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MULTIPLE POSITIVE SOLUTIONS FOR SCHRÖDINGER-POISSON SYSTEMS INVOLVING CONCAVE-CONVEX NONLINEARITIES

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Abstract. In this article, we study the existence of multiple positive solutions for Schrödinger-Poisson systems involving concave-convex nonlinearities and sign-changing weight potentials. With the help of Nehari manifold and Ljusternik-Schnirelmann category theory, we investigate how the coefficient $g(x)$ of the critical nonlinearity affects the number of positive solutions. Furthermore, we obtain a relationship between the number of positive solutions and the topology of the global maximum set of g.

1. Introduction

In present article, we study the existence of multiple positive solutions to the Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + l(x)\phi u = f_\lambda(x)u^{q-1} + g(x)u^5, \quad x \in \Omega, \\ -\Delta\phi = l(x)u^2, \quad x \in \Omega, \\ \phi = u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary and $1 < q < 2$. Moreover, l and g are continuous functions on $\overline{\Omega}$. The function $f_\lambda(x) = \lambda f_+ + f_-$, where $\lambda > 0$ is a small parameter and $f_\pm = \pm\max\{\pm f(x), 0\}$.

In recent years, the nonlinear Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + V(x)u + l(x)\phi u = f(x,u), \\ -\Delta\phi = l(x)u^2, \end{gathered} \tag{1.2}$$

has been widely investigated and it is well known that it has a strong physical meaning because they appear in quantum mechanics models (see [4, 18]) and in semiconductor theory [19, 21]. In particular, system (1.2) was introduced in [2, 3] as a model describing solitary waves, for nonlinear stationary equations of Schrödinger type interacting with an electrostatic field, and are usually known as Schrödinger-Poisson systems. We refer to [2] for more details on physical aspects. Many researches have been devoted to the study of (1.2) in the recent literature, see for example, [12, 13, 15, 22, 23, 25] and the references therein.

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On a bounded domain, Azzollini [1] studied the Schrödinger-Poisson system

$$\begin{aligned} -\Delta u + \varepsilon q\phi f(u) &= \eta|u|^{p-1}, \quad x \in \Omega,\\ -\Delta\phi &= 2qF(u), \quad x \in \Omega,\\ \phi = u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < p < 5$, $q > 0$, $\varepsilon, \eta = \pm 1$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $F(t) = \int_0^t f(s)ds$. By using the method of a cut-off function and variational arguments, the authors proved the existence and multiplicity results based on f a subcritical growth condition and they also considered the existence and nonexistence results under the critical case. Recently, Lei et al. [14] considered the Schrödinger-Poisson system

$$\begin{aligned} -\Delta u + \lambda\phi u &= \lambda u^{q-1} + u^5, \quad x \in \Omega,\\ -\Delta\phi &= u^2, \quad x \in \Omega,\\ \phi = u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary and $\lambda > 0$ is a real parameter, $1 < q < 2$. By using the Ekelands variational principle and the Mountain Pass Theorem, they proved that (1.4) has at least two positive solutions provided λ enough small.

Under the assumption $l(x) \neq 0$, (1.1) can be regarded as a perturbation problem of the problem

$$\begin{aligned} -\Delta u &= f_\lambda(x)u^{q-1} + g(x)u^5, \quad x \in \Omega,\\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.5}$$

It is well known that the existence of positive solutions of (1.5) is affected by the topology of the global maximum set of g. This has been the focus of a great deal of research by several authors. In particular, f_λ and g satisfy the following assumptions:

(A1) There exist k points $a^1, a^2, \dots, a^k$ in Ω such that

$$g(a^i) = \max_{x\in\Omega} g(x) = 1 \text{ for } 1 \le i \le k,$$

and for a positive number ρ with $\rho > 3$ such that $g(x) - g(a^i) = O(|x - a^i|^\rho)$ as $x \to a^i$ uniformly in i.

(A2) Choosing $\rho_0 > 0$ such that

$$\overline{B_{\rho_0}(a^i)} \bigcap \overline{B_{\rho_0}(a^j)} = \emptyset \quad \text{for } i \neq j \text{ and } 1 \le i, j \le k,$$

and $\cup_{i=1}^k \overline{B_{\rho_0}(a^i)} \subset \Omega$, where $\overline{B_{\rho_0}(a^i)} = \{x \in \mathbb{R}^3; |x - a^i| \le \rho_0\}$.

(A3) $f_\lambda(x), g(x) > 0$ for $x \in \cup_{i=1}^k \overline{B_{\rho_0}(a^i)}$.

Fan [6] proved that (1.5) admits at least $k+1$ positive solutions when f_λ is small enough. Lin [20] Li and Wu [16] also proved a similar result. There are several generalizations of this result, we refer to [7, 8, 17].

A natural question now is whether the same existence results as [15-20] occur for problem (1.1). Motivated by this idea, we aim to investigate how the coefficient $g(x)$ of the critical nonlinearity affects the number of positive solutions of (1.1) in this work. We consider the relationship between the number of positive solutions and the topology of global maximum set of g by the idea of category. Moreover, we should point out that the appearance of the poisson equation prevents us from using the variational methods that used in [6, 7, 8, 16, 17, 20] in a standard way.

To state our main result, we introduce precise conditions on l, f_λ and g:

- (A4) $l(x), g(x) > 0$ on Ω.
- (A5) There exist a non-empty closed set $M = \{z \in \overline{\Omega} : g(z) = \max_{x\in\overline{\Omega}} g(x) = 1\}$ and a positive number $\rho > 3$ such that $g(z) - g(x) = O(|x-z|^\rho)$ as $x \to z$ and uniformly in $z \in M$.
- (A6) $f_\lambda(x) > 0$ for $x \in M$.

Remark 1.1. Let $M_r = \{x \in \mathbb{R}^3; dist(x, M) < r\}$ for $r > 0$. Then by (A4)–(A6), there exist $C_0, r_0 > 0$ such that

$$\begin{gathered} f_\lambda(x) > 0, \quad \forall x \in M_{r_0} \subset \Omega, \\ g(z) - g(x) \le C_0|x-z|^\rho \quad \forall x \in B_{r_0}(z) \end{gathered}$$

uniformly in $z \in M$, where $B_{r_0}(z) = \{x \in \mathbb{R}^3 : |x-z| < r_0\}$.

The main result of this work in the following theorem.

Theorem 1.2. *Assume* (A4)–(A6) *hold. Then for each $\delta < r_0$, there exists $\Lambda_\delta > 0$ such that if $\lambda \in (0, \Lambda_\delta)$,* (1.1) *has at least* $\operatorname{cat}_{M_\delta}(M) + 1$ *distinct positive solutions, where* cat *means the Ljusternik-Schnirelmann category (see* [24]*).*

Remark 1.3. Suppose (A1)–(A3) hold. By Theorem 1.2, we obtain that (1.1) has at least $k+1$ positive solutions when λ is small enough.

Remark 1.4. Suppose $l(x) = f(x) \equiv \lambda$ and $g(x) \equiv 1$, Then Theorem 1.2 is the result of the recent paper [14]. We should point out that the condition that $l(x)$ is small enough is important in [14]. However, we do not need this condition due to our precise estimates in this paper. Moreover, we assume that $f_\lambda(x)$ maybe sign-changing in this work. Lei and Suo obtained that (1.1) has at least two positive solutions in [14], while we will obtain a relationship between the number of positive solutions and the topology of global maximum set of g in this paper.

This article is organized as follows. In Section 2, we give some preliminary results and obtain the first positive solution of (1.1). In Section 3, we present some technical results and useful estimates which are crucial in the proof of Theorem 1.2. In Section 4, we use the Ljusternik-Schnirelmann category theory to prove Theorem 1.2. Throughout this paper we denote by $\to$ (resp. $\rightharpoonup$) the strong (resp. weak) convergence. We will use $C, C_0, C_1, C_2, \dots$ to denote various positive constants.

