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EXISTENCE OF POSITIVE SOLUTIONS FOR A NONLINEAR QUADRATIC INTEGRAL EQUATION

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Abstract. In this article, we study the existence of positive solutions for the nonlinear quadratic integral equation

$$x(t) = g(t, x(t)) \int_{-\infty}^{t} a(t, t-s) f(s, x(s)) ds, \quad t \in \mathbb{R}.$$

By using fixed point theory on cones, we prove the existence and uniqueness of bounded and continuous solution with positive infimum. An example illustrates the abstract result.

1. Introduction

The direct impetus of this paper comes from two sources. The first source is the literature on the existence of positive solutions for the equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s)) ds, \quad t \in \mathbb{R}, \tag{1.1}$$

which is a model for the spread of some infectious disease (cf. [6]). In fact, many authors have studied the existence of positive solutions, especially periodic and almost periodic solutions, of (1.1) and its variants (see, e.g., [1, 2, 3, 4, 5, 11, 12, 14, 19, 22] and references therein). There are several interesting works on generalized variants of equation (1.1). For example, Torrejón [22] studied the integral equation

$$x(t) = \int_{t-\tau(t)}^{t} f(s, x(s)) ds, \quad t \in \mathbb{R},$$

where the delay is state-dependent. Ait Dads and Ezzinbi [1] considered the neutral integral equation

$$x(t) = \gamma x(t-\tau) + (1-\gamma) \int_{t-\tau}^{t} f(s, x(s)) ds, \quad t \in \mathbb{R}. \tag{1.2}$$

Ait Dads and Ezzinbi [2] investigated the infinite delay integral equation

$$x(t) = \int_{-\infty}^{t} a(t-s) f(s, x(s)) ds, \quad t \in \mathbb{R}. \tag{1.3}$$

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Afterwards, Ait Dads, Cieutat, and Lhachimi [4] generalized equation (1.3), i.e., they discussed the following more general infinite delay integral equation

$$x(t)=\int_{-\infty}^{t} a(t,t-s)f(s,x(s))ds, \quad t\in\mathbb{R}. \tag{1.4}$$

In fact, (1.1) is also a special case of (1.4). This is because, if

$$a(t,s)=\begin{cases} 1, & s\in[0,\tau],\ t\in\mathbb{R},\\ 0, & s>\tau,\ t\in\mathbb{R},\end{cases}$$

then equation (1.4) recovers equation (1.1). In fact, it is still of great interest for several authors to work on this direction (see, e.g., [11, 5]). As noted in [4] and [5], these variants of (1.1) include many important integral and functional equations that arise in biomathematics.

The second source of this paper comes from the fact that quadratic functional integral equations are one of the most attractive and interesting research area of integral equations and functional integral equations. In fact, as noted in some earlier literature (see, e.g., [20] and references therein), the nonlinear quadratic functional integral equations has been applied to, for example, the theory of radiative transfer, kinetic theory of gases, the theory of neutron transport, the traffic theory, plasma physics, and numerous branches of mathematical physics. There is a lot of literature on the existence of solutions for quadratic functional integral equations. We refer the reader to [20, 18, 10, 21, 8, 17, 7, 16, 13] for some of recent results.

Motivated by the above works, in this paper, we study the nonlinear quadratic integral equation

$$x(t)=g(t,x(t))\int_{-\infty}^{t} a(t,t-s)f(s,x(s))ds, \quad t\in\mathbb{R}, \tag{1.5}$$

where f,g,a satisfy some conditions stated in Section 3.

2. Preliminaries

Let E and F be two metric spaces. We denote by $C(E,F)$ the space of continuous functions, and by $BC(E,F)$ the space of continuous and bounded functions defined on E with values in F. Let $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers, and $\mathbb{R}^+$ the set of nonnegative real numbers. In the case $E=\mathbb{R}$ and $F=\mathbb{R}^+$, for every $x,y\in BC(\mathbb{R},\mathbb{R}^+)$, we denote the distance between x and y by

$$\|x-y\|=\sup_{t\in\mathbb{R}}|x(t)-y(t)|.$$

We denote by $L^1(\mathbb{R}^+)$ the space of Lebesgue measurable functions on $\mathbb{R}^+$ with norm

$$\|x\|_{L^1(\mathbb{R}^+)}=\int_0^{+\infty}|x(t)|dt.$$

Now, we recall some basic notation about cone (for more details see [9]). Let X be a real Banach space, and θ be the zero element in X. A closed convex set K in X is called a cone if the following conditions are satisfied:

(1) if $x\in K$, then $\lambda x\in K$ for any $\lambda\geq 0$,
(2) if $x\in K$ and $-x\in K$, then $x=\theta$.

