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TRAVELLING SOLITARY WAVES FOR BOSON STARS

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Abstract. In this article, we study the pseudo-relativistic Hartree equation

$$i\partial_t\psi = (\sqrt{-\Delta+m^2}-m)\psi - (\frac{e^{-\mu|x|}}{4\pi|x|} * |\psi|^2)\psi, \quad \text{on } \mathbb{R}^3,$$

which describes the dynamics of pseudo-relativistic boson stars with rest mass $m > 0$ in the mean-field limit. Based on Ekeland variational principle, concentration-compactness lemma and Gagliardo-Nirenberg inequality, we prove existence of travelling solitary waves under the critical stellar mass. In addition to their existence, we obtain orbital stability by using a general idea presented in Cazenave and Lions [2].

1. Introduction

In this article, we consider the pseudo-relativistic Hartree equation

$$\begin{gathered} i\partial_t\psi = (\sqrt{-\Delta+m^2}-m)\psi - \Phi\psi, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ -\Delta\Phi + \mu^2\Phi = |\psi|^2, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \end{gathered} \tag{1.1}$$

where $\psi(x,t)$ and $\Phi(x,t)$ are complex-valued wave functions, m denotes the relativistic particle of mass, when $m \geq 0$ and when $\mu > 0$. It is straightforward to solve the second equation of (1.1) and obtain the expression

$$\Phi(x) = \frac{e^{-\mu|x|}}{4\pi|x|} * |\psi|^2 = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-\mu|x-y|}}{|x-y|}\psi(y)dy, \tag{1.2}$$

where the symbol $*$ stands for convolution of functions on $\mathbb{R}^3$. By substituting the expression of $\Phi(x)$ into the first equation of (1.1), we obtain the nonlocal nonlinear Schrödinger equation

$$i\partial_t\psi = (\sqrt{-\Delta+m^2}-m)\psi - (\frac{e^{-\mu|x|}}{4\pi|x|} * |\psi|^2)\psi, (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{1.3}$$

Recently, many authors have studied the pseudo-relativistic Hartree equation. In 2006, Elgart and Schlein [3] studied the nonlocal nonlinear Schrödinger equation

$$i\partial_t\psi = (\sqrt{-\Delta+m^2}-m)\psi - (\frac{1}{|x|} * |\psi|^2)\psi, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{1.4}$$

Equation (1.4) arises as an effective dynamical description for an N-body quantum system of relativistic bosons with two-body interaction given by Newtonian gravity.

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In 2007, Lenzmann [11] proved local and global well-posedness for equation (1.4) by using Kato's inequality, a priori estimates and conservation of charge and energy. Fröhlich and Lenzmann [6], Lenzmann and Lewin [8] obtained the existence of finite-time blow up solution. In particular, the idea of a mathematical model of pseudo-relativistic boson stars dates back to the works of Lieb and Thirring [15] and of Lieb and Yau [16], where the corresponding N-body Hamiltonian and its relation to the Hartree energy functional are discussed.

Because of the focusing nature of the nonlinearity in equation (1.4), there exist solitary wave solutions. Based on rearrangement inequalities and variational arguments, Lieb and Yau [16] proved the existence of ground state solitary wave solutions for equation (1.4). Lenzmann [10] obtained the uniqueness of ground states. In 2018, Guo and Zeng [7] proved existence of ground state solitary wave solutions, and presented a detailed analysis of the behavior of ground states.

In this article, we focus on the existence and properties of travelling solitary wave solutions for (1.3). More precisely, we consider solutions of the form

$$\psi(x,t) = e^{it\omega}\varphi_v(x - vt), \tag{1.5}$$

with $\omega \in \mathbb{R}$ and travelling velocity $v \in \mathbb{R}^3$ such that $|v| < 1$ (i.e., below the speed of light in our units). Substituting ansatz (1.5) in (1.3), we have

$$(\sqrt{-\Delta + m^2} - m)\varphi_v + i(v \cdot \nabla)\varphi_v - (\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_v|^2)\varphi_v = -\omega\varphi_v, \tag{1.6}$$

with $\omega \in \mathbb{R}$. Using the Ekeland variational principle, concentration-compactness lemma and Gagliardo-Nirenberg inequality, we obtain existence of travelling solitary wave solutions under the critical stellar mass.

We point out that, since equation (1.6) is not the Lorentz covariant, travelling solitary wave solutions can not be directly obtained from solitary waves at rest (that is, when $v = 0$ in (1.5)) and then applying a Lorentz boost. We also obtain the existence of ground state travelling solitary wave solutions for (1.6) under the critical stellar mass. Apart from the existence of travelling solitary wave solutions, we are also concerned with properties such as "orbital stability".

This article is organized as follows. In Section 2, we set up the variational structure for equation (1.6), and state our main theorems. In Section 3 and 4, we obtain existence of travelling solitary wave solutions for (1.6). In Section 5, we prove orbital stability of travelling solitary waves.

2. Preliminaries and statement of main results

In this section, we introduce some basic notation and lemmas which will be used in subsequent sections. Let $L^p(\mathbb{R}^3)$ denote the usual Lebesgue space for $p \geq 1$. We define the Fourier transform for $f \in S(\mathbb{R}^3)$ (i.e., Schwartz functions) by

$$(\mathcal{F}f)(k) = \hat{f}(k) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x)e^{-ikx}dx,$$

where $\mathcal{F}$ extends to $S'(\mathbb{R}^3)$ (i.e., the space of tempered distributions) by duality.

We introduce the operator $(1 - \Delta)^{1/2}$ via its multiplier $(1 + |k|^2)^{1/2}$ in Fourier space, i.e., we set $(1 - \Delta)^{1/2} f = \mathcal{F}^{-1}[(1 + |k|^2)^{1/2}\mathcal{F}f]$. Likewise, we define the operator $\sqrt{-\Delta + m^2}$ through its multiplier $\sqrt{|k|^2 + m^2}$ in the Fourier space.

