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GENERALIZED SPREADING SPEEDS FOR LATTICE DIFFERENTIAL EQUATIONS WITH TIME AND SPACE DEPENDENCE

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Abstract. This article concerns the spatial spreading speeds for lattice differential equations with general time and space dependence. Firstly, we give the concept of spreading speed intervals. Then, under the suitable assumptions we show the existence and properties of spreading speed intervals.

1. Introduction

In this article we explore the spatial spreading speeds for the lattice differential equation

$$\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + f_i(t, u_i(t)), \quad i \in \mathbb{Z},\ t \in \mathbb{R}. \tag{1.1}$$

Let

$$X := l^\infty(\mathbb{Z}) = \{u = \{u_i\}_{i\in\mathbb{Z}} : \sup_{i\in\mathbb{Z}} |u_i| < \infty\}$$

equipped with norm $\|u\|_\infty = \sup_{i\in\mathbb{Z}} |u_i|$. For given $u^1, u^2 \in X$, we define $u^1 < u^2$ ($u^1 \le u^2$) if $u^1_i < u^2_i$ ($u^1_i \le u^2_i$) for each $i \in \mathbb{Z}$. We make an assumption on $f(t,s) := \{f_i(t,s)\}_{i\in\mathbb{Z}}$:

$$f(t,s) := \{f_i(t,s)\}_{i\in\mathbb{Z}} \in C^1(\mathbb{R}^2, X).$$

Under the assumption above, for any given $u^0 \in X$ and $t_0 \in \mathbb{R}$, (1.1) has a unique (local) solution, denoted by $u(t; t_0, u^0, f)$ with $u(t_0; t_0, u^0, f) = u^0$. We denote by $u(t; t_0, j, u^0(j'), f)$ the solution of the space shifted equation of (1.1),

$$\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + f_{i+j}(t, u_i(t)), \quad i \in \mathbb{Z},\ t \in \mathbb{R} \tag{1.2}$$

with initial condition $u_i(t_0; t_0, j, u^0(j'), f) = u^0_{i+j'}$ for $u^0 \in X$. Note that for any $(i,t) \in \mathbb{Z}\times\mathbb{R}$, $u_i(t; t_0, j, u^0(j'), f) = u_{i+j}(t; t_0, u^0(j'-j), f)$ and $u(t; t_0, 0, u^0(0), f) = u(t; t_0, u^0, f)$.

Equation (1.1) comes directly from many biological models in patchy environments [14, 13], which describes the growth of population or biological invasion process. In fact, it is also the spatially discrete version of the reaction diffusion equation

$$u_t = u_{xx} + f(t, x, u). \tag{1.3}$$

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Since the pioneer works [5, 8] for the evolutionary take-over of a habitat by a fitter genotype in 1937, there have been many studies on the spatial spreading dynamics in spatially and temporal homogeneous media or spatially and/or temporal periodic media. Because of the existence of various heterogeneities in many natural environments, it is of great importance to investigate the spatial spreading and front propagating dynamics for reaction diffusion equations in temporal and/or spatially heterogeneous environment.

The spatial spreading and front propagating dynamics of KPP models (1.3) in time almost periodic and space periodic media were studied in [1, 2, 7]. Shen [11] explored the spatial spreading speeds for two-dimensional discrete KPP models in time recurrent and space independent environments. In the case where the nonlinearity of (1.1) are homogeneous in space variable, that is, $f_i(t,s) \equiv f_j(t,s)$ for all $i,j \in \mathbb{Z}$, the spatial spreading speeds has been characterized in [3]. After then, under the assumption that there is a transition wave of (1.1), Cao and Shen [4] established a method to test the stability and uniqueness of it. However, there are few investigation on spatial spreading speeds for lattice differential equations with general time and space dependence.

The objective of this article is to study the spatial spreading speeds for lattice differential equation in general heterogeneous media. We need the following assumptions:

(H1) There exist $0 < \tilde{m}_0 < \tilde{M}_0$ such that for all $(i,t) \in \mathbb{Z} \times \mathbb{R}$ and $s \geq 0$,
$$\partial_s f_i^-(t,0)s - \tilde{M}_0 s^2 \leq f_i^-(t,s) \leq f_i(t,s) \leq f_i^+(t,s) \leq \partial_s f_i^+(t,0)s - \tilde{m}_0 s^2,$$
where $f_i^{\pm}(t,s)$ $(i \in \mathbb{Z})$ are C^1 in $t \in \mathbb{R}$, and are C^2 in $s \in \mathbb{R}$ with $\frac{\partial f_i^{\pm}}{\partial s}(t,s)$ and $\frac{\partial^2 f_i^{\pm}}{\partial s^2}(t,s)$ being bounded uniformly in $(t,s) \in \mathbb{R}^2$. Moreover, $f_i^{\pm}(t,s) = f_{i+N}^{\pm}(t,s)$ (N is a positive integer) for all $i \in \mathbb{Z}$ and $(t,s) \in \mathbb{R}^2$.

For the sake of simplicity, we denote $(1.1)^-$ and $(1.1)^+$ (resp. $(1.2)^-$ and $(1.2)^+$) as the equation (1.1) (resp. (1.2)) with f being replaced by f^- and f^+ respectively. Let $u(t;t_0,u^0,f^-)$ and $u(t;t_0,u^0,f^+)$ (resp. $u(t;t_0,j,u^0(j'),f^-)$ and $u(t;t_0,j,u^0(j'),f^+)$) be the solution of $(1.1)^-$ and $(1.1)^+$ (resp. $(1.2)^-$ and $(1.2)^+$) respectively, where $u^0 \in X$ and
$$u(t_0;t_0,u^0,f^-) = u^0, \quad u(t_0;t_0,u^0,f^+) = u^0$$
(resp.
$$u_i(t_0;t_0,j,u^0(j'),f^-) = u^0_{i+j'}, \quad u_i(t_0;t_0,j,u^0(j'),f^+) = u^0_{i+j'}, \quad i \in \mathbb{Z}).$$
If no confusion occurs, we may write the solution $u(t;t_0,u^0,f)$ of (1.1) and the solution $u(t;t_0,j,u^0(j'),f)$ of (1.2) as $u(t;t_0,u^0)$ and $u(t;t_0,j,u^0(j'))$ respectively.

A solution $\hat{u}^+(t) = \{\hat{u}_i^+(t)\}_{i\in\mathbb{Z}}$ of $(1.1)^-$ (resp. $\check{u}^+(t) = \{\check{u}_i^+(t)\}_{i\in\mathbb{Z}}$ of $(1.1)^+$) is called a entirely positive solution if it is a solution of $(1.1)^-$ (resp. $(1.1)^+$) for $t \in \mathbb{R}$ and $\hat{u}^+_{\rm inf} := \inf_{t\in\mathbb{R},\, i\in\mathbb{Z}} \hat{u}_i^+(t) > 0$ (resp. $\check{u}^+_{\rm inf} := \inf_{t\in\mathbb{R},\, i\in\mathbb{Z}} \check{u}_i^+(t) > 0$).

A solution $\hat{u}^+(t)$ (resp. $\check{u}^+(t)$) is globally stable in the sense that for any $u^0 \in X$ with $\inf_{i\in\mathbb{Z}} u_i^0 > 0$,
$$\|u(t+t_0;t_0,u^0,f^-) - \hat{u}^+(t+t_0)\|_\infty \to 0 \quad \text{as } t \to \infty,$$
$$(\text{resp. } \|u(t+t_0;t_0,u^0,f^+) - \check{u}^+(t+t_0)\|_\infty \to 0 \quad \text{as } t \to \infty)$$
uniformly in $t_0 \in \mathbb{R}$.