A cone K induces a partial ordering $\leq$ in X by

$$x \leq y \Leftrightarrow y - x \in K.$$

For any given $u, v \in K$ with $u \leq v$,

$$[u, v] := \{x \in X : u \leq x \leq v\}.$$

A cone K is called normal if there exists a constant $k > 0$ such that

$$\theta \leq x \leq y \Rightarrow \|x\| \leq k\|y\|,$$

where $\|\cdot\|$ is the norm on X. We denote by K° the interior of K. A cone K is called a solid cone if $K^\circ \neq \emptyset$.

Lemma 2.1 ([4])**.** *Suppose that the function $t \mapsto a(t, \cdot)$ is in $BC(\mathbb{R}, L^1(\mathbb{R}^+))$ and $f \in BC(\mathbb{R}, \mathbb{R})$. Then $F \in BC(\mathbb{R}, \mathbb{R})$, where*

$$F(t) = \int_{-\infty}^{t} a(t, t-s) f(s) ds, \quad t \in \mathbb{R}.$$

Theorem 2.2 ([11])**.** *Let K be a normal solid cone in a real Banach space X, $D : K \to K$ be a linear operator, and A, B be two operators from $K^\circ \times K^\circ \times K^\circ$ to K° with*

$$A(x, y, z) = B(x, y, z) + D(x), \quad x, y, z \in K^\circ.$$

Assume that the following conditions hold:

(1) *for every $x, y, z \in K^\circ$, $B(\cdot, y, z)$ is increasing in K°, $B(x, \cdot, z)$ is decreasing in K°, and $B(x, y, \cdot)$ is decreasing in K°;*
(2) *there exists a function $\varphi : (0, 1) \times K^\circ \times K^\circ \to (0, +\infty)$ such that for every $x, y, z \in K^\circ$ and $t \in (0, 1), \varphi(t, x, y) > t$ and*

$$B(tx, t^{-1}y, z) \geq \varphi(t, x, y) B(x, y, z);$$

(3) *there exist $x_0, y_0 \in K^\circ$ with $x_0 \leq y_0$, $A(x_0, y_0, x_0) \geq x_0$ and $A(y_0, x_0, y_0) \leq y_0$ such that*

$$\inf_{x,y \in [x_0, y_0]} \varphi(t, x, y) > t \tag{2.1}$$

for all $t \in (0, 1)$;
(4) *there exists a constant $L > 0$ such that for all $x, y, z_1, z_2 \in K^\circ$ with $z_1 \geq z_2$,*

$$B(x, y, z_1) - B(x, y, z_2) \geq -L(z_1 - z_2).$$

Then A has a unique fixed point $x^* \in [x_0, y_0]$, i.e., $A(x^*, x^*, x^*) = x^*$. In addition, if (2.1) is strengthened to the case for all $u, v \in K^\circ$ with $u \leq v$,

$$\inf_{x,y \in [u, v]} \varphi(t, x, y) > t$$

for all $t \in (0, 1)$. Then x^* is the unique fixed point of A in K°.

In this paper, we utilize the following corollary of Theorem 2.2:

Corollary 2.3. *Let K be a normal solid cone in a real Banach space X and A be an operator from K° to K° satisfying the following conditions:*

(1) *A is increasing in K°;*
(2) *there exists a function $\varphi : (0, 1) \to (0, \infty)$ such that for every $x \in K^\circ$ and $\lambda \in (0, 1), \varphi(\lambda) > \lambda$ and*

$$A(\lambda x) \geq \varphi(\lambda) A(x);$$

(3) *there exist* $x_0, y_0 \in K^\circ$ *with* $x_0 \le y_0$ *such that* $A(x_0) \ge x_0$ *and* $A(y_0) \le y_0$. *Then* A *has a unique fixed point* x^* *in* K°.

