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PERIODIC SOLUTIONS FOR SEASONAL SIQRS MODELS WITH NONLINEAR INFECTION TERMS

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Abstract. In this work, we considered a family of SIRS models for a fatal disease, with seasonal variation in the contact rate and isolation control strategies. We establish the existence of periodic orbits of seasonal SIQRS disease, by using Leray-Schauder degree theory. Examples related to the seasonal variation in respiratory syncytial virus infection are included.

1. Introduction

Some infectious diseases confer temporal acquired immunity. This types of diseases can be modeled by classical susceptible-infectious-recovered-susceptible (SIRS) models. The SIRS type epidemic system and its related extensions have been used to model several topics such as malaria [5], Japanese encephalitis [20], cholera [21], respiratory syncytial virus [29], and others.

In the classic SIRS model, the incidence rate is assumed to be mass action incidence with bilinear interactions given βSI, where S and I are the numbers of infectious and susceptible individuals, respectively. The parameter β is the transmission rate. However, there are many reasons for using non-linear incidence rates, such as saturated function $\beta SI/(1+kI)$ has been first proposed by Capasso and Serio [4]. SIRS models with non-linear incidence rates are numerous in the literature [14, 16, 17, 22, 30, 28]. For instance, Liu and coworkers [16, 17] showed that the incidence rate $\beta S^p I^q$, with p and q are positive constants, can exhibit qualitatively different dynamical behaviors, including Hopf bifurcations, saddle-node bifurcations, and homoclinic loop bifurcations. To incorporate the effect of behavioral changes, some scholars [16, 22] used a non-linear incidence rate given $\beta SI^l/(1+kI^h)$ with k, l, $h>0$. In this incidence rate, the term βI^l measures the infection force of the disease and $1/(1+kI^h)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infectious individuals. Xiao and Ruan [30] investigated a non-monotone incidence rate, which is a particular case with $l=1$, $h=2$. In [3] used the incidence rate $\beta \ln(1+kI)S$ with $k>0$. Korobeinikov [14] used

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Lyapunov functions to proved global stability of the disease-free and endemic equilibrium states of SIRS models with non-linear transmission rate of a very general form $f(S, I)$.

The SIRS models have also been extended to incorporate control strategies, such as vaccination [8, 24], treatment [15], quarantine or isolation measures [10, 24]. Sun and Yang [24] developed an SIRS model with both vaccination and isolation control strategies. This system can exhibit different qualitative behaviors depending on the value of the vaccination-isolation reproductive number. Li and Cui [15] considered an SIRS model with nonlinear incidence rate and treatment, and it can undergo a Hopf bifurcation at the positive equilibrium. Gumel and his colleagues [8] proposed an SIRS model subject to an imperfect vaccine with waning natural and vaccine-induced immunity. The model undergoes backward bifurcation when the vaccination reproductive number is less than unity. Recently, Huang and coworkers [10] analyzed an SIRS epidemic model with quarantine and vaccination on complex heterogeneous networks. In particular, some conditions for global stability of the unique endemic equilibrium are obtained by using a monotone iterative technique.

A few studies have considered including in the classic SIRS model in the seasonal process. On the one hand, some works are focused to study the stability properties of the systems [11, 12, 18, 25, 26]. Thieme [25, 26] derived threshold results for the global stability of disease-free equilibrium and disease persistence for an SIRS model with very general time-heterogeneous coefficients. Greenhalgh and Moneim [11] analyzed the global stability of disease-free equilibrium of a model with general seasonal variation in the contact rate. Jódar and coworkers [12] studied the existence and the behavior of periodic solutions of a generalized model with general time-heterogeneous coefficients, by using a continuation theorem based on coincidence degree theory. In the above models have been considered that the constant population size; instead, Liu and Zhang [18] considered a model with density dependent birth rate and assumed that the total number of the population is governed by a logistic equation. On the other hand, a few seasonal models have focused on the estimation of parameters using epidemiological data. In particular, Weber and coworkers [29] used SIRS and MSEIRS models to interpret the pattern of seasonal epidemics of respiratory syncytial virus (RSV) disease observed in Gambia, Florida, Finland and Singapore, and they estimated the parameters for RSV infection. For an excellent review of seasonal epidemiological models, see [2]. In this paper, we consider a seasonal SIQRS model with disease-induced mortality.

The organization of this article is as follows. In Section 2, we formulate a family SIRS models incorporating a seasonal variation in the contact rate, isolation control strategies and disease-induced death. In Section 3, we discuss and establish the conditions for the existence of periodic orbits. In Section 4, we use epidemiologically realistic range of parameter values to present some numerical simulations. Lastly, in Section 4, we provide a few concluding remarks.