Also, we can define $\hat{u}^+(t;j) = \{\hat{u}_i^+(t;j) = \hat{u}_{i+j}^+(t)\}_{i\in\mathbb{Z}}$ and $\check{u}^+(t;j) = \{\check{u}_i^+(t;j) = \check{u}_{i+j}^+(t)\}_{i\in\mathbb{Z}}$ as the globally stable entirely positive solution of $(1.2)^-$ and $(1.2)^+$ respectively.

Cao and Shen [4, Proposition 2.1] obtained the existence, uniqueness and global stability of uniformly bounded entirely positive solutions for $(1.1)^\pm$ being represented in the form

$$\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + u_i\tilde{f}_i^\pm(t, u_i(t)), \quad i \in \mathbb{Z},\ t \in \mathbb{R}$$

where $\tilde{f}_i^\pm(t,s)$ $(i \in \mathbb{Z})$ are of mono-stable or Fisher-KPP type and satisfy the regularity assumption in (H1); for some $M_0 > 0$, $\tilde{f}_i^\pm(t,s) < 0$ when $s \geq M_0$; $\frac{\partial \tilde{f}_i^\pm}{\partial s}(t,s) < 0$ for $s \geq 0$; and

$$\liminf_{t-t_0\to\infty} \frac{1}{t-t_0}\int_{t_0}^{t} \inf_{i\in\mathbb{Z}} \tilde{f}_i^\pm(\tau,0)d\tau > 0.$$

This ensures the validity of the assumption (H2) in the following. It is not clear to us if such solutions still exist without hypotheses above. Therefore, we make the further assumptions on $f^\pm$ as follows:

- (H2) There are the unique uniformly bounded globally stable positive solution $\hat{u}^+(t)$ and $\check{u}^+(t)$ with $\hat{u}^+_{\rm inf} \leq \check{u}^+_{\rm inf}$ of $(1.1)^\pm$ respectively.

This paper is organized as follows. In Section 2, we introduce some important definitions and state our main results. Section 3 is devoted to investigating spatial spreading speeds for general time and space dependent lattice differential equations (1.1).

2. Preliminaries and statement of main results

In this section, we present some standard notation and state the main results of this article. Let

$$X_1 = \{u^0 \in X : 0 \leq u^0 < \hat{u}^+_{\rm inf},\ 0 < \liminf_{i\to-\infty} u_i^0 < \hat{u}^+_{\rm inf},\ u_i^0 = 0\ (i \gg 1)\}.$$

For a given function $t \to u(t) \in X$ and $c \in \mathbb{R}$, we define

$$\limsup_{i\geq ct, t\to\infty} u_i(t) = \limsup_{t\to\infty} \sup_{i\in\mathbb{Z}, i\geq ct} u_i(t), \quad \liminf_{i\leq ct, t\to\infty} u_i(t) = \liminf_{t\to\infty} \inf_{i\in\mathbb{Z}, i\leq ct} u_i(t).$$

For (1.1) involving not globally stable positive solution, we make the following definitions:

$$c^*_{\rm sup} = \inf\big\{c : \forall u^0 \in X_1,\ \limsup_{i\geq ct,t\to\infty} u_i(t+t_0;t_0,u^0) = 0 \text{ uniformly in } t_0 \in \mathbb{R}\big\},$$

$$\check{c}^*_{\rm sup} = \inf\big\{c : \forall u^0 \in X_1,\ \limsup_{i\geq ct,t\to\infty} u_i(t+t_0;t_0,u^0,f^+) = 0 \text{ uniformly in } t_0 \in \mathbb{R}\big\},$$

$$c^*_{\rm inf} = \sup\big\{c : \forall u^0 \in X_1,\ \liminf_{i\leq ct,t\to\infty} (u_i(t+t_0;t_0,u^0) - \hat{u}_i^+(t+t_0)) \geq 0 \text{ uniformly in } t_0 \in \mathbb{R}\big\},$$

$$\hat{c}^*_{\rm inf} = \sup\big\{c : \forall u^0 \in X_1,\ \liminf_{i\leq ct,t\to\infty} (u_i(t+t_0;t_0,u^0,f^-) - \hat{u}_i^+(t+t_0)) = 0 \text{ uniformly in } t_0 \in \mathbb{R}\big\}.$$

In fact, $[c^*_{\rm inf}, c^*_{\rm sup}]$ can be called the spreading speed interval of (1.1). We then state our main results. The next theorem is about spreading properties.

Theorem 2.1.

(1) *$\hat{c}^*_{\rm inf}$ and $\check{c}^*_{\rm sup}$ are finite.*

(2) *If $c < \hat{c}^*_{\rm inf}$, then for any $u^0 \in X_1$,*

$$\liminf_{i\le ct, t\to\infty} (u_i(t+t_0;t_0,j,u^0) - \hat{u}^+_i(t+t_0;j)) \ge 0 \tag{2.1}$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \mathbb{Z}$. In particular, $\hat{c}^*_{\rm inf} \le c^*_{\rm inf}$.

(3) *If $c > \check{c}^*_{\rm sup}$, then for any $u^0 \in X_1$,*

$$\limsup_{i\ge ct, t\to\infty} u(t+t_0;t_0,j,u^0) = 0 \tag{2.2}$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \mathbb{Z}$. In particular, $\check{c}^*_{\rm sup} \ge c^*_{\rm sup}$.

(4) *Assume that $0 \le u^0 < \hat{u}^+_{\rm inf}(u^0 \in X)$ with $u^0_i = 0$ for $i \gg 1$, then for any $c > \check{c}^*_{\rm sup}$,*

$$\limsup_{i\ge ct, t\to\infty} u_i(t+t_0;t_0,j,u^0) = 0$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \mathbb{Z}$.

Let $g(t) \in L^\infty(\mathbb{R})$. Define

$$g_{\rm inf} = \liminf_{t\ge s, t-s\to\infty} \frac{1}{t-s}\int_s^t g(\tau)d\tau, \quad g_{\rm sup} = \limsup_{t\ge s, t-s\to\infty} \frac{1}{t-s}\int_s^t g(\tau)d\tau,$$
$$\lfloor g\rfloor_T = \inf_{k\in\mathbb{Z}} \frac{1}{T}\int_{(k-1)T}^{kT} g(\tau)d\tau, \quad \lceil g\rceil_T = \sup_{k\in\mathbb{Z}} \frac{1}{T}\int_{(k-1)T}^{kT} g(\tau)d\tau.$$

Introduce the $N\times N$ matrix $C_\lambda := [c_{\lambda;i,j}]$ defined by

$$\begin{gathered} c_{\lambda;j,j}(t) = -2, \quad j\in\{1,\dots,N\},\\ c_{\lambda;j,j+1} = e^{-\lambda}, \quad j\in\{1,\dots,N-1\},\\ c_{\lambda;j+1,j} = e^{\lambda}, \quad j\in\{1,\dots,N-1\},\\ c_{\lambda;1,N} = e^{\lambda}, \quad c_{\lambda;N,1} = e^{-\lambda},\\ c_{\lambda;i,j} = 0, \quad |i-j|\ge 2\ (i,j)\notin\{(1,N),(N,1)\} \end{gathered}$$

for $\lambda \in \mathbb{R}$. Furthermore, call $D_{\rm inf}(f^-) := [a_{i,j}]$ the diagonal $N\times N$ matrix defined by $a_{i,i} = \partial_s f^-_i(t,0)_{\rm inf}$ for all $i\in\{1,\dots,N\}$; call $D_{\rm sup}(f^+) := [b_{i,j}]$ the diagonal $N\times N$ matrix defined by $b_{i,i} = \partial_s f^+_i(t,0)_{\rm sup}$ for all $i\in\{1,\dots,N\}$. Lastly, call $\lfloor D\rfloor_T(f^-) := [c_{i,j}]$ the diagonal $N\times N$ matrix defined by $c_{i,i} = \lfloor\partial_s f^-_i(t,0)\rfloor_T$ for all $i\in\{1,\dots,N\}$; call $\lceil D\rceil_T(f^+) := [d_{i,j}]$ the diagonal $N\times N$ matrix defined by $d_{i,i} = \lceil\partial_s f^+_i(t,0)\rceil_T$ for all $i\in\{1,\dots,N\}$.

