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DECAY ESTIMATES FOR NONLINEAR WAVE EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT. We study the long-time behavior of solutions to a particular class of nonlinear wave equations that appear in models for waves traveling in a non-homogeneous gas with variable damping. Specifically, decay estimates for the energy of such solutions are established. We find three different regimes of energy decay depending on the exponent of the absorption term $|u|^{p-1}u$ and show the existence of two critical exponents $p_1(n,\alpha,\beta) = 1 + (2-\beta)/(n-\alpha)$ and $p_2(n,\alpha) = (n+\alpha)/(n-\alpha)$. For $p > p_1(n,\alpha,\beta)$, the decay of solutions of the nonlinear equation coincides with that of the corresponding linear problem. For $p_1(n,\alpha,\beta) > p$, the solution decays much faster. The other critical exponent $p_2(n,\alpha)$ further divides this region into two subregions with different decay rates. Deriving the sharp decay of solutions even for the linear problem with potential $a(x)$ is a delicate task and requires serious strengthening of the multiplier method. Here we use a modification of an approach of Todorova and Yordanov to derive the exact decay of the nonlinear equation.

1. INTRODUCTION

In this article, we consider a class of nonlinear partial differential equations used to model waves traveling in a non-homogenous medium with both space-dependent friction coefficient and bulk modulus, which accounts for varying temperature in the medium. Further, we include a non-linear term which causes the system to be much more complicated than previous efforts that only focused on more homogeneous cases. Until now, the significant issues in these equations have only been investigated separately. Combining the variable coefficients with the nonlinearity results in a much more general problem.

Particularly, our nonlinearity is defocusing. If it were focusing, with a negative coefficient, the behavior of solutions would be drastically different. For such nonlinearity and supercritical nonlinear exponents, small initial data solutions are global, but alternatively, for subcritical exponents, the solutions blow up in finite time for any initial data positive in average, as shown in [4]. On the other hand, with a defocusing nonlinearity, the global existence of the solution is a classical result, but the asymptotic behavior of the energy is still under investigation.

Past works dealing with similar equations are easily separated into the linear and nonlinear cases. Papers [1, 3, 5, 6, 8] each study nonlinear cases with absorption

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nonlinearities. None of these papers estimate asymptotic decay rates for the case with the variable coefficients: they each assume the bulk modulus is constant, which simplifies the calculations and results.

Kenigson [7] considered a linear case similar to (2.1) but with only a space-time dependent damping coefficient, and in [2], the Laplacian is split by the variable bulk modulus, but with a critical difference to our equations: theirs are linear. The nonlinearity raises many issues that must be managed very delicately, so including it is a worthy expansion of the problem.

Mathematically, the primary difficulties arise in dealing with the energy terms that come from the interactions among the nonlinearity and the variable coefficients. Using an advanced weighted multiplier method developed by Todorova and Yordanov [2], we overcome these issues.

This article is organized such that in section 2, we state the precise problem with certain assumptions and the primary result. In section 3, we derive the main inequalities for the weighted energy followed by the corresponding inequalities for the unweighted energy. In section 4, we define the weights ϕ and θ and derive the main energy decay inequality. In section 5, we prove theorems concerning the rate of decay for cases of either supercritical or subcritical nonlinear exponents. Lastly, in section 6, we prove the main Theorem, 2.3.

2. Mathematical preliminaries

Consider the dissipative non-linear wave equation

$$\begin{gathered} u_{tt} - \operatorname{div}(b(x)\nabla u) + a(x)u_t + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^n,\ t > 0 \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^n), \quad u_t(0,x) = u_1(x) \in L^2(\mathbb{R}^n), \end{gathered} \tag{2.1}$$

where $n \geq 3$, $1 < p < (n+2)/(n-2)$, $a \in C(\mathbb{R}^n)$, and $b \in C^1(\mathbb{R}^n)$ are positive, and u_0 and u_1 have compact support such that $u_0(x) = 0$ and $u_1(x) = 0$ for $|x| > R$. In addition, we require that $a(x)$ and $b(x)$ behave in such a way that

$$b_0(1+|x|)^\beta \leq b(x) \leq b_1(1+|x|)^\beta, \tag{2.2}$$

$$a_0(1+|x|)^{-\alpha} \leq a(x) \leq a_1(1+|x|)^{-\alpha} \tag{2.3}$$

where $\alpha, \beta \in \mathbb{R}$ and a_0, a_1, b_0, b_1 are positive constants and

$$\alpha < 1, \quad 0 \leq \beta < 2, \quad 2\alpha + \beta < 2. \tag{2.4}$$

The restrictions on these exponents are natural because as $\beta \to 2^-$ and $\alpha \to 0^+$, the decay approaches infinity (see Theorem 2.3). Further, if $\alpha \geq 1$, there is no longer decay, but the energy instead dissipates to some positive constant per [9].

First, we create what we hope will be an approximate solution to (2.1), using the following conjecture.

Conjecture 2.1. *Under the assumptions* (2.2) *and* (2.3)*, there exists a subsolution* $A(x)$ *which satisfies a related differential inequality*

$$\operatorname{div}(b(x)\nabla A(x)) \geq a(x) \tag{2.5}$$

with the following properties, for $d_0, d_1 > 0$*:*

$$d_0(1+|x|)^{2-\alpha-\beta} \leq A(x) \leq d_1(1+|x|)^{2-\alpha-\beta} \tag{2.6}$$

$$\mu = \liminf_{|x|\to\infty} \frac{a(x)A(x)}{b(x)|\nabla A(x)|^2} > 0. \tag{2.7}$$

There are multiple cases with only mild restrictions on a and b that grant existence of such subsolutions. These cases and that $\mu = \frac{n-a}{2-\alpha-\beta}$ are dealt with in [2]. For now, we assume A exists, which we then use to construct

$$\sigma(x) = (\mu - \delta)A(x) + \sigma_0, \tag{2.8}$$

where $\delta \in (0, \mu/2)$ and σ_0 is a positive constant sufficiently large to make $\sigma \geq 0$.