2. Formulation of seasonal SIQRS model

In most classic disease transmission models, the total population is assumed to be constant. Anderson and May [1] formulated one of the first epidemiological models where the total population is not constant. This model that includes recruitment rate of susceptible individuals into the community and that the disease produces non-negligible death in the infectious class. The model is given by the following

coupled system of ordinary differential equations:

$$\begin{aligned} S' &= \Lambda - \beta SI - dS + \xi R, \\ I' &= \beta SI - (\gamma + d + \alpha)I, \\ R' &= \gamma I - (d + \xi)R, \end{aligned} \tag{2.1}$$

Individuals are susceptible ($S(t)$), then infectious ($I(t)$), then recovered with temporary immunity ($R(t)$), and then susceptible again when the immunity is lost. Λ is the recruitment rate of susceptibles, d is the natural death rate and α is the disease-induced mortality. The parameter β is the disease transmission coefficient, γ describes the rate that the infectious population becomes recovered and ξ denotes the rate which recovered individuals return to the susceptible statue due to loss of immunity. All parameters are positive.

Quarantine and isolation are control strategies used to stop or limit the spread of infectious diseases. The basic idea of the isolation consists in separate sick individuals who have a communicable disease from those who are healthy. SIRS diseases are commonly recurrent due to social and weather-related factors, that is to say, disease vary according to an almost regular annual cycle. For these reason, it is important to incorporate a isolation control strategies and a time-varying rate of contact into the SIRS model (2.1).

We consider the following family of SIRS models for a fatal disease, with seasonal variation in the contact rate and isolation control strategies:

$$\begin{aligned} S' &= \Lambda - \beta(t)Sf(I) - dS + \xi R \\ I' &= \beta(t)Sf(I) - (\gamma + \delta + d + \alpha_1)I, \\ Q' &= \delta I - (\epsilon + d + \alpha_2)Q, \\ R' &= \gamma I + \epsilon Q - (d + \xi)R, \end{aligned} \tag{2.2}$$

where S is the number of individuals in the susceptible class, I is the number of individuals who are infectious but not isolated, Q is the number of individuals who are isolated (Q as in quarantine), and R is the number of individuals who are recovered. The parameter δ is the rate constant for individuals leaving the infectious class I for the isolated class Q, ϵ is the rate that the infectious-isolated class becomes recovered, and α_1 and α_2 represent the extra disease-related death rate constants in classes I and Q, respectively. The other parameters are the same as in the previous model (2.1). The parameters are positive constants. In short, we call these models as seasonal SIQRS models.

The seasonal force of infection is given by $\beta(t)f(I)$, we make the following assumptions:

(A1) $\beta(t)$ is a non-constant continuous T-periodic function.
(A2) $f(0) = 0, f(I) > 0$ and $f'(I) > 0$ for all $I \geq 0$.
(A3) $f(I)$ is concave; i.e. $f''(I) \leq 0$.

Remark: As was observed in [7, 27], the condition (A3) implies that f is uniformly sublinear; i.e.

$$If'(I) \leq f(I). \tag{2.3}$$

An example of a seasonally forced function is

$$\beta(t) = b_0(1 + b_1 \cos(2\pi f(t + \phi))),$$

where $b_0 \geq 0$ is the baseline transmission parameter, b_1 measures the amplitude of the seasonal variation in transmission, f is the frequency of seasonal cycles and ϕ is the phase shift.

An SIRS model with constant population and force of infection $\beta(t)f(I) = b_0(1 + b_1\cos(2\pi t))I$, and without isolation was studied in [11]. The case of an SIQRS model with β-constant and $f(I) = I$ was recently considered in [10]. Our model includes a wide variety of cases, which have not been considered in literature, such as, the following non-linear forces of infection $\beta(t)f(I) = \beta(t)I/(1 + kI)$ and $\beta(t)f(I) = \beta(t)\ln(1 + kI)$ with $k > 0$ [3].

We define the total population size as $N(t) = S(t) + I(t) + Q(t) + R(t)$, implies $N(t)' = \Lambda - dN(t) - \alpha_1 I(t) - \alpha_2 Q(t)$. We can see that in the absence of disease, the population size N converges to the equilibrium Λ/d. The differential equation for N implies that solutions of (2.2) starting in $\mathbb{R}^4_+$ either approach, enter, or remain in the subset of $\mathbb{R}^4$ defined by

$$\Sigma := \{(S, I, Q, R) : S \geq 0, I \geq 0, Q \geq 0, Q \geq 0, S + I + Q + R \leq \Lambda/d\},$$

Thus it suffices to consider solutions in the region Σ.