2. Preliminaries

Throughout this article by $|\cdot|_r$ we denote the L^r-norm. On the space $H_0^1(\Omega)$ we consider the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}.$$

Let S be the best Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ given by

$$S := \inf\Big\{\int_\Omega |\nabla u|^2 dx; u \in H_0^1(\Omega), |u|_6 = 1\Big\}.$$

It is well known that S is independent of Ω and is never achieved except when $\Omega = \mathbb{R}^3$. Moreover, S is achieved by the function

$$v_\varepsilon(x) = \frac{(3\varepsilon)^{1/4}}{(\varepsilon + |x|^2)^{1/2}}, \quad \text{for any } \varepsilon > 0. \tag{2.1}$$

We obtain that

$$\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx = \int_{\mathbb{R}^3} |v_\varepsilon|^6 dx = S^{3/2}. \tag{2.2}$$

For every $u \in H_0^1(\Omega)$, the Lax-Milgram theorem implies that there exists a unique solution $\phi_u \in H_0^1(\Omega)$ for the second equation of (1.1). We substitute ϕ_u into the first equation of (1.1), then (1.1) transforms into the equation

$$\begin{gathered} -\Delta u + l(x)\phi_u u = f_\lambda(x)u^{q-1} + g(x)u^5, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{2.3}$$

We can easily proved that $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of (1.1) if and only if u solves (2.3) and $\phi = \phi_u$. The energy functional associated with (2.3) is defined by

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_\Omega l(x)\phi_u u^2 dx - \frac{1}{q}\int_\Omega f_\lambda(x)|u|^q dx - \frac{1}{6}\int_\Omega g(x)|u|^6 dx.$$

Moreover, if $u \in H_0^1(\Omega)$ is called a weak solution of (2.3), then (u, ϕ_u) is a solution of (1.1) and

$$\langle J'_\lambda(u), v\rangle = \int_\Omega \nabla u \nabla v\, dx + \int_\Omega l(x)\phi_u uv\, dx - \int_\Omega f_\lambda(x)u^{q-1}v\, dx - \int_\Omega g(x)u^5 v\, dx = 0$$

for all $v \in H_0^1(\Omega)$. At first, we introduce the following lemma (see [2, 14]).

Lemma 2.1. *For every $u \in H_0^1(\Omega)$, there exists a unique $\phi_u \in H_0^1(\Omega)$ solution of*

$$\begin{gathered} -\Delta\phi = l(x)u^2, \quad x \in \Omega, \\ \phi = 0, \quad x \in \partial\Omega, \end{gathered}$$

and

- (i) $\|\phi_u\|^2 = \int_\Omega l(x)\phi_u u^2 dx$.
- (ii) $\phi_u \geq 0$. *Moreover,* $\phi_u > 0$ *when* $u \neq 0$.
- (iii) *For each* $t \neq 0$, $\phi_{tu} = t^2\phi_u$.
- (iv) $\int_\Omega l(x)\phi_u u^2 dx = \|\phi_u\|^2 \leq S^{-1}|u|_{12/5}^4$.
- (v) *Assume that* $u_n \rightharpoonup u$ *in* $H_0^1(\Omega)$, *then* $\phi_{u_n} \to \phi_u$ *in* $H_0^1(\Omega)$ *and*
$$\int_\Omega l(x)\phi_{u_n}u_n v\, dx \to \int_\Omega l(x)\phi_u uv\, dx$$
for every $v \in H_0^1(\Omega)$.
- (vi) *Set* $L(u) = \int_\Omega l(x)\phi_u u^2 dx$ *then* $L : H_0^1(\Omega) \to H_0^1(\Omega)$ *is* C^1 *and*
$$\langle L'(u), v\rangle = 4\int_\Omega l(x)\phi_u uv dx, \quad \forall v \in H_0^1(\Omega).$$

As J_λ is not bounded from below on $H_0^1(\Omega)$, we consider the behaviors of J_λ on the Nehari manifold

$$N_\lambda := \{u \in H_0^1(\Omega)\backslash\{0\} : \langle J'_\lambda(u), u\rangle = 0\},$$

where $\langle,\rangle$ denotes the usual duality between $H_0^1(\Omega)$ and H^{-1}. Clearly, $u \in N_\lambda$ if and only if

$$\|u\|^2 + \int_\Omega l(x)\phi_u u^2 dx = \int_\Omega f_\lambda(x)|u|^q dx + \int_\Omega g(x)|u|^6 dx. \tag{2.4}$$

On the Nehari manifold N_λ, by (2.4), Sobolev and Young inequalities, it follows

$$\begin{aligned} J_\lambda(u) &= J_\lambda(u) - \frac{1}{4}\langle J_\lambda'(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 + \frac{1}{12}\int_\Omega g(x)|u|^6 dx - \big(\frac{1}{q} - \frac{1}{4}\big)\int_\Omega f_\lambda(x)|u|^q dx \\ &\geq \frac{1}{4}\|u\|^2 - \lambda\big(\frac{1}{q} - \frac{1}{4}\big)C|f_+|_\infty \|u\|^q \\ &\geq \frac{1}{4}\|u\|^2 - \frac{1}{4}\|u\|^2 - D\lambda^{\frac{2}{2-q}} \\ &= -D\lambda^{\frac{2}{2-q}}, \end{aligned} \tag{2.5}$$

where D denotes a positive constant independent of $u \in H_0^1(\Omega)$. Let

$$\psi_\lambda(u) := \langle J_\lambda'(u), u\rangle = \|u\|^2 + \int_\Omega l(x)\phi_u u^2 dx - \int_\Omega f_\lambda(x)|u|^q dx - \int_\Omega g(x)|u|^6 dx. \tag{2.6}$$

Then for $u \in N_\lambda$, we have

$$\begin{aligned} \langle \psi_\lambda'(u), u\rangle &= 2\|u\|^2 + 4\int_\Omega l(x)\phi_u u^2 dx - q\int_\Omega f_\lambda(x)|u|^q dx - 6\int_\Omega g(x)|u|^6 dx \\ &= (4-q)\int_\Omega f_\lambda(x)|u|^q dx - 2\|u\|^2 - 2\int_\Omega g(x)|u|^6 dx \\ &= (2-q)\|u\|^2 + (4-q)\int_\Omega l(x)\phi_u u^2 dx + (q-6)\int_\Omega g(x)|u|^6 dx. \end{aligned} \tag{2.7}$$

As in [6, 7, 8, 9, 16, 20], we split N_λ into three parts:

$$\begin{aligned} N_\lambda^+ &= \{u \in N_\lambda; \langle \psi_\lambda'(u), u\rangle > 0\}, \\ N_\lambda^0 &= \{u \in N_\lambda; \langle \psi_\lambda'(u), u\rangle = 0\}, \\ N_\lambda^- &= \{u \in N_\lambda; \langle \psi_\lambda'(u), u\rangle < 0\}. \end{aligned}$$

Then we have the following results.

Lemma 2.2. *Suppose that u_0 is a local minimizer for J_λ on N_λ and $u_0 \notin N_\lambda^0$. Then $J_\lambda'(u_0) = 0$.*

Proof. If u_0 is a local minimizer for J_λ on N_λ, then u_0 is a solution of the optimization problem

$$\text{minimize } J_\lambda(u) \text{ subject to } \{u \in H_0^1(\Omega)\backslash\{0\}; \psi_\lambda(u) = 0\}.$$

Hence by the theory of Lagrange multipliers, there exists a $\theta \in \mathbb{R}$ such that $J_\lambda'(u_0) = \theta\psi_\lambda'(u_0)$ in H^{-1}. Thus $\langle J_\lambda'(u_0), u_0\rangle = \theta\langle \psi_\lambda'(u_0), u_0\rangle$. Moreover, because of $u_0 \notin N_\lambda^0$, we obtain $\langle \psi_\lambda'(u_0), u_0\rangle \neq 0$, and so $\theta = 0$. □

Motivated by Lemma 2.2, we will obtain conditions for $N_\lambda^0 = \emptyset$.

Lemma 2.3. *There exists $\Lambda_1 > 0$ such that $N_\lambda^0 = \emptyset$ for $\lambda \in (0, \Lambda_1)$.*

Proof. Suppose that $N_\lambda^0 \neq \emptyset$ for all $\lambda > 0$. If $u \in N_\lambda^0$, then from (2.6)-(2.7) and Sobolev inequality, we obtain

$$2\|u\|^2 \leq 2\|u\|^2 + 2\int_\Omega g(x)|u|^6 dx = (4-q)\int_\Omega f_\lambda(x)|u|^q dx \leq \lambda(4-q)CS^{-\frac{q}{2}}\|u\|^q$$

and

$$\|u\|^2 \leq \frac{6-q}{2-q}\int_\Omega g(x)|u|^6 dx \leq \frac{6-q}{2-q}S^{-3}\|u\|^6.$$

Thus we obtain

$$C_1 \leq \|u\| \leq \lambda^{\frac{1}{2-q}}C_2,$$

where $C_1, C_2 > 0$ and are independent of the choice of u and λ. For λ is sufficient small, this is a contradiction. Hence, there exists $\Lambda_1 > 0$ such that for $\lambda \in (0, \Lambda_1)$, we have $N_\lambda^0 = \emptyset$. □

Now we can write $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ and define $\alpha_\lambda = \inf_{u\in N_\lambda} J_\lambda(u)$, $\alpha_\lambda^+ = \inf_{u\in N_\lambda^+} J_\lambda(u)$ and $\alpha_\lambda^- = \inf_{u\in N_\lambda^-} J_\lambda(u)$.

Lemma 2.4. *We have the following statements:*

(i) $\alpha_\lambda^+ < 0$.

