathrm{d}t,
\end{aligned}\tag{3.15}
$$

which implies

$$
\begin{aligned}
A^1_\epsilon&\le\int_0^\tau\int_\Omega\mu\nabla(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla\mathbf{u}+(\mu+\nu)\operatorname{div}(\mathbf{u}_\epsilon-\mathbf{u})\operatorname{div}\mathbf{u}+\varrho_\epsilon(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla p'(\varrho)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\theta\int_0^\tau\|\nabla(\mathbf{u}_\epsilon-\mathbf{u})\|^2_{L^2(\Omega)}\,\mathrm{d}t+C(\theta)\int_0^\tau\mathcal{E}_\epsilon(t)\,\mathrm{d}t
\end{aligned}\tag{3.16}
$$

where we have here used the Poincare's inequality.

From (3.16), we obtain

$$
\begin{aligned}
A^1_\epsilon+A^2_\epsilon+A^3_\epsilon&\le\theta\|\nabla(\mathbf{u}_\epsilon-\mathbf{u})\|^2_{L^2(\Omega;\mathbb{R}^3)}+C(\theta)\int_0^\tau\mathcal{E}_\epsilon(t)\,\mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega\operatorname{div}\mathbf{u}\Big(\varrho_\epsilon^\gamma-\varrho^\gamma-\gamma\varrho^{\gamma-1}(\varrho_\epsilon-\varrho)\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\le\theta\|\nabla(\mathbf{u}_\epsilon-\mathbf{u})\|^2_{L^2(\Omega;\mathbb{R}^3)}+C(\theta)\int_0^\tau\mathcal{E}_\epsilon(t)\,\mathrm{d}t
\end{aligned}\tag{3.17}
$$

while

$$
\int_0^\tau\int_\Omega(\varrho-\varrho_\epsilon)(\partial_tp'(\varrho)+\nabla p'(\varrho)\cdot\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t=-\int_0^\tau\int_\Omega\operatorname{div}\mathbf{u}(\varrho_\epsilon-\varrho)\gamma\varrho^{\gamma-1}\,\mathrm{d}x\,\mathrm{d}t.
$$

Step III. In the continuity equation (1.6), dividing by ϱ gives

$$
\partial_t\ln\varrho+\operatorname{div}\mathbf{u}+\mathbf{u}\cdot\nabla\ln\varrho=0,
$$

which implies $A^4_\epsilon=0$.

The terms of A^5_ϵ, A^6_ϵ, can be estimated as follows:

$$
\begin{aligned}
A^5_\epsilon\le&C\epsilon\|\epsilon\nabla G_\epsilon\|_{L^\infty(0,T;L^2(\Omega))}\Big(\|\nabla(\mathbf{u}\cdot\nabla\ln\varrho)\|_{L^\infty(0,T;L^2(\Omega))}\\
&+\|\ln\varrho\|_{W^{1,\infty}(0,T;H^1(\Omega))}\Big)
\end{aligned}\tag{3.18}
$$

and

$$A_\epsilon^6 \le C\epsilon \|\epsilon\nabla G_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega))}\|\nabla \mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega))}. \tag{3.19}$$

Step IV. Finally, it remains to estimate $A_\epsilon^7, A_\epsilon^8$. Using Y as a test function to the equation (2.5), we obtain the following weak formulation of the equation (2.5),

$$\begin{aligned} &-\int_\Omega \varrho_\epsilon Y_\epsilon Y \,\mathrm{d}x \\ &= -\int_\Omega \varrho_{0,\epsilon} Y_{0,\epsilon} Y_0 \,\mathrm{d}x - \int_0^\tau \int_\Omega \varrho_\epsilon Y_\epsilon \partial_t Y \,\mathrm{d}x\,\mathrm{d}t + d\int_0^\tau \int_\Omega \nabla Y_\epsilon \cdot \nabla Y \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \varrho_\epsilon Y_\epsilon \mathbf{u}_\epsilon \cdot \nabla Y \,\mathrm{d}x\,\mathrm{d}t + \kappa \int_0^\tau \int_\Omega \varrho_\epsilon Y_\epsilon Y \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.20}$$