3. Existence of periodic orbits

The system always has a disease-free equilibrium $(S_0, I_0, Q_0, R_0) = (\Lambda/d, 0, 0, 0)$. We define the basic reproductive number $\mathcal{R}_0$ for system (2.2) when β is constant as

$$\mathcal{R}_0 = \frac{\beta(\Lambda/d)f'(0)}{\gamma + \delta + d + \alpha_1},$$

which is a product of the number Λ/d of susceptibles at the disease-free equilibrium, the transmission coefficient β, and the average residence time $1/(\gamma + \delta + d + \alpha_1)$ in the infectious individuals class. Thus $\mathcal{R}_0$ is the average number of secondary infections that occur when one infectious individual is introduced into a completely susceptible population.

Motivated by this, we consider $\mathcal{R}_0$ for system (2.2) as

$$\mathcal{R}_0 := \frac{\overline{\beta}(\Lambda/d)f'(0)}{\gamma + \delta + d + \alpha_1}, \quad \text{where } \overline{\beta} := \frac{1}{T}\int_0^T \beta(t)dt.$$

We write

$$\beta(t) = \overline{\beta} + \beta_0(t), \quad \text{where } \int_0^T \beta_0(t)dt = 0.$$

The proof of the existence of periodic orbits for systems (2.2) will be done in two steps. First, we consider the case $\xi = 0$,

$$\begin{aligned} S' &= \Lambda - \beta(t)Sf(I) - dS, \\ I' &= \beta(t)Sf(I) - (\gamma + \delta + d + \alpha_1)I, \\ Q' &= \delta I - (\epsilon + d + \alpha_2)Q, \\ R' &= \gamma I + \epsilon Q - dR, \end{aligned} \tag{3.1}$$

and prove the existence of solutions on this system. Then, we construct an homotopy between (3.1) and (2.2).

For $\lambda \in [0,1]$ we define the homotopy

$$\begin{aligned} S' &= \Lambda - \beta_\lambda S f(I) - dS, \\ I' &= \beta_\lambda S f(I) - (\gamma + \delta + d + \alpha_1) I, \\ Q' &= \delta I - (\epsilon + d + \alpha_2) Q, \\ R' &= \gamma I + \epsilon Q - dR, \end{aligned} \tag{3.2}$$

where $\beta_\lambda := \overline{\beta} + \lambda \beta_0(t)$.

To show the existence of a positive periodic solution, we shall use the Leray-Schauder degree theory. For this, we extend the work done in [13] and establish suitable modifications to describe system (3.1). To do so, we need to reformulate the problem in a functional setting in the following way.

For $l = 0, 1$ we consider the Banach spaces

$$\begin{aligned} \mathcal{C}^l_T :=& \big\{(S,I,Q,R) : S,I,Q,R \in C^l(\mathbb{R},\mathbb{R}), S(t+T) = S(t), I(t+T) = I(t), \\ & Q(t+T) = Q(t), R(t+T) = R(t)\big\}. \end{aligned}$$

Let $L : \mathcal{C}^1_T \to \mathcal{C}^0$ and $N_\lambda : \mathcal{C}^0_T \to \mathcal{C}^0_T$ be the operators given by

$$L(S,I,Q,R) := (S' + dS,\ I' + (\gamma+\delta+d+\alpha_1)I,\ Q' + (\epsilon+d+\alpha_2)Q,\ R' + dR), \tag{3.3}$$

and

$$N_\lambda(S,I,Q,R) := (\Lambda - \beta_\lambda S f(I),\ \beta_\lambda S f(I),\ \delta I,\ \gamma I + \epsilon Q).$$

Since L is invertible we define

$$F_\lambda(S,I,Q,R) := (S,I,Q,R) - L^{-1} \circ N_\lambda(S,I,Q,R)\,. \tag{3.4}$$

Since $\mathcal{C}^1_T$ is compactly embedded in $\mathcal{C}^0_T$, we can think of L^{-1} as going from $\mathcal{C}^0_T$ to $\mathcal{C}^0_T$, therefore $L^{-1} \circ N_\lambda : \mathcal{C}^0_T \to \mathcal{C}^0_T$ is a compact operator. In a similar fashion, we can consider $F_\lambda : \mathcal{C}^0_T \to \mathcal{C}^0_T$. Thus, (3.4) is a functional reformulation of problem (3.2); in particular, periodic solutions of (3.2) correspond to zeroes of F_λ.