3. Main results

In this section, we study the nonlinear integral equation

$$x(t) = g(t, x(t)) \int_{-\infty}^{t} a(t, t-s) f(s, x(s)) ds, \quad t \in \mathbb{R} \tag{3.1}$$

under the following assumptions:

- (H1) $f \in BC(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$ such that for every $s \in \mathbb{R}$, $f(s, \cdot)$ is increasing in $\mathbb{R}^+$.
- (H2) There exists $\alpha \in (0,1)$ such that
$$f(s, \lambda x) \ge \lambda^\alpha f(s, x)$$
for all $x \ge 0, \lambda \in (0,1)$ and $s \in \mathbb{R}$.
- (H3) a is a function from $\mathbb{R} \times \mathbb{R}^+$ to $\mathbb{R}_+$, and the function $t \mapsto a(t, \cdot)$ is in $BC(\mathbb{R}, L^1(\mathbb{R}^+))$.
- (H4) $g \in BC(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$ such that for every $t \in \mathbb{R}$, $g(t, \cdot)$ is increasing in $\mathbb{R}^+$.
- (H5) There exists $L_g > 0$ such that
$$|g(t, x_1) - g(t, x_2)| \le L_g |x_1 - x_2|$$
for all $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^+$.
- (H6) There exists $\beta \in (0, 1-\alpha)$ such that
$$g(t, \lambda x) \ge \lambda^\beta g(t, x)$$
for all $x \ge 0, \lambda \in (0,1)$ and $t \in \mathbb{R}$.
- (H7) There exists a constant $c > 0$ such that
$$\inf_{t \in \mathbb{R}} g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, c) ds \ge c.$$

Theorem 3.1. *Let* (H1)–(H7) *hold and* $L_g M_f D < 1$, *where*

$$M_f = \sup\{|f(t,x)| : t \in \mathbb{R},\ x \in \mathbb{R}^+\}, \quad D = \sup_{t \in \mathbb{R}} \int_0^{+\infty} |a(t,s)| ds.$$

Then equation (3.1) *has a unique solution with positive infimum in* $BC(\mathbb{R}, \mathbb{R}^+)$.

Proof. Let

$$K = \{y \in BC(\mathbb{R}, \mathbb{R}^+) : y(t) \ge 0, \forall t \in \mathbb{R}\}.$$

Then

$$K^\circ = \{y \in BC(\mathbb{R}, \mathbb{R}^+) : \text{there exists } \xi > 0 \text{ such that } y(t) \ge \xi, \forall t \in \mathbb{R}\}.$$

It is easy to verify that K is a normal and solid cone in $BC(\mathbb{R}, \mathbb{R}^+)$.

For $y \in BC(\mathbb{R}, \mathbb{R}^+)$, define an operator A_y on $BC(\mathbb{R}, \mathbb{R}^+)$ by

$$(A_y x)(t) = g(t, x(t)) \int_{-\infty}^{t} a(t, t-s) f(s, y(s)) ds, \quad x \in BC(\mathbb{R}, \mathbb{R}^+),\ t \in \mathbb{R}. \tag{3.2}$$

It is not difficult to verify that A_y is an operator from $BC(\mathbb{R}, \mathbb{R}^+)$ into itself. Moreover, by a direct calculations, for every $x_1, x_2 \in BC(\mathbb{R}, \mathbb{R}^+)$, we can get

$$\|A_y(x_1) - A_y(x_2)\| \le L_g M_f D \|x_1 - x_2\|.$$

Thus, by the classical Banach contraction principle, we conclude that A_y has a unique fixed point, which we denote by x_y, in $BC(\mathbb{R},\mathbb{R}^+)$.

Now, we define an operator A on $BC(\mathbb{R},\mathbb{R}^+)$ by

$$(Ay)(t) = x_y(t) = (A_y x_y)(t) = g(t, x_y(t)) \int_{-\infty}^{t} a(t, t-s) f(s, y(s)) ds,\ t \in \mathbb{R},$$

where x_y is the unique fixed point of A_y. Next, let us show that A satisfies all the assumptions of Corollary 2.3. We divide the remaining of the proof into four steps.