(ii) *there exists* $\Lambda_2 \in (0, \Lambda_1)$ *such that* $\alpha_\lambda^- > d_0$ *for some* $d_0 > 0$ *and* $\lambda \in (0, \Lambda_2)$.

In particular, $\alpha_\lambda^+ = \inf_{u\in N_\lambda} J_\lambda(u)$ *for all* $\lambda \in (0, \Lambda_2)$.

Proof. (i) Let $u \in N_\lambda^+$, then we have

$$(2-q)\|u\|^2 + (4-q)\int_\Omega l(x)\phi_u u^2 dx > (6-q)\int_\Omega g(x)|u|^6 dx.$$

Thus,

$$\begin{aligned}
J_\lambda(u) &= J_\lambda(u) - \frac{1}{q}\langle J_\lambda'(u), u\rangle \\
&= \big(\frac{1}{2} - \frac{1}{q}\big)\|u\|^2 + \big(\frac{1}{4} - \frac{1}{q}\big)\int_\Omega l(x)\phi_u u^2 dx + \big(\frac{1}{q} - \frac{1}{6}\big)\int_\Omega g(x)|u|^6 dx \\
&< \frac{q-2}{4q}\|u\|^2 + \frac{q-4}{4q}\int_\Omega l(x)\phi_u u^2 dx + \frac{6-q}{6q}\int_\Omega g(x)|u|^6 dx \\
&< -\frac{6-q}{12q}\int_\Omega g(x)|u|^6 dx < 0.
\end{aligned}$$

Thus $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

(ii) Let $u \in N_\lambda^-$, then we obtain from (2.7) that

$$\begin{aligned}
(2-q)\|u\|^2 &\leq (2-q)\|u\|^2 + (4-q)\int_\Omega l(x)\phi_u u^2 dx \\
&< (6-q)\int_\Omega |u|^6 dx \leq (6-q)S^{-3}\|u\|^6.
\end{aligned}$$

This implies

$$\|u\| \geq \big(\frac{2-q}{6-q}S^3\big)^{1/4}, \tag{2.8}$$

for any $u \in N_\lambda^-$. From (2.5), we obtain that

$$J_\lambda(u) \geq \|u\|^q\Big(\frac{1}{4}\|u\|^{2-q} - \lambda\big(\frac{1}{q} - \frac{1}{4}\big)C|f_+|_\infty\Big) \tag{2.9}$$

Since $1 < q < 2$, (2.7) and (2.8) implies that there exists $\Lambda_2 \in (0, \Lambda_1)$ such that $\alpha_\lambda^- > d_0$ for some $d_0 > 0$ and $\lambda \in (0, \Lambda_2)$. $\square$

For each $u \in H_0^1(\Omega)$ with $\int_\Omega g(x)|u|^6 dx > 0$, we write

$$t_{\max} = \Big(\frac{\int_\Omega l(x)\phi_u u^2 dx + \sqrt{\big(\int_\Omega l(x)\phi_u u^2 dx\big)^2 + 4\|u\|^2 \int_\Omega g(x)|u|^6 dx}}{2\int_\Omega g(x)|u|^6 dx} \Big)^{1/2}.$$

Then we have the following Lemma.

Lemma 2.5. *For each $u \in H_0^1(\Omega)$ with $\int_\Omega g(x)|u|^6 dx > 0$, there exists $\Lambda_3 \in (0, \Lambda_2)$ such that we have the following results:*

(i) If $\int_\Omega f_\lambda |u|^q dx \le 0$, then there is a unique $t^- = t^-(u) > t_{\max}$ such that $t^- u \in N_\lambda^-$ and $J_\lambda(tu)$ is increasing on $(0, t^-)$ and decreasing on (t^-, ∞). Moreover, $J_\lambda(t^- u) = \sup_{t \ge 0} J_\lambda(tu)$.

(ii) If $\int_\Omega f_\lambda |u|^q dx > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max} < t^-$ such that $t^- u \in N_\lambda^-, t^+ u \in N_\lambda^+, J_\lambda(tu)$ is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^-, ∞). Moreover, $J_\lambda(t^+ u) = \inf_{0 \le t \le t_{\max}} J_\lambda(tu); J_\lambda(t^- u) = \sup_{t \ge t^+} J_\lambda(tu)$.

Proof. Fix $u \in H_0^1(\Omega)$ with $\int_\Omega g(x)|u|^6 dx > 0$. Let

$$s(t) = t^{2-q}\|u\|^2 + t^{4-q} \int_\Omega l(x)\phi_u u^2 dx - t^{6-q} \int_\Omega g(x)|u|^6 dx,$$

for $t \ge 0$. We have $s(0) = 0$, and $s(t) \to -\infty$ as $t \to \infty$. The function $s(t)$ achieves its maximum at $t_{\max}$, increasing in $[0, t_{\max})$ and decreasing in $(t_{\max}, \infty)$. Moreover, we obtain

$$\begin{aligned}
s(t_{\max}) &\ge \max_{t \ge 0} \Big(t^{2-q}\|u\|^2 - t^{6-q} \int_\Omega g(x)|u|^6 dx \Big) \\
&= \Big(\frac{(2-q)\|u\|^2}{(6-q)\int_\Omega g(x)|u|^6 dx} \Big)^{\frac{2-q}{4}} \|u\|^2 \\
&\quad - \Big(\frac{(2-q)\|u\|^2}{(6-q)\int_\Omega g(x)|u|^6 dx} \Big)^{\frac{6-q}{4}} \int_\Omega g(x)|u|^6 dx \\
&= \|u\|^q \Big[\big(\frac{2-q}{6-q}\big)^{\frac{2-q}{4}} - \big(\frac{2-q}{6-q}\big)^{\frac{6-q}{4}} \Big] \Big(\frac{\|u\|^6}{\int_\Omega g(x)|u|^6 dx} \Big)^{\frac{2-q}{4}} \\
&\ge \|u\|^q \big(\frac{4}{6-q}\big) \big(\frac{2-q}{6-q}\big)^{\frac{2-q}{4}} D(S),
\end{aligned} \tag{2.10}$$

where $D(S) > 0$ is a constant depends on S. We consider two cases now.

(i) $\int_\Omega f_\lambda |u|^q dx \le 0$. There is a unique $t^- > t_{\max}$ such that $s(t^-) = \int_\Omega f_\lambda |u|^q dx$ and $s'(t^-) < 0$, which implies $t^- u \in N_\lambda^-$. Because of $t > t_{\max}$, we have

$$(2-q)\|tu\|^2 + (4-q) \int_\Omega l(x)\phi_{(tu)}(tu)^2 dx - (6-q) \int_\Omega g(x)|tu|^6 dx < 0$$

and

$$\begin{aligned}
&\frac{d}{dt} J_\lambda(tu)|_{t=t^-} \\
&= \Big\{ t\|u\|^2 + t^3 \int_\Omega l(x)\phi_u u^2 dx - t^{q-1} \int_\Omega f_\lambda |u|^q dx - t^5 \int_\Omega g(x)|u|^6 dx \Big\}|_{t=t^-} = 0.
\end{aligned}$$

Thus $J_\lambda(tu)$ is increasing on $(0,t^-)$ and decreasing on (t^-,∞). Moreover, $J_\lambda(t^-u) = \sup_{t\geq 0} J_\lambda(tu)$.

(ii) $\int_\Omega f_\lambda|u|^q dx > 0$. By (2.10), we know that there exists $\Lambda_3 > 0$ such that

$$s(0) = 0 < \int_\Omega \lambda f_+|u|^q dx \leq \lambda C|f_+|_\infty S^{-\frac{q}{2}}\|u\|^q \tag{2.11}$$

$$< \|u\|^q\big(\frac{4}{6-q}\big)\big(\frac{2-q}{6-q}\big)^{\frac{2-q}{4}} D(S) \leq s(t_{\max}) \tag{2.12}$$

for $\lambda \in (0,\Lambda_3)$. It follows that there are a unique t^+ and a unique t^- such that for $0 < t^+ < t_{\max} < t^-$, and we obtain

$$s(t^+) = \int_\Omega f_\lambda|u|^q dx = s(t^-)$$

and $s'(t^+) > 0 > s'(t^-)$.

Similarly as in case (i), we have $t^+u \in N_\lambda^+$, $t^-u \in N_\lambda^-$, and $J_\lambda(t^-u) \geq J_\lambda(tu) \geq J_\lambda(t^+u)$ for each $t \in [t^+,t^-]$. Furthermore, we can get $J_\lambda(t^+u) \leq J_\lambda(tu)$ for each $t \in [0,t^+]$. In other words, $J_\lambda(tu)$ is decreasing on $(0,t^+)$, increasing on (t^+,t^-) and decreasing on (t^-,∞) again. Moreover,

$$J_\lambda(t^+u) = \inf_{0\leq t\leq t_{\max}} J_\lambda(tu); \quad J_\lambda(t^-u) = \sup_{t\geq t^+} J_\lambda(tu).$$

This completes the proof. □

Next we establish that J_λ satisfies the $(PS)_c$-condition for $c \in (-\infty, \alpha_\lambda^+ + \frac{1}{3}S^{3/2})$.