We consider the open sets

$$\begin{aligned} D &:= \{(S,I,Q,R) \in \mathcal{C}^0_T : S > 0, I > 0, Q > 0, R > 0, S + I + Q + R < \Lambda/d\} \\ G &:= \{(S,I,Q,R) \in D : \min_{[0,T]} S(t) < r(\Lambda/d)\}, \end{aligned}$$

for a fixed $0 < r < 1$. For our main result we will assume $\frac{1}{\mathcal{R}_0} < \frac{f'(\frac{\Lambda}{d})}{f'(0)}$, so we choose r such that

$$\frac{1}{\mathcal{R}_0} < r\frac{f'(\frac{\Lambda}{d})}{f'(0)} \ \text{ and } \ \frac{\Lambda}{d + \overline{\beta} f(I_1)} < r(\frac{\Lambda}{d}). \tag{3.5}$$

Recall that the existence of a solution for F_1 in G via Leray-Schauder degree is guaranteed if $deg(F_0, G) \neq 0$ and F_λ is an admissible homotopy i.e. $0 \notin F_\lambda(\partial G), \forall \lambda \in [0,1]$. The next result says that F_λ is admissible.

Lemma 3.1. *If $\mathcal{R}_0 > \frac{f'(0)}{f'(\frac{\Lambda}{d})}$, then for any $\lambda \in [0,1]$ there are no solutions (S,I,Q,R) of (3.2) on ∂G.*

Proof. First note that the arguments in [13, Lemma 1] prove that (S_0, I_0, Q_0, R_0) is the only solution of (3.2) entirely contained in ∂D for any $\lambda \in [0,1]$. So, if $(S,I,Q,R) \in \partial G$, then $(S,I,Q,R) \notin \partial D$ so

$$(S,I,Q,R) \in D \quad \text{and} \quad S(t) \geq r(\Lambda/d), \quad \forall t. \tag{3.6}$$

By integrating the second equation in (3.2) on the interval $[0,T]$, we have that

$$\int_0^T \frac{I'}{I}dt + (\gamma+\delta+d+\alpha_1)T = \int_0^T \beta_\lambda S\frac{f(I)}{I}dt,$$

but $\int_0^T \frac{I'}{I}dt = 0$ because I is T-periodic and by (2.3) we obtain

$$\gamma+\delta+d+\alpha_1 = \frac{1}{T}\int_0^T \beta_\lambda S f'(I)dt,$$

by using the hypothesis (A3), $f'(\frac{\Lambda}{d}) \le f'(I) \le f'(0)$ and inequality (3.6) one obtain

$$\gamma+\delta+d+\alpha_1 \ge \frac{1}{T}\int_0^T \beta_\lambda S f'(\frac{\Lambda}{d})dt \ge r(\Lambda/d)\overline{\beta}\frac{f'(\frac{\Lambda}{d})}{f'(0)}f'(0),$$

Now from our hypothesis we obtain

$$\gamma+\delta+d+\alpha_1 > \overline{\beta}(\Lambda/d)f'(0)\frac{1}{\mathcal{R}_0} = \gamma+\delta+d+\alpha_1, \tag{3.7}$$

which is a contradiction. □

Lemma 3.2. *When $\lambda = 0$ and $\mathcal{R}_0 > 1$, the system (3.2) has exactly two periodic orbits in $\mathcal{C}^1$ being these: $S_0 = \Lambda/d, I_0 = 0, Q_0 = 0, R_0 = 0$ and the second satisfies*

$$S_1 = \frac{\Lambda}{d+\overline{\beta}f(I_1)}, \quad Q_1 = \frac{\delta I_1}{\epsilon+d+\alpha_2}, \quad R_1 = \Big(\gamma + \frac{\delta\epsilon}{\epsilon+d+\alpha_2}\Big)\frac{I_1}{d}$$

and I_1 is the unique solution of:

$$\frac{\overline{\beta}\Lambda f(I)}{d+\overline{\beta}f(I)} - (\gamma+\delta+d+\alpha_1)I = 0, \tag{3.8}$$

these in fact are critical points.

Proof. It is clear that the system admits an unique infection-free equilibrium state $(S_0 = \Lambda/d, I_0 = 0, Q_0 = 0, R_0 = 0)$. Since f satisfies 2.3 the existence and uniqueness of the endemic point follows the same lines as in Dénes and Röst in [7, Lemma 2.3]. and Röst in [7] or the arguments in [14, Theorem 2.1]. □

Proposition 3.3. *If $\mathcal{R}_0 > 1$ then for the open set G, we have that* $\deg(F_0, G) \neq 0$.