We employ the Sobolev space $H^{1/2}(\mathbb{R}^3)$ of fractional order 1/2 defined by

$$H^{1/2}(\mathbb{R}^3) = \{f \in S'(\mathbb{R}^3) : (1-\Delta)^{1/4} f \in L^2(\mathbb{R}^3)\}, \tag{2.1}$$

and equipped with the norm $\|f\|_{H^{1/2}(\mathbb{R}^3)} = \|(1-\Delta)^{1/4} f\|_{L^2(\mathbb{R}^3)}$. In 2007, Lenzmann [11] investigated the local and global well posedness of the Cauchy problem for equation (1.3).

Lemma 2.1. *For any $\psi(x,0) = \psi_0(x) \in H^{1/2}(\mathbb{R}^3)$, there exists a unique global solution $\psi(x,t) \in C(\mathbb{R}, H^{1/2}(\mathbb{R}^3)) \cap C^1(\mathbb{R}, H^{-\frac{1}{2}}(\mathbb{R}^3))$, provided that*

$$\|\psi_0(x)\|_{L^2(\mathbb{R}^3)} \le \|Q(x)\|_{L^2(\mathbb{R}^3)},$$

where $Q(x) \in H^{1/2}(\mathbb{R}^3)$ is a strictly positive solution of

$$\sqrt{-\Delta} Q - (\frac{1}{4\pi|x|} * |Q|^2) Q = -Q. \tag{2.2}$$

We defined the charge $M(\psi(x,t))$ by

$$M(\psi(x,t)) = \int_{\mathbb{R}^3} |\psi(x,t)|^2 dx,$$

and the energy associated with (1.3) by

$$E(\psi(x,t)) = \frac{1}{2}\langle \psi, (\sqrt{-\Delta + m^2} - m)\psi\rangle - \frac{1}{4}\int_{\mathbb{R}^3} (\frac{e^{-\mu|x|}}{4\pi|x|} * |\psi|^2)|\psi|^2 dx. \tag{2.3}$$

Lenzmann [11] proved that the solution $\psi(x,t)$ obtained in Lemma 2.1 conserves both the charge $M(\psi(x,t))$ and the energy $E(\psi(x,t))$, i.e.,

$$M(\psi(x,t)) = M(\psi_0(x)) \quad \text{and} \quad E(\psi(x,t)) = E(\psi_0(x)). \tag{2.4}$$

For equation (1.6), we define the functional $E_v : H^{1/2}(\mathbb{R}^2) \to \mathbb{R}$ by

$$\begin{aligned} E_v(\varphi_v) :=& \frac{1}{2}\langle \varphi_v, (\sqrt{-\Delta + m^2} - m)\varphi_v\rangle + \frac{i}{2}\langle \varphi_v, (v\cdot\nabla)\varphi_v\rangle \\ &- \frac{1}{4}\int_{\mathbb{R}^3} (\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_v|^2)|\varphi_v|^2 dx, \end{aligned} \tag{2.5}$$

and the charge functional $\mathcal{N} : H^{1/2}(\mathbb{R}^2) \to \mathbb{R}$ by

$$\mathcal{N}(\varphi_v) := \int_{\mathbb{R}^3} |\varphi_v|^2 dx = \|\varphi_v\|^2_{L^2(\mathbb{R}^3)}. \tag{2.6}$$

It is straightforward to verify that $E_v \in C^1(H^{1/2}(\mathbb{R}^3), \mathbb{R})$ and $\mathcal{N} \in C^1(H^{1/2}(\mathbb{R}^3), \mathbb{R})$.

Lemma 2.2 ([4])**.** *For any $v \in \mathbb{R}^3$ with $|v| < 1$, there exists an optimal constant S_v such that*

$$\int_{\mathbb{R}^3} (\frac{1}{4\pi|x|} * |\varphi_v|^2)|\varphi_v|^2 dx \le S_v \langle \varphi_v, (\sqrt{-\Delta} + iv\cdot\nabla)\varphi_v\rangle\langle\varphi_v, \varphi_v\rangle, \tag{2.7}$$

holds for all $\varphi_v \in H^{1/2}(\mathbb{R}^3)$. Moreover, we have

$$S_v = \frac{2}{\langle Q_v, Q_v\rangle},$$

where $Q_v \in H^{1/2}(\mathbb{R}^3), Q_v \neq 0$ is an optimizer for (2.6) and it satisfies

$$\sqrt{-\Delta} Q_v + i(v\cdot\nabla)Q_v - (\frac{1}{4\pi|x|} * |Q_v|^2)Q_v = -Q_v.$$

In addition, the following estimates hold:

$$S_{v=0} \le S_v \le (1-|v|)^{-1} S_{v=0}.$$

Definition 2.3. We say that N^* is a critical stellar mass if

$$N_v^* = \|Q_v\|^2_{L^2(\mathbb{R}^3)} = \frac{2}{S_v},$$

where Q_v is obtained in Lemma 2.2.

We consider the following minimization problem

$$I_v(N) = \inf\{E_v(\varphi_v) : \varphi_v \in H^{1/2}(\mathbb{R}^3),\ \mathcal{N}(\varphi_v) = N\}, \tag{2.8}$$

where $E_v, \mathcal{N}$ is defined by (2.5) and (2.6), $N > 0, v \in \mathbb{R}^3$, with $|v| < 1$ denote given parameters. Any minimizer $\varphi_v \in H^{1/2}(\mathbb{R}^3)$ for minimization problem (2.8) is referred to as ground state solution of equation (1.6). Concerning existence of ground states, we have the following theorems.