We also use $\frac{1}{2}|Y|^2$ as a test function to the continuity equation (1.1) to deduce that

$$\begin{aligned} \frac{1}{2}\int_\Omega \varrho_\epsilon |Y|^2 \,\mathrm{d}x = \frac{1}{2}\int_\Omega \varrho_{0,\epsilon}|Y_0|^2 \,\mathrm{d}x + \int_0^\tau \int_\Omega \varrho_\epsilon Y \partial_t Y \,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^\tau \int_\Omega \varrho_\epsilon \mathbf{u}_\epsilon \cdot Y \nabla Y \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.21}$$

Adding the above two equations (3.7) and (3.21) and using (2.7), the sum of A_ϵ^7 and A_ϵ^8 yield that

$$A_\epsilon^7 + A_\epsilon^8 \le \sum_{j=1}^3 B_\epsilon^j \tag{3.22}$$

where equation (1.8) provides

$$\begin{gathered} B_\epsilon^1 = \frac{1}{2}\int_\Omega \varrho_{0,\epsilon}(Y_{0,\epsilon} - Y_0)^2 \,\mathrm{d}x \\ B_\epsilon^2 = -d\int_0^\tau \int_\Omega \Big(|\nabla Y_\epsilon|^2 - \nabla Y_\epsilon \cdot \nabla Y + \frac{\varrho_\epsilon}{\varrho}(Y_\epsilon - Y)\Delta Y\Big) \,\mathrm{d}x\,\mathrm{d}t \\ B_\epsilon^3 = -\kappa \int_0^\tau \int_\Omega \varrho_\epsilon (Y_\epsilon - Y)^2 \,\mathrm{d}x\,\mathrm{d}t. \end{gathered}$$

It is easily seen to show that

$$B_\epsilon^1 + B_\epsilon^3 \le \mathcal{E}_\epsilon(0) + C\int_0^\tau \mathcal{E}_\epsilon(t)\,\mathrm{d}t.$$

For the term of B_ϵ^2, we have to estimate

$$\int_0^\tau \int_\Omega \frac{\varrho_\epsilon}{\varrho}(Y_\epsilon - Y)\Delta Y \,\mathrm{d}x\,\mathrm{d}t$$

which can be divided as follows:

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \frac{\varrho_\epsilon}{\varrho}(Y_\epsilon - Y)\Delta Y \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_\Omega \frac{\varrho_\epsilon - \varrho}{\varrho}(Y_\epsilon - Y)\Delta Y \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega (Y_\epsilon - Y)\Delta Y \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_\Omega \Big[\frac{\varrho_\epsilon - \varrho}{\varrho}\Big]_{\rm ess}(Y_\epsilon - Y)\Delta Y \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega [\frac{\varrho_\epsilon - \varrho}{\varrho}]_{\rm res}(Y_\epsilon - Y)\Delta Y \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega (Y_\epsilon - Y)\Delta Y \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C(\theta)\int_0^\tau \mathcal{E}_\epsilon(t)\,\mathrm{d}t + \theta\|\nabla(Y_\epsilon - Y)\|^2_{L^1(0,T;L^2(\Omega))} + \int_0^\tau \int_\Omega (Y_\epsilon - Y)\Delta Y \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.23}
$$