Proof. Since $\mathcal{R}_0 > 1$ by Lemma 3.2, (S_1, I_1, Q_1, R_1) is the unique periodic solution of $F_0(S,I,Q,R) = 0$ in G. So to establish the degree $deg(F_0, G) \neq 0$ we need only to prove that $DF_0(S_1, I_1, Q_1, R_1)$ is invertible. We have that F_0 is a compact perturbation of the identity, so by the Fredholm alternative it is enough to prove that the

$$\ker(DF_0(S_1, I_1, Q_1, R_1)) = \{0\}.$$

Consider $(U,V,W,Z) \in \mathcal{C}^0$ so that $(U,V,W,Z) \in \ker(DF_0(S_1, I_1, Q_1, R_1))$, by the definition of F_0, we obtain that $L(U,V,W,Z) = DN_0(S_1, I_1, Q_1, R_1)(U,V,W,Z)$, since

$$N_0(S,I,Q,R) = (\Lambda - \bar{\beta}Sf(I), \bar{\beta}Sf(I)S, \delta I, \gamma I + \epsilon Q).$$

Then, we obtain

$$\begin{aligned} &DN_0(S_1, I_1, Q_1, R_1)(U,V,W,Z) \\ &= (-\bar{\beta}(Uf(I_1) + S_1 f'(I_1)V), \bar{\beta}(Uf(I_1) + S_1 f'(I_1)V), \delta V, \gamma V + \epsilon W). \end{aligned}$$

Using the definition

$$L(U,V,W,Z) = (U' + dU, V' + (\gamma+\delta+d+\alpha_1)V, W' + (\epsilon+d+\alpha_2)W, Z' + dZ),$$

we obtain

$$\begin{aligned}(U',V',W',Z') &= (-(dU + \bar\beta U f(I_1) + \bar\beta S_1 f'(I_1)V), \bar\beta U f(I_1) - (\bar\beta S_1 f'(I_1)\\ &\quad + (\gamma+\delta+d+\alpha_1))V, -(\epsilon+d+\alpha_2)W + \delta V, -dZ + \gamma V + \epsilon W).\end{aligned}$$

Rewriting in matrix form, we have

$$\begin{pmatrix} U' \\ V' \\ W' \\ Z' \end{pmatrix} = A \begin{pmatrix} U \\ V \\ W \\ Z \end{pmatrix}. \tag{3.9}$$

where

$$A = \begin{pmatrix} -(d+\bar\beta f(I_1)) & -\bar\beta S_1 f'(I_1) & 0 & 0 \\ \bar\beta f(I_1) & \bar\beta S_1 f'(I_1) - (\delta+\gamma+d+\alpha_1) & 0 & 0 \\ 0 & \delta & -(\epsilon+d+\alpha_2) & 0 \\ 0 & \gamma & \epsilon & -d \end{pmatrix}$$

The the characteristic polynomial of this matrix is

$$p(\lambda) = (\lambda+d)(\lambda+\epsilon+d\alpha_2)(\lambda^2 - \mathrm{Tr}(B)\lambda + \det(B)), \tag{3.10}$$

where

$$B = \begin{pmatrix} -(d+\bar\beta f(I_1)) & -\bar\beta S_1 f'(I_1) \\ \bar\beta f(I_1) & \bar\beta S_1 f'(I_1) - (\delta+\gamma+d+\alpha_1) \end{pmatrix}. \tag{3.11}$$

To prove that the characteristic polynomial is Hurwitz, it is sufficient to prove that $\mathrm{Tr}(B) < 0$ and $\det(B) > 0$. In this way,

$$\begin{aligned}\mathrm{Tr}(B) &= \bar\beta S_1 f'(I_1) - \bar\beta f(I_1) - (\delta+\gamma+2d+\alpha_1), \quad f(I_1) \geq I_1 f'(I_1)\\ &\leq \bar\beta S_1 \frac{f(I_1)}{I_1} - \bar\beta f(I_1) - (\delta+\gamma+2d+\alpha_1), \quad \text{substituting } S_1 \text{ and (3.8)}\\ &\leq (\delta+\gamma+d+\alpha_1) - \bar\beta f(I_1) - (\delta+\gamma+2d+\alpha_1), < 0.\end{aligned}$$

In (3.11), we add the first row to the second row to obtain

$$\begin{aligned}\det(B) &= (d+\bar\beta f(I_1))(\delta+\gamma+d+\alpha_1) - \bar\beta d S_1 f'(I_1), \quad f(I_1) \geq I_1 f'(I_1)\\ &\geq (d+\bar\beta f(I_1))(\delta+\gamma+d+\alpha_1) - \bar\beta d S_1 \frac{f(I_1)}{I_1}\\ &= (d+\bar\beta f(I_1))(\delta+\gamma+d+\alpha_1) - d(\delta+\gamma+d+\alpha_1) > 0.\end{aligned}$$

substituting S_1 and (3.8).