Theorem 2.4. *Suppose $m > 0$ is sufficiently large, then*

- (a) *If $0 < N < N_v^*$, there exists at least one minimizer for* (2.8)*, i.e.,* (1.6) *has at least a ground state solution;*
- (b) *If $N > N_v^*$, no minimizer exists for* (2.8)*, i.e., there is no solution for equation* (1.6)*.*

Theorem 2.5. *Suppose $m > 0$ is sufficiently large, $N = N_v^*$ and*

$$\liminf_{|x|\to 0} \frac{1 - e^{-\mu|x|}}{4\pi|x|} \ge \frac{2m}{N_v^*}.$$

Then there exists at least one minimizer for (2.8)*. Hence* (1.6) *has at least a ground state solution at the critical stellar mass $N = N_v^*$.*

Theorems 2.4 and 2.5 imply that the existence of minimizer depends greatly on m and N. In particular, Theorem 2.5 shows that the existence of minimizer may occur at the critical stellar mass $N = N_v^*$. On the other hand, we address orbital stability of travelling solitary waves

$$\psi(x,t) = e^{it\omega}\varphi_v(x - vt), \tag{2.9}$$

where $\varphi_v \in H^{1/2}(\mathbb{R}^3)$ is a ground state solution for equation (1.6).

Theorem 2.6. *Suppose $m > 0$ is sufficiently large.*

- (a) *If $0 < N < N_v^*$, or*
- (b) *If $N = N_v^*$ and $\liminf_{|x|\to 0} \frac{1-e^{-\mu|x|}}{4\pi|x|} \ge \frac{2m}{N_v^*}$.*

Let $G_{v,N}$ denote the corresponding set of ground states, i.e.,

$$G_{v,N} = \{\varphi_v \in H^{1/2}(\mathbb{R}^3) : I_v(N) = E_v(\varphi_v),\ \mathcal{N}(\varphi_v) = N\},$$

which is non-empty by Theorems 2.4 and 2.5.

Then the travelling solitary waves given in (2.9)*, with $\varphi_v \in G_{v,N}$ are stable in the following sense. For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\inf_{\varphi_v \in G_{v,N}} \|\psi_0(x) - \varphi_v\|_{H^{1/2}(\mathbb{R}^3)} \le \delta \text{implies} \sup_{t\ge 0} \inf_{\varphi_v\in G_{v,N}} \|\psi(t) - \varphi_v\|_{H^{1/2}(\mathbb{R}^3)} \le \varepsilon.$$

Here $\psi(x,t)$ denotes the solution of equation (1.3) *with initial condition $\psi_0(x) \in H^{1/2}(\mathbb{R}^3)$.*

3. Proof of Theorem 2.4

To reach this end we use the concentration-compactness lemma and variational arguments.

Lemma 3.1. *Suppose $m > 0$ is sufficiently large. Then*

$$E_v(\varphi_v) \geq \frac{1}{2}(1 - \frac{N}{N_v^*})\langle \varphi_v, (\sqrt{-\Delta} + iv \cdot \nabla)\varphi_v \rangle - \frac{1}{2}mN, \tag{3.1}$$

for all $\varphi_v \in H^{1/2}(\mathbb{R}^3)$ with $\mathcal{N}(\varphi_v) = N$, where N_v^* is defined by definition 2.3.

Moreover, we have $I_v(N) \geq -\frac{1}{2}mN$ for $0 < N < N_v^*$, and $I_v(N) = -\infty$ for $N > N_v^*$.

Proof. Since $m > 0$ is sufficiently large, we have the operator inequality

$$\sqrt{-\Delta + m^2} \geq \sqrt{-\Delta}. \tag{3.2}$$

By (2.5) and (3.2), we have

$$\begin{aligned} 2E_v(\varphi_v) &\geq \langle \varphi_v, (\sqrt{-\Delta} + iv \cdot \nabla)\varphi_v \rangle - \frac{1}{2}\int_{\mathbb{R}^3} (\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_v|^2)|\varphi_v|^2 dx - mN \\ &\geq \langle \varphi_v, (\sqrt{-\Delta} + iv \cdot \nabla)\varphi_v \rangle - \frac{1}{2}\int_{\mathbb{R}^3} (\frac{1}{4\pi|x|} * |\varphi_v|^2)|\varphi_v|^2 dx - mN. \end{aligned}$$

From Lemma 2.2, we have

$$\begin{aligned} 2E_v(\varphi_v) &\geq \langle \varphi_v, (\sqrt{-\Delta} + iv \cdot \nabla)\varphi_v \rangle - \frac{S_v}{2}N\langle \varphi_v, (\sqrt{-\Delta} + iv \cdot \nabla)\varphi_v \rangle - mN \\ &= (1 - \frac{S_v}{2}N)\langle \varphi_v, (\sqrt{-\Delta} + iv \cdot \nabla)\varphi_v \rangle - mN \\ &= (1 - \frac{N}{N_v^*})\langle \varphi_v, (\sqrt{-\Delta} + iv \cdot \nabla)\varphi_v \rangle - mN. \end{aligned}$$

Hence, inequality (3.1) is proved. Furthermore, that $I_v(N) \geq -\frac{1}{2}mN$ for $N < N_v^*$ is a consequence of (3.1). To see that $I_v(N) = -\infty$ when $N > N_v^*$. Indeed, we define a L^2-norm preserving rescalings

