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FRACTIONAL SCHRÖDINGER-POISSON SYSTEMS WITH WEIGHTED HARDY POTENTIAL AND CRITICAL EXPONENT

YU SU, HAIBO CHEN, SENLI LIU, XIANWEN FANG

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Abstract. In this article we consider the fractional Schrödinger-Poisson system

$$\begin{gathered}(-\Delta)^s u-\mu\frac{\Phi(x/|x|)}{|x|^{2s}}u+\lambda\phi u=|u|^{2_s^*-2}u,\quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=u^2,\quad \text{in } \mathbb{R}^3,\end{gathered}$$

where $s\in(0,3/4)$, $t\in(0,1)$, $2t+4s=3$, $\lambda>0$ and $2_s^*=6/(3-2s)$ is the Sobolev critical exponent. By using perturbation method, we establish the existence of a solution for λ small enough.

1. Introduction

In this article we consider the fractional Schrödinger-Poisson system

$$\begin{gathered}(-\Delta)^s u-\mu\frac{\Phi(x/|x|)}{|x|^{2s}}u+\lambda\phi u=|u|^{2_s^*-2}u,\quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=u^2,\quad \text{in } \mathbb{R}^3,\end{gathered}\tag{1.1}$$

where $s\in(0,3/4)$, $t\in(0,1)$, $2t+4s=3$, $\lambda>0$ and $2_s^*=6/(3-2s)$ is the Sobolev critical exponent. The function Φ and the parameter μ satisfy the condition

(A1) If $0\leqslant\Phi\in L^{\frac{3}{2s}}(\mathbb{S}^2)$ then $\mu\in(0,\Lambda(\Phi))$, where $\Lambda(\Phi)$ is defined in Lemma 2.1 below.

In quantum mechanics a zero spin relativistic particle of charge e and mass m in the Coulomb field of an infinitely heavy nucleus of charge Z is described by the Hamiltonian (see e.g. [23, 28])

$$H(p,x)=(p^2+m^2)^{1/2}-\frac{Ze^2}{|x|}.$$

Fall and Felli [16], extended the study of $H(p,x)$ to

$$\tilde{H}(p,x)=(p^2+m^2)^s-\frac{\Phi(\frac{x}{|x|})}{|x|^{2s}}.$$

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They considered the operator

$$\tilde{H}(i\nabla, x) = (-\Delta + m^2)^s - \frac{\Phi(\frac{x}{|x|})}{|x|^{2s}}.$$

For $\Phi \equiv 0$, the operator $\tilde{H}(i\nabla, x)$ becomes $(-\Delta + m^2)^s$, for more details, we refer to [2, 3]. In this article, we study $\tilde{H}(i\nabla, x)$, i. e. when $m = 0$.

In nonrelativistic molecular physics, the interaction between an electric charge and the dipole moment $\mathbf{D} \in \mathbb{R}^N$ of a molecule is described by an inverse square potential with an anisotropic coupling strength. In particular the Schrödinger equation for the wave function of an electron interacting with a polar molecule (supposed to be point-like) can be written as

$$H = -\frac{\hbar}{2m}\Delta + e\frac{x \cdot \mathbf{D}}{|x|^3} - E,$$

where $\mathbf{D}$ is the dipole moment of the molecule, e and m denote the charge and the mass of the electron (see [26]). We consider the operator

$$L_{\mathbf{d}} := -\Delta - \frac{2me|\mathbf{D}|}{\hbar}\frac{x \cdot \mathbf{d}}{|x|^3},$$

in $\mathbb{R}^N$, where $N \geqslant 3$, being $|\mathbf{D}|$ the magnitude of the dipole moment $\mathbf{D}$, and $\mathbf{d} = \frac{\mathbf{D}}{|\mathbf{D}|}$ denotes the orientation of $\mathbf{D}$. The Laplace operator with dipole-type potential (purely angular multiples of radial inverse square potentials):

$$\mathcal{L}_\Phi := -\Delta - \mu\frac{\Phi(x/|x|)}{|x|^2},$$

in $\mathbb{R}^N$, where $N \geqslant 3$. We consider the more general class of operator $\tilde{H}(i\nabla, x)$ with zero mass,

$$\mathcal{L}_{s,\Phi} := (-\Delta)^s - \mu\frac{\Phi(x/|x|)}{|x|^{2s}}.$$

If $\Phi \equiv 0$, then $\mathcal{L}_{s,\Phi}$ becomes the fractional Laplacian operator $(-\Delta)^s$. If $\Phi \equiv 1$, then $\mathcal{L}_{s,\Phi}$ is the fractional Laplacian operator with Hardy potential $(-\Delta)^s - \frac{\mu}{|x|^{2s}}$. If $s = 1$, then $\mathcal{L}_{s,\Phi}$ becomes $\mathcal{L}_\Phi$.

From the mathematical point of view, a reason of interest in potentials of the type $\Phi(\frac{x}{|x|})/|x|^{2s}$ relies in their criticality with respect to the differential operator $(-\Delta)^s$; indeed, they have the same homogeneity as the fractional Laplacian $(-\Delta)^s$, hence they cannot be regarded as a lower order perturbation term. We mention that the operator with singular potentials have been widely studied, see for example [1, 13, 14, 15, 17, 18, 19, 20, 21, 34, 36, 38, 39] and references therein.