Thus the inequalities are valid, and the characteristic polynomial of system (3.9) is Hurwitz. Therefore the linear system (3.9) has no periodic orbits different to the trivial solution. □

Theorem 3.4. *If $\mathcal{R}_0 > f'(0)/f'(\Lambda/d)$, then the system* (3.1) *admits a non-trivial periodic solution.*

Proof. Using the invariance of the Leray-Schauder degree under homotopy, by Lemma 3.1 and Proposition 3.3 we obtain that $\deg(F_1, G) \neq 0$, then the system (3.1) admits a non-trivial periodic solution, which proves Theorem 3.4. □

We now establish the existence of periodic solutions in the case of system (2.2). For $\tau \in [0,1]$ we define the homotopy

$$\begin{aligned} S' &= \Lambda - \beta(t)Sf(I) + \xi\tau R \\ I' &= \beta(t)Sf(I) - (\gamma+\delta+d+\alpha_1)I, \\ Q' &= \delta I - (\epsilon+d+\alpha_2)Q, \\ R' &= \gamma I + \epsilon Q - (d+\xi\tau)R, \end{aligned} \tag{3.12}$$

We consider the operators $M_\tau : \mathcal{C}^0 \to \mathcal{C}^0$ and $L_\tau : \mathcal{C}^1 \to \mathcal{C}^0$ given by

$$\begin{aligned} &M_\tau(S,I,Q,R) := (\Lambda + \xi\tau R - \beta(t)Sf(I),\ \beta(t)Sf(I)),\ \delta I,\ \gamma I + \epsilon Q), \\ &L_\tau(S,I,Q,R) \\ &:= (S' + dS,\ I' + (\gamma+\delta+d+\alpha_1)I,\ Q' + (\epsilon+d+\alpha_2)Q,\ R' + (d+\xi\tau)R), \end{aligned}$$

we define

$$H_\tau(S,I,Q,R) := (S,I,Q,R) - L_\tau^{-1} \circ M_\tau(S,I,Q,R). \tag{3.13}$$

Thus, (3.13) is a functional reformulation of problem (2.2); in particular, periodic solutions of (2.2) correspond to zeroes of H_τ. Note that $H_0 = F_1$, therefore $deg(H_0, G) \neq 0$. Recall that the existence of a solution for H_1 in G is guaranteed via Leray-Schauder degree if $deg(H_0, G) \neq 0$ and H_τ is an admissible homotopy i.e. $0 \notin H_\tau(\partial G), \forall \tau \in [0,1]$. So we need only establish that H_τ is an admissible homotopy, note that the same proof of lemma 3.1 applies to obtain the following result.

Lemma 3.5. *If $\mathcal{R}_0 > f'(0)/f'(\Lambda/d)$, then for any $\tau \in [0,1]$ there are no solutions (S,I,Q,R) of (3.12) on ∂G.*

Hence H_τ is an admissible homotopy, as required. By combining our observations we obtain the following theorem.

Theorem 3.6. *If $\mathcal{R}_0 > f'(0)/f'(\Lambda/d)$, then there is at least one T-periodic orbit of (2.2) whose components are positive.*

4. Applications

In the previous section, we analyzed the existence of periodic solutions of a seasonal SIQRS models for a fatal disease. The object of this section is to show numerical evidence of the existence of periodic solutions of two SIQRS models. A representative example of an SIRS-type disease is the RSV infection. This disease is a common cause of acute respiratory illness in children and older adults, with a cyclic annual pattern. We will use the estimated parameters for the country of Finland given in [29].

4.1. SIQRS models with bilinear incidence.

We proposed an SIQRS model with bilinear incidence rate, isolation control strategy and seasonal variation in the contact rate of period $T = 1$ year:

$$\begin{aligned} S' &= \Lambda - \beta(t)SI - dS + \xi R, \\ I' &= \beta(t)SI - (\gamma+\delta+d+\alpha_1)I, \\ Q' &= \delta I - (\epsilon+d+\alpha_2)Q, \\ R' &= \gamma I + \epsilon Q - (d+\xi)R, \end{aligned} \tag{4.1}$$

where the initial conditions are: $S(0) = 2554$, $I(0) = 50$, $Q(0) = 0$ and $R(0) = 0$. The seasonally forced function is determined by $\beta(t) = 0.0169(1 + 0.36\cos(2\pi(t + 0.60)))$.

Figure 1 shows the effect of the presence or absence ($\delta = \epsilon = 0$) of isolation in the time plots of system (4.1). We observed that the presence of the isolation control strategy decreases the reproductive number from $R_0^{SIRS} = 1.222$ to $R_0^{\mathrm{SIQRS}} = 1.023$, and consequently also the number of infected cases decreases.