Step 1. A is an operator from K° to K°. It is easy to verify that A is an operator from K° to $BC(\mathbb{R},\mathbb{R}^+)$. Fix $y \in K^\circ$. There exists $\xi > 0$ such that $y(t) \geq \xi$ for all $t \in \mathbb{R}$. Thus, we have

$$\begin{aligned}\inf_{t\in\mathbb{R}} (Ay)(t) &= \inf_{t\in\mathbb{R}} g(t, x_y(t)) \int_{-\infty}^{t} a(t, t-s) f(s, y(s)) ds \\ &\geq \inf_{t\in\mathbb{R}} g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, \xi) ds.\end{aligned}$$

Using (H7), there exists a constant $c > 0$ such that

$$\inf_{t\in\mathbb{R}} g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, c) ds \geq c.$$

If $\xi \geq c$, we deduce that

$$\inf_{t\in\mathbb{R}} g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, \xi) ds \geq \inf_{t\in\mathbb{R}} g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, c) ds \geq c > 0.$$

If $0 < \xi < c$, we obtain

$$\begin{aligned}g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, \xi) ds &= g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, \frac{\xi}{c} \cdot c) ds \\ &\geq (\frac{\xi}{c})^\alpha g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, c) ds \\ &\geq \frac{\xi}{c} g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, c) ds.\end{aligned}$$

Then

$$\begin{aligned}\inf_{t\in\mathbb{R}} g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, \xi) ds &\geq \frac{\xi}{c} \inf_{t\in\mathbb{R}} g(t, 0) \int_{-\infty}^{t} a(t, t-s) f(s, c) ds \\ &\geq \frac{\xi}{c} \cdot c = \xi > 0.\end{aligned}$$

Thus, we conclude that

$$\inf_{t\in\mathbb{R}} (Ay)(t) > 0.$$

By the above proof, we know that A is an operator from K° to K°.

Step 2. A is increasing in K°. Let $y_1, y_2 \in K^\circ$ and $y_1 \leq y_2$. By the property of partial ordering of cone K, we have $A(y_1) \leq A(y_2) \Leftrightarrow A(y_2) - A(y_1) \in K$. Thus, to prove that A is increasing in K°, we only need to prove that $A(y_2) - A(y_1) \in K$. It is easy to know that $A(y_2) - A(y_1) \in BC(\mathbb{R},\mathbb{R}^+)$.

By Step 1, we know that A_{y_1} and A_{y_2} are both contraction mappings satisfying

$$\|A_{y_i}(x_1) - A_{y_i}(x_2)\| \le L_g M_f D \|x_1 - x_2\|, \quad i = 1, 2,$$

for all $x_1, x_2 \in BC(\mathbb{R}, \mathbb{R}^+)$. Fix an arbitrary $\gamma_0 \in BC(\mathbb{R}, \mathbb{R}^+)$ and define two sequences $\{\gamma_n^1\}$ and $\{\gamma_n^2\}$ in $BC(\mathbb{R}, \mathbb{R}^+)$ as follows

$$\begin{gathered}
\gamma_1^1 = A_{y_1}\gamma_0, \quad \gamma_n^1 = A_{y_1}\gamma_{n-1}^1, \quad n = 2, 3, \dots; \\
\gamma_1^2 = A_{y_2}\gamma_0, \quad \gamma_n^2 = A_{y_2}\gamma_{n-1}^2, \quad n = 2, 3, \dots.
\end{gathered}$$

Note that x_{y_1} and x_{y_2} are fixed points of A_{y_1} and A_{y_2}, respectively, we conclude that

$$\lim_{n\to\infty} \gamma_n^1 = x_{y_1}, \quad \lim_{n\to\infty} \gamma_n^2 = x_{y_2}.$$