Let us mention that σ ultimately plays a crucial role in the definitions of the multiplier weights. The idea is to imitate similar methods used to approximate the solutions of linear equations with constant coefficients with the diffusion phenomenom, through which, it is shown that the solution of the linear dissipative equation

$$u_{tt} - \Delta u + u_t = 0$$

has similar large time behavior to the solution of the diffusion equation

$$w_t - \Delta w = 0.$$

In the linear case with constant coefficients, (2.5) becomes the Poisson equation $\Delta A = 1$ with radial, nonnegative solution $A(x) = \frac{|x|^2}{2n}$. This A is then used to construct the Gaussian approximate solution $w(x,t) = t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$. In our case, with variable coefficients and a nonlinearity, the diffusion phenomenon has not been proven, but we still use σ to construct our approximate solution

$$\phi(x,t) = t^{-m} e^{-\frac{\sigma(x)}{t}},$$

with suitable parameter m to optimize decay, in the hopes that ϕ and u will have similar large t behavior.

Before we begin, it is important to note that $\operatorname{suppt}(u(t,x))$ is contained in the set

$$\{x \in \mathbb{R}^n : |x| \leq [(1+R)^{(2-\beta)/2} + t\sqrt{b_1}]^{2/(2-\beta)}\}.$$

Notice that the speed of propagation is finite, but variable due to the coefficient b. This and the following proposition are proven in [2].

Proposition 2.2. *Define*

$$g(t) := \inf\{a(x) : x \in \operatorname{suppt} u(\cdot, t)\}, \tag{2.9}$$

$$G(t) := \sup\{A(x) : x \in \operatorname{suppt} u(\cdot, t)\}, \tag{2.10}$$

$$\gamma = \begin{cases} \frac{2\alpha}{2-\beta}, & \text{if } \beta \leq 2 - 2\alpha, \\ 0, & \text{if } \alpha \leq 0. \end{cases} \tag{2.11}$$

Then

$$g(t) \geq g_0 t^{-\gamma} \quad \text{if } t \geq t_0, \tag{2.12}$$

$$G(t) \leq G_0 t^{2-\gamma} \quad \text{if } t \geq t_0, \tag{2.13}$$

where g_0 and G_0 are positive constants.

At first, we factor out our approximate solution from our solution u in the hopes that the new equation will have more timid large time behavior. We then apply a strengthened multiplier method using weights designed for our problem, progressing through the proofs by simply placing sufficient conditions on the arbitrary weights in order to grant us important mathematical qualities of the solution of the new equation. Estimating the energy decay of this altered problem gives us a strict energy decay estimate for the original problem, and lastly, we verify that weights with such qualities actually exist.

After doing all this, we obtain the following theorem concerning the decay rates of the energy,

$$E(t)=\frac{1}{2}\int u_t^2+b|\nabla u|^2dx+\frac{1}{p+1}\int |u|^{p+1}dx.$$

Theorem 2.3. *The energy of the solution to* (2.1) *satisfies, for some* $c>0$,

$$E(t)\le ct^{-m-1},$$

where, for $\delta>0$,

$$m=\begin{cases}\frac{2}{p-1}-\delta & \text{if } 1<p\le 1+\frac{2-\beta}{n-\alpha},\\ \frac{2}{p-1}+\frac{\alpha\frac{p+1}{p-1}-n}{2-\alpha-\beta}-\delta & \text{if } 1+\frac{2-\beta}{n-\alpha}<p\le\frac{n+\alpha}{n-\alpha},\\ \frac{n-\alpha}{2-\alpha-\beta}-\delta & \text{if } \frac{n+\alpha}{n-\alpha}<p<\frac{n+2}{n-2}.\end{cases}$$

Remark 2.4. Notice that as the nonlinear exponent p becomes larger, the nonlinearity affects the decay less. For large p, the optimal decay corresponds with the decay of the linear equation, derived in [2].

3. Weighted energy

To begin, we factor out the approximate solution $\phi(x,t)=t^{-m}e^{-\frac{\sigma(x)}{t}}$, granted by imitating the diffusion phenomenon of linear equations, from u. Setting $u=v\phi$ gives a new partial differential equation with respect to v. Ideally, this new equation will have simpler large time behavior. We obtain

$$v_{tt}-\hat{b}_1\Delta v-\hat{b}_2\cdot\nabla v+\hat{a}_1v_t+\hat{a}_2v+\phi^{p-1}|v|^{p-1}=0 \tag{3.1}$$

with new coefficients

$$\begin{gathered}\hat{b}_1(x)=b(x),\quad \hat{b}_2(x,t)=\nabla b(x)+2b(x)\phi(x,t)^{-1}\nabla\phi(x,t)\\ \hat{a}_1(x,t)=a(x)+2\phi(x,t)^{-1}\phi_t(x,t)\\ \hat{a}_2(x,t)=\phi(x,t)^{-1}(\phi_{tt}(x,t)-\operatorname{div}(b(x)\nabla\phi(x,t))+a(x)\phi_t(x,t)).\end{gathered} \tag{3.2}$$

Now we apply a strengthened multiplier method using weights that will later be specifically designed for our problem. Note that despite one weight's being named ϕ, we do not mean to necessarily imply a connection between the weight and the approximate solution right now. This naming is used only in foresight that the two are actually the same.

Proposition 3.1. *Using multipliers* ϕv *and* θv_t *and adding the resulting equations together gives the weighted energy identity* $\frac{d}{dt}E_w+F+G+H=0$, *where*

$$E_w=\frac{1}{2}\int\theta(v_t^2+b|\nabla v|^2)+2\phi v_tv+(\hat{a}_2\theta+\phi_t+a\phi)v^2+\frac{\phi^{p-1}\theta|v|^{p+1}}{p+1}dx,$$

$$\begin{aligned}F=&\frac{1}{2}\int(-\theta_t+2(a+2\phi^{-1}\phi_t)\theta-2\phi)v_t^2dx+\int b(\nabla\theta-2\theta\phi^{-1}\nabla\phi)\cdot v_t\nabla v dx\\ &+\frac{1}{2}\int b(-\theta_t+2\phi)|\nabla v|^2dx,\end{aligned}$$

$$G=\frac{1}{2}\int[\hat{a}_2\phi-(\hat{a}_2\theta)_t]v^2dx,$$

$$H = \int [\phi^p - \frac{1}{1+p}(\theta\phi^{p-1})_t]|v|^{p+1}dx.$$

Proof. Using the finite speed of propagation and elementary calculus allows us to integrate by parts over the compact support. Doing so straightforwardly yields the desired result. □

We now seek to bound the weighted energy E_w so that the unweighted energy will be decaying. In order to proceed, we place conditions on our weights θ and ϕ. The proof that there are weights satisfying these conditions will come later.