while

$$
\begin{aligned}
&\int_0^\tau \int_\Omega [\frac{\varrho_\epsilon - \varrho}{\varrho}]_{\rm ess}(Y_\epsilon - Y)\Delta Y \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C\int_0^\tau \|\Delta Y\|_{L^3(\Omega)}\|[\varrho_\epsilon - \varrho]_{\rm ess}\|_{L^2(\Omega)}\|Y_\epsilon - Y\|_{L^6(\Omega)}\,\mathrm{d}t \\
&\leq C(\theta)\int_0^\tau \mathcal{E}_\epsilon(t)\,\mathrm{d}t + \theta\|Y_\epsilon - Y\|^2_{L^1(0,T;L^6(\Omega))} \\
&\leq C(\theta)\int_0^\tau \mathcal{E}_\epsilon(t)\,\mathrm{d}t + \theta\|\nabla(Y_\epsilon - Y)\|^2_{L^1(0,T;L^2(\Omega))}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^\tau \int_\Omega [\frac{\varrho_\epsilon - \varrho}{\varrho}]_{\rm res}(Y_\epsilon - Y)\Delta Y \,\mathrm{d}x\,\mathrm{d}t &\leq C\int_0^\tau \int_\Omega \Big([\varrho_\epsilon^\gamma]_{\rm res} + [1]_{\rm res}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\leq C\int_0^\tau \mathcal{E}_\epsilon(t)\,\mathrm{d}t.
\end{aligned}
$$

Using (3.23), we obtain

$$
B_\epsilon^2 \leq C(\theta)\int_0^\tau \mathcal{E}_\epsilon(t)\,\mathrm{d}t - (d-\theta)\int_0^\tau \int_\Omega |\nabla(Y_\epsilon - Y)|^2\,\mathrm{d}x\,\mathrm{d}t
$$

for small $\theta > 0$.

Summing up all estimates in Step II, Step III, and Step IV and taking a small suitable $\theta > 0$, we obtain the relative entropy

$$
\begin{aligned}
&\mathcal{E}_\epsilon(\tau) + C\int_0^\tau \int_\Omega \big(\mu|\nabla\mathbf{u}_\epsilon - \nabla\mathbf{u}|^2 + (\mu+\nu)|\operatorname{div}\mathbf{u}_\epsilon - \operatorname{div}\mathbf{u}|^2\big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ C\int_0^\tau \int_\Omega |\nabla Y_\epsilon - \nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t \leq C\mathcal{E}_\epsilon(0) + C\epsilon.
\end{aligned} \tag{3.24}
$$

Step V. Complete the proof. For the estimates of the initial data, we use the assumptions (2.9)-(2.11) to obtain

$$
\begin{aligned}
\int_\Omega \varrho_{0,\epsilon}|\mathbf{u}_{0,\epsilon} - \mathbf{u}_0|^2\,\mathrm{d}x &\leq C\int_\Omega |\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon} - \mathbf{u}_0|^2\,\mathrm{d}x + C\int_\Omega |\sqrt{\varrho_{0,\epsilon}} - \sqrt{\varrho_0}|^2\,\mathrm{d}x \\
&\leq C\int_\Omega |\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon} - \mathbf{u}_0|^2\,\mathrm{d}x + C\int_\Omega |\varrho_{0,\epsilon} - \varrho_0|^2\,\mathrm{d}x \leq C\epsilon,
\end{aligned}
$$

$$\int_\Omega \varrho_{0,\epsilon}|Y_{0,\epsilon}-Y_0|^2\,\mathrm{d}x \le \int_\Omega (\varrho_{0,\epsilon}-\varrho_0)|Y_{0,\epsilon}-Y_0|^2\,\mathrm{d}x + \int_\Omega \varrho_0|Y_{0,\epsilon}-Y_0|^2\,\mathrm{d}x \le C\epsilon,$$

$$\int_\Omega h(\varrho_{0,\epsilon})\,\mathrm{d}x \le C\epsilon,$$

$$\int_\Omega \big[\exp V_{0,\epsilon}\ln\big(\exp V_{0,\epsilon}/\varrho_0\big) - \exp V_{0,\epsilon}+\varrho_0\big]\,\mathrm{d}x \le \int_\Omega (\exp V_{0,\epsilon}-\varrho_0)^2 \le C\epsilon.$$