Lemma 2.6. *For $\lambda \in (0,\Lambda_3)$, J_λ satisfies the $(PS)_c$-condition for $c \in (-\infty, \alpha_\lambda^+ + \frac{1}{3}S^{3/2})$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(PS)_c$-sequence for J_λ and $c \in (-\infty, \alpha_\lambda^+ + \frac{1}{3}S^{3/2})$. Since

$$\begin{aligned} o(\|u_n\|) + \alpha_\lambda^+ + \frac{1}{3}S^{3/2} &= J_\lambda(u_n) - \frac{1}{4}\langle J_\lambda'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \frac{1}{12}\int_\Omega g(x)|u_n|^6 dx - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)\int_\Omega f_\lambda|u|^q dx \\ &\geq \frac{1}{4}\|u_n\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{4}\right) C\|u_n\|^q, \end{aligned}$$

we obtain that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Thus, there exist a subsequence still denoted by $\{u_n\}$ and $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. By the compactness of Sobolev embedding, we obtain

$$\begin{gathered} \int_\Omega f_\lambda|u_n|^q dx = \int_\Omega f_\lambda|u|^q dx + o(1); \\ \|u_n - u\|^2 = \|u_n\|^2 - \|u\|^2 + o(1); \\ \int_\Omega g|u_n - u|^6 dx = \int_\Omega g|u_n|^6 dx - \int_\Omega g|u|^6 dx + o(1). \end{gathered}$$

Moreover, we obtain from Lemma 2.1 that

$$\int_\Omega l(x)\phi_{u_n} u_n^2 dx \to \int_\Omega l(x)\phi_u u^2 dx,$$

$$\int_\Omega l(x)\phi_{u_n}u_n u dx \to \int_\Omega l(x)\phi_u u^2 dx,$$

as $n \to \infty$. Then we can obtain $J'_\lambda(u) = 0$ in H^{-1}. Since $J_\lambda(u_n) = c + o(1)$ and $J'_\lambda(u_n) = o(1)$ in H^{-1}, we deduce that

$$\frac{1}{2}\|u_n - u\|^2 - \frac{1}{6}\int_\Omega g|u_n - u|^6 dx = c - J_\lambda(u) + o(1) \tag{2.13}$$

and

$$\begin{aligned} o(1) &= \langle J'_\lambda(u_n), u_n - u\rangle = \langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \\ &= \|u_n - u\|^2 - \int_\Omega g|u_n - u|^6 dx + o(1). \end{aligned}$$

Now we may assume that

$$\|u_n - u\|^2 \to a \quad \text{and} \quad \int_\Omega g|u_n - u|^6 dx \to a \quad \text{as } n \to \infty,$$

for some $a \in [0, +\infty)$.

Suppose $a \neq 0$ and notice the fact $g \leq 1$, using the Sobolev embedding theorem and passing to the limit as $n \to \infty$, we have $a \geq Sa^{1/3}$, i.e.,

$$a \geq S^{3/2}. \tag{2.14}$$

Then by (2.10)-(2.13) and $u \in N_\lambda \cup \{0\}$,

$$c = J_\lambda(u) + \frac{a}{N} \geq \alpha_\lambda^+ + \frac{1}{3}S^{3/2},$$

which contradicts the definition of c. Hence $a = 0$, i.e., $u_n \to u$ strongly in $H_0^1(\Omega)$. □

Next we obtain the existence of a local minimizer for J_λ on N_λ^+.

Theorem 2.7. *For each $\lambda \in (0, \Lambda_3)$, J_λ has a minimizer u_λ^+ in N_λ^+ which satisfies:*

(i) *u_λ^+ is a positive solution of* (1.1)*;*
(ii) *$J_\lambda(u_\lambda^+) = \alpha_\lambda^+$;*
(iii) *$J_\lambda(u_\lambda^+) \to 0$ as $\lambda \to 0$;*
(iv) *$\|u_\lambda^+\| \to 0$ as $\lambda \to 0$.*

Proof. Similarly as [9, Lemma 4.7], we can obtain a $(PS)_{\alpha_\lambda^+}$-sequence for J_λ defined by $\{u_n\} \subset N_\lambda$. By Lemma 2.6, there exists a subsequence still denoted by $\{u_n\}$ and $u_\lambda^+ \in H_0^1(\Omega)$ such that $u_n \to u_\lambda^+$ in $H_0^1(\Omega)$ as $n \to \infty$. Since $N_\lambda^0 = \emptyset$, we deduce that $u_\lambda^+ \in N_\lambda^+$ and $J_\lambda(u_\lambda^+) = \alpha_\lambda^+ < 0$. Note that $J_\lambda(u_n) = J_\lambda(|u_n|)$, we obtain that $u_\lambda^+ \geq 0$ and $u_\lambda^+ \not\equiv 0$. Recalling that $\phi_{u_\lambda^+} > 0$ and $\phi_{u_\lambda^+} \in C^0(\overline{\Omega})$, then the strong maximum principle suggests that $u_\lambda^+ > 0$ in Ω. Then we can obtain the assertion (i) and (ii).

By (2.5), we have

$$0 > J_\lambda(u_\lambda^+) \geq -D\lambda^{\frac{2}{2-q}}.$$

This implies $J_\lambda(u_\lambda^+) \to 0$ as $\lambda \to 0^+$. We obtain (iii).

Now we show (iv). Since $u_\lambda^+ \in N_\lambda^+$ and (2.6), we know

$$\|u_\lambda^+\|^2 \leq \frac{4-q}{2}\int_\Omega f_\lambda |u|^q dx \leq \lambda C|f_+|_\infty \|u_\lambda^+\|^q. \tag{2.15}$$

Moreover, because J_λ is coercive and bounded from below on N_λ, $\{u_\lambda^+\}_\lambda$ is bounded in $H_0^1(\Omega)$. It follows from (2.15) that

$$\|u_\lambda^+\|^{2-q} \le C\lambda^{\frac{1}{2-q}}.$$

Then $\|u_\lambda^+\| \to 0$, as $\lambda \to 0^+$. $\square$

3. Technical results

In this Section, we will recall and prove some lemmas which are crucial in the proof of the main theorem.

For $b > 0$, we define

$$J_\infty^b(u) = \frac{1}{2}\|u\|^2 - \frac{b}{6}\int_\Omega g|u|^6 dx,$$
$$N_\infty^b = \{u \in H_0^1(\Omega)\backslash\{0\}; \langle (J_\infty^b)'(u), v\rangle = 0\}.$$

Lemma 3.1. *For each $u \in N_\lambda^-$, we have*

(i) *There is a unique t_u^b such that $t_u^b u \in N_\infty^b$ and*
$$\max_{t\ge 0} J_\infty^b(tu) = J_\infty^b(t_u^b u) = \frac{1}{3} b^{-1/2}\Big(\frac{\|u\|^6}{\int_\Omega g|u|^6 dx}\Big)^{1/2}.$$

(ii) *For $\mu \in (0,1)$, there is a unique t_u^1 such that $t_u^1 u \in N_\infty^1$. Moreover,*
$$J_\infty^1(t_u^1 u) \le (1-\mu)^{-3/2}\Big(J_\lambda(u) + \frac{2-q}{2q}\mu^{\frac{q}{q-2}}\lambda^{\frac{2}{2-q}}C\Big).$$

Proof. (i) For each $u \in N_\lambda^-$, let

$$h(t) = J_\infty^b(tu) = \frac{t^2}{2}\|u\|^2 - \frac{b}{6}t^6\int_\Omega g|u|^6 dx.$$

We have $h(t) \to -\infty$ as $t \to \infty$,

$$h'(t) = t\|u\|^2 - bt^5\int_\Omega g|u|^6 dx,$$
$$h''(t) = t\|u\|^2 - 5bt^4\int_\Omega g|u|^6 dx.$$

Set

$$t_u^b = \Big(\frac{\|u\|^2}{\int_\Omega bg|u|^6 dx}\Big)^{1/4} > 0.$$

Then $h'(t_u^b) = 0$, $t_u^b u \in N_\infty^b$ and $h''(t_u^b) = -4\|u\|^2 < 0$. Hence there is a unique $t_u^b > 0$ such that $t_u^b u \in N_\infty^b$ and

$$\max_{t\ge 0} J_\infty^b(tu) = J_\infty^b(t_u^b u) = \frac{1}{3} b^{-1/2}\Big(\frac{\|u\|^6}{\int_\Omega g|u|^6 dx}\Big)^{-1/2}.$$