Proposition 3.2. *Assume that $\theta > 0$ and $\phi > 0$ are C^1-functions such that*

(A1) $-\theta_t + \phi \geq 0$,

(A2) $(-\theta_t + 2\theta(a + 2\phi^{-1}\phi_t) - 2\phi)(-\theta_t + 2\phi) \geq b|\nabla\theta - 2\theta\phi^{-1}\nabla\phi|^2$.

Then $F \geq 0$ for $t \geq t_0$.

The above proposition follows directly from the quadratic form of F in v_t and ∇v.

Proposition 3.3. *Assume that θ and ϕ satisfy conditions* (A1) *and* (A2) *in Proposition 3.2, and in addition satisfy the following two conditions:*

(A3) $(p+1)\phi^p - (\theta\phi^{p-1})_t \geq \phi^p$,

(A4) $\hat{a}_2\phi - (\hat{a}_2\theta)_t \geq k_0 C^-\phi$ *with* $C^-(x,t)$ *such that* $\int_{t_0}^\infty \int \phi^{-1}|C^-|^{\frac{p+1}{p-1}}\,dx\,dt < \infty$ *and* $k_0 > 0$.

Then, for $t \geq t_0$ and some $k \geq 0$, we have $G + H \geq -k\int \phi^{-1}|C^-|^{\frac{p+1}{p-1}}dx$.

Proof. Using (A3), (A4), and Hölder's inequality with exponents $\frac{p+1}{p-1}$ and $\frac{p+1}{2}$ yields

$$\begin{aligned}
G + H &= \frac{1}{2}\int [\hat{a}_2\phi - (\hat{a}_2\theta)_t]v^2 dx + \int [\phi^p - \frac{1}{p+1}(\theta\phi^{p-1})_t]|v|^{p+1}dx \\
&\geq \frac{k_0}{2}\int C^-\phi v^2 dx + \int \frac{1}{p+1}\phi^p|v|^{p+1}dx \\
&\geq -k_0\frac{p-1}{2(p+1)}\int \phi^{-1}|C^-|^{\frac{p+1}{p-1}}dx.
\end{aligned}$$

Letting $k = k_0(p-1)/(2p+2)$ completes the proof. □

These first four conditions, when applied to the weighted energy identity $\frac{d}{dt}E_w + F + G + H = 0$, give us a constant upper bound on the weighted energy, which is one step in ensuring the decay of the unweighted energy, as follows.

Theorem 3.4. *If conditions* (A1)–(A4) *hold, then*

$$E_w(t) \leq E_w(t_0) + k\int_{t_0}^\infty \int \phi^{-1}|C^-|^{\frac{p+1}{p-1}}\,dx\,dt < \infty.$$

Proof. Using Propositions 3.2 and 3.3 in the weighted energy equality gives that

$$\frac{d}{dt}E_w = -F - (G+H) \leq -(G+H) \leq k\int \phi^{-1}|C^-|^{\frac{p+1}{p-1}}dx.$$

Finally, integrating both sides from t_0 to ∞ and using the integral inequality in (A4) leaves us with a finite bound on the weighted energy as claimed. □

We now need to eliminate the terms that lack an obvious sign by imposing three more conditions, and also, for our next theorem, we will need the following lemma from [2].

Lemma 3.5. *If $f \in C([t_0,\infty))$ is a positive function, then*
$$e^{-\int_{t_0}^t f(s)ds}\int_{t_0}^t e^{\int_{t_0}^s f(\tau)d\tau}ds \le \max_{s\in[t_0,t]}\frac{1}{f(s)}.$$

Theorem 3.6. *Assume that conditions* (A1)–(A4) *hold and that*

(A5) $C^- \le \hat a_2$ *satisfying*
$$\sup_{t\ge t_0}\int \theta\phi^{-2}|C^-|^{\frac{p+1}{p-1}}dx < \infty.$$

Then for $t \ge t_0$ and for some $k_0, k_1 > 0$, we have that
$$\int \phi v^2 dx \le k_0 + k_1 t^{2\alpha/(2-\beta)}.$$

Proof. Recall that the weighted energy is
$$E_w = \frac{1}{2}\int \theta(v_t^2 + b|\nabla v|^2) + 2\phi v_t v + (\hat a_2\theta + \phi_t + a\phi)v^2 + \frac{1}{p+1}\phi^{p-1}\theta|v|^{p+1}dx.$$
As shown in Theorem 3.4,
$$E_w(t) \le b_0 := E_w(t_0) + k\int_{t_0}^\infty\int \phi^{-1}|C^-|^{\frac{p+1}{p-1}}\,dx\,dt < \infty.$$
After rearranging terms in E_w, we have that
$$\begin{aligned}
&\frac{d}{dt}\int \phi v^2 dx + \int a(x)\phi v^2 dx\\
&\le 2b_0 + \int -\hat a_2\theta v^2 dx - \int \frac{2}{p+1}\phi^{p-1}\theta|v|^{p+1}dx\\
&\le 2b_0 + 2c_1\int \theta\phi^{-2}|C^-|^{\frac{p+1}{p-1}}dx \quad \text{(per Young's inequality)}\\
&\le 2b_0 + c_2 = c_0
\end{aligned}$$
by condition (A5).