$$Q_v^*(x) \mapsto a^{3/2}Q_v(ax),$$

with $a > 0$, $Q_v(x)$ is defined by (2.7). By Lemma 2.2 and (3.2), we have

$$\begin{aligned} E_v(Q_v^*(x)) &\leq \frac{1}{2}\langle Q_v^*(x), (\sqrt{-\Delta} + iv \cdot \nabla)Q_v^*(x) \rangle \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3} (\frac{e^{-\mu|x|}}{4\pi|x|} * |Q_v^*(x)|^2)|Q_v^*(x)|^2 dx \\ &= \frac{Na}{2N_v^*}\langle Q_v(x), (\sqrt{-\Delta} + iv \cdot \nabla)Q_v(x) \rangle \\ &\quad - \frac{N^2 a}{4(N_v^*)^2}\int_{\mathbb{R}^3} (\frac{e^{-\mu|\frac{x}{a}|}}{4\pi|x|} * |Q_v(x)|^2)|Q_v(x)|^2 dx \\ &= \frac{Na}{4N_v^*}\int_{\mathbb{R}^3} (\frac{1}{4\pi|x|} * |Q_v(x)|^2)|Q_v(x)|^2 dx \\ &\quad - \frac{N^2 a}{4(N_v^*)^2}\int_{\mathbb{R}^3} (\frac{e^{-\mu|\frac{x}{a}|}}{4\pi|x|} * |Q_v(x)|^2)|Q_v(x)|^2 dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{Na}{4N_v^*}[(1-\frac{N}{N_v^*})\int_{\mathbb{R}^3}(\frac{1}{4\pi|x|} * |Q_v(x)|^2)|Q_v(x)|^2dx \\
&\quad + \frac{N}{N_v^*}\int_{\mathbb{R}^3}(\frac{1-e^{-\mu|\frac{x}{a}|}}{4\pi|x|} * |Q_v(x)|^2)|Q_v(x)|^2dx].
\end{aligned}
$$

Hence, when $a \to \infty$, we find that

$$
\begin{aligned}
I_v(N) &\le E_v(Q_v^*(x)) \\
&\le \frac{Na}{4N_v^*}[(1-\frac{N}{N_v^*})\int_{\mathbb{R}^3}(\frac{1}{4\pi|x|} * |Q_v(x)|^2)|Q_v(x)|^2dx + o(1)] \to -\infty,
\end{aligned}
$$

for $N > N_v^*$. Therefore, $I_v(N) = -\infty$ when $N > N_v^*$. □

Remark 3.2. By Lemma 3.1, we deduce that any minimizing sequence for (2.8) is bounded in $H^{1/2}(\mathbb{R}^3)$ whenever $0 < N < N_v^*$. Indeed, we note that $\sqrt{-\Delta} + iv\cdot\nabla \ge (1-|v|)\sqrt{-\Delta}$ holds. Hence, we see that $\sup_n\langle\varphi_{v,n}, \sqrt{-\Delta}\varphi_{v,n}\rangle \le C < \infty$ thanks to (3.1).

Lemma 3.3. *Suppose $m > 0$ is sufficiently large. Then we have*

$$
I_v(N) < -\frac{1}{2}(1-\sqrt{1-v^2})mN + I_v^1(N), \tag{3.3}
$$

and $I_v^1(N) < 0$, where

$$
I_v^1(N) = \inf\{E_v^1(\varphi_v) : \varphi_v \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3}|\varphi_v|^2dx = N\}, \tag{3.4}
$$

$$
E_v^1(\varphi_v) = \frac{\sqrt{1-v^2}}{4m}\int_{\mathbb{R}^3}|\nabla\varphi_v|^2dx - \frac{1}{4}\int_{\mathbb{R}^3}(\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_v|^2)|\varphi_v|^2dx. \tag{3.5}
$$

Proof. We select an spherically symmetric function, $\rho(x) \in H^1(\mathbb{R}^3)$ with

$$
\int_{\mathbb{R}^3}|\rho(x)|^2dx = N,
$$

and we introduce the one-parameter family $\rho_b(x) = e^{ib|v|z}\rho(x)$ with $b > 0$. Here and in what follows, we assume that v is parallel to the z-axis. One checks that

$$
\frac{i}{2}\langle\rho_b, (v\cdot\nabla)\rho_b\rangle = -\frac{bv^2}{2}N.
$$

Hence, we obtain

$$
\begin{aligned}
E_v(\rho_b) &\le \frac{1}{4b}(b^2v^2N + \langle\rho, -\Delta\rho\rangle + (m^2 + b^2N)) - \frac{1}{2}mN - \frac{1}{2}v^2bN \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3}(\frac{1}{4\pi|x|}e^{-\mu|x|} * |\rho|^2)|\rho|^2dx.
\end{aligned}
$$

Let $b = \frac{m}{\sqrt{1-v^2}}$, we have

$$
E_v(\rho_b) \le -\frac{1}{2}(1-\sqrt{1-v^2})mN + E_v^1(\rho).
$$

On the other hand, we define $\rho_c = c^{3/2}\rho(cx), c > 0$ with $\int_{\mathbb{R}^3}|\rho_c(x)|^2dx = N$, then we have

$$
E_v^1(\rho_c) = \frac{c^2\sqrt{1-v^2}}{4m}\int_{\mathbb{R}^3}|\nabla\rho|^2dx - \frac{c}{4}\int_{\mathbb{R}^3}(\frac{1}{4\pi|x|}e^{-\mu|x|} * |\rho|^2)|\rho|^2dx.
$$

Hence, we can choose $c = 1$ and $\rho_1 > 0$ is a suitable test function. Then we have $E_v^1(\rho_1) < 0$, when $m > 0$ is sufficiently large. So, we have $I_v^1(N) \le E_v^1(\rho_1) < 0$, and the proof is complete. $\square$

Proof of Theorem 2.4. (1) Let $\{\varphi_{v,n}(x)\}$ be a minimizing sequence for (2.8). By Remark 3.2, we obtain that $\{\varphi_{v,n}(x)\}$ is a bounded sequence in $H^{1/2}(\mathbb{R}^3)$. Now, we apply concentration-compactness lemma [13], and conclude that a suitable subsequence $\{\varphi_{v,n_k}(x)\}$ satisfies "vanishing", "dichotomy" or "compactness".

Suppose that $\{\varphi_{v,n_k}(x)\}$ satisfies "vanishing". Then we conclude that

$$\lim_{k\to\infty}\int_{\mathbb{R}^3}(\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_{v,n_k}(x)|^2)|\varphi_{v,n_k}(x)|^2dx = 0.$$

Hence, we have $I_v(N) \ge -\frac{1}{2}(1-\sqrt{1-v^2})mN$, which is contradicts (3.3). Hence, "vanishing" can not occur.