The matrices $C_\lambda + D_{\rm inf}(f^-)$, $C_\lambda + D_{\rm sup}(f^+)$, $C_\lambda + \lfloor D\rfloor_T(f^-)$ and $C_\lambda + \lceil D\rceil_T(f^+)$ have principle eigenvalue $M_{\rm inf}(f^-;\lambda)$, $M_{\rm sup}(f^+;\lambda)$, $\lfloor M\rfloor_T(f^-;\lambda)$ and $\lceil M\rceil_T(f^+;\lambda)$ respectively (see [6, Lemma 2.1]).

Theorem 2.2. *(1) Let $u^0 \in X$ with $0 \le u^0_i \le \hat{u}^+_{\rm inf}$ for $i\in\mathbb{Z}$ and $\gamma' \in \mathbb{R}$ be given. If*

$$\liminf_{j\le\gamma' t, t\to\infty} u_j(t+t_0;t_0,u^0,f^-) > 0$$

uniformly in $t_0 \in \mathbb{R}$*, then for any* $\gamma < \gamma'$*,*

$$\liminf_{j \le \gamma t, t\to\infty} (u_j(t+t_0; t_0, u^0, f^-) - \hat{u}_j^+(t+t_0)) = 0 \tag{2.3}$$

uniformly in $t_0 \in \mathbb{R}$*. Moreover,*

$$\liminf_{j \le \gamma t, t\to\infty} (u_j(t+t_0; t_0, u^0) - \hat{u}_j^+(t+t_0)) \ge 0$$

uniformly in $t_0 \in \mathbb{R}$*.*

(2) One has

$$c^- := \inf_{\lambda>0} \frac{M_{\inf}(f^-;\lambda)}{\lambda} \le c^*_{\inf} \le c^*_{\sup} \le c^+ := \inf_{\lambda>0} \frac{M_{\sup}(f^+;\lambda)}{\lambda}.$$

Remark 2.3. *If* $\partial_s f_i^{\pm}(t,0)$ *are unique ergodic for each* $i \in \mathbb{Z}$*, namely the limit*

$$\lim_{t\ge t_0, t-t_0\to\infty} \frac{1}{t-t_0}\int_{t_0}^t \partial_s f_i^{\pm}(\tau,0)d\tau, \quad i \in \mathbb{Z}$$

exist (see [11, 12] *for details), and*

$$\lim_{t\ge t_0, t-t_0\to\infty} \frac{1}{t-t_0}\int_{t_0}^t \partial_s f_i^{-}(\tau,0)d\tau = \lim_{t\ge t_0, t-t_0\to\infty} \frac{1}{t-t_0}\int_{t_0}^t \partial_s f_i^{+}(\tau,0)d\tau, \quad i \in \mathbb{Z}.$$

Thus $c^- = c^*_{\inf} = c^*_{\sup} = c^+$*. In this case,* c^- *is called the spreading speed of* (1.1)*. This implies that under well-fitted perturbation, there still exists the spreading speed for lattice mono-stable equations with time and space periodic dependence.*

3. Spreading speeds

In this section, we investigate the fundamental properties of (generalized) spatial spreading speeds of (1.1) and prove Theorems 2.1 and 2.2. First we present some important lemmas to be used in the proofs of the main results. In the following, we establish a basic comparison principle for $(1.1)^{\pm}$ in terms of sub-solutions and super-solutions. A function $v(t) \in C([t_0,T), X)$ is called a super-solution or sub-solution of $(1.1)^{\pm}$ if $v(t)$ is absolutely continuous in $t \in [t_0, T)$ and

$$\dot{v}_i(t) \ge v_{i+1}(t) - 2v_i(t) + v_{i-1}(t) + f_i^{\pm}(t, v_i(t)) \quad \text{for } i \in \mathbb{Z},\ t \in [t_0,T)$$

or

$$\dot{v}_i(t) \le v_{i+1}(t) - 2v_i(t) + v_{i-1}(t) + f_i^{\pm}(t, v_i(t)) \quad \text{for } i \in \mathbb{Z},\ t \in [t_0,T).$$

Similarly, we can also define a function $v(t;j) \in C([t_0,T), X)$ which is super-solution or sub-solution of $(1.2)^{\pm}$, respectively.

Lemma 3.1. *(1) Suppose that* $v^1(t)$ *and* $v^2(t)$ *are bounded super-solution and sub-solution of* $(1.1)^-$ *on* $[t_0,T)$ *respectively, and* $v^1(t_0) \ge v^2(t_0)$*, then* $v^1(t) \ge v^2(t)$ *for* $t \in [t_0,T)$*. In particular, for any* $u^1, u^2 \in X$ *with* $u^1 \le u^2$*, we have*

$$u_i(t; t_0, u^1, f^-) \le u_i(t; t_0, u^2), \quad i \in \mathbb{Z},$$

where $t > t_0$ *is such that both* $u(t; t_0, u^1, f^-)$ *and* $u(t; t_0, u^2)$ *exist.*

(2) Suppose that $w^1(t)$ *and* $w^2(t)$ *are bounded super-solution and sub-solution of* $(1.1)^+$ *on* $[t_0,T)$ *respectively, and* $w^1(t_0) \ge w^2(t_0)$*, then* $w^1(t) \ge w^2(t)$ *for* $t \in [t_0,T)$*. In particular, for any* $u^1, u^2 \in X$ *with* $u^1 \le u^2$*, we have*

$$u_i(t; t_0, u^2, f^+) \ge u_i(t; t_0, u^1), \quad i \in \mathbb{Z},$$

where $t > t_0$ *is such that both* $u(t; t_0, u^2, f^+)$ *and* $u(t; t_0, u^1)$ *exist.*

Proof. We prove (1), while (2) can be proved similarly. Let $w_i(t) = e^{ct}(v_i^1(t) - v_i^2(t))$, where c is a constant to be determined later. Then

$$\begin{aligned}\dot{w}_i(t) &= ce^{ct}(v_i^1(t) - v_i^2(t)) + e^{ct}(\dot{v}_i^1(t) - \dot{v}_i^2(t)) \\ &\geq w_{i+1}(t) + w_{i-1}(t) + (a_i(t) - 2 + c)w_i(t)\end{aligned} \tag{3.1}$$

for all $i \in \mathbb{Z}$ and $t \in [t_0, T)$, where

$$a_i(t) = \int_0^1 \frac{\partial f_i^-}{\partial s}(t, \tau v_i^1(t) + (1-\tau)v_i^2(t))d\tau \quad \text{for } i \in \mathbb{Z},\ t \in [t_0, T).$$