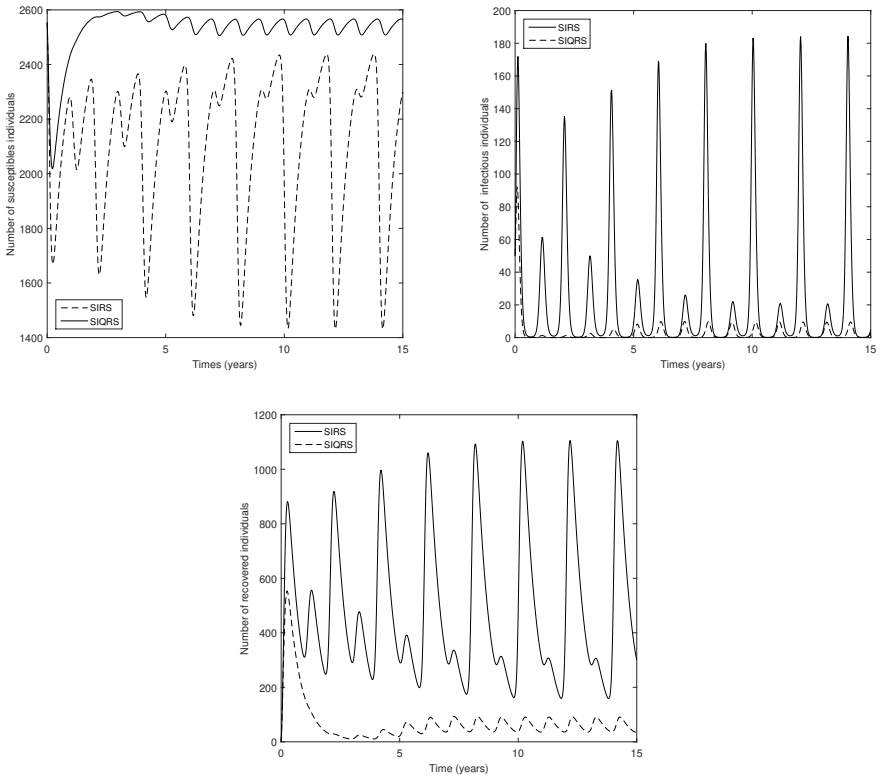

FIGURE 1. Time plots for the seasonal SIRS model (4.1) with presence (dashed line) or absence (solid line) isolation control strategy. The parameter values are as in [29], except the following parameters: $\Lambda = 3.3852$ people/year, $\delta = 7$/year, $\epsilon = 54$/year, $b_0 = 0.0169$ year/person and $\alpha_1 = \alpha_2 = 0.0$/year. These are: $d = 0.0013$/year, $\gamma = 36$/year, $\xi = 1.8$/year, $b_1 = 0.36$ and $\phi = 0.60$ year.