Thus we obtain

$$\mathcal{E}_\epsilon(0) \le C\epsilon$$

and the Gronwall inequality gives the results (**??**)–(**??**) in Theorem 2.4. Indeed,

$$\int_\Omega |\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon - \sqrt{\varrho}\mathbf{u}|^2\,\mathrm{d}x \le C\int_\Omega \varrho_\epsilon|\mathbf{u}_\epsilon-\mathbf{u}|^2\,\mathrm{d}x + C\int_\Omega |\sqrt{\varrho_\epsilon}-\sqrt{\varrho}|^2\,\mathrm{d}x \le C\epsilon,$$

$$\int_\Omega \Big([\varrho_\epsilon-\varrho]^2_{\rm ess} + [\varrho_\epsilon-\varrho]^\gamma_{\rm res}\Big)\,\mathrm{d}x \le \int_\Omega h(\varrho_\epsilon)\,\mathrm{d}x \le C\epsilon,$$

$$\int_\Omega |\sqrt{\exp V_\epsilon} - \sqrt{\varrho}|^2\,\mathrm{d}x \le C\epsilon,$$

then we obtain

$$\begin{aligned}
&\int_\Omega |Y_\epsilon - Y|^2\,\mathrm{d}x \\
&\le C\int_\Omega (\varrho-\varrho_\epsilon)|Y_\epsilon-Y|^2\,\mathrm{d}x + \int_\Omega \varrho_\epsilon|Y_\epsilon-Y|^2\,\mathrm{d}x \\
&= C\int_\Omega [\varrho-\varrho_\epsilon]_{\rm ess}|Y_\epsilon-Y|^2\,\mathrm{d}x + C\int_\Omega [\varrho-\varrho_\epsilon]_{\rm res}|Y_\epsilon-Y|^2\,\mathrm{d}x + \int_\Omega \varrho_\epsilon|Y_\epsilon-Y|^2\,\mathrm{d}x \\
&\le C(\theta)\Big(\int_\Omega [\varrho-\varrho_\epsilon]^2_{\rm ess}\,\mathrm{d}x + \int_\Omega [\varrho-\varrho_\epsilon]^\gamma_{\rm ess}\,\mathrm{d}x\Big) + \theta\int_\Omega |Y_\epsilon-Y|^2\,\mathrm{d}x \\
&\quad + \int_\Omega \varrho_\epsilon|Y_\epsilon-Y|^2\,\mathrm{d}x
\end{aligned}$$

which implies

$$\int_\Omega |Y_\epsilon - Y|^2\,\mathrm{d}x \le C\epsilon$$

by taking small suitable number $\theta > 0$. Hence the proof of Theorem 2.4 is complete.

Acknowledgments. This research was supported by the research fund of Dong-A University.

References

[1] D. Bresch, B. Desjardins, B. Ducomet; *Quasi-neutral limit for a viscous capillary model of plasma.* Ann. Inst. H. Poincaré Anal. Non Linéaire, 22 (2005), no. 1, 1–9.

[2] L. Chen, D. Donatelli, P. Marcati; *Incompressible type limit analysis of a hydrodynamic model for charge-carrier transport* SIAM J. Math. Anal., 45 (2013), no. 3, 915–933.

[3] S. Cordier, E. Grenier; *Quasineutral limit of an Euler-Poisson system arising from plasma physics*, Comm. Partial Differential Equations, 25 (2000), no. 5-6, 1099–1113.

[4] D. Donatelli, E. Feireisl, A. Novotný; *Scale analysis of a hydrodynamic model of plasma.* Math. Models Methods Appl. Sci., 25 (2015), no. 2, 371–394.

[5] D. Donatelli, P. Marcati; *The quasineutral limit for the Navier-Stokes-Fourier-Poisson system*, in Hyperbolic conservation laws and related analysis with applications, 193-206, Springer Proc. Math. Stat., 49, Springer, Heidelberg, 2014.