Let $p_i(t) = a_i(t) - 2 + c$. Since $v^1(t)$ and $v^2(t)$ are bounded on $t \in [t_0, T)$, then there is a $c > 0$ such that

$$\inf_{i \in \mathbb{Z}, t \in [t_0, T]} p_i(t) > 0.$$

In the following, one claims that $w_i(t) \geq 0$ for $i \in \mathbb{Z}$ and $t \in [t_0, T)$. Denote $p_0 = \sup_{i \in \mathbb{Z}, t \in [t_0, T)} p_i(t)$. It is obviously sufficient to prove the claim for $i \in \mathbb{Z}$ and $t \in [t_0, t_0 + T_0]$ with $T_0 = \frac{1}{2}\min\{T - t_0, \frac{1}{p_0+2}\}$. Assume, towards contradiction, that there exists $\tilde{i} \in \mathbb{Z}$ and $\tilde{t} \in [t_0, t_0 + T_0]$ such that $w_{\tilde{i}}(\tilde{t}) < 0$. Thus

$$w_{\inf} = \inf_{i \in \mathbb{Z}, t \in [t_0, t_0 + T_0]} w_i(t) < 0.$$

Hence, we can find some sequences $i_n \in \mathbb{Z}$ and $t_n \in [t_0, t_0 + T_0]$ such that

$$w_{i_n}(t_n) \to w_{\inf} \quad \text{as } n \to \infty.$$

From (3.1) and the fundamental theorem of calculus for Lebesgue integrals, we obtain

$$\begin{aligned}w_{i_n}(t_n) - w_{i_n}(t_0) &\geq \int_{t_0}^{t_n} [w_{i_n+1}(t) + w_{i_n-1}(t) + p_{i_n}(t)w_{i_n}(t)]dt \\ &\geq \int_{t_0}^{t_n} [2w_{\inf} + p_{i_n}(t)w_{\inf}]dt \\ &\geq T_0(2 + p_0)w_{\inf} \quad \text{for } n \geq 1.\end{aligned}$$

Recall that $w_{i_n}(t_0) \geq 0$, then $w_{i_n}(t_n) \geq T_0(2 + p_0)w_{\inf}$ for $n \geq 1$. It follows that

$$w_{\inf} \geq T_0(2 + p_0)w_{\inf} > \frac{1}{2}w_{\inf} \quad \text{as } n \to \infty,$$

which is a contradiction to $w_{\inf} < 0$. Therefore, $v^1(t) \geq v^2(t)$ for $t \in [t_0, T)$. □

Remark 3.2. Clearly, the results in Lemma 3.1 are still valid for $(1.2)^{\pm}$.

Let $\eta(s) = \frac{1}{2}(1 + \tanh\frac{s}{2})$ for $s \in \mathbb{R}$. Observe that

$$\eta'(s) = \eta(s)(1 - \eta(s)) \quad \text{and} \quad \eta''(s) = \eta(s)(1 - \eta(s))(1 - 2\eta(s)), \quad s \in \mathbb{R}.$$

In addition, there exists a constant $M > 0$ such that, for any $s', s'' \in \mathbb{R}$ with $|s' - s''| \leq 1$,

$$\Big|\frac{\eta'(s')}{\eta'(s'')}\Big| \leq M. \tag{3.2}$$

Without loss of generality, we may assume that $f_i^-(t, s) = 0$ (resp. $f_i^+(t, s) = 0$) for $s \ll 0$ and $i \in \mathbb{Z}$. For otherwise, let $\zeta(\cdot) \in C^\infty(\mathbb{R})$ be such that $\zeta(s) = 1$ for $s \geq 0$ and $\zeta(s) = 0$ for $s \ll 0$. We replace $f_i^-(t, u_i)$ (resp. $f_i^+(t, u_i)$) by $f_i^-(t, u_i)\zeta(u_i)$

(resp. $f_i^+(t,u_i)\zeta(u_i)$). Hence, we may also assume that there is a $u^- \in X$ with $u^- < 0$, such that for any $t_0 \in \mathbb{R}$ and any u^0 with $u^- \le u^0 \le 0$,

$$u^- \le u(t;t_0,u^0,f^-) \le 0 \quad \text{for } t \ge t_0 \tag{3.3}$$
$$(\text{resp. } u^- \le u(t;t_0,u^0,f^+) \le 0 \quad \text{for } t \ge t_0).$$

Lemma 3.3. *There is $C_0 > 0$ such that for any $\alpha_\pm$ with $u^- \le \alpha_- \le 0 < \alpha_+ \le \hat{u}^+_{\inf}$, $C \ge C_0$, $t_0 \in \mathbb{R}$ and $j \in \mathbb{Z}$, the following holds:*

(1) Let

$$\begin{aligned} &v_i^\pm(t;t_0,j)\\ &= u_i(t;t_0,j,\alpha_\pm,f^-)\eta(i+C(t-t_0)) + u_i(t;t_0,j,\alpha_\mp,f^-)(1-\eta(i+C(t-t_0))). \end{aligned}$$

Then $v^+(t;t_0,j) = \{v_i^+(t;t_0,j)\}_{i\in\mathbb{Z}}$ and $v^-(t;t_0,j) = \{v_i^-(t;t_0,j)\}_{i\in\mathbb{Z}}$ are super- and sub-solutions of $(1.2)^-$ on $[t_0,\infty)$, respectively.

(2) Let

$$\begin{aligned} &w_i^\pm(t;t_0,j)\\ &= u_i(t;t_0,j,\alpha_\mp,f^-)\eta(i-C(t-t_0)) + u_i(t;t_0,j,\alpha_\pm,f^-)(1-\eta(i-C(t-t_0))). \end{aligned}$$

Then $w^+(t;t_0,j) = \{w_i^+(t;t_0,j)\}_{i\in\mathbb{Z}}$ and $w^-(t;t_0,j) = \{w_i^-(t;t_0,j)\}_{i\in\mathbb{Z}}$ are super- and sub-solutions of $(1.2)^-$ on $[t_0,\infty)$, respectively.

Proof. Without loss of generality, one can assume that $j = 0$. We prove that $v^+(t;t_0,0)$ is a super-solution of $(1.1)^-$. The other statements can be proven similarly.

First of all, owing to Taylor expansion, for any $i \in \mathbb{Z}$, one has

$$\begin{aligned} &f_i^-\big(t,u_i(t;t_0,\alpha_+,f^-)\big)\eta(i+C(t-t_0))\\ &+ f_i^-(t,u_i(t;t_0,\alpha_-,f^-))(1-\eta(i+C(t-t_0)))\\ &- f_i^-(t,u_i(t;t_0,\alpha_+,f^-)\eta(i+C(t-t_0))\\ &+ u_i(t;t_0,\alpha_-,f^-)(1-\eta(i+C(t-t_0))))\\ &= f_i^-(t,u_i(t;t_0,\alpha_+,f^-) - u_i(t;t_0,\alpha_-,f^-) + u_i(t;t_0,\alpha_-,f^-))\eta(i+C(t-t_0))\\ &\quad + f_i^-(t,u_i(t;t_0,\alpha_-,f^-))(1-\eta(i+C(t-t_0)))\\ &\quad - f_i^-(t,(u_i(t;t_0,\alpha_+,f^-) - u_i(t;t_0,\alpha_-,f^-))\eta(i+C(t-t_0)) + u_i(t;t_0,\alpha_-,f^-))\\ &= \Big(\frac{\partial f_i^-}{\partial s}(t,u_i^*(t) + u_i(t;t_0,\alpha_-,f^-)) - \frac{\partial f_i^-}{\partial s}(t,u_i^*(t)\eta(i+C(t-t_0))\\ &\quad + u_i(t;t_0,\alpha_-,f^-))\Big)\cdot(u_i(t;t_0,\alpha_+,f^-) - u_i(t;t_0,\alpha_-,f^-))\eta(i+C(t-t_0))\\ &= \frac{\partial^2 f_i^-}{\partial s^2}(t,u_i^{**}(t))u_i^*(t)(u_i(t;t_0,\alpha_+,f^-) - u_i(t;t_0,\alpha_-,f^-))\eta'(i+C(t-t_0)), \end{aligned}$$

where $u_i^*(t)$ and $u_i^{**}(t)$ are between $u_i(t;t_0,\alpha_-,f^-)$ and $u_i(t;t_0,\alpha_+,f^-)$.