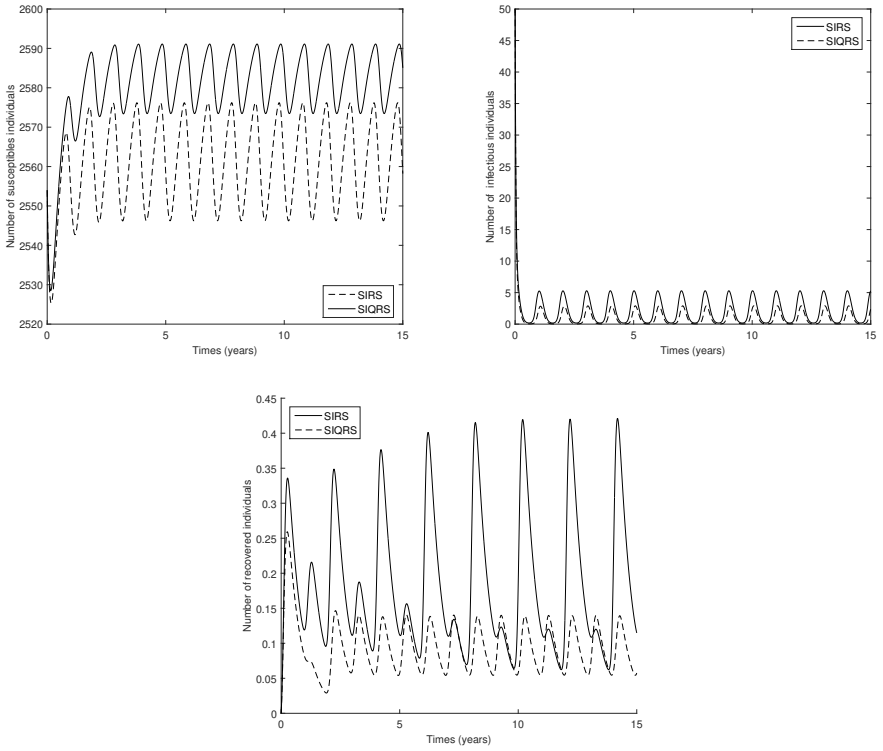

FIGURE 2. Time plots for the seasonal SIRS model (4.2) with presence (dashed line) or absence (solid line) isolation control strategy. The parameters chosen are the following: $\delta = 4$/year and $k = 0.048$ year/person, and the other parameters are the same as in the SIQRS model with bilinear incidence rate. It is clear that the inequality $\mathcal{R}_0^{\mathrm{SIQRS}} > f'(0)/f'(1) = 1.0983$ is satisfied.

4.2. **SIQRS models with saturated incidence rate.** We proposed the following SIQRS model with saturated incidence rate:

$$\begin{aligned} S' &= \Lambda - \beta(t)\frac{IS}{1+kI} - dS + \xi R, \\ I' &= \beta(t)\frac{IS}{1+kI} - (\gamma + \delta + d + \alpha_1)I, \\ Q' &= \delta I - (\epsilon + d + \alpha_2)Q, \\ R' &= \gamma I + \epsilon Q - (d + \xi)R, \end{aligned} \tag{4.2}$$

where $1/(1 + kI)$ measures the psychological or inhibitory from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infected individuals. The seasonally forced function is $\beta(t) = 44(1+0.36cos(2\pi(t+0.60)))$, and initial conditions are $S(0) = 0.9808$, $I(0) = 0.0192$,

$Q(0) = 0$ and $R(0) = 0$. Similarly, we observed that the isolation of symptomatic individuals decreases the reproductive number from $\mathcal{R}_0^{SIRS} = 1.222$ to $\mathcal{R}_0^{\text{SIQRS}} = 1.099$. (See Figure 2).

5. Conclusions

In this paper, we studied a non-autonomous epidemic system that describes the isolation control measures for infectious diseases with temporary immunity and seasonal cycles. This seasonal variation may be due to social and weather-related factors.

A classic problem in differential equations is to prove the existence of periodic orbits. Usually, it is difficult to prove the existence of periodic orbits of ODE systems in dimensions greater than or equal to three. We used the Leray-Schauder degree theory to prove the existence of a non-trivial periodic endemic solution.

In our work, we established analytically the existence of a periodic endemic orbit in a family of *SIRS* epidemic model with isolation control strategy and seasonal variation in a wide class of nonlinear infection force $\beta(t)f(I)$. This seasonal force of infection includes various special nonlinear functions $f(I)$ and seasonally forced functions $\beta(t)$. If $f(I) = I$, then the incidence rate $\beta(t)f(I)S$ becomes a classic bilinear form, and if $f(I) = I/(1 + kI)$ then the incidence rate describes saturated effects, which is proposed in [4]. In addition, $f(I) = \ln(1 + kI)$, then the incidence rate is one of the form proposed in Briggs and Godfray [3]. The most common seasonal function is $\beta(t) = b_0(1 + b_1 \cos(2\pi(t + \phi)))$, and we proposed another sinusoidal form $\beta(t) = b_0(1-b_1 \sin(2\pi f(t+\phi)))$, where f is the frequency of seasonal cycles.

A example of an SIRS-type disease is the respiratory syncytial virus infection. We used estimated parameters given in [29] of this disease to show numerical evidence of the existence of such periodic endemic solutions by means of numerical simulations of two SIQRS epidemic models.

Finally, we analyzed the relationship between the *SIQRS* model and its basic reproductive number, $\mathcal{R}_0$. On the one hand, we observed that the $\mathcal{R}_0$ is independent of the following parameters: recovery rate (ϵ) and disease-related death rate (α_2) of isolated individuals Q, and the loss of immunity rate (ξ) of recovered individuals R. It is not surprising that the recovery rate of isolated individuals does not affect the threshold quantity $\mathcal{R}_0$, since the *SIQRS* model assumed that people in the isolation class Q do not infect others and people are not infectious when they move out of the isolation class. On the other hand, the $\mathcal{R}_0$ do depend on the parameter δ, which governs the transfer rate out of the infectious class into the isolation class. In practice, the parameter δ is the most easily controlled, that is the isolation rate of infectious individuals. We can see that the effective infectious period $1/(\gamma + \delta + d + \alpha_1)$ and the basic reproductive number $\mathcal{R}_0$ decrease as the isolation rate constant δ increases. By means of this result, we showed that seasonal epidemic outbreaks can be controlled.

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