(ii) For $\mu \in (0,1)$, we have

$$\begin{aligned}
\int_\Omega \lambda f_+|t_u^b u|^q dx &\le \lambda C\|t_u^b u\|^q \\
&\le \frac{2-q}{2}(\lambda C\mu^{-\frac{q}{2}})^{\frac{2}{2-q}} + \frac{q}{2}\Big(\mu^{\frac{q}{2}}\|t_u^b u\|^q\Big)^{2/q} \\
&= \frac{2-q}{2}\mu^{\frac{q}{q-2}}C\lambda^{\frac{2}{2-q}} + \frac{q\mu}{2}\|t_u^b u\|^2.
\end{aligned}$$

Then letting $b = \frac{1}{1-\mu}$, by part (i), we have

$$
\begin{aligned}
J_\lambda(u) &= \max_{t\geq 0} J_\lambda(tu) \geq J_\lambda(t_u^{\frac{1}{1-\mu}} u)\\
&\geq \frac{1-\mu}{2}\|(t_u^{\frac{1}{1-\mu}} u)\|^2 + \frac{1}{4}(t_u^{\frac{1}{1-\mu}})^4 \int_\Omega l(x)\phi_u u^2 dx\\
&\quad - \frac{1}{6}(t_u^{\frac{1}{1-\mu}})^6 \int_\Omega g|u|^6 dx - \frac{2-q}{2q}\mu^{\frac{q}{q-2}} C\lambda^{\frac{2}{2-q}}\\
&\geq (1-\mu) J_\infty^{\frac{1}{1-\mu}}(t_u^{\frac{1}{1-\mu}} u) - \frac{2-q}{2q}\mu^{\frac{q}{q-2}} C\lambda^{\frac{2}{2-q}}\\
&= (1-\mu)^{3/2}\frac{1}{3}\Big(\frac{\|u\|^6}{\int_\Omega g|u|^6 dx}\Big)^{-1/2} - \frac{2-q}{2q}\mu^{\frac{q}{q-2}} C\lambda^{\frac{2}{2-q}}\\
&= (1-\mu)^{3/2} J_\infty^1(t_u^1 u) - \frac{2-q}{2q}\mu^{\frac{q}{q-2}} C\lambda^{\frac{2}{2-q}}.
\end{aligned}
$$

This completes the proof. □

Let $\eta(x) \in C_0^\infty(\mathbb{R}^3)$ be a radially symmetric function with $0 \leq \eta \leq 1$, $|\nabla\eta| \leq C$, and

$$
\eta(x) = \begin{cases} 1, & \text{if } |x| \leq \frac{r_0}{2},\\ 0, & \text{if } |x| \geq r_0. \end{cases}
$$

For any $z \in M$, we define

$$
\omega_{\varepsilon,z}(x) = \eta(x-z)v_\varepsilon(x-z).
$$

where $v_\varepsilon(x)$ is given by (2.1). From the same arguments of [24] we know that

$$
\int_\Omega |\nabla\omega_{\varepsilon,z}|^2 dx = S^{\frac{3}{2}} + O(\varepsilon^{1/2}) \tag{3.1}
$$

and

$$
\begin{gathered}
C_1\varepsilon^{q/4} \leq \int_\Omega |\omega_{\varepsilon,z}|^q dx \leq C_2\varepsilon^{q/4}, \quad 1 \leq q < 3,\\
C_3\varepsilon^{q/4}|\ln\varepsilon| \leq \int_\Omega |\omega_{\varepsilon,z}|^q dx \leq C_4\varepsilon^{q/4}|\ln\varepsilon|, \quad q = 3,\\
C_5\varepsilon^{(6-q)/4} \leq \int_\Omega |\omega_{\varepsilon,z}|^q dx \leq C_6\varepsilon^{(6-q)/4}, \quad 3 < q < 6.
\end{gathered}
$$

Lemma 3.2. *We have*

$$
\int_\Omega g|\omega_{\varepsilon,z}|^6 dx = S^{\frac{3}{2}} + O(\varepsilon^{3/2})
$$

For a proof of the above lemma, see [10, Lemma 3.1].

Lemma 3.3. *There exists $\varepsilon_0 > 0$ small enough such that for $\varepsilon \in (0, \varepsilon_0)$, we have $\sigma(\varepsilon_0) > 0$ and*

$$
\sup_{t\geq 0} J_\lambda(u_\lambda^+ + t\omega_{\varepsilon,z}) < \alpha_\lambda^+ + \frac{1}{3}S^{3/2} - \sigma(\varepsilon_0)
$$

uniformly for $z \in M$. Furthermore, there exists $t_z^- > 0$ such that

$$
u_\lambda^+ + t_z^-\omega_{\varepsilon,z} \in N_\lambda^-, \quad \forall z \in M.
$$

Proof. It is easy to see that

$$\lim_{t\to 0} J_\lambda(u_\lambda^+ + t\omega_{\varepsilon,z}) = \alpha_\lambda^+ < 0 \quad \text{and} \quad \lim_{t\to\infty} J_\lambda(u_\lambda^+ + t\omega_{\varepsilon,z}) = -\infty,$$

for all $\varepsilon > 0$ small enough. Thus, there exists $t_0 > 0$ small enough and $t_1 > 0$ large enough such that

$$J_\lambda(u_\lambda^+ + t\omega_{\varepsilon,z}) < \alpha_\lambda^+ + \frac{1}{3}S^{3/2}, \quad \text{for } t \in (0, t_0] \cup [t_1, +\infty).$$

We only need to prove that

$$J_\lambda(u_\lambda^+ + t\omega_{\varepsilon,z}) < \alpha_\lambda^+ + \frac{1}{3}S^{3/2}, \quad \text{for } t \in [t_0, t_1].$$

It is easy to see that for $1 < q < 2$ it holds

$$(a+b)^q \geq a^q + qa^{q-1}b, \quad (a+b)^6 \geq a^6 + b^6 + 6a^5b + 6ab^5, \quad \text{for } a, b \geq 0. \tag{3.2}$$

Since u_λ^+ is a solution of (1.1), it holds

$$\begin{aligned} &\int_\Omega \nabla u_\lambda^+ \nabla \omega_{\varepsilon,z} dx + \int_\Omega l(x)\phi_{u_\lambda^+} u_\lambda^+ \omega_{\varepsilon,z} dx \\ &- \int_\Omega f_\lambda(x)(u_\lambda^+)^{q-1}\omega_{\varepsilon,z} dx - \int_\Omega g(x)(u_\lambda^+)^5 \omega_{\varepsilon,z} dx = 0. \end{aligned} \tag{3.3}$$

It follows from Theorem 2.7 and (3.2)-(3.3) that

$$\begin{aligned} &J_\lambda(u_\lambda^+ + t\omega_{\varepsilon,z}) \\ &= J_\lambda(u_\lambda^+) + \frac{t^2}{2}\|\omega_{\varepsilon,z}\|^2 + t\int_\Omega [\nabla u_\lambda^+ \nabla \omega_{\varepsilon,z} + l\phi_{u_\lambda^+} u_\lambda^+ \omega_{\varepsilon,z} \\ &\quad - g(u_\lambda^+)^5 \omega_{\varepsilon,z} - f_\lambda (u_\lambda^+)^{q-1}\omega_{\varepsilon,z}] dx \\ &\quad + \frac{1}{4}\int_\Omega l[\phi_{u_\lambda^+ + t\omega_{\varepsilon,z}}(u_\lambda^+ + t\omega_{\varepsilon,z})^2 - \phi_{u_\lambda^+}(u_\lambda^+)^2 - 4\phi_{u_\lambda^+} u_\lambda^+ (t\omega_{\varepsilon,z})] dx \\ &\quad - \frac{1}{6}\int_\Omega g[(u_\lambda^+ + t\omega_{\varepsilon,z})^6 - (u_\lambda^+)^6 - 6(u_\lambda^+)^5 t\omega_{\varepsilon,z}] dx \\ &\quad - \frac{1}{q}\int_\Omega f_\lambda [(u_\lambda^+ + t\omega_{\varepsilon,z})^q - (u_\lambda^+)^q - q(u_\lambda^+)^{q-1} t\omega_{\varepsilon,z}] dx \\ &\leq \alpha_\lambda^+ + k(t) + h(t), \end{aligned} \tag{3.4}$$

where

$$k(t) = \frac{t^2}{2}\|\omega_{\varepsilon,z}\|^2 - \frac{t^6}{6}\int_\Omega g(\omega_{\varepsilon,z})^6 dx - t^5 \int_\Omega g u_\lambda^+ (\omega_{\varepsilon,z})^5 dx,$$

$$h(t) = \frac{1}{4}\int_\Omega l[\phi_{u_\lambda^+ + t\omega_{\varepsilon,z}}(u_\lambda^+ + t\omega_{\varepsilon,z})^2 - \phi_{u_\lambda^+}(u_\lambda^+)^2 - 4\phi_{u_\lambda^+} u_\lambda^+ (t\omega_{\varepsilon,z})] dx.$$