Next, for any $i \in \mathbb{Z}$, a straightforward calculation then gives

$$\begin{aligned} &\dot{v}_i^+(t;t_0) - v_{i+1}^+(t;t_0) + 2v_i^+(t;t_0) - v_{i-1}^+(t;t_0) - f_i^-(t,v_i^+(t;t_0))\\ &= \eta'(i+C(t-t_0))\Big\{C(u_i(t;t_0,\alpha_+,f^-) - u_i(t;t_0,\alpha_-,f^-))\\ &\quad + [u_{i+1}(t;t_0,\alpha_+,f^-) - u_{i+1}(t;t_0,\alpha_-,f^-)]\\ &\quad \times[\eta(i+C(t-t_0)) - \eta(i+1+C(t-t_0))][\eta'(i+C(t-t_0))]^{-1} \end{aligned}$$

$$
\begin{aligned}
&+ [u_{i-1}(t;t_0,\alpha_+,f^-) - u_{i-1}(t;t_0,\alpha_-,f^-)][\eta(i+C(t-t_0)) \\
&- \eta(i-1+C(t-t_0))][\eta'(i+C(t-t_0))]^{-1} \\
&- \frac{\partial^2 f_i^-}{\partial s^2}(t,u_i^{**}(t))u_i^*(t)(u_i(t;t_0,\alpha_+,f^-) - u_i(t;t_0,\alpha_-,f^-))\Big\}.
\end{aligned}
$$

we know from (3.2) that for all $i \in \mathbb{Z}$ and $t \geq t_0$,

$$
\Big|\frac{\eta(i+1+C(t-t_0)) - \eta(i+C(t-t_0))}{\eta'(i+C(t-t_0))}\Big| \leq M.
$$

By (H2), there is a $\delta_0 > 0$ such that

$$
u_i(t;t_0,\alpha_+,f^-) - u_i(t;t_0,\alpha_-,f^-) \geq \delta_0 \quad \text{for all } i \in \mathbb{Z} \text{ and } t \geq t_0.
$$

Therefore, there is a $C_0 > 0$ such that for any $\alpha_\pm$ with $u^- \leq \alpha_- \leq 0 < \alpha_+ \leq \hat{u}^+_{\rm inf}$, $C \geq C_0$ and $t_0 \in \mathbb{R}$, $v^+(t;t_0,0) = \{v_i^+(t;t_0,0)\}_{i\in\mathbb{Z}}$ is a super-solution of $(1.1)^-$ on $[t_0,\infty)$. This completes the proof. □

Remark 3.4. We can give super- and subsolutions of $(1.2)^+$ in a similar way to Lemma 3.3.

Proof of Theorem 2.1. (1) Let $\alpha_- = 0 < \alpha_+ \leq \hat{u}^+_{\rm inf}$ be given constants. There is $u^{0,*} \in X_1$ such that

$$
\alpha_-\eta(i) + \alpha_+(1-\eta(i)) \geq u_i^{0,*}, \quad i \in \mathbb{Z}.
$$

Then by Remarks 3.2 and 3.4, there is $C_1 > 0$ such that

$$
\begin{aligned}
w_i^+(t+t_0;t_0,j) =& u_i(t+t_0;t_0,j,\alpha_-,f^+)\eta(i-C_1t) \\
&+ u_i(t+t_0;t_0,j,\alpha_+,f^+)(1-\eta(i-C_1t)) \\
\geq& u_i(t+t_0;t_0,j,u^{0,*},f^+)
\end{aligned}
$$

for $t \geq 0$, $t_0 \in \mathbb{R}$, and $i,j \in \mathbb{Z}$. Therefore, one has that for any $C > C_1$,

$$
\limsup_{i\geq Ct, t\to\infty} u_i(t+t_0;t_0,j,u^{0,*},f^+) = 0 \tag{3.4}
$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \mathbb{Z}$.

For any $u^0 \in X_1$, by (H2) and Remark 3.4, there are $T > 0$ and $i_T \in \mathbb{Z}$ such that

$$
u_{i+i_T}(t_0;t_0-T,u^{0,*},f^+) \geq u_i^0, \quad i \in \mathbb{Z}
$$

for all $t_0 \in \mathbb{R}$. It then follows that

$$
u_{i+i_T}(t+t_0;t_0-T,-i_T,u^{0,*},f^+) \geq u_i(t+t_0;t_0,u^0,f^+)
$$

for any $t_0 \in \mathbb{R}$, $t \geq 0$, $i \in \mathbb{Z}$. This together with (3.4) implies that for any $C' > C$ and $u^0 \in X_1$,

$$
\begin{aligned}
0 =& \limsup_{i+i_T\geq Ct, t\to\infty} u_{i+i_T}(t+t_0;t_0-T,-i_T,u^{0,*},f^+) \\
\geq& \limsup_{i\geq C't, t\to\infty} u_{i+i_T}(t+t_0;t_0-T,-i_T,u^{0,*},f^+) \\
\geq& \limsup_{i\geq C't, t\to\infty} u_i(t+t_0;t_0,u^0,f^+) \geq 0
\end{aligned}
$$

uniformly in $t_0 \in \mathbb{R}$. Therefore, $\check{c}^*_{\rm sup} \leq C_1$.

Now let $\hat{u}^+_{\rm inf} > \alpha_+ > 0 > \alpha_- \ge u^-$ be given constants, where u^- satisfies (3.3). There is $u^{0,**} \in X_1$ such that

$$\alpha_-\eta(i) + \alpha_+(1-\eta(i)) \le u^{0,**}_i, \quad i \in \mathbb{Z}.$$

Then by Remark 3.2 and Lemma 3.3,

$$\begin{aligned} v^-_i(t+t_0;t_0,j) &= u_i(t+t_0;t_0,j,\alpha_-,f^-)\eta(i+C_0t) \\ &\quad + u_i(t+t_0;t_0,j,\alpha_+,f^-)(1-\eta(i+C_0t)) \\ &\le u_i(t+t_0;t_0,j,u^{0,**},f^-) \end{aligned}$$

for $t \ge 0$, $t_0 \in \mathbb{R}$ and $i,j \in \mathbb{Z}$. This implies that for $C < -C_0$,

$$\liminf_{i \le Ct, t\to\infty} (u_i(t+t_0;t_0,j,u^{0,**},f^-) - \hat{u}^+_i(t+t_0;j)) = 0 \tag{3.5}$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \mathbb{Z}$.