Using the finite speed of propagation and (2.12), we find a lower bound on $a(x)$ for $x \in \operatorname{suppt}(u)$:
$$g_0 t^{-2\alpha/(2-\beta)} \le a(x).$$
This gives the ordinary differential inequality
$$\frac{d}{dt}\int \phi v^2 dx + g_0 t^{-2\alpha/(2-\beta)}\int \phi v^2 dx \le c_0,$$
which can be solved to show that
$$\int \phi v^2 dx \le e^{-\int_{t_0}^t g_0 s^{-\gamma}ds}\Big[c_1 + c_0\int_{t_0}^t e^{\int_{t_0}^s g_0\tau^{-\gamma}d\tau}ds\Big]$$
for $t > t_0$ and γ as defined in (2.11). Because of Lemma 3.5 and the fact that $t^{-\gamma}$ is decreasing,
$$\int \phi v^2 dx \le e^{-\int_{t_0}^t g_0 s^{-\gamma}ds}\Big[c_1 + c_0\int_{t_0}^t e^{\int_{t_0}^s g_0\tau^{-\gamma}d\tau}ds\Big]$$

$$
\begin{aligned}
&\le c_0 e^{-\int_{t_0}^t g_0 s^{-\gamma} ds} + \max_{s\in[t_0,t]} cs^\gamma \\
&\le k_0 + k_1 t^\gamma .
\end{aligned}
$$

Thus, by the definition of γ, $\int \phi v^2 dx \le k_0 + k_1 t^{2\alpha/(2-\beta)}$ for some positive constants k_0 and k_1, as claimed. □

Using the previous theorem, we can eliminate some unsigned terms.

Lemma 3.7. *Given conditions* (A1)–(A5),

$$
\int (\theta - \phi\epsilon t^\gamma)(v_t^2 + b|\nabla v|^2) + a\phi v^2 + \frac{\theta\phi^{p-1}|v|^{p+1}}{p+1} dx \le b_1 + \int (\phi\epsilon^{-1}t^{-\gamma} - \phi_t)v^2 dx
$$

where $\epsilon = (2k_1)^{-1}$ *and* b_1 *is a positive constant.*

Proof. First, consider the unsigned term $2\phi v_t v$ in the weighted energy

$$
\begin{aligned}
|2\phi v_t v| &= 2\phi|\epsilon^{-1/2}\epsilon^{1/2}t^{\gamma/2}t^{-\gamma/2}v_t v| \\
&\le \phi\epsilon t^\gamma v_t^2 + \phi\epsilon^{-1}t^{-\gamma}v^2 \quad \text{(by Young's inequality)} \\
&\le \phi\epsilon t^\gamma v_t^2 + \phi\epsilon^{-1}t^{-\gamma}v^2 + \phi\epsilon t^\gamma b(x)|\nabla v|^2 .
\end{aligned}
$$

Using this inequality, we then estimate $2\phi v_t v$ from below,

$$
2\phi v_t v \ge -|2\phi v_t v| \ge -(\phi\epsilon t^\gamma)(v_t^2 + b|\nabla v|^2) - \phi\epsilon^{-1}t^{-\gamma}v^2 .
$$

Further, by Theorem 3.4 and the previous inequality,

$$
\begin{aligned}
b_0 &\ge \frac{1}{2}\int \theta(v_t^2 + b|\nabla v|^2) + 2\phi v_t v + (\hat{a}_2\theta + \phi_t + a\phi)v^2 + \frac{1}{p+1}\phi^{p-1}\theta|v|^{p+1} dx \\
&\ge \int (\theta - \phi\epsilon t^\gamma)(v_t^2 + b|\nabla v|^2) + (\phi_t - \frac{\phi}{\epsilon t^\gamma})v^2 + a\phi v^2 - |C^-|\theta v^2 \\
&\quad + \frac{2\theta\phi^{p-1}|v|^{p+1}}{p+1} dx .
\end{aligned}
$$

Rearranging terms gives

$$
\begin{aligned}
&\int (\theta - \phi\epsilon t^\gamma)(v_t^2 + b|\nabla v|^2) + a\phi v^2 dx \\
&\le b_0 + \int (\frac{\phi}{\epsilon t^\gamma} - \phi_t)v^2 + |C^-|\theta v^2 - \frac{2\theta\phi^{p-1}|v|^{p+1}}{p+1} dx .
\end{aligned}
$$

Note that, by Young's inequality and (A5), for some positive c_3, c_4,

$$
\begin{aligned}
\int |C^-|\theta v^2 dx &\le \frac{1}{p+1}\int \theta\phi^{p-1}|v|^{p+1} + c_3\theta\phi^{-2}|C^-|^{\frac{p+1}{p-1}} dx \\
&= \int \frac{1}{p+1}\theta\phi^{p-1}|v|^{p+1} dx + c_4 .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int (\theta - \phi\epsilon t^\gamma)(v_t^2 + b|\nabla v|^2) + a\phi v^2 + \frac{1}{p+1}\theta\phi^{p-1}|v|^{p+1} dx \\
&\le b_1 + \int (\phi\epsilon^{-1}t^{-\gamma} - \phi_t)v^2 dx ,
\end{aligned}
$$

as claimed. □

The previous lemma has removed most of the unsigned terms from E_w. We now need to simplify the factor $(\theta - \phi\epsilon t^\gamma)$ and bound the resulting integral by a constant. The two following conditions guarantee these results.

Theorem 3.8. *Assume* (A1)–(A5) *and*

(A6) $\phi \le k_1 t^{-\gamma}\theta$,
(A7) $\phi_t \ge -k_1 t^{-\gamma}\phi$

for some positive constant k_1 and sufficiently large t. Then

$$\int \frac{1}{2}\theta(v_t^2 + b|\nabla v|^2)dx \le C, \tag{3.3}$$

$$\int a\phi v^2 dx \le C, \tag{3.4}$$

$$\int \frac{1}{p+1}\theta\phi^{p-1}|v|^{p+1}dx \le C \tag{3.5}$$

for some $C > 0$.

Proof. Using (A6),

$$\phi \le k_1 t^{-\gamma}\theta \ \Rightarrow\ \frac{1}{2k_1}t^\gamma\phi \le \frac{1}{2}\theta \ \Rightarrow\ \theta - \epsilon t^\gamma\phi \ge \frac{1}{2}\theta,\ \text{letting } \epsilon = (2k_1)^{-1}.$$

Thus, by Lemma 3.7,

$$\begin{aligned}
&\int \frac{1}{2}\theta(v_t^2 + b|\nabla v|^2) + a\phi v^2 + \frac{1}{p+1}\theta\phi^{p-1}|v|^{p+1}dx \\
&\le b_1 + \int (\phi\epsilon^{-1}t^{-\gamma} - \phi_t)v^2 dx.
\end{aligned}$$