Note that

$$\begin{aligned} \int_\Omega g u_\lambda^+ (\omega_{\varepsilon,z})^5 dx &= \int_\Omega g u_\lambda^+ (\eta(x-z) v_\varepsilon(x-z))^5 dx \\ &\geq C\int_{B_{2\rho}} \frac{(3\varepsilon)^{5/4}}{(\varepsilon + |x|^2)^{5/2}} dx \\ &\geq C\varepsilon^{1/4}\int_0^\rho \frac{r^2}{(1+r^2)^{\frac{5}{2}}} dr \\ &\geq C\varepsilon^{1/4}, \end{aligned}$$

for some $C > 0$, we have

$$\begin{aligned} k(t) &\le \frac{t^2}{2}\|\omega_{\varepsilon,z}\|^2 - \frac{t^6}{6}\int_\Omega g(\omega_{\varepsilon,z})^6 dx - C\varepsilon^{1/4} \\ &\le \frac{1}{3}\Big(\frac{\|\omega_{\varepsilon,z}\|^2}{\big(\int_\Omega g(\omega_{\varepsilon,z})^6 dx\big)^{1/3}}\Big)^{3/2} - C\varepsilon^{1/4} \\ &\le \frac{1}{3}\left(\frac{S^{3/2}+O(\varepsilon^{1/2})}{(S^{3/2}+O(\varepsilon^{3/2}))^{1/3}}\right)^{3/2} - C\varepsilon^{1/4} \\ &\le \frac{1}{3}S^{3/2} + O(\varepsilon^{1/2}) - C\varepsilon^{1/4}. \end{aligned} \tag{3.5}$$

We claim that

$$h(t) \le C\varepsilon^{1/2}, \quad \text{for } t \in [t_0, t_1]. \tag{3.6}$$

In fact, by calculations we arrive at

$$\begin{aligned} h(t) &= \frac{1}{4}\int_\Omega l[\phi_{u_\lambda^+ + t\omega_{\varepsilon,z}}(u_\lambda^+ + t\omega_{\varepsilon,z})^2 - \phi_{u_\lambda^+}(u_\lambda^+)^2 - 4\phi_{u_\lambda^+}u_\lambda^+(t\omega_{\varepsilon,z})]dx \\ &= t\int_\Omega l\omega_{\varepsilon,z}u_\lambda^+\phi_{t\omega_{\varepsilon,z}}dx + \frac{t^2}{2}\int_\Omega l\phi_{u_\lambda^+}(\omega_{\varepsilon,z})^2 dx + \frac{t^2}{4}\int_\Omega l\phi_{t\omega_{\varepsilon,z}}(\omega_{\varepsilon,z})^2 dx \\ &\quad + t^2\int_{\Omega\times\Omega}\frac{1}{|x-y|}l(y)u_\lambda^+(y)\omega_{\varepsilon,z}(y)l(x)u_\lambda^+(x)\omega_{\varepsilon,z}(x)\,dx\,dy. \end{aligned} \tag{3.7}$$

Using the Hölder inequality, (3.1) and the fact that $t \in [t_0, t_1]$, we obtain that

$$\int_\Omega l\omega_{\varepsilon,z}u_\lambda^+\phi_{t\omega_{\varepsilon,z}}dx \le |l|_\infty|\phi_{\omega_{\varepsilon,z}}|_6|u_\lambda^+|_{12/5}|\omega_{\varepsilon,z}|_{12/5} \le C|\omega_{\varepsilon,z}|_{12/5}^3 \le C\varepsilon^{3/4}; \tag{3.8}$$

$$\int_\Omega l\phi_{t\omega_{\varepsilon,z}}(\omega_{\varepsilon,z})^2 dx \le |l|_\infty|\phi_{t\omega_{\varepsilon,z}}|_6|\omega_{\varepsilon,z}|_{12/5}^2 \le C|\omega_{\varepsilon,z}|_{12/5}^4 \le C\varepsilon; \tag{3.9}$$

$$\int_\Omega l\phi_{u_\lambda^+}(\omega_{\varepsilon,z})^2 dx \le |l|_\infty|\phi_{u_\lambda^+}|_6|\omega_{\varepsilon,z}|_{12/5}^2 \le C\varepsilon^{1/2}. \tag{3.10}$$

Moreover, by [9, Lemma 2, P.31], it holds

$$\begin{aligned} &\int_{\Omega\times\Omega}\frac{1}{|x-y|}l(y)u_\lambda^+(y)\omega_{\varepsilon,z}(y)l(x)u_\lambda^+(x)\omega_{\varepsilon,z}(x)\,dx\,dy \\ &\le \Big(\int_\Omega |l(x)u_\lambda^+(x)\omega_{\varepsilon,z}(x)|^{6/5}dx\Big)^{5/3} \\ &\le C|u_\lambda^+|_{12/5}^2|\omega_{\varepsilon,z}|_{12/5}^2 \le C\varepsilon^{1/2}. \end{aligned} \tag{3.11}$$

It follows from (3.7)-(3.11) that (3.6) holds. We deduce from (3.4)-(3.6) that

$$J_\lambda(u_\lambda^+ + t\omega_{\varepsilon,z}) < \alpha_\lambda^+ + \frac{1}{3}S^{3/2} + C\varepsilon^{1/2} - C\varepsilon^{1/4},$$

for $t \in [t_0, t_1]$. Consequently, there exists $\varepsilon_0 > 0$ small enough such that for $\varepsilon \in (0, \varepsilon_0)$, we have $\sigma(\varepsilon_0) > 0$ and

$$\sup_{t\ge 0} J_\lambda(u_\lambda^+ + t\omega_{\varepsilon,z}) < \alpha_\lambda^+ + \frac{1}{3}S^{3/2} - \sigma(\varepsilon_0) \quad \text{uniformly in } z \in M.$$

Now, we prove that there exists $t_z^- > 0$ such that

$$u_\lambda^+ + t_z^-\omega_{\varepsilon,z} \in N_\lambda^-, \text{ for all } z \in M.$$

Let

$$U_1 = \big\{u \in H_0^1(\Omega)\backslash\{0\}; \frac{1}{\|u\|}t^-\big(\frac{u}{\|u\|}\big) > 1\big\} \cup \{0\};$$
$$U_2 = \big\{u \in H_0^1(\Omega)\backslash\{0\}; \frac{1}{\|u\|}t^-\big(\frac{u}{\|u\|}\big) < 1\big\}.$$

Then N_λ^- disconnects $H_0^1(\Omega)$ into two connected components U_1 and U_2. Moreover, $H_0^1(\Omega)\backslash N_\lambda^- = U_1 \cup U_2$. For each $u \in N_\lambda^+$, we have

$$1 < t_{\max} < t^-(u).$$

Since $t^-(u) = \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right)$, then $N_\lambda^+ \subset U_1$. In particular, $u_\lambda^+ \in U_1$. We claim that we can find a constant $c > 0$ such that

$$0 < t^-\Big(\frac{u_\lambda^+ + t\omega_{\varepsilon,z}}{\|u_\lambda^+ + t\omega_{\varepsilon,z}\|}\Big) < c \quad \text{for each } t \ge 0 \text{ and } z \in M.$$

Otherwise, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and

$$t^-\Big(\frac{u_\lambda^+ + t_n\omega_{\varepsilon,z}}{\|u_\lambda^+ + t_n\omega_{\varepsilon,z}\|}\Big) \to \infty \quad \text{as } n \to \infty.$$

Let

$$v_n = \frac{u_\lambda^+ + t_n\omega_{\varepsilon,z}}{\|u_\lambda^+ + t_n\omega_{\varepsilon,z}\|}.$$

Since $t^-(v_n)v_n \in N_\lambda^- \subset N_\lambda$ and by the Lesbesgue dominated convergence theorem,

$$\begin{aligned}
\int_\Omega g|v_n|^6 dx &= \frac{1}{\|u_\lambda^+ + t_n\omega_{\varepsilon,z}\|^6}\int_\Omega g|u_\lambda^+ + t_n\omega_{\varepsilon,z}|^6 dx \\
&= \frac{1}{\|\frac{u_\lambda^+}{t_n} + \omega_{\varepsilon,z}\|^6}\int_\Omega g|\frac{u_\lambda^+}{t_n} + \omega_{\varepsilon,z}|^6 dx \\
&\to \frac{\int_\Omega g|\omega_{\varepsilon,z}|^6 dx}{\|\omega_{\varepsilon,z}\|^6} > 0, \text{ as } n \to \infty,
\end{aligned}$$

we have

$$\begin{aligned}
J_\lambda(t^-(v_n)v_n) &= \frac{1}{2}[t^-(v_n)]^2 + \frac{(t^-(v_n))^4}{4}\int_\Omega l\phi_{v_n}v_n^2 dx - \frac{[t^-(v_n)]^q}{q}\int_\Omega f_\lambda|v_n|^q dx \\
&\quad - \frac{[t^-(v_n)]^6}{6}\int_\Omega g|v_n|^6 dx \to -\infty \quad \text{as } n \to \infty.
\end{aligned}$$