Suppose that $\{\varphi_{v,n_k}(x)\}$ satisfies "dichotomy", we have

$$I_v(N) \ge I_v(r) + I_v(N-r) \quad \text{for } 0 < r < N \text{ and } 0 < N < N_v^*. \tag{3.6}$$

By Lemma 3.3 and the same method as employed in [4, Lemma 2.3], we have

$$I_v(N) < I_v(r) + I_v(N-r) \quad \text{for } 0 < r < N \text{ and } 0 < N < N_v^*. \tag{3.7}$$

So, inequality (3.6) contradicts the strict subadditivity condition (3.7), and "dichotomy" can not occur.

By the discussion so far, we conclude that there exists a subsequence $\{\varphi_{v,n_k}(x)\}$ and $\{y_k\} \subset \mathbb{R}^3$ such that the subsequence $\tilde{\varphi}_{v,n} = \varphi_{v,n_k}(\cdot + y_k)$ satisfies

$$\tilde{\varphi}_{v,n} \to \varphi_v \quad \text{strongly in } H^{1/2}(\mathbb{R}^3) \quad \text{as} k \to \infty$$

and

$$\int_{\mathbb{R}^3}(\frac{e^{-\mu|x|}}{4\pi|x|} * |\tilde{\varphi}_{v,n}|^2)|\tilde{\varphi}_{v,n}|^2dx \to \int_{\mathbb{R}^3}(\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_v|^2)|\varphi_v|^2dx, \text{as } k\to\infty,$$

for some $\varphi_v \in H^{1/2}(\mathbb{R}^3)$. We therefore conclude that $\int_{\mathbb{R}^3}|\varphi_v|^2dx = N$ and $I_v(N) = E_v(\varphi_v)$ by the weak lower semicontinuity. This implies that (1) of Theorem 2.4 holds.

(2) Clearly, there is no minimizer if $N > N_v^*$. Since in this case, we have $I_v(N) = -\infty$ by Lemma 3.1. $\square$

4. Critical stellar mass

In this section, we prove the existence of minimizers for (2.8) at $N = N_v^*$. We consider the manifold

$$M = \big\{\varphi_v : \varphi_v \in H^{1/2}(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3}|\varphi_v|^2dx = N_v^* big\},$$

and define the metric

$$d(\varphi_v,\phi) = \|\varphi_v - \phi\|_{H^{1/2}(\mathbb{R}^3)}, \varphi_v, \phi \in M.$$

So that (M,d) is a complete metric space. For the minimizing problem

$$I_v(N_v^*) := \inf\{E_v(\varphi_v) : \varphi_v \in H^{1/2}(\mathbb{R}^3), \mathcal{N}(\varphi_v) = N_v^*\}.$$

By Ekeland variational principle [18], we obtain that there exists a minimizing sequence $\{\varphi_{v,n}(x)\}$ of $I_v(N_v^*)$ such that

$$I_v(N_v^*) \leq E_v(\varphi_{v,n}) \leq I_v(N_v^*) + \frac{1}{n},, \tag{4.1}$$

$$E_v(\phi) \geq E_v(\varphi_{v,n}) - \frac{1}{n}\|\varphi_{v,n} - \phi\|_{H^{1/2}(\mathbb{R}^3)}, \quad \text{for any } \phi \in M. \tag{4.2}$$

By applying (4.1) and (4.2), we shall prove that

$$\{\varphi_{v,n}(x)\} \text{ is uniformly bounded in } M. \tag{4.3}$$

Indeed, if (4.3) holds, by using the same arguments of (1) in the proof of Theorem 2.4, we obtain the existence of minimizers for (2.8) at critical stellar mass $N = N_v^*$.

In the rest of this section we derive claim (4.3). On the contrary, suppose (4.3) is false, then there exists a subsequence $\{\varphi_{v,n_k}(x)\}$, such that $\|\varphi_{v,n_k}(x)\|_{H^{1/2}(\mathbb{R}^3)} \to \infty$ as $n \to \infty$, and we shall finally derive a contradiction.

Lemma 4.1. *Suppose $m > 0$ is sufficiently large, and we define*

$$\tilde{\eta}_n(x) = \lambda_n^{3/2}\varphi_{v,n}(\lambda_n x), \eta_n(x) = \tilde{\eta}_n(x + y_{\lambda_n}), \tag{4.4}$$

$$\lambda_n^{-1} := \int_{\mathbb{R}^3} (\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_{v,n}|^2)|\varphi_{v,n}|^2 dx. \tag{4.5}$$

Then there exist positive constants R and C satisfying

$$\liminf_{\lambda_n \to 0} \int_{B_R(0)} |\eta_n(x)|^2 dx \geq C > 0. \tag{4.6}$$

Proof. From (4.1)and (4.2), we have

$$\begin{aligned} 0 &\leq \frac{1}{2}\langle \varphi_{v,n}, (\sqrt{-\Delta} + iv\cdot\nabla)\varphi_{v,n}\rangle - \frac{1}{4}\int_{\mathbb{R}^3}(\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_{v,n}|^2)|\varphi_{v,n}|^2 dx \\ &\leq I_v(N_v^*) + \frac{1}{n} + \frac{1}{2}mN_v^*. \end{aligned} \tag{4.7}$$

Hence, we have

$$\frac{1}{2}\langle \varphi_{v,n}, (\sqrt{-\Delta} + iv\cdot\nabla)\varphi_{v,n}\rangle \to \infty, \quad \text{as } n \to \infty, \tag{4.8}$$

$$\frac{1}{4}\int_{\mathbb{R}^3}(\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_{v,n}|^2)|\varphi_{v,n}|^2 dx \leq I_v(N_v^*) + \frac{1}{n} + \frac{1}{2}mN_v^* \to \infty, \quad \text{as } n \to \infty. \tag{4.9}$$