Further, using (A7), we have that

$$\begin{aligned}
\epsilon^{-1}t^{-\gamma}\phi - \phi_t &\le \epsilon^{-1}t^{-\gamma}\phi + k_1 t^{-\gamma}\phi \\
&= 2k_1 t^{-\gamma}\phi + k_1 t^{-\gamma}\phi \\
&= 3k_1 t^{-\gamma}\phi, \quad \text{for sufficiently large t.}
\end{aligned}$$

The previous inequalities and Theorem 3.6 give, for $t \ge t_0$,

$$\begin{aligned}
&\int \frac{1}{2}\theta(v_t^2 + b|\nabla v|^2) + a\phi v^2 + \frac{1}{p+1}\theta\phi^{p-1}|v|^{p+1}dx \\
&\le b_1 + \int 3k_1 t^{-\gamma}\phi v^2 dx \\
&\le b_1 + 3k_1 t^{-\gamma}\int \phi v^2 dx \le C.
\end{aligned}$$

Note that now each term under the integral on the left hand side is positive, so we have accomplished our goal of removing the unsigned terms. We can therefore use the upper bound C for each term individually. □

We now reintroduce the actual solution u by substituting $v = u\phi^{-1}$ back into the estimates obtained above. One last condition is needed to ensure that doing so preserves a constant upper bound.

Theorem 3.9. *Given conditions* (A1)–(A7) *and*

(A8) $\theta\phi^{-3}(\phi_t^2 + b|\nabla\phi|^2) \le k_2 a(x)$ *for some $k_2 > 0$,* *we have that*

$$
\begin{gathered}
\int a\phi^{-1}u^2dx \le K,\\
\int \theta\phi^{-2}|u|^{p+1}dx \le K,\\
\int \theta\phi^{-2}(u_t^2 + b|\nabla u|^2)dx \le K
\end{gathered}
$$

for some positive K.

Proof. Using $v = u\phi^{-1}$ in (3.4) and (3.5) immediately gives two simple results:

$$
\begin{gathered}
\int a\phi^{-1}u^2dx \le K,\\
\int \theta\phi^{-2}|u|^{p+1}dx \le K.
\end{gathered} \tag{3.6}
$$

Applying $v = u\phi^{-1}$ to (3.3) is a bit more complicated. Notice that

$$
\begin{gathered}
v = u\phi^{-1}, \quad v_t = u_t\phi^{-1} - u\phi^{-2}\phi_t,\\
v_t^2 = u_t^2\phi^{-2} - 2uu_t\phi^{-3}\phi_t + u^2\phi^{-4}\phi_t^2,\\
v_t^2 = \Big(\frac{1}{2}u_t^2\phi^{-2} - 3u^2\phi^{-4}\phi_t^2\Big) + \Big(\frac{1}{2}u_t^2\phi^{-2} + 4u^2\phi^{-4}\phi_t^2 - 2uu_t\phi^{-3}\phi_t\Big).
\end{gathered} \tag{3.7}
$$

Working toward producing the unweighted energy, we estimate v_t^2 and $|\nabla v|^2$ from below, starting with

$$
\begin{aligned}
2uu_t\phi^{-3}\phi_t &= (2u\phi^{-1}\phi_t)u_t\phi^{-2}\\
&\le \Big(\frac{1}{2}u_t^2 + \frac{1}{2}4u^2\phi^{-2}\phi_t^2\Big)\phi^{-2}\\
&\le \frac{1}{2}u_t^2\phi^{-2} + 4u^2\phi^{-4}\phi_t^2.
\end{aligned}
$$

Using this in (3.7), we have

$$
v_t^2 \ge \frac{1}{2}u_t^2\phi^{-2} - 3\phi^{-4}\phi_t^2u^2. \tag{3.8}
$$

A similar method can be used to show that

$$
|\nabla v|^2 \ge \frac{1}{2}|\nabla u|^2\phi^{-2} - 3\phi^{-4}|\nabla\phi|^2u^2. \tag{3.9}
$$

Finally, using (3.8) and (3.9) in (3.3) yields

$$
\begin{aligned}
2C &\ge \int \theta(v_t^2 + b|\nabla v|^2)dx\\
&\ge \int \frac{1}{2}u_t^2\phi^{-2}\theta - 3\phi^{-4}\phi_t^2u^2\theta + \frac{1}{2}b|\nabla u|^2\phi^{-2}\theta - 3b\phi^{-4}|\nabla\phi|^2\theta u^2dx.
\end{aligned}
$$

Rearranging terms gives

$$
\frac{1}{2}\int \theta\phi^{-2}(u_t^2 + b|\nabla u|^2)dx \le 2C + 3\int \theta\phi^{-4}u^2(\phi_t^2 + b|\nabla\phi|^2)dx.
$$

By (A8), $\theta\phi^{-4}(\phi_t^2 + b|\nabla\phi|^2)u^2 \le k_2a(x)\phi^{-1}u^2$. Using this, we obtain

$$
\int \theta\phi^{-2}(u_t^2 + b|\nabla u|^2)dx \le 4C + 6\int \theta\phi^{-4}u^2(\phi_t^2 + b|\nabla\phi|^2)dx
$$

$$\leq 4C + 6\int k_2 a\phi^{-1}u^2 dx \leq K$$

per (3.6). ☐

4. Definitions of ϕ and θ

Recall the conditions sufficient for energy decay for sufficiently large t:

(i) $-\theta_t + \phi \geq 0$,
(ii) $(-\theta_t + 2\theta(a + 2\phi^{-1}\phi_t) - 2\phi)(-\theta_t + 2\phi) \geq b|\nabla\theta - 2\theta\phi^{-1}\nabla\phi|^2$,
(iii) $(p+1)\phi^p - (\theta\phi^{p-1})_t \geq \phi^p$,
(iv) $\hat{a}_2\phi - (\hat{a}_2\theta)_t \geq k_0 C^-\phi$ where $k_0 > 0$ and

$$\int_{t_0}^{\infty}\int \phi^{-1}|C^-|^{\frac{p+1}{p-1}}\,dx\,dt < \infty$$

(v) $C^- \leq \hat{a}_2$ satisfying $\sup_{t\geq t_0}\int \theta\phi^{-2}|C^-|^{\frac{p+1}{p-1}}dx < \infty$,
(vi) $\phi \leq k_1 t^{-\gamma}\theta$,
(vii) $\phi_t \geq -k_1 t^{-\gamma}\phi$ for some $k_1 > 0$,
(viii) $\theta\phi^{-3}(\phi_t^2 + b|\nabla\phi|^2) \leq k_2 a(x)$ for some $k_2 > 0$.