For any $u^{0,*} \in X_1$, by (H2) and Lemma 3.3, there are $T > 0$ and $i_T \in \mathbb{Z}$ such that

$$\hat{u}^+_{i-i_T}(T+t_0) \ge u_{i-i_T}(T+t_0;t_0,u^{0,*},f^-) \ge u^{0,**}_i, \quad i \in \mathbb{Z}$$

for any $t_0 \in \mathbb{R}$. This implies that

$$\begin{aligned} \hat{u}^+_i(t;-i_T) &\ge u_i(t;T+t_0,-i_T,u(T+t_0;t_0,-i_T,u^{0,*}(-i_T),f^-),f^-) \\ &\ge u_i(t;T+t_0,-i_T,u^{0,**},f^-), \quad i \in \mathbb{Z} \end{aligned}$$

and hence

$$\begin{aligned} \hat{u}^+_i(t) &\ge u_i(t;T+t_0,u(T+t_0;t_0,u^{0,*},f^-),f^-) \\ &\ge u_{i+i_T}(t;T+t_0,-i_T,u^{0,**},f^-), \quad i \in \mathbb{Z} \end{aligned}$$

for $t \ge T+t_0$. This and (3.5) imply that for any $C' < C$,

$$\begin{aligned} 0 &\ge \liminf_{i \le C't, t\to\infty} (u_{i+i_T}(t+T+t_0;T+t_0,-i_T,u^{0,**},f^-) - \hat{u}^+_i(t+T+t_0)) \\ &\ge \liminf_{i+i_T \le Ct, t\to\infty} (u_{i+i_T}(t+T+t_0;T+t_0,-i_T,u^{0,**},f^-) - \hat{u}^+_i(t+T+t_0)) = 0 \end{aligned}$$

uniformly in $t_0 \in \mathbb{R}$. Therefore,

$$\liminf_{i \le C't, t\to\infty} (u_i(t+t_0;t_0,u^{0,*},f^-) - \hat{u}^+_i(t+t_0)) = 0$$

uniformly in $t_0 \in \mathbb{R}$. Therefore, $\hat{c}^*_{\rm inf} \ge -C_0$.

As a conclusion, $\hat{c}^*_{\rm inf}$ and $\check{c}^*_{\rm sup}$ are finite.

(2) Assume $c < \hat{c}^*_{\rm inf}$. Let $c < c_1 < \hat{c}^*_{\rm inf}$. For any $u^0 \in X_1$, note that there is $u^{0,*} \in X_1$ such that $u^0_{i-j} \ge u^{0,*}_i$ for all $i \in \mathbb{Z}$ and $j \in \{1,\dots,N\}$. By Lemma 3.1, we have

$$u_{i+j}(t;t_0,u^0(-j),f^-) \ge u_{i+j}(t;t_0,u^{0,*},f^-), \quad i \in \mathbb{Z},\ t > t_0$$

for all $t_0 \in \mathbb{R}$ and $j \in \{1,\dots,N\}$. By the assumption,

$$\liminf_{i \le c_1t, t\to\infty} (u_i(t+t_0;t_0,u^{0,*}f^-) - \hat{u}^+_i(t+t_0)) = 0$$

uniformly in $t_0 \in \mathbb{R}$. This implies that

$$\liminf_{i+j \le c_1t, t\to\infty} (u_{i+j}(t+t_0;t_0,u^{0,*},f^-) - \hat{u}^+_{i+j}(t+t_0)) = 0$$

and then

$$\liminf_{i+j \le c_1t, t\to\infty} (u_{i+j}(t+t_0;t_0,u^0(-j),f^-) - \hat{u}^+_{i+j}(t+t_0)) = 0$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \{1, \dots, N\}$. Hence

$$\liminf_{i \le ct, t \to \infty} (u_i(t+t_0; t_0, j, u^0, f^-) - \hat{u}_i^+(t+t_0; j)) = 0$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \{1, \dots, N\}$. By the periodicity of f^- with respect to i, it then follows that

$$\liminf_{i \le ct, t \to \infty} (u_i(t+t_0; t_0, j, u^0, f^-) - \hat{u}_i^+(t+t_0; j)) = 0$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \mathbb{Z}$. This and Remark 3.2 imply that the limit in (2.1) is uniform in $t_0 \in \mathbb{R}$ and $j \in \mathbb{Z}$.

(3) Assume $c > \check{c}^*_{\rm sup}$. Let $c > c_1 > \check{c}^*_{\rm sup}$. For any $u^0 \in X_1$, note that there is $u^{0,*} \in X_1$ such that $u_i^{0,*} \ge u^0_{i-j}$ for all $i \in \mathbb{Z}$ and $j \in \{1, \dots, N\}$. By Lemma 3.1,

$$u_{i+j}(t+t_0; t_0, u^0(-j), f^+) \le u_{i+j}(t+t_0; t_0, u^{0,*}, f^+), \quad i \in \mathbb{Z},\ t > t_0$$

for all $t_0 \in \mathbb{R}$ and $j \in \{1, \dots, N\}$. By the assumptions,

$$\limsup_{i \ge c_1 t, t \to \infty} u_i(t+t_0; t_0, u^{0,*}, f^+) = 0$$

uniformly in $t_0 \in \mathbb{R}$. This implies that

$$\limsup_{i+j \ge c_1 t, t \to \infty} u_{i+j}(t+t_0; t_0, u^{0,*}, f^+) = 0$$

and then

$$\limsup_{i+j \ge c_1 t, t \to \infty} u_{i+j}(t+t_0; t_0, u^0(-j), f^+) = 0$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \{1, \dots, N\}$. Hence

$$\limsup_{i \ge ct, t \to \infty} u_i(t+t_0; t_0, j, u^0, f^+) = 0$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \{1, \dots, N\}$. This and the periodicity of f^+ with respect to i and Remark 3.2 implies that the limit in (2.2) is uniform in $t_0 \in \mathbb{R}$ and $j \in \mathbb{Z}$.

(4) Take any $c > \check{c}^*_{\rm sup}$ and fix it. First, for given $0 \le u^0 < \hat{u}^+_{\rm inf}$ ($u^0 \in X$) satisfying $u^0_i = 0$ for $i \gg 1$, there is $\tilde{u}^0 \in X_1$ such that $u^0 \le \tilde{u}^0$. Then

$$0 \le u(t+t_0; t_0, j, u^0) \le u(t+t_0; t_0, j, \tilde{u}^0)$$

for $t > 0$ and $t_0 \in \mathbb{R}$. It then follows from (3) that

$$0 \le \limsup_{i \ge ct, t \to \infty} u_i(t+t_0; t_0, j, u^0) \le \limsup_{i \ge ct, t \to \infty} u_i(t+t_0; t_0, j, \tilde{u}^0) = 0$$

uniformly in $t_0 \in \mathbb{R}$ and $j \in \mathbb{Z}$. The proof is complete. □

Lemma 3.5. *For given $\gamma' < c^-$, there is $T > 0$ such that $\gamma' < \inf_{\lambda > 0} \frac{\lfloor M \rfloor_T(f^-;\lambda)}{\lambda}$.*

Proof. By [10, Proposition 3.1], we have

$$\partial_s f_i^-(t,0)_{\rm inf} = \lim_{T \to \infty} \inf_{t \in \mathbb{R}} \frac{1}{T} \int_t^{t+T} \partial_s f_i^-(\tau, 0) d\tau,\ i \in \mathbb{Z}.$$

This and [6, Lemma 2.1] imply the statement of the lemma. □

Lemma 3.6. *For a given $C > 0$, consider*

$$\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + u_i(t)(\lfloor \partial_s f_i^-(t,0)\rfloor_T - Cu_i(t)), \quad i \in \mathbb{Z}. \tag{3.6}$$

Then

$$c^* := \inf_{\lambda>0} \frac{\lfloor M\rfloor_T(f^-;\lambda)}{\lambda}$$

is the spreading speed of (3.6) *which coincides with the minimal speed.*

The above lemma follows from [6, Lemma 2.1] and [9, Theorems 4.1, 4.2].