For each $t \in \mathbb{R}$, by (3.2), (H1) and (H4), we have

$$\begin{aligned}
\gamma_1^2(t) = (A_{y_2}\gamma_0)(t) &= g(t, \gamma_0(t)) \int_{-\infty}^t a(t, t-s) f(s, y_2(s)) ds \\
&\ge g(t, \gamma_0(t)) \int_{-\infty}^t a(t, t-s) f(s, y_1(s)) ds \\
&= (A_{y_1}\gamma_0)(t) = \gamma_1^1(t),
\end{aligned}$$

and

$$\begin{aligned}
\gamma_2^2(t) = (A_{y_2}\gamma_1^2)(t) &= g(t, \gamma_1^2(t)) \int_{-\infty}^t a(t, t-s) f(s, y_2(s)) ds \\
&\ge g(t, \gamma_1^1(t)) \int_{-\infty}^t a(t, t-s) f(s, y_1(s)) ds \\
&= (A_{y_1}\gamma_1^1)(t) = \gamma_2^1(t), \quad t \in \mathbb{R}.
\end{aligned}$$

By induction, we can deduce that $\gamma_n^2 \ge \gamma_n^1$, $n = 1, 2, \dots$, and thus

$$x_{y_2} = \lim_{n\to\infty} \gamma_n^2 \ge \lim_{n\to\infty} \gamma_n^1 = x_{y_1}. \tag{3.3}$$

Then, by (H1), (H3), (H4) and (3.3), we obtain

$$\begin{aligned}
&A(y_2)(t) - A(y_1)(t) \\
&= g(t, x_{y_2}(t)) \int_{-\infty}^t a(t, t-s) f(s, y_2(s)) ds - g(t, x_{y_1}(t)) \int_{-\infty}^t a(t, t-s) f(s, y_1(s)) ds \\
&= g(t, x_{y_2}(t)) \int_{-\infty}^t a(t, t-s) f(s, y_2(s)) ds - g(t, x_{y_1}(t)) \int_{-\infty}^t a(t, t-s) f(s, y_2(s)) ds \\
&\quad + g(t, x_{y_1}(t)) \int_{-\infty}^t a(t, t-s) f(s, y_2(s)) ds - g(t, x_{y_1}(t)) \int_{-\infty}^t a(t, t-s) f(s, y_1(s)) ds \\
&= [g(t, x_{y_2}(t)) - g(t, x_{y_1}(t))] \int_{-\infty}^t a(t, t-s) f(s, y_2(s)) ds \\
&\quad + g(t, x_{y_1}(t)) \int_{-\infty}^t a(t, t-s) [f(s, y_2(s)) - f(s, y_1(s))] ds \\
&\ge 0, \quad t \in \mathbb{R}.
\end{aligned}$$

Therefore, we infer that $A(y_2) - A(y_1) \in K$, which means that A is increasing in K°.

Step 3. A satisfies assumption (2) in Corollary 2.3. Fix $y \in K^\circ$ and $\lambda \in (0,1)$. Taking an arbitrary $\gamma_0 \in BC(\mathbb{R}, \mathbb{R}^+)$, we define two sequences $\{\gamma_n\}$ and $\{\gamma_n'\}$ as follows

$$\begin{gathered}\gamma_1 = A_y\gamma_0, \quad \gamma_n = A_y\gamma_{n-1}, \quad n = 2, 3, \dots, \\ \gamma_1' = A_{\lambda y}\gamma_0, \quad \gamma_n' = A_{\lambda y}\gamma_{n-1}', \quad n = 2, 3, \dots.\end{gathered}$$

As in to step2, we have

$$\lim_{n\to\infty} \gamma_n = x_y, \quad \lim_{n\to\infty} \gamma_n' = x_{\lambda y}.$$

Using (3.2), (H2) and (H6), for $t \in \mathbb{R}$, we have

$$\begin{aligned}\gamma_1'(t) &= (A_{\lambda y}\gamma_0)(t) = g(t, \gamma_0(t)) \int_{-\infty}^t a(t, t-s) f(s, \lambda y(s))ds \\ &\geq \lambda^\alpha g(t, \gamma_0(t)) \int_{-\infty}^t a(t, t-s) f(s, y(s))ds \\ &= \lambda^\alpha (A_y\gamma_0)(t) = \lambda^\alpha \gamma_1(t),\end{aligned}$$