We propose the following definitions of the weights ϕ and θ, and then ensure the sufficient conditions are met:

$$\begin{gathered}\phi(x,t) = t^{-m}e^{-\frac{\sigma(x)}{t}}\\ \theta(x,t) = \frac{3}{4}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1}\phi(x,t),\end{gathered} \tag{4.1}$$

where $\sigma(x)$ is defined in (2.8). The constants $\frac{3}{4}$ and 6 are chosen for technical reasons, ensuring that the eight conditions are satisfied. With these choices of weights, we have a crucial theorem.

Theorem 4.1. *Given conditions* (i)-(viii) *and ϕ and θ defined as in* (4.1)*, we have*

$$E(t) := \frac{1}{2}\int u_t^2 + b|\nabla u|^2 dx + \frac{1}{p+1}\int |u|^{p+1}dx \leq ct^{-m-1}$$

for some $c > 0$.

Proof. First we look at $\theta\phi^{-2}$:

$$\theta\phi^{-2} = \frac{3}{4}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1}\phi^{-1}.$$

Using this and Theorem 3.9, we obtain

$$\begin{aligned}&\int \theta\phi^{-2}(u_t^2 + b|\nabla u|^2 + |u|^{p+1})dx\\ &= \int \frac{3}{4}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1}\phi^{-1}(u_t^2 + b|\nabla u|^2 + |u|^{p+1})dx\\ &= \int \frac{3}{4}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1}t^m e^{\frac{\sigma(x)}{t}}(u_t^2 + b|\nabla u|^2 + |u|^{p+1})dx \leq K_0.\end{aligned}$$

Rearranging terms yields

$$\int \Big(6 + \frac{\sigma(x)}{t}\Big)^{-1}e^{\frac{\sigma(x)}{t}}(u_t^2 + b|\nabla u|^2 + |u|^{p+1})dx \leq K_1 t^{-m-1}.$$

Furthermore, based on a Taylor's series approximation,

$$e^{\frac{\sigma(x)}{t}} \geq \epsilon\big(6 + \frac{\sigma(x)}{t}\big)$$

for ϵ sufficiently small. This leaves

$$\int \frac{1}{2}(u_t^2 + b|\nabla u|^2) + \frac{1}{p+1}|u|^{p+1}dx = E(t) \leq ct^{-m-1}. \qquad \square$$

Now we must address that the weights satisfy the eight conditions. That ϕ and θ satisfy (i), (ii), and (vi)-(viii) is shown in [2], while condition (iii) is proven in [1]. This leaves (iv) and (v). The integrals in these remaining conditions relate m to C^-. We now consider two choices of C^-, which admit separate values of m for different values of the nonlinear exponent p. We will define these values for m in the next sections, and as we do, we will also ensure the weights satisfy conditions (iv) and (v).

5. Nonlinear exponent

5.1. **Supercritical case.**

Theorem 5.1. *By choosing $C^- = 0$ and by choosing $m = \mu - \delta = \frac{n-\alpha}{2-\alpha-\beta} - \delta$, for small $\delta > 0$, the weights ϕ and θ as in* (4.1) *satisfy conditions* (iv) *and* (v).

Proof. With $C^- = 0$, the integrals in conditions (iv) and (v) are trivially satisfied. Further, recall that

$$\begin{aligned} \hat{a}_2 &= \phi^{-1}(\phi_{tt} - \operatorname{div}(b\nabla\phi) + a\phi_t) \quad \text{as defined in (3.2)} \\ &= \Big(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\Big)^2 + \Big(\frac{m}{t^2} - \frac{2\sigma(x)}{t^3}\Big) + a\Big(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\Big) \\ &\quad - \Big(\frac{b|\nabla\sigma(x)|^2}{t^2} - \frac{\operatorname{div}(b\nabla\sigma(x))}{t}\Big) \\ &= \frac{\operatorname{div}(b\nabla\sigma(x)) - am}{t} + \frac{a\sigma - b|\nabla\sigma|^2}{t^2} + \Big(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\Big)^2 + \frac{m}{t^2} - \frac{2\sigma}{t^3}. \end{aligned} \tag{5.1}$$

To continue, we convert the conditions on $A(x)$ in Conjecture 2.1 to conditions on $\sigma(x)$, as per [2]:

$$\begin{gathered} \operatorname{div}(b(x)\nabla\sigma(x)) \geq (m+\delta)a(x), \\ 0 < \sigma(x) \leq (1+|x|)^{2-\alpha-\beta}, \\ \big(1 - \frac{\delta}{2\mu}\big)a(x)\sigma(x) \geq b(x)|\nabla\sigma(x)|^2. \end{gathered} \tag{5.2}$$

Further estimating from below, using (5.2) in (5.1),

$$\begin{aligned} \hat{a}_2 &\geq \frac{(\mu-\delta)a - am}{t} + \frac{a\sigma + a\sigma\big(\frac{\delta}{2\mu} - 1\big)}{t^2} - \frac{2\sigma}{t^3} \\ &\geq \frac{a(\mu - m - \delta)}{t} + \frac{\sigma(g_0 t^{-\gamma}\frac{\delta}{2\mu} - 2t^{-1})}{t^2} \\ &\geq \frac{a(\mu - m - \delta)}{t} \end{aligned}$$

because $0 \leq \gamma < 1$, and t is sufficiently large. Thus, for (v) $\hat{a}_2 \geq C^- = 0$ to be satisfied, we require $m \leq \mu - \delta$. Hence, choosing $m = \mu - \delta$ is sufficient.