This contradicts that J_λ is bounded below on N_λ and the claim is proved. Let

$$t_\lambda = \frac{|c^2 - \|u_\lambda^+\|^2|^{1/2}}{\|\omega_{\varepsilon,z}\|} + 1\,.$$

Then

$$\begin{aligned}
\|u_\lambda^+ + t_\lambda\omega_{\varepsilon,z}\|^2 &= \|u_\lambda^+\|^2 + t_\lambda^2\|\omega_{\varepsilon,z}\|^2 + 2t_\lambda\langle u_\lambda^+, \omega_{\varepsilon,z}\rangle \\
&> \|u_\lambda^+\|^2 + |c^2 - \|u_\lambda^+\|^2| + 2t_\lambda\int_\Omega u_\lambda^+\omega_{\varepsilon,z} dx \\
&> c^2 > \Big[t^-\Big(\frac{u_\lambda^+ + t_\lambda\omega_{\varepsilon,z}}{\|u_\lambda^+ + t_\lambda\omega_{\varepsilon,z}\|}\Big)\Big]^2,
\end{aligned}$$

that is $u_\lambda^+ + t_\lambda \omega_{\varepsilon,z} \in U_2$. Thus there exists $0 < t_z^- < t_\lambda$ such that $u_\lambda^+ + t_z^- \omega_{\varepsilon,z} \in N_\lambda^-$. $\square$

Lemma 3.4. *We have*
$$\inf_{u\in N_\infty^1} J_\infty^1(u) = \inf_{u\in N^\infty} J^\infty(u) = \frac{1}{3}S^{3/2},$$
where
$$J^\infty(u) = \frac{1}{2}\|u\|^2 - \frac{1}{6}\int_\Omega |u|^6 dx, \quad N^\infty = \{u \in H_0^1(\Omega)\backslash\{0\}; \langle (J^\infty)'(u), u\rangle = 0\}.$$

Proof. From [24], we have
$$\inf_{u\in N^\infty} J^\infty(u) = \frac{1}{3}S^{3/2}.$$
Thus it suffices to show that $\inf_{u\in N_\infty^1} J_\infty^1(u) = \frac{1}{3}S^{3/2}$. Since
$$\max_{t\geq 0}\Big(\frac{a}{2}t^2 - \frac{b}{6}t^6\Big) = \frac{1}{3}\Big(\frac{a}{b^{1/3}}\Big)^{3/2}$$
for any $a > 0$ and $b > 0$, by (3.1) and Lemma 3.2 we deduce that
$$\sup_{t\geq 0} J_\infty^1(t\omega_{\varepsilon,z}) = \frac{1}{3}\Big(\frac{\|\omega_{\varepsilon,z}\|^2}{\big(\int_\Omega g|\omega_{\varepsilon,z}|^6 dx\big)^{1/3}}\Big)^{3/2} = \frac{1}{3}S^{3/2} + O(\varepsilon^{1/2}).$$
Then we obtain
$$\inf_{u\in N_\infty^1} J_\infty^1(u) \leq \frac{1}{3}S^{3/2}, \quad \text{as } \varepsilon \to 0^+.$$
Since $g \leq 1$, for each $u \in H_0^1(\Omega)\backslash\{0\}$, we have
$$\sup_{t\geq 0} J^\infty(tu) \leq \sup_{t\geq 0} J_\infty^1(tu).$$
Hence
$$\begin{aligned}\frac{1}{3}S^{3/2} &= \inf_{u\in N^\infty} J^\infty(u) = \inf_{u\in H_0^1(\Omega)\backslash\{0\}} \sup_{t\geq 0} J^\infty(tu)\\ &\leq \inf_{u\in H_0^1(\Omega)\backslash\{0\}} \sup_{t\geq 0} J_\infty^1(tu) = \inf_{u\in N_\infty^1} J_\infty^1(u) \leq \frac{1}{3}S^{3/2}.\end{aligned}$$
This completes the proof. $\square$

4. Proof of Theorem 1.2

In this section, we use the idea of category to get positive solutions of (1.1) and give the proof of Theorem 1.2. Initially, we state the following two propositions related to category theory.

Proposition 4.1 ([5, Theorem 2.1])**.** *Let R be a $C^{1,1}$ complete Riemannian manifold (modelled on a Hilbert space) and assume $F \in C^1(R,\mathbb{R})$ bounded from below. Let $-\infty < \inf_R F < a < b < +\infty$. Suppose that F satisfies (PS)-condition on the sublevel $\{u \in R; F(u) \leq b\}$ and that a is not a critical level for F. Then*
$$\sharp\{u \in F^a; \nabla F(u) = 0\} \geq \operatorname{cat}_{F^a}(F^a),$$
where $F^a \equiv \{u \in R; F(u) \leq a\}$.

Proposition 4.2 ([5, Lemma 2.2]). *Let Q, Ω^+ and Ω^- be closed sets with $\Omega^- \subset \Omega^+$; let $\phi : Q \to \Omega^+$, $\varphi : \Omega^- \to Q$ be two continuous maps such that $\phi \circ \varphi$ is homotopically equivalent to the embedding $j : \Omega^- \to \Omega^+$. Then $\operatorname{cat}_Q(Q) \geq \operatorname{cat}_{\Omega^+}(\Omega^-)$.*

The proof of Theorem 1.2 is based on Propositions 4.1 and 4.2. To argue further, we need to introduce the following Lemma.

Lemma 4.3. *Let $\{u_n\} \subset H_0^1(\Omega)$ be a nonnegative function sequence with $|u_n|_6 = 1$ and $\int_\Omega |\nabla u_n|^2 dx \to S$. Then there exists a sequence $(y_n, \theta_n) \in \mathbb{R}^3 \times \mathbb{R}^+$ such that*

$$v_n(x) := \theta_n^{1/2} u_n(\theta_n x + y_n)$$

contains a convergent subsequence denoted again by $\{v_n\}$ such that $v_n \to v$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ with $v(x) > 0$ in $\mathbb{R}^3$. Moreover, we have $\theta_n \to 0$ and $y_n \to y \in \overline{\Omega}$.

For a proof of the above lemmas, see See Willem [24]. Next we define the continuous map $\Phi : H_0^1(\Omega)\backslash G \to \mathbb{R}^N$ by

$$\Phi(u) := \frac{\int_\Omega x|u - u_\lambda^+|^6 dx}{\int_\Omega |u - u_\lambda^+|^6 dx},$$

where $G = \{u \in H_0^1(\Omega); \int_\Omega |u - u_\lambda^+|^6 dx = 0\}$. Then we have the following lemma.

Lemma 4.4. *For each $0 < \delta < r_0$, there exist $\Lambda_\delta, \delta_0 > 0$ such that if $u \in N_\infty^1$, $J_\infty^1(u) < \frac{1}{3}S^{3/2} + \delta_0$ and $\lambda < \Lambda_\delta$, then $\Phi(u) \in M_\delta$.*

Proof. Suppose the contrary. Then there exists a sequence $\{u_n\} \subset N_\infty^1$ such that $J_\infty^1(u_n) = \frac{1}{3}S^{3/2} + o(1)$, $\lambda \to 0^+$, and

$$\Phi(u_n) \notin M_\delta \quad \forall n.$$

It is easy to show that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and there is a sequence $\{t_n^\infty\} \subset \mathbb{R}^+$ such that $\{t_n^\infty u_n\} \in N^\infty$ and

$$\frac{1}{3}S^{3/2} \leq J^\infty(t_n^\infty u_n) \leq J_\infty^1(t_n^\infty u_n) \leq J_\infty^1(u_n) = \frac{1}{3}S^{3/2} + o(1). \tag{4.1}$$

We obtain $t_n^\infty = 1 + o(1)$ as $n \to \infty$ and

$$\begin{aligned} \lim_{n\to\infty} J^\infty(u_n) &= \lim_{n\to\infty} \frac{1}{3}\|u_n\|^2 = \lim_{n\to\infty} \frac{1}{3}\int_\Omega |u_n|^6 dx \\ &= \lim_{n\to\infty} \frac{1}{3}\int_\Omega g|u_n|^6 dx = \frac{1}{3}S^{3/2} + o(1). \end{aligned} \tag{4.2}$$