i.e., $\gamma_1' \geq \lambda^\alpha \gamma_1$. Moreover, we have

$$\begin{aligned}\gamma_2'(t) &= (A_{\lambda y}\gamma_1')(t) = g(t, \gamma_1'(t)) \int_{-\infty}^t a(t, t-s) f(s, \lambda y(s))ds \\ &\geq \lambda^\alpha g(t, \lambda^\alpha \gamma_1(t)) \int_{-\infty}^t a(t, t-s) f(s, y(s))ds \\ &\geq \lambda^\alpha \lambda^{\alpha\beta} g(t, \gamma_1(t)) \int_{-\infty}^t a(t, t-s) f(s, y(s))ds \\ &= \lambda^{\alpha(1+\beta)} (A_y\gamma_1)(t) = \lambda^{\alpha(1+\beta)} \gamma_2(t),\end{aligned}$$

i.e., $\gamma_2' \geq \lambda^{\alpha(1+\beta)} \gamma_2$. We also have

$$\begin{aligned}\gamma_3'(t) &= (A_{\lambda y}\gamma_2')(t) = g(t, \gamma_2'(t)) \int_{-\infty}^t a(t, t-s) f(s, \lambda y(s))ds \\ &\geq \lambda^\alpha \lambda^{\alpha\beta(1+\beta)} g(t, \gamma_2(t)) \int_{-\infty}^t a(t, t-s) f(s, y(s))ds \\ &= \lambda^{\alpha(1+\beta+\beta^2)} (A_y\gamma_2)(t) \\ &= \lambda^{\alpha(1+\beta+\beta^2)} \gamma_3(t),\end{aligned}$$

i.e., $\gamma_3' \geq \lambda^{\alpha(1+\beta+\beta^2)} \gamma_3$. In general, we have

$$\gamma_n' \geq \lambda^{\alpha(1+\beta+\dots+\beta^{n-1})} \gamma_n = \lambda^{\frac{\alpha(1-\beta^n)}{1-\beta}} \gamma_n \geq \lambda^{\frac{\alpha}{1-\beta}} \gamma_n,$$

which yields

$$x_{\lambda y} = \lim_{n\to\infty} \gamma_n' \geq \lambda^{\frac{\alpha}{1-\beta}} \lim_{n\to\infty} \gamma_n = \lambda^{\frac{\alpha}{1-\beta}} x_y.$$

Then, for every $t \in \mathbb{R}$, we have

$$\begin{aligned}A(\lambda y)(t) &= g(t, x_{\lambda y}(t)) \int_{-\infty}^t a(t, t-s) f(s, \lambda y(s))ds \\ &\geq \lambda^\alpha \lambda^{\frac{\alpha\beta}{1-\beta}} g(t, x_y(t)) \int_{-\infty}^t a(t, t-s) f(s, y(s))ds\end{aligned}$$

$$= \lambda^{\frac{\alpha}{1-\beta}}(Ay)(t),$$

i.e., $A(\lambda y) \geq \lambda^{\frac{\alpha}{1-\beta}} Ay$. In addition, it is easy to verify that

$$\lambda^{\frac{\alpha}{1-\beta}} > \lambda, \quad \lambda \in (0,1),$$

since $\beta \in (0, 1-\alpha)$.

Step 4. A satisfies assumption (3) of Corollary 2.3. Applying (H7), there exists a constant $c > 0$ such that

$$\inf_{t\in\mathbb{R}} g(t,0) \int_{-\infty}^{t} a(t,t-s) f(s,c) ds \geq c. \tag{3.4}$$

Letting $x_0(t) = c$ for all $t \in \mathbb{R}$, we have $x_0 \in K^\circ$. By (3.4), we have

$$\begin{aligned} A(x_0)(t) &= g(t, x_{x_0}(t)) \int_{-\infty}^{t} a(t,t-s) f(s, x_0(s)) ds \\ &\geq g(t,0) \int_{-\infty}^{t} a(t,t-s) f(s,c) ds \\ &\geq \inf_{t\in\mathbb{R}} g(t,0) \int_{-\infty}^{t} a(t,t-s) f(s,c) ds \\ &\geq c = x_0(t), \quad t \in \mathbb{R}, \end{aligned}$$

i.e., $A(x_0) \geq x_0$. Moreover, let $y_0(t) = \max\{M_g M_f D, c\}$ for all $t \in \mathbb{R}$, where $M_g = \sup\{|g(t,x)| : t \in \mathbb{R},\ x \in \mathbb{R}^+\}$. We have

$$\begin{aligned} A(y_0)(t) &= g(t, x_{y_0}(t)) \int_{-\infty}^{t} a(t,t-s) f(s, y_0(s)) ds \\ &\leq M_g M_f \int_{-\infty}^{t} a(t,t-s) ds \\ &= M_g M_f \int_{0}^{+\infty} a(t,s) ds \\ &\leq M_g M_f \sup_{t\in\mathbb{R}} \int_{0}^{+\infty} a(t,s) ds \\ &= M_g M_f D \leq y_0(t), \quad t \in \mathbb{R}, \end{aligned}$$

i.e., $A(y_0) \leq y_0$.