Similarly, we have that $(\hat{a}_2)_t \le 0$:

$$
\begin{aligned}
-(\hat{a}_2)_t &= -\Big(\frac{\operatorname{div}(b\nabla\sigma(x)) - am}{t} + \frac{a\sigma - b|\nabla\sigma|^2}{t^2} + \Big(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\Big)^2 + \frac{m}{t^2} - \frac{2\sigma}{t^3}\Big)_t \\
&= \frac{\operatorname{div}(b\nabla\sigma(x)) - am}{t^2} + \frac{2a\sigma - 2b|\nabla\sigma|^2}{t^3} \\
&\quad + 2\Big(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\Big)\Big(\frac{-m}{t^2} + \frac{2\sigma(x)}{t^3}\Big) + \frac{2m}{t^3} + \frac{6\sigma}{t^4} \\
&\ge \frac{(\mu-\delta)a - am}{t^2} + \frac{2a\sigma + 2a\sigma\big(\frac{\delta}{2\mu} - 1\big)}{t^3} - \frac{6\sigma(m-1)}{t^4} \\
&\ge \frac{a(\mu - m - \delta)}{t^2} + \frac{\sigma(g_0 t^{-\gamma}\frac{\delta}{2\mu} - (m-1)t^{-1})}{t^3} \\
&\ge \frac{a(\mu - m - \delta)}{t^2} \ge 0
\end{aligned}
$$

because $0 \le \gamma < 1$ and $m = \mu - \delta$.

By the prior inequality and (i),

$$
\hat{a}_2\phi - (\hat{a}_2\theta)_t = \hat{a}_2(\phi - \theta_t) - (\hat{a}_2)_t\theta \ge 0,
$$

which shows that the first part of (v) is satisfied. Therefore, with $C^- = 0$ and $m = \frac{n-\alpha}{2-\alpha-\beta} - \delta = \mu - \delta$, conditions (iv) and (v) hold, and we obtain a powerful energy decay estimate. □

5.2. **Subcritical case.** Now we choose a different C^- that is slightly negative, making $\hat{a}_2$ no longer necessarily nonnegative, which allows larger values of m for smaller p.

Theorem 5.2. *There exist some positive constants k and c_1 such that by choosing*

$$
C^- := \begin{cases} -c_1 t^{-1}(1+|x|)^{-\alpha} & \text{if } 1+|x| \le kt^\eta, \\ 0, & \text{if } 1+|x| > kt^\eta, \end{cases} \tag{5.3}
$$

where $\eta = \frac{1}{2-\alpha-\beta}$, and by choosing

$$
m = \frac{2}{p-1} + \frac{\alpha\frac{p+1}{p-1} - n}{2-\alpha-\beta} - \delta, \tag{5.4}
$$

for small $\delta > 0$, the weights ϕ and θ as in (4.1) *satisfy conditions* (iv) *and* (v).

Proof. Note that

$$
\hat{a}_2 = \frac{\operatorname{div}(b\nabla\sigma(x)) - am}{t} + \frac{a\sigma - b|\nabla\sigma|^2}{t^2} + \Big(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\Big)^2 + \frac{m}{t^2} - \frac{2\sigma}{t^3}.
$$

Using (5.2), we obtain

$$
\begin{aligned}
\hat{a}_2 &\ge \frac{(\mu - \delta - m)a}{t} + \frac{a\sigma - b|\nabla|^2}{t^2} - \frac{2\sigma}{t^3} \\
&\ge (\mu - \delta - m)t^{-1}(1+|x|)^{-\alpha} + \frac{\delta}{2\mu}t^{-2}a\sigma - 2t^{-3}\sigma \\
&= (\mu - \delta - m)t^{-1}(1+|x|)^{-\alpha} + \frac{\delta}{2\mu}t^{-2}(1+|x|)^{2-2\alpha-\beta} - 2t^{-3}\sigma.
\end{aligned}
$$

Because $\frac{\delta}{2\mu}$ is positive, $\frac{\delta}{2\mu}t^{-2}a\sigma$ will absorb $-2t^{-3}\sigma$, for sufficiently large t. Furthermore, because we want $m > \mu - \delta$, for $t \geq t_0$,

$$\hat{a}_2 \geq -c_1 t^{-1}(1+|x|)^{-\alpha} + c_2 t^{-2}(1+|x|)^{2-2\alpha-\beta} \text{ for some } c_1,\, c_2 > 0.$$

Through an almost identical calculation, we have, for $t \geq t_0$,

$$t(\hat{a}_2)_t \leq c_3 t^{-1}(1+|x|)^{-\alpha} - c_4 t^{-2}(1+|x|)^{2-2\alpha-\beta} \text{ for some } c_3,\, c_4 > 0.$$

We are now ready to define the slightly negative lower bound of $\hat{a}_2$ for small p. Let

$$C^- := \begin{cases} -c_1 t^{-1}(1+|x|)^{-\alpha} & \text{if } 1+|x| \leq kt^{\eta}, \\ 0, & \text{if } 1+|x| > kt^{\eta}, \end{cases}$$

where $\eta = 1/(2-\alpha-\beta)$ and $k = \big(c_1/c_2\big)^{\eta}$. By construction, C^- satisfies the first part of condition (v), that $\hat{a}_2 \geq C^-$. Condition (iv) requires that $\hat{a}_2\phi - (\hat{a}_2\theta)_t \geq k_0 C^-\phi$ for some positive k_0. Noting that $\hat{a}_2 \geq C^-$ and $-t(\hat{a}_2)_t \geq c_0 C^-$ for large t and some $c_0 > 0$, we have that, for large t,

$$\begin{aligned} \hat{a}_2\phi - (\hat{a}_2\theta)_t &= \hat{a}_2\phi - t(\hat{a}_2)_t t^{-1}\theta - \hat{a}_2\theta_t \\ &= \hat{a}_2(\phi - \theta_t) - t(\hat{a}_2)_t t^{-1}\theta \\ &\geq C^-(\phi - \theta_t + c_0 t^{-1}\theta). \end{aligned}$$

Because $C^- \leq 0$ and $\phi > 0$, it only remains to be shown that $-\theta_t + c_0 t^{-1}\theta \leq c\phi$ for some $c > 0$. Consider

$$\theta\phi^{-1} = \frac{3}{4}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1}\phi\phi^{-1} = \frac{3t}{4}\Big(6 + \frac{\sigma(x)}{t}\Big)^{-1} \leq \frac{3t}{4}6^{-1} = \frac{t}{8}. \tag{5.5}$$

Hence, $c_0 t^{-1}\theta \leq c_0\phi/8$. Now consider

$$\begin{aligned} -\theta_t &= -\frac{3}{4}\Big(\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1}\phi(x,t)\Big)_t \\ &= -\frac{3}{4}\Big(\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-2}\Big(\frac{6}{t^2} + 2\frac{\sigma(x)}{t^3}\Big)\phi(x,t) + \Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1}\phi_t(x,t)\Big) \\ &\leq -\frac{3}{4}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1}\phi_t(x,t) \\ &= -\frac{3}{4}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1}\Big(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\Big)\phi \leq \frac{3\phi}{4}. \end{aligned}$$

Finally, $-\theta_t + c_0 t^{-1}\theta \leq c\phi$ with $c = \frac{3}{4} + \frac{c_0}{8}$, which means

$$\hat{a}_2\phi - (\hat{a}_2\theta)_t \geq C^-(\phi - \theta_t + c_0 t^{-1}\theta) \geq (c+1)C^-\phi = k_0 C^-\phi$$

where $k_0 = c+1$.