Proof of Theorem 2.2. (1) Let

$$\delta_0 = \liminf_{t_0\in\mathbb{R}, j\le\gamma' t, t\to\infty} u_j(t+t_0;t_0,u^0,f^-).$$

Then there is $T > 0$ such that

$$u_j(t+t_0;t_0,u^0,f^-) \ge \frac{\delta_0}{2} \quad \forall t_0 \in \mathbb{R},\ j \le \gamma' t,\ t \ge T.$$

Assume that there is $\gamma_0 < \gamma'$ such that (2.3) does not hold. Then there are $\epsilon_0 > 0$, $t_{0;n} \in \mathbb{R}$, $j_n \in \mathbb{Z}$, and $t_n > 0$ such that $j_n \le \gamma_0 t_n$, $t_n \to \infty$ and

$$u_{j_n}(t_n + t_{0;n}; t_{0;n}, u^0, f^-) - \hat{u}^+(t_n + t_{0;n}) \le -\epsilon_0. \tag{3.7}$$

Let $\tilde{u}^0 = \{\tilde{u}^0_j\}_{j\in\mathbb{Z}}$ where $\tilde{u}^0_j = \frac{\delta_0}{2}$ for all $j \in \mathbb{Z}$. By (H2), there is $\tilde{T} \ge T$ such that

$$|u_j(t+t_0;t_0,\tilde{u}^0,f^-) - \hat{u}^+_j(t+t_0)| \le \frac{\epsilon_0}{2} \tag{3.8}$$

for any $t_0 \in \mathbb{R}$, $j \in \mathbb{Z}$, $t \ge \tilde{T}$. Let $\tilde{u}^n = \{\tilde{u}^n_j\}_{j\in\mathbb{Z}}$ be given by

$$\tilde{u}^n_j = \begin{cases} \delta_0/2 & j \le (\gamma' - \gamma_0)(t_n - \tilde{T}),\\ 0, & \text{otherwise.} \end{cases}$$

Next we claim that for each $t > 0$,

$$\lim_{n\to\infty} u_j(t+t_0;t_0,\tilde{u}^n,f^-) = u_j(t+t_0;t_0,\tilde{u}^0,f^-)$$

uniformly for j in bounded subsets of $\mathbb{Z}$ and $t_0 \in \mathbb{R}$. Let $v^n_j(t+t_0;t_0) = u_j(t+t_0;t_0,\tilde{u}^0,f^-) - u_j(t+t_0;t_0,\tilde{u}^n,f^-)$. Then $v^n(t+t_0;t_0)$ satisfies

$$\dot{v}^n_j(t+t_0) = v^n_{j+1}(t+t_0) - 2v^n_j(t+t_0) + v^n_{j-1}(t+t_0) + a^n_j(t+t_0)v^n_j(t+t_0), \quad j \in \mathbb{R},\ t > 0,$$

where

$$a^n_j(t+t_0;t_0) = \int_0^1 \frac{\partial f_j^-}{\partial s}(t, \tau u_j(t+t_0;t_0,\tilde{u}^0,f^-) + (1-\tau)u_j(t+t_0;t_0,\tilde{u}^n,f^-))d\tau.$$

Observe that $(a^n(t+t_0;t_0))_{n\in\mathbb{Z}}$ is uniformly bounded. Then we can find $M > 0$ such that

$$\|a^n(t+t_0;t_0)\| \le M \quad \forall n \in \mathbb{Z},\ t > 0,\ t_0 \in \mathbb{R}.$$

We consider

$$\begin{gathered} \dot{V}^n_j(t+t_0) = V^n_{j+1}(t+t_0) - 2V^n_j(t+t_0) + V^n_{j-1}(t+t_0) + MV^n_j(t+t_0),\\ j \in \mathbb{R},\ t > 0,\\ V^n(t_0) = v^n(t_0). \end{gathered} \tag{3.9}$$

Giving the definition of discrete Fourier transform and inverse Fourier transform [15] as follows:

$$\hat{V}^n(t+t_0,\omega) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-i(\omega j)} V_j^n(t+t_0), \quad \omega \in [-\pi,\pi],\ t>0,\ t_0 \in \mathbb{R} \tag{3.10}$$

and

$$V_j^n(t+t_0) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i(\omega j)} \hat{V}^n(t+t_0,\omega) d\omega, \quad j \in \mathbb{Z},\ t>0,\ t_0 \in \mathbb{R}, \tag{3.11}$$

where i is the imaginary unit. Applying the discrete Fourier transform (3.10) to (3.9) yields

$$\frac{\partial}{\partial t}\hat{V}^n(t+t_0,\omega) = (e^{i\omega} + e^{-i\omega} - 2 + M)\hat{V}^n(t+t_0,\omega) \quad \text{for } t>0.$$

This equation can be solved easily as

$$\hat{V}^n(t+t_0,\omega) = \exp[(e^{i\omega} + e^{-i\omega} - 2 + M)t]\hat{V}^n(t_0,\omega) \quad \text{for } t>0.$$

Using the inverse discrete Fourier transform (3.11) we obtain

$$\begin{aligned} V_j^n(t+t_0) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} V_k^n(t_0) e^{(M-2)t} \int_{-\pi}^{\pi} e^{i\omega(j-k)} e^{2(\cos\omega)t} d\omega \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} V_k^n(t_0) e^{(M-2)t} \int_{-\pi}^{\pi} \cos((j-k)\omega) e^{2(\cos\omega)t} d\omega \quad t>0,\ t_0 \in \mathbb{R}. \end{aligned}$$

Note that

$$1 = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-2t} \int_{-\pi}^{\pi} \cos((j-k)\omega) e^{2(\cos\omega)t} d\omega \quad \forall t>0$$

and by [16, Lemma 2.1],

$$\int_{-\pi}^{\pi} \cos((j-k)\omega) e^{2(\cos\omega)t} d\omega > 0 \quad \forall t>0.$$

This and Lemma 3.1 imply that

$$\begin{aligned} 0 &\le v_j^n(t+t_0;t_0) \le V_j^n(t+t_0;t_0) \\ &= \frac{\delta_0}{4\pi} e^{(M-2)t} \sum_{k=[J^n]+1-j}^{\infty} \int_{-\pi}^{\pi} \cos(k\omega) e^{2(\cos\omega)t} d\omega \to 0 \quad \text{as } n \to \infty \end{aligned}$$

uniformly for j in bounded subsets of $\mathbb{Z}$ and $t_0 \in \mathbb{R}$, where $J^n := [(\gamma'-\gamma_0)(t_n-\tilde{T})] \in \mathbb{Z}$ and $J^n \to +\infty$ as $n \to \infty$. Hence the claim holds. By the above claim, we have

$$\lim_{n\to\infty} u_j(t_n+t_{0;n}; t_{0;n}+t_n-\tilde{T}, \tilde{u}^n, f^-) = u_j(t_n+t_{0;n}; t_{0;n}+t_n-\tilde{T}, \tilde{u}^0, f^-) \tag{3.12}$$

uniformly for j in bounded subsets of $\mathbb{Z}$. Observe that

$$\begin{aligned} &u_{j_n}\big(t_n+t_{0;n}; t_{0;n}, u^0, f^-\big) \\ &= u_{j_n}\Big(t_n+t_{0;n}; t_n+t_{0;n}-\tilde{T}, u\big(t_n+t_{0;n}-\tilde{T}; t_{0;n}, u^0, f^-\big), f^-\Big) \\ &= u_0\Big(t_n+t_{0;n}; t_n+t_{0;n}-\tilde{T}, j_n, u\big(t_n+t_{0;n}-\tilde{T}; t_{0;n}, j_n, u^0(j_n), f^-\big), f^-\Big) \\ &\ge u_0\Big(t_n+t_{0;n}; t_n+t_{0;n}-\tilde{T}, \tilde{u}^n, f^-\Big) \quad \text{for } n \gg 1. \end{aligned}$$

This together with (3.8) and (3.12) imply that

$$u_{j_n}(t_n+t_{0;n};t_{0;n},u^0,f^-) > \hat{u}^+_{j_n}(t_n+t_{0;n})-\epsilon_0$$

for $n\gg 1$, which contradicts to (3.7). (1) is thus proved.