From the definition of ${\lambda_n}^{-1}$ in (4.5), we have $\lambda_n \to 0$ as $n \to \infty$. By (4.7), it follows from that there exists a constant $K > 0$ independent of n such that

$$\begin{aligned} 0 < K{\lambda_n}^{-1} &\leq \frac{1}{2}\langle \varphi_{v,n}, (\sqrt{-\Delta} + iv\cdot\nabla)\varphi_{v,n}\rangle \\ &\leq \frac{1}{K}{\lambda_n}^{-1} + I_v(N_v^*) + \frac{1}{2}mN_v^*, \quad \text{as } n \to \infty. \end{aligned} \tag{4.10}$$

Using (4.5) and (4.10), we have

$$\int_{\mathbb{R}^3}(\frac{e^{-\mu|\lambda_n x|}}{4\pi|x|} * |\tilde{\eta}_n(x)|^2)|\tilde{\eta}_n(x)|^2 dx = \lambda_n \int_{\mathbb{R}^3}(\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_{v,n}|^2)|\varphi_{v,n}|^2 dx = 1, \tag{4.11}$$

$$K \le \frac{1}{2}\langle \tilde{\eta}_n, (\sqrt{-\Delta} + iv\cdot\nabla)\tilde{\eta}_n\rangle \le \frac{1}{K} + \lambda_n(I_v(N_v^*) + \frac{1}{2}mN_v^*). \tag{4.12}$$

Claim: There exists a sequence $\{y_{\lambda_n}\}$ and positive constant R and C such that

$$\liminf_{\lambda_n\to 0}\int_{B_R(y_{\lambda_n})} |\tilde{\eta}_n|^2 dx \ge C > 0.$$

Indeed, suppose that (4) is false. A proof similar to [4, Lemma A.2] then gives that

$$\int_{\mathbb{R}^3} (\frac{e^{-\mu|\lambda_n x|}}{4\pi|x|} * |\tilde{\eta}_n(x)|^2)|\tilde{\eta}_n(x)|^2 dx \to 0, \quad \text{as } n\to\infty,$$

which contradicts (4.11). Hence, the claim holds, and (4.6) is proved. □

Lemma 4.2. *Suppose $m > 0$ is sufficiently large, and $\eta_n \in H^{1/2}(\mathbb{R}^3)$ be defined by (4.4). Then we have $\eta_n \to \eta_0$ strongly in $L^p(\mathbb{R}^3)$ for all $p \in [2,3)$, where η_0 satisfies the nonlinear equation*

$$(\sqrt{-\Delta} + iv\cdot\nabla)\eta_0(x) + \frac{1}{N_v^*}\eta_0(x) - (\frac{1}{4\pi|x|} * |\eta_0(x)|^2)|\eta_0(x)| = 0, \text{on}\mathbb{R}^3. \tag{4.13}$$

Proof. For any $u(x) \in C_c^\infty(\mathbb{R}^3)$, we define

$$\tilde{u}(x) = \lambda_n^{-\frac{1}{2}} u(\frac{x-\lambda_n y_{\lambda_n}}{\lambda_n}), \quad j(\alpha,\sigma) = \frac{1}{2}\int_{\mathbb{R}^3} |\varphi_{v,n} + \alpha\varphi_{v,n} + \sigma\tilde{u}|^2 dx - \frac{N_v^*}{2}.$$

Then the function $j(\alpha,\sigma)$ satisfies

$$j(0,0) = 0, \quad \frac{\partial j(0,0)}{\partial\alpha} = \int_{\mathbb{R}^3} |\varphi_{v,n}|^2 dx = N_v^*, \quad \frac{\partial j(0,0)}{\partial\sigma} = \int_{\mathbb{R}^3} \varphi_{v,n}\tilde{u}(x)dx.$$

Using the implicit function theorem in [18], we obtain that there exist $c > 0$ and a function $\alpha(\sigma) \in C^1((-c,c),\mathbb{R})$, where $|\sigma| > 0$ is sufficiently small, such that

$$\alpha(0) = 0, \alpha'(0) = -\frac{1}{N_v^*}\int_{\mathbb{R}^3} \varphi_{v,n}\tilde{u}(x)dx \quad \text{and} \quad j(\alpha(\sigma),\sigma) = 0.$$

Therefore, $\varphi_{v,n} + \alpha(\sigma)\varphi_{v,n} + \sigma\tilde{u} \in M$, where $\alpha \in (-c,c)$. From (4.3), we have

$$E_v(\varphi_{v,n} + \alpha(\sigma)\varphi_{v,n} + \sigma\tilde{u}) - E_v(\varphi_{v,n}) \ge -\frac{1}{n}\|\alpha(\sigma)\varphi_{v,n} + \sigma\tilde{u}\|_{H^{1/2}(\mathbb{R}^3)}, \tag{4.14}$$

and so we have

$$|\langle E_v'(\varphi_{v,n}), \alpha'(0)\varphi_{v,n} + \tilde{u}\rangle| \le \frac{1}{n}\|\alpha'(0)\varphi_{v,n} + \tilde{u}\|_{H^{1/2}(\mathbb{R}^3)}. \tag{4.15}$$

On the other hand,

$$\begin{aligned}\frac{1}{2}\langle E_v'(\varphi_{v,n}), \tilde{u}\rangle &= \int_{\mathbb{R}^3} u(\sqrt{-\Delta + m^2{\lambda_n}^2} - m\lambda_n)\eta_n(x)dx + \int_{\mathbb{R}^3} u(x)i(v\cdot\nabla)\eta_n(x)dx \\ &\quad - \int_{\mathbb{R}^3} (\frac{e^{-\mu|\lambda_n x|}}{4\pi|x|} * |\eta_n(x)|^2)\eta_n(x)u(x)dx.\end{aligned} \tag{4.16}$$