Lastly, the integral inequalities of (iv) and (v) must be satisfied. To verify them, consider ϕ^{-1}. Recall, per (5.2),

$$\begin{aligned} \sigma(x) &\leq (1+|x|)^{2-\alpha-\beta} \\ &\leq (kt)^{\eta(2-\alpha-\beta)} \\ &= (kt)^{(2-\alpha-\beta)/(2-\alpha-\beta)} = kt \end{aligned} \tag{5.6}$$

on suppt(C^-). Thus, on suppt(C^-),

$$\phi^{-1} = t^m e^{\frac{\sigma(x)}{t}} \leq t^m e^k \leq ct^m \tag{5.7}$$

for some positive constant c. Using (5.5), (5.7), and the compact support of C^-,

$$\begin{aligned}\sup_{t\ge t_0}\int \theta\phi^{-2}|C^-|^{\frac{p+1}{p-1}}dx &\le \sup_{t\ge t_0} ct^{m+1}\int |C^-|^{\frac{p+1}{p-1}}dx\\ &\le \sup_{t\ge t_0} ct^{m+1-\frac{p+1}{p-1}}\int_0^{kt^\eta}\int_{\partial B(0,s)} s^{-\alpha\frac{p+1}{p-1}}d\sigma_{n-1}ds\\ &\le \sup_{t\ge t_0} ct^{m+1-\frac{p+1}{p-1}+\frac{-\alpha\frac{p+1}{p-1}+n}{2-\alpha-\beta}},\end{aligned}$$

which must be finite to satisfy (v). Therefore, we must have

$$m+1-\frac{p+1}{p-1}-\frac{\alpha\frac{p+1}{p-1}-n}{2-\alpha-\beta}=m-\frac{2}{p-1}-\frac{\alpha\frac{p+1}{p-1}-n}{2-\alpha-\beta}\le 0.$$

The integral in condition (iv) is even easier to verify and gives another restriction on m that is essentially the same.

$$m-\frac{p-1}{p+1}-\frac{\alpha\frac{p+1}{p-1}-n}{2-\alpha-\beta}<-1. \tag{5.8}$$

This inequality is strict because there is an integral from t_0 to ∞ rather that a supremum over $t\ge t_0$. Hence, if m satisfies this inequality, both conditions (iv) and (v) are true, and setting

$$m=\frac{2}{p-1}+\frac{\alpha\frac{p+1}{p-1}-n}{2-\alpha-\beta}-\delta,$$

for δ small, gives a powerful energy decay estimate for this choice of C^-. □

6. Main result

In the previous two subsections, we derived two separate estimates of the energy decay. We now determine which estimates give faster decay as the nonlinear exponent p varies.

Proof of Theorem 2.3. To begin, recall from Theorem 4.1 that

$$E(t)\le ct^{-m-1}.$$

Thus, as m increases, the decay becomes faster. Define

$$\begin{aligned}m_0&=\frac{2}{p-1}-\delta,\\ m_1&=\frac{2}{p-1}+\frac{\alpha\frac{p+1}{p-1}-n}{2-\alpha-\beta}-\delta,\\ m_2&=\frac{n-\alpha}{2-\alpha-\beta}-\delta.\end{aligned}$$

From (5.4), notice that if we have $\alpha\frac{p+1}{p-1}-n\le 0$, then

$$m\le\frac{2}{p-1}+\frac{\alpha\frac{p+1}{p-1}-n}{2-\alpha-\beta}-\delta\le\frac{2}{p-1}-\delta=m_0,$$

and choosing $m = m_0$ still satisfies (5.8). Thus, a transition between decay rates m_0 and m_1 occurs when $\alpha\frac{p+1}{p-1} - n = 0$. Solving for p gives the first threshold,

$$p_1 = \frac{n+\alpha}{n-\alpha}.$$

Furthermore, comparing the two decay rates derived in the previous two chapters gives a second threshold. As shown in Theorem 5.1, when $C^- = 0$,

$$m = \mu - \delta = \frac{n-\alpha}{2-\alpha-\beta} - \delta.$$

Setting this equal to m_1 gives

$$\mu = \frac{n-\alpha}{2-\alpha-\beta} = \frac{2}{p-1} + \frac{\alpha\frac{p+1}{p-1} - n}{2-\alpha-\beta} = m_1 + \delta.$$

Thus, we obtain the second threshold for p,

$$p_2 = 1 + \frac{2-\beta}{n-\alpha}.$$

Combining all the previous, we obtain the optimal value of m, a function of p as follows:

$$m = \begin{cases} \frac{2}{p-1} - \delta & \text{if } 1 < p \leq 1 + \frac{2-\beta}{n-\alpha}, \\ \frac{2}{p-1} + \frac{\alpha\frac{p+1}{p-1} - n}{2-\alpha-\beta} - \delta & \text{if } 1 + \frac{2-\beta}{n-\alpha} < p \leq \frac{n+\alpha}{n-\alpha}, \\ \frac{n-\alpha}{2-\alpha-\beta} - \delta & \text{if } \frac{n+\alpha}{n-\alpha} < p < \frac{n+2}{n-2}, \end{cases}$$

which completes the proof. □

In conclusion, we used a strengthened multiplier method developed by Todorova and Yordanov in [2] to convert our equation into a similar equation which permits simpler long-time behavior of solutions. After recovering the energy estimates for the original solution, we showed the exponents used in the variable coefficients' bounds and the nonlinearity's exponent all interact to create three distinct regimes of energy decay, which are consistent with prior, less general results.

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