Now, all conditions of Corollary 2.3 are satisfied and thus A has a unique fixed point y in K°, which means that (3.1) has a unique solution with positive infimum in $BC(\mathbb{R}, \mathbb{R}^+)$. □

4. An example

In this section, we present an example to illustrate our main result obtained in the previous Section.

Example 4.1. Let

$$f(s,x) = \frac{(\sin s + 2)(x^{1/3} + 1)}{x^{1/3} + 2}$$

for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^+$,

$$g(t,x) = \frac{(\sin t + 2)[(x+1)^{1/2} + 2]}{9\pi[(x+1)^{1/2} + 3]}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^+$, and

$$a(t,s) = \frac{1}{1+s^2}$$

for all $t \in \mathbb{R}$ and $s \in \mathbb{R}^+$.

Now, we show that f, a and g satisfy assumptions (H1)–(H7). It is easy to see that $f \in BC(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$. Moreover,

$$\begin{aligned} 0 < f(s,x) &= \frac{(\sin s + 2)(x^{1/3}+1)}{x^{1/3}+2} \\ &\le \frac{(\sin s + 2)(x^{1/3}+2)}{x^{1/3}+2} \\ &= \sin s + 2 \le 3 \end{aligned}$$

for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^+$, which means that $M_f \le 3$.

Letting $0 \le x_1 \le x_2$, we have

$$\begin{aligned} f(s,x_1) - f(s,x_2) &= \frac{(\sin s + 2)(x_1^{1/3}+1)}{x_1^{1/3}+2} - \frac{(\sin s + 2)(x_2^{1/3}+1)}{x_2^{1/3}+2} \\ &= (\sin s + 2)\Big[\frac{x_1^{1/3} - x_2^{1/3}}{(x_1^{1/3}+2)(x_2^{1/3}+2)}\Big] \le 0. \end{aligned}$$

Thus, $f(s,\cdot)$ is increasing in $\mathbb{R}^+$ for all $s \in \mathbb{R}$. So (H1) holds.

There exists $\alpha = 1/3 \in (0,1)$ such that

$$f(s,\lambda x) = \frac{(\sin s + 2)(\lambda^{1/3}x^{1/3}+1)}{\lambda^{1/3}x^{1/3}+2} \ge \frac{(\sin s + 2)(\lambda^{1/3}x^{1/3}+\lambda^{1/3})}{x^{1/3}+2} = \lambda^{1/3} f(s,x)$$

for all $x \ge 0, \lambda \in (0,1)$ and $s \in \mathbb{R}$. Obviously, $\lambda^{1/3} > \lambda$. Thus, the assumption (H2) holds.

For each $t \in \mathbb{R}$, we have

$$\int_0^{+\infty} \frac{1}{1+s^2} ds = \frac{\pi}{2} < +\infty.$$

Therefore, $a(t,\cdot) \in L^1(\mathbb{R}^+)$. It is not difficult to see that the map $t \mapsto a(t,\cdot)$ is in $BC(\mathbb{R}, L^1(\mathbb{R}_+))$. Thus, (H3) holds. Also, we have

$$D = \sup_{t\in\mathbb{R}} \int_0^{+\infty} \frac{1}{1+s^2} ds = \frac{\pi}{2}.$$

We have $g \in BC(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$, and for $0 \le x_1 \le x_2$,

$$\begin{aligned} g(t,x_1) - g(t,x_2) &= \frac{(\sin t + 2)[(x_1+1)^{1/2}+2]}{9\pi[(x_1+1)^{1/2}+3]} - \frac{(\sin t + 2)[(x_2+1)^{1/2}+2]}{9\pi[(x_2+1)^{1/2}+3]} \\ &= \frac{\sin t + 2}{9\pi}\Big[\frac{(x_1+1)^{1/2} - (x_2+1)^{1/2}}{[(x_1+1)^{1/2}+3][(x_2+1)^{1/2}+3]}\Big] \le 0. \end{aligned}$$

Thus, $g(t,\cdot)$ is increasing in $\mathbb{R}^+$ for all $t \in \mathbb{R}$ and (H4) holds.