[6] D. Donatelli, P. Marcati; *A quasineutral type limit for the Navier-Stokes-Poisson system with large data.* Nonlinearity 21 (2008), no. 1, 135–148.

[7] D. Donatelli, K. Trivisa; *A multidimensional model for the combustion of compressible fluids. Arch. Ration. Mech. Anal.*, **185**(3) (2007), 379–408.
[8] D. Donatelli, K. Trivisa; *From the dynamics of gaseous stars to the incompressible Euler equations*. J. Differential Equations, **245**(5) (2008), 1356–1389.
[9] D. Donatelli, K. Trivisa; *On the motion of a viscous compressible radiative-reacting gas* Comm. Math. Phys., **265**(2) (2006), 463–491.
[10] E. Feireisl, A. Novotný; *Singular Limits in Thermodynamics of Viscous Fluids*, Birkhauser-Verlag, Basel, 2009.
[11] E. Feireisl, P. Zhang; *Quasi-neutral limit for a model of viscous plasma*, Arch. Ration. Mech. Anal., 197 (2010), no. 1, 271–295.
[12] D. Gérard-Varet, D. Han-Kwan, F. Rousset; *Quasineutral limit of the Euler-Poisson system for ions in a domain with boundaries II*. J. Éc. polytech. Math., 1 (2014), 343–386.
[13] D. Gérard-Varet, D. Han-Kwan, F. Rousset; *Quasineutral limit of the Euler-Poisson system for ions in a domain with boundaries*. Indiana Univ. Math. J. 62 (2013), no. 2, 359–402.
[14] Q. Ju, F. Li, H. Li; *The quasineutral limit of compressible Navier-Stokes-Poisson system with heat conductivity and general initial data*. J. Differential Equations 247 (2009), no. 1, 203–224.
[15] Q. Ju, F. Li, S. Wang; *Convergence of the Navier-Stokes-Poisson system to the incompressible Navier-Stokes equations*. J. Math. Phys. 49 (2008), no. 7, 073515, 8 pp.
[16] Q. Ju, Y. Li, S. Wang; *Rate of convergence from the Navier-Stokes-Poisson system to the incompressible Euler equations*, J. Math. Phys. 50 (2009), no. 1, 013533, 12 pp.
[17] Q. Ju, Y. Li; *Asymptotic limits of the full Navier-Stokes-Fourier-Poisson system*, J. Differential Equations, 254 (2013), no. 6, 2587–2602.
[18] Y. Li, Q. Ju, W.-Q Xu; *Quasi-neutral limit of the full Navier-Stokes-Fourier-Poisson system*, J. Differential Equations, 258 (2015), no. 11, 3661–3687.
[19] Marion, Martine, R. Temam; *Global existence for fully nonlinear reaction-diffusion systems describing multicomponent reactive flows*, J. Math. Pures Appl., (9) 104 (2015), no. 1, 102–138.
[20] X. Pu, B. Guo; *Quasineutral limit of the pressureless Euler-Poisson equation for ions*, Quart. Appl. Math., 74 (2016), no. 2, 245–273.
[21] S. Wang; *Quasineutral limit of Euler-Poisson system with and without viscosity*, Comm. Partial Differential Equations, 29 (2004), no. 3-4, 419–456.
[22] S. Wang, S. Jiang; *The convergence of the Navier-Stokes-Poisson system to the incompressible Euler equations*, Comm. Partial Differential Equations, 31 (2006), no. 4-6, 571–591.
[23] A. I. Vol'pert, S. I. Hudjaev; *The Cauchy problem for composite systems of nonlinear differential equations*, Math. USSR SB. 16 (1972), 504–528.

Young-Sam Kwon
Department of Mathematics, Dong-A University, Busan 604-714, Korea
Email address: ykwon7210@gmail.com