By setting $\mu_n = \langle E_v'(\varphi_{v,n}), \varphi_{v,n}\rangle$, combining (4.14), (4.15) and (4.16), we have

$$|\mu_n\lambda_n + 1| = |\mu_n\lambda_n + \lambda_n\int_{\mathbb{R}^3} (\frac{e^{-\mu|x|}}{4\pi|x|} * |\varphi_{v,n}|^2)|\varphi_{v,n}|^2 dx| \to 0, \quad \text{as } n\to\infty,$$

$$\|\alpha'(0)\varphi_{v,n} + \tilde{u}\|_{H^{1/2}(\mathbb{R}^3)} \le C\lambda_n^{1/2}, \tag{4.17}$$

$$\alpha'(0) = -\frac{\lambda_n}{N_v^*}\int_{\mathbb{R}^3}\eta_n(x)u(x)dx.$$

Thus, estimates (4.15)–(4.17), yield

$$\begin{aligned}
&|\int_{\mathbb{R}^3} u(\sqrt{-\Delta+m^2{\lambda_n}^2}-m\lambda_n)\eta_n(x)dx+\int_{\mathbb{R}^3} u(x)i(v\cdot\nabla)\eta_n(x)dx\\
&-\frac{\mu_n\lambda_n}{N_v^*}\int_{\mathbb{R}^3}\eta_n(x)u(x)dx-\int_{\mathbb{R}^3}(\frac{e^{-\mu|\lambda_n x|}}{4\pi|x|}*|\eta_n(x)|^2)\eta_n(x)u(x)dx| \qquad (4.18)\\
&=|\langle E_v'(\varphi_{v,n}),\tilde u\rangle+\mu_n\alpha'(0)|=|\langle E_v'(\varphi_{v,n}),\alpha'(0)\varphi_{v,n}+\tilde u\rangle|\le\frac{C\lambda_n^{1/2}}{n}.
\end{aligned}$$

From this equality and Lemma 4.1, we have that $\eta_n\rightharpoonup\eta_0\neq 0$ in $H^{1/2}(\mathbb{R}^3)$ and

$$0\le\int_{\mathbb{R}^3}|\eta_0|^2dx\le\liminf_{\lambda_n\to\infty}\int_{\mathbb{R}^3}|\eta_n|^2dx=N_v^*. \qquad (4.19)$$

On the other hand, by the Pohozaev identity [17], we derive from (4.13) that

$$\frac{1}{N_v^*}\int_{\mathbb{R}^3}|\eta_0|^2dx=\int_{\mathbb{R}^3}\eta_0(\sqrt{-\Delta}+iv\cdot\nabla)\eta_0(x)dx=\frac12\int_{\mathbb{R}^3}(\frac{1}{4\pi|x|}*|\eta_0(x)|^2)|\eta_0(x)|^2dx.$$

Furthermore, this and (2.7) imply that

$$\frac{N_v^*}{2}\le\frac{\int_{\mathbb{R}^3}\eta_0(\sqrt{-\Delta}+iv\cdot\nabla)\eta_0(x)dx\int_{\mathbb{R}^3}|\eta_0|^2dx}{\int_{\mathbb{R}^3}(\frac{1}{4\pi|x|}*|\eta_0(x)|^2)|\eta_0(x)|^2dx}=\frac12\int_{\mathbb{R}^3}|\eta_0|^2dx.$$

This and (**??**) indicate that $\eta_n\to\eta_0$ strongly in $L^p(\mathbb{R}^3)$ for all $p\in[2,3)$. In view of the $H^{1/2}(\mathbb{R}^3)$ boundness, the proof is complete. □

Proof of Theorem 2.5. By Lemma 4.2, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\int_{\mathbb{R}^3}(\frac{1-e^{-\mu|x|}}{4\pi|x|}*|\varphi_{v,n}(x)|^2)|\varphi_{v,n}(x)|^2dx\\
&=\liminf_{n\to\infty}\int_{\mathbb{R}^3}(\frac{1-e^{-\mu|\lambda_n x|}}{4\pi|\lambda_n x|}*|\eta_n(x)|^2)|\eta_n(x)|^2dx\\
&\ge\int_{\mathbb{R}^3}\liminf_{n\to\infty}(\frac{1-e^{-\mu|\lambda_n x|}}{4\pi|\lambda_n x|}*|\eta_n(x)|^2)|\eta_n(x)|^2dx\\
&\ge\frac{2m}{N_v^*}(N_v^*)^2=2mN_v^*.
\end{aligned}$$

By (2.7), we have

$$\begin{aligned}
I_v(N_v^*)&=\liminf_{n\to\infty}E_v(\varphi_{v,n})\\
&=-\frac12 mN_v^*+\liminf_{n\to\infty}\{\frac12\langle\varphi_{v,n},(\sqrt{-\Delta}+iv\cdot\nabla)\varphi_{v,n}\rangle\\
&\quad-\frac14\int_{\mathbb{R}^3}(\frac{1}{4\pi|x|}*|\varphi_{v,n}(x)|^2)|\varphi_{v,n}(x)|^2dx\\
&\quad+\frac12\langle\varphi_{v,n},(\sqrt{-\Delta+m^2}-\sqrt{-\Delta})\varphi_{v,n}\rangle\\
&\quad+\frac14\int_{\mathbb{R}^3}(\frac{1-e^{-\mu|x|}}{4\pi|x|}*|\varphi_{v,n}(x)|^2)|\varphi_{v,n}(x)|^2dx\}
\end{aligned}$$

$$\geq -\frac{1}{2}mN_v^* + \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^3}(\frac{1-e^{-\mu|x|}}{4\pi|x|} * |\varphi_{v,n}(x)|^2)|\varphi_{v,n}(x)|^2dx \geq 0,$$

which contradicts Lemma 3.1. This verifies that (4.3) holds, and the proof is complete. □

5. Orbital stability

In this section, we prove the orbital stability of travelling solitary wave by a general idea which is introduced by [1, 2].