Let

$$U_n = \frac{u_n}{\left(\int_\Omega |u_n|^6 dx\right)^{1/6}}.$$

We see that $\int_\Omega |U_n|^6 dx = 1$. Furthermore, it follows from (4.2) that

$$\lim_{n\to\infty} \|U_n\|^2 = S.$$

By Lemma 4.3, there is a sequence $\{(x_n, \varepsilon_n)\} \subset \mathbb{R}^3 \times \mathbb{R}^+$ such that $\varepsilon_n \to 0$, $x_n \to x_0 \in \overline{\Omega}$ and $\omega_n(x) = \varepsilon_n^{1/2} U_n(\varepsilon_n x + x_n) \to \omega$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ with $\omega > 0$ as $n \to \infty$. Then by (4.2),

$$1 = o(1) + \int_\Omega g|U_n|^6 dx = \varepsilon_n^{-3}\int_\Omega g\big|\omega_n\big(\frac{x - x_n}{\varepsilon_n}\big)\big|^6 dx + o(1) = g(x_0),$$

as $n \to \infty$, which implies $x_0 \in M$. By the Lebesgue dominated convergence theorem again, we have

$$\begin{aligned}
\Phi(u_n) &= \frac{\int_\Omega x|u_n - u_\lambda^+|^6 dx}{\int_\Omega |u_n - u_\lambda^+|^6 dx} && (4.3)\\
&= \frac{\int_\Omega x|u_n|^6 dx}{\int_\Omega |u_n|^6 dx} + o(1), && (4.4)\\
&= \frac{\varepsilon_n^{-3}\int_\Omega x\big|\omega_n\big(\frac{x-x_n}{\varepsilon_n}\big)\big|^6 dx}{\varepsilon_n^{-3}\int_\Omega \big|\omega_n\big(\frac{x-x_n}{\varepsilon_n}\big)\big|^6 dx} + o(1) && (4.5)\\
&\to x_0 \in M \quad \text{as } n \to \infty,\ \lambda \to 0 && (4.6)
\end{aligned}$$

which is a contradiction. □

Lemma 4.5. *There exists $\Lambda_\delta > 0$ small enough such that if $\lambda < \Lambda_\delta$ and $u \in N_\lambda^-$ with $J_\lambda(u) < \frac{1}{3}S^{3/2} + \frac{\delta_0}{2}$ (δ_0 is given in Lemma 4.4), then $\Phi(u) \in M_\delta$.*

Proof. By Lemma 3.1, for $\mu \in (0,1)$, there is a unique t_u^1 such that $t_u^1 u \in N_\infty^1$ and

$$J_\infty^1(t_u u) \le (1-\mu)^{-3/2}(J_\lambda(u) + C\mu^{\frac{q}{q-2}}\lambda^{\frac{2}{2-q}}).$$

Thus there exists $\Lambda_\delta > 0$ small enough such that if $\lambda < \Lambda_\delta$ and $J_\lambda(u) < \frac{1}{N}S_{\alpha,\beta}^{N/2} + \frac{\delta_0}{2}$,

$$J_\infty^1(t_u^1 u) \le \frac{1}{3}S^{3/2} + \delta_0.$$

By Lemma 4.4 and $\|u_\lambda^+\| \to 0$ as $\lambda \to 0$, we complete the proof. □

Now we denote $c_\lambda := \alpha_\lambda^+ + \frac{1}{3}S^{3/2} - \sigma(\varepsilon_0)$ and consider the filtration of the manifold of N_λ^- as follows:

$$N_\lambda^-(c_\lambda) := \{u \in N_\lambda^-; J_\lambda(u) \le c_\lambda\}.$$

Then $\mathrm{cat}_{M_\delta}(M)$ critical points of J_λ will be obtained from $N_\lambda^-(c_\lambda)$ in the following. At first, we show that a critical point of J_λ restrict on N_λ^- is in fact a critical point of J_λ in $H_0^1(\Omega)$.

Lemma 4.6. *If u is a critical point of J_λ on N_λ^-, then it is a critical point of J_λ in $H_0^1(\Omega)$.*

Proof. If $u \in N_\lambda^-$, then $\langle J_\lambda'(u), u\rangle = 0$. On the other hand,

$$J_\lambda'(u) = \tau\psi_\lambda'(u)$$

for some $\tau \in \mathbb{R}$, where ψ_λ is defined in (2.6). Thus we have

$$0 = \tau\langle\psi_\lambda'(u), u\rangle,$$

which combined with the definition of N_λ^- imply that $\tau = 0$, i.e. $J_\lambda'(u) = 0$. □

In the succeeding text, we denote by $J_{N_\lambda^-}$ the restriction of J_λ on N_λ^- and show that $J_{N_\lambda^-}$ satisfies (PS)-condition on $N_\lambda^-(c_\lambda)$.

Lemma 4.7. *Any sequence $\{u_n\} \subset N_\lambda^-$ such that $J_{N_\lambda^-}(u_n) \to \beta \in (-\infty, c_\lambda]$ and $J_{N_\lambda^-}'(u_n) \to 0$ contains a convergent subsequence.*

Proof. By hypothesis there exists a sequence $\{\theta_n\} \subset \mathbb{R}$ such that

$$J'_\lambda(u_n) = \theta_n \psi'_\lambda(u_n) + o(1).$$

Recall that $u_n \in N^-_\lambda$ and so

$$\langle \psi'_\lambda(u_n), u_n\rangle < 0.$$

If $\langle \psi'_\lambda(u_n), u_n\rangle \to 0$, we from (2.7) deduce that

$$\begin{aligned}
&J_\lambda(u_n)\\
&= J_\lambda(u_n) - \frac{1}{q}\langle J'_\lambda(u_n), u_n\rangle\\
&= \big(\frac{1}{2} - \frac{1}{q}\big)\|u_n\|^2 + \big(\frac{1}{4} - \frac{1}{q}\big)\int_\Omega l(x)\phi_{u_n} u_n^2 dx - \big(\frac{1}{q} - \frac{1}{6}\big)\int_\Omega g(x)|u_n|^6 dx\\
&= \frac{q-2}{2q}\|u_n\|^2 + \frac{q-4}{4q}\int_\Omega l(x)\phi_{u_n} u_n^2 dx - \frac{6-q}{6q}\int_\Omega g(x)|u_n|^6 dx\\
&= \frac{q-2}{3q}\|u_n\|^2 + \frac{q-4}{12q}\int_\Omega l(x)\phi_{u_n} u_n^2 dx \le 0.
\end{aligned}$$

Hence we arrive at a contradiction to that $\alpha^-_\lambda > 0$ (Lemma 2.4 (ii)). Thus we may assume that $\langle \psi'_\lambda(u_n), u_n\rangle \to l < 0$. Because $\langle J'_\lambda(u_n), u_n\rangle = 0$, we conclude that $\theta_n \to 0$ and, consequently, $J'_\lambda(u_n) \to 0$. Using this information we have

$$J_\lambda(u_n) \to \beta \in (-\infty, c_\lambda] \text{ and } J'_\lambda(u_n) \to 0,$$

so by Lemma 2.6, the proof is complete. □

Lemma 4.8. *Let $\delta, \Lambda_\delta > 0$ be as in Lemmas 4.4 and 4.5. Then for $\lambda < \Lambda_\delta$, J_λ has at least* $\mathrm{cat}_{M_\delta}(M)$ *critical points in $N^-_\lambda(c_\lambda)$.*

Proof. For $z \in M$, by Lemma 3.3, we can define

$$F(z) = u^+_\lambda + t^-_z \omega_{\varepsilon,z} \in N^-_\lambda(c_\lambda).$$

Furthermore, J_λ satisfies (PS)-condition on $N^-_\lambda(c_\lambda)$. Moreover, it follows from Lemma 4.5 that $\Phi(N^-_\lambda(c_\lambda)) \subset M_\delta$ for $\lambda < \Lambda_\delta$. Define $\xi : [0,1] \times M \to M_\delta$ by

$$\xi(\theta, z) = \Phi\left(u^+_\lambda + t^-_z \omega_{(1-\theta)\varepsilon,z}\right) \in N^-_\lambda(c_\lambda).$$

Then straightforward calculations provide that $\xi(0,z) = \Phi\circ F(z)$ and $\lim_{\theta\to 1^-}\xi(\theta,z) = z$. Hence $\Phi \circ F$ is homotopic to the inclusion $j : M \to M_\delta$. Combining Lemma 4.7 with Propositions 4.1 and 4.2, we obtain that $J_{N^-_\lambda(c_\lambda)}$ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $N^-_\lambda(c_\lambda)$. By Lemma 4.6, we know that J_λ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $N^-_\lambda(c_\lambda)$. □

Proof of Theorem 1.2. By Theorem 2.7 and Lemma 4.8, applying $N^+_\lambda \bigcap N^-_\lambda = \emptyset$ and the strong maximum principle, we obtain the statement of Theorem 1.2. □

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