The value $L_g = 1/3\pi$ satisfies $L_g M_f D < 1$, and

$$|g(t,x_1) - g(t,x_2)| = \Big|\frac{(\sin t + 2)[(x_1+1)^{1/2}+2]}{9\pi[(x_1+1)^{1/2}+3]} - \frac{(\sin t + 2)[(x_2+1)^{1/2}+2]}{9\pi[(x_2+1)^{1/2}+3]}\Big|$$

$$
\begin{aligned}
&\le \frac{1}{3\pi}\Big|\frac{(x_1+1)^{1/2}+2}{(x_1+1)^{1/2}+3}-\frac{(x_2+1)^{1/2}+2}{(x_2+1)^{1/2}+3}\Big| \\
&\le \frac{1}{3\pi}|(x_1+1)^{1/2}-(x_2+1)^{1/2}| \\
&\le \frac{1}{3\pi}|x_1-x_2|.
\end{aligned}
$$

for all $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^+$. Thus, (H5) holds.

Letting $\beta = 1/2 \in (0, 1-\alpha)$, we have

$$
\begin{aligned}
g(t,\lambda x) &= \frac{(\sin t+2)[(\lambda x+1)^{1/2}+2]}{9\pi[(\lambda x+1)^{1/2}+3]} \\
&\ge \frac{(\sin t+2)[(\lambda x+\lambda)^{1/2}+2\lambda^{1/2}]}{9\pi[(x+1)^{1/2}+3]} = \lambda^{1/2} g(t,x)
\end{aligned}
$$

for all $x \ge 0, \lambda \in (0,1)$ and $t \in \mathbb{R}$. Thus, (H6) holds.

When $c = \frac{1}{48} \approx 0.020833 > 0$ we have

$$
\begin{aligned}
&g(t,0)\int_{-\infty}^{t} a(t,t-s)f(s,c)ds \\
&= \Big(\frac{\sin t+2}{12\pi}\Big)\int_{-\infty}^{t}\Big(\frac{1}{1+(t-s)^2}\Big)\Big(\frac{(\sin s+2)(c^{1/3}+1)}{c^{1/3}+2}\Big)ds \\
&= \Big(\frac{\sin t+2}{12\pi}\Big)\Big(\frac{c^{1/3}+1}{c^{1/3}+2}\Big)\int_{0}^{+\infty}\frac{\sin(t-s)+2}{1+s^2}ds \\
&\ge \frac{1}{12\pi}\Big(\frac{c^{1/3}+1}{c^{1/3}+2}\Big)\int_{0}^{+\infty}\frac{1}{1+s^2}ds = \frac{c^{1/3}+1}{24c^{1/3}+48}
\end{aligned}
$$

for all $t \in \mathbb{R}$. Thus, we have

$$
\inf_{t\in\mathbb{R}} g(t,0)\int_{-\infty}^{t} a(t,t-s)f(s,c)ds \ge \frac{c^{1/3}+1}{24c^{1/3}+48} \approx 0.023353 > c,
$$

i.e., (H7) holds. Thus, Theorem 3.1 yields that the quadratic integral equation

$$
\begin{aligned}
x(t) = &\frac{(\sin t+2)[(x(t)+1)^{1/2}+2]}{9\pi[(x(t)+1)^{1/2}+3]} \\
&\times \int_{-\infty}^{t}\Big(\frac{1}{1+(t-s)^2}\Big)\Big(\frac{(\sin s+2)[(x(s))^{1/3}+1]}{(x(s))^{1/3}+2}\Big)ds,
\end{aligned}
$$

for $t \in \mathbb{R}$, has a unique solution with positive infimum in $BC(\mathbb{R}\times\mathbb{R}^+)$.

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