Proof of Theorem 2.6. (1) We choose $\delta > 0$ sufficiently small such that

$$\inf_{\varphi_v \in G_{v,N}} \|\psi_0(x) - \varphi_v(x)\|_{H^{1/2}(\mathbb{R}^3)} \leq \delta.$$

By Lemma 2.1, we have that the corresponding solution $\psi(t)$ exists for all times $t \geq 0$ with $0 < N < N_v^*$.

Now, argue by contradiction, we assume that orbital stability does not hold. Then this means that there exist $\varepsilon_0 > 0$, a sequence of initial value $\{\psi_n(0)\} \in H^{1/2}(\mathbb{R}^3)$ and $\{t_n\} \in \mathbb{R}$ with

$$\inf_{\varphi_v \in G_{v,N}} \|\psi_n(0) - \varphi_v\|_{H^{1/2}(\mathbb{R}^3)} \to 0 \quad \text{as } n \to \infty, \tag{5.1}$$

and some $\varepsilon_0 > 0$ such that

$$\inf_{\varphi_v \in G_{v,N}} \|\psi_n(t_n) - \varphi_v\|_{H^{1/2}(\mathbb{R}^3)} > \varepsilon_0, \quad \text{as } n \geq 0, \tag{5.2}$$

where $\{\psi_n(t_n)\}$ denotes the solution to equation (1.3) with initial datum $\{\psi_n(0)\}$.

Note that (5.1) implies that $\mathcal{N}(\psi_n(0)) \to N$ as $n \to \infty$. Since $0 < N < N_v^*$, we can assume that $\mathcal{N}(\psi_n(0)) < N_v^*$ holds for all $n \geq 0$, which guarantees that $\{\psi_n(t)\}$ exists globally in time. Define

$$\alpha_n = \psi_n(t_n) \text{in} H^{1/2}(\mathbb{R}^3).$$

By conservation of $\mathcal{N}(\psi(t))$ and $E_v(\psi(t))$, we have

$$\lim_{n\to\infty} E_v(\alpha_n) = I_v(N) \quad \text{and} \quad \lim_{n\to\infty} \mathcal{N}(\alpha_n) = N.$$

Next, we consider the rescaled sequence

$$\widetilde{\alpha_n} = a_n\alpha_n, \quad \text{where} \quad a_n = \sqrt{\frac{N}{\mathcal{N}(\alpha_n)}}.$$

Using Remark 3.2, we deduce that

$$\|\widetilde{\alpha_n} - \alpha_n\|_{H^{1/2}(\mathbb{R}^3)} \leq C|1 - a_n| \to 0, \quad \text{as } n \to \infty.$$

By continuity of $E_v(\varphi_v)$, we deduce that

$$\lim_{n\to\infty} E_v(\widetilde{\alpha_n}) = I_v(N), \quad \lim_{n\to\infty} \mathcal{N}(\widetilde{\alpha_n}) = N, \quad \text{for all } n \geq 0.$$

Therefore, $\{\widetilde{\alpha_n}\}$ is a minimizing sequence for (2.8), and we have a contradiction.

(2) By Theorem 2.5, the solution $\psi(t)$ of equation (1.3) is a global solution in $H^{1/2}(\mathbb{R}^3)$, when $N = N_v^*$ and $\liminf_{|x|\to 0} \frac{1-e^{-\mu|x|}}{4\pi|x|} \geq \frac{2m}{N_v^*}$. In a similar way, this completes the proof of Theorem 2.6. □

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References

[1] T. Cazenave, A. Haraux, Y. Martel; *An introduction to semilinear evolution equations*, Oxford university press, New York, 1998.
[2] T. Cazenave, P. L. Lions; *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Commun. Math. Phys., 85 (1982), 549-561.
[3] A. Elgart, B. Schlein; *Mean field dynamics of Boson stars*, Comm. Pure. Appl. Math., 60 (2006), 500-545.
[4] J. Fröhlich, B. L. G. Jonsson, E. Lenzmann; *Boson stars as solitary waves*, Commun. Math. Phys., 274 (2007), 1-30.
[5] J. Fröhlich, B. L. G. Jonsson, E. Lenzmann; *Effective dynamics for Boson stars*, Nonlinearity, 32 (2007), 1031-1075.
[6] J. Fröhlich, E. Lenzmann; *Blowup for nonlinear wave equations describing boson stars*, Comm. Pure. Appl. Math., 60 (2007), 1691-1705.
[7] Y. Guo, X. Zeng; *Ground states of pseudo-relativistic boson stars under the critical stellar mass*, Ann. Inst. Henri Poincare, 34 (2017), 1611-1632.
[8] E. Lenzmann, M. Lewin; *On singularity formation for the L^2-critical Boson star equation*, Nonlinearity 24 (2011), 3515-3540.
[9] E. Lenzmann; *Nonlinear dispersive equations describing boson stars*, Ph.D., ETH, (2006).
[10] E. Lenzmann; *Uniqueness of ground states for pseudo-relativistic Hartree equations*, Analysis PDE, 2 (2009), 1-28.
[11] E. Lenzmann; *Well-posedness for semi-relativistic Hartree equations of critical type*, Math. Phys. Anal. Geom., 10 (2007), 43-64.
[12] E. Lieb, M. Loss; *Analysis*, (Second edition), AMS, Providence, RI, 2001.
[13] P. L. Lions; *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1.* Ann. Inst. Henri Poincare, 1984, 1(2), 109-145.
[14] E. H. Lieb; *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Applied Mathematics, 57 (1997), 93-105.
[15] E. H. Lieb, W. E. Thirring; *Gravitational collapse in quantum mechanics with relativistic kinetic energy*, Annals of Physics, 155 (1984), 494-512.
[16] E. H. Lieb, H. T. Yau; *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Commun. Math. Phys., 112 (1987), 147-174.
[17] X. Ros-Oton, J. Serra; *The Pohozaev identity for the fractional Laplacian.* Arch. Rat. Mech. Anal., 213 (2014), 587-628.
[18] M. Struwe; *Variational Methods*, (Second edition), Springer-Verlag, Berlin, 1996.

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