(2) First we prove that for any given $\gamma'<c^-$ and $u^0\in X_1$,

$$\liminf_{i\le\gamma' t,t\to\infty} u_i\left(t+t_0;t_0,u^0,f^-\right)>0. \tag{3.13}$$

For the given $\gamma'<c^-$, let $T>0$ be as in Lemma 3.5. Then we have $\gamma'<\inf_{\lambda>0}\frac{\lfloor M\rfloor_T(f^-;\lambda)}{\lambda}$. Note that from the proof of [10, Lemma 3.2], we can get that for given $T>0$, there is $G_i(t)\in W^{1,\infty}(\mathbb{R})$ such that

$$\operatorname{ess\,inf}_{t\in\mathbb{R}}(G_i'(t)+\partial_s f_i^-(t,0))=\lfloor\partial_s f_i^-(t,0)\rfloor_T\quad\forall\, i\in\mathbb{Z}.$$

Call $v_i(t)=u_i\left(t+t_0;t_0,u^0,f^-\right)e^{G_i(t)}$ ($i\in\mathbb{Z}$). Then $v_i(t)$ is absolutely continuous in and differentiable in $t\in\mathbb{R}$ and satisfies

$$\begin{aligned}
\dot v_i(t)&=\dot u_i\left(t+t_0;t_0,u^0\right)e^{G_i(t)}+G_i'(t)u_i\left(t+t_0;t_0,u^0\right)e^{G_i(t)}\\
&=v_{i+1}(t)+v_{i-1}(t)-2v_i(t)+f_i^-\left(t,u_i\left(t+t_0;t_0,u^0,f^-\right)\right)e^{G_i(t)}+G_i'(t)v_i(t)\\
&\ge v_{i+1}(t)+v_{i-1}(t)-2v_i(t)\\
&\quad+v_i(t)\left(\partial_s f_i^-(t,0)-\tilde M_0u_i\left(t+t_0;t_0,u^0,f^-\right)+G_i'(t)\right)\\
&\ge v_{i+1}(t)+v_{i-1}(t)-2v_i(t)\\
&\quad+v_i(t)\left(\lfloor\partial_s f_i^-(t,0)\rfloor_T-\tilde M_0u_i\left(t+t_0;t_0,u^0,f^-\right)\right)\\
&=v_{i+1}(t)+v_{i-1}(t)-2v_i(t)+v_i(t)\left(\lfloor\partial_s f_i^-(t,0)\rfloor_T-\tilde M_0e^{-G_i(t)}v_i(t)\right)\\
&\ge v_{i+1}(t)+v_{i-1}(t)-2v_i(t)+v_i(t)\left(\lfloor\partial_s f_i^-(t,0)\rfloor_T-\tilde M v_i(t)\right),\ i\in\mathbb{Z}
\end{aligned}$$

where $\tilde M=\tilde M_0\sup_{i\in\mathbb{Z},t\in\mathbb{R}}e^{-G_i(t)}$. By Lemmas 3.5 and 3.6,

$$\liminf_{i\le\gamma' t,t\to\infty}v_i(t)>0.$$

This implies (3.13).

For any $\gamma<c^-$, let $\gamma'\in(\gamma,c^-)$. Then (3.13) and (1) imply

$$\liminf_{i\le\gamma t,t\to\infty}\left(u_i(t+t_0;t_0,u^0)-\hat u_i^+(t+t_0)\right)\ge 0.$$

Thus $c^-\le c^*_{\rm inf}$.

Next we prove that for any $\gamma>c^+$ and $u^0\in X_1$,

$$\limsup_{i\ge\gamma t,t\to\infty}u_i(t+t_0;t_0,u^0)=0. \tag{3.14}$$

For the given $\gamma>c^+$, there is $\tilde T>0$ such that

$$\gamma>\inf_{\mu>0}\frac{\lceil M\rceil_{\tilde T}(f^+;\lambda)}{\lambda}. \tag{3.15}$$

Then by [10, Lemma 3.2], there is $\tilde G_i(t)\in W^{1,\infty}(\mathbb{R})$ such that

$$-\lceil\partial_s f_i^+(t,0)\rceil_{\tilde T}=\inf_{k\in\mathbb{Z}}\frac{1}{\tilde T}\int_{(k-1)\tilde T}^{k\tilde T}(-\partial_s f_i^+(\tau,0))d\tau=\operatorname{ess\,inf}_{t\in\mathbb{R}}(-\tilde G_i'(t)-\partial_s f_i^+(t,0))$$

ll $i \in \mathbb{Z}$. Let $\tilde{v}_i(t) = u_i\left(t+t_0; t_0, u^0\right) e^{\tilde{G}_i(t)}$. By (H1), $f_i(t,u_i) \le \partial_s f_i^+(t,0)u_i - \tilde{m}_0 u_i^2$ for all $i \in \mathbb{Z}$. Then $\tilde{v}_i(t)$ is absolutely continuous in $t \in \mathbb{R}$ and satisfies

$$\begin{aligned}
\dot{\tilde{v}}_i(t) &= \dot{u}_i(t+t_0;t_0,u^0)e^{\tilde{G}_i(t)} + \tilde{G}_i'(t)u_i(t+t_0;t_0,u^0)e^{\tilde{G}_i(t)} \\
&= \tilde{v}_{i+1}(t) + \tilde{v}_{i-1}(t) - 2\tilde{v}_i(t) + f_i\left(t, u_i\left(t+t_0;t_0,u^0\right)\right)e^{\tilde{G}_i(t)} + \tilde{v}_i(t)\tilde{G}_i'(t) \\
&\le \tilde{v}_{i+1}(t) + \tilde{v}_{i-1}(t) - 2\tilde{v}_i(t) + \tilde{v}_i(t)\Big(\partial_s f_i^+(t,0) - \tilde{m}_0 u_i\big(t+t_0;t_0,u^0\big) + \tilde{G}_i'(t)\Big) \\
&\le \tilde{v}_{i+1}(t) + \tilde{v}_{i-1}(t) - 2\tilde{v}_i(t) + \tilde{v}_i(t)\left(\lceil \partial_s f_i^+(t,0)\rceil_{\tilde{T}} - \tilde{m}_0 u_i(t+t_0;t_0,u^0)\right) \\
&= \tilde{v}_{i+1}(t) + \tilde{v}_{i-1}(t) - 2\tilde{v}_i(t) + \tilde{v}_i(t)\left(\lceil \partial_s f_i^+(t,0)\rceil_{\tilde{T}} - \tilde{m}_0 e^{-\tilde{G}_i(t)}\tilde{v}_i(t)\right) \\
&\le \tilde{v}_{i+1}(t) + \tilde{v}_{i-1}(t) - 2\tilde{v}_i(t) + \tilde{v}_i(t)\left(\lceil \partial_s f_i^+(t,0)\rceil_{\tilde{T}} - \tilde{m}\tilde{v}_i(t)\right), \quad i \in \mathbb{Z}
\end{aligned}$$

where $\tilde{m} = \tilde{m}_0 \inf_{i\in\mathbb{Z}, t\in\mathbb{R}} e^{-\tilde{G}_i(t)}$. By Lemma 3.6 and (3.15),

$$\limsup_{i\ge\gamma t, t\to\infty} \tilde{v}_i(t) = 0.$$

This implies (3.14). Hence $c^+ \ge c^*_{\sup}$. The proof of Theorem 2.2 is now complete. □

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