On the other hand, if $\mu = 0$ and $\lambda = 1$, then system (1.1) becomes the system:

$$\begin{gathered} (-\Delta)^s u + \phi u = |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.2}$$

For this system there are four cases: (i) $s = t = 1$; (ii) $s = 1$ and $t \neq 1$; (iii) $s = t \neq 1$; (iv) $s \neq 1$, $t \neq 1$, and $2t + 4s > 3$.

(i) For $s = t = 1$, system (1.2) reduces to the Schrödinger–Poisson system

$$\begin{gathered} -\Delta u + \phi u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.3}$$

The Schrödinger-Poisson system arises in many mathematical physical context, such as in quantum electrodynamics, to describe the interaction between a charge particle interacting with the electromagnetic field, and in semiconductor theory, in nonlinear optics and in plasma physics (see [8] for more details in the physics aspects). System (1.3) has been investigated by many researchers, see for example: [6, 11] for $|u|^{p-2}u$ $(4 < p < 6)$, [47] for $|u|^{p-2}u$ $(3 < p < 4)$, [40] for $|u|^{p-2}u$ $(1 < p < 2)$.

(ii) For $s = 1$ and $t \neq 1$, Mercuri, Moroz and Van Schaftingen [32] studied the Schrödinger–Poisson–Slater equation

$$-\Delta u + (I_\alpha * |u|^q)\,|u|^{q-2}u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

where $p > 1$, $q > 1$, $\alpha \in (0, N)$, and $I_\alpha = \frac{\Gamma(\frac{3-\alpha}{2})}{2^\alpha \pi^{\frac{3}{2}} \Gamma(\frac{\alpha}{2})} \cdot \frac{1}{|x|^{3-\alpha}}$ is the Riesz potential. Equation (1.4) is formally equivalent to the following system (see [32, Page 146, Line 10]):

$$\begin{gathered} -\Delta u + \phi |u|^{q-2}u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^{\alpha/2}\phi = |u|^q, \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.5}$$

The existence of nonnegative ground state solution of (1.4) was established when the parameters satisfy: either

$$q > 3 + \alpha \text{ and } \frac{2(\alpha + 2q)}{\alpha + 2} < p < 6,$$

or

$$q < 3 + \alpha \text{ and } 6 < p < \frac{2(\alpha + 2q)}{\alpha + 2}.$$

See [32, Theorem 3]. Moreover, some qualitative properties of the solutions were estudied. Li, Gao and Zhu [27] proved the existence of sign-changing solution of Kirchhoff type system with Hartree-type nonlinearity

$$\begin{gathered} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + \lambda V(x)u + \phi |u|^{q-2}u = f(u), \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^{\alpha/2}\phi = l|u|^q, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.6}$$

where $a > 0$, $b, l \geqslant 0$, $\alpha \in (0, 3)$, $q \in [2, 3 + \alpha)$, $\lambda > 0$ is a parameter, and the functions V and f satisfy suitable assumptions. For recent works, we refer to [4] and the references therein.

(iii) For $s = t \neq 1$, the author in [43] applied the fountain theorem to prove the existence of infinitely many solutions to the system

$$\begin{gathered} (-\Delta)^s u + V(x)u + \phi u = f(x, u), \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^s \phi = \gamma_s u^2, \quad \text{in } \mathbb{R}^3. \end{gathered}$$

(iv) For $s \neq 1$, $t \neq 1$ and $2t + 4s > 3$, Zhang, do Ó and Squassina [46] investigated the system

$$\begin{gathered} (-\Delta)^s u + \lambda \phi u = f(u), \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = \lambda u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.7}$$

where f is a nonlinearity of Berestycki-Lions type. The authors proved that system (1.7) admits a positive radial solution if $\lambda > 0$ small enough. Liu and Zhang [30] proved the existence of positive ground state solution for $\varepsilon > 0$ sufficiently small to the system

$$\begin{gathered}\varepsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = f(u) + |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^3,\\ \varepsilon^{2t}(-\Delta)^t\phi = u^2, \quad \text{in } \mathbb{R}^3,\end{gathered}$$

where $V \in C^1(\mathbb{R}^3, \mathbb{R}^+)$ and f is subcritical. For more details and recent works, we refer to [5, 31, 37, 41, 42] and the references therein.

In [30, 41, 42, 46] the authors considered the existence of solution for system (1.7) in which $2t + 4s > 3$. Therefore, it is natural to ask whether system (1.7) admits a solution for $2t + 4s = 3$. To the best of our knowledge, there is no result on such question in the current literature. Motivated by the above facts, we study the existence of solution for system (1.1).

1.1. **Statement of results.** We introduce the energy functional associated with system (1.1) by

$$I_\lambda(u) = \frac{1}{2}\|u\|_\Phi^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\mathrm{d}x\mathrm{d}y - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x.$$

where the norm $\|\cdot\|_\Phi$ is defined in Section 2. There is a difficulty in applying variational methods to the functional I_λ.

(Difficulty) Since $2_s^* < 4$, we know that the functional I_λ does not satisfy Mountain Pass Theorem, the boundedness of the Palais-Smale sequence for I_λ is hard to obtain. To overcome this difficult, by using the perturbation method in [12, 25], for λ small enough, we look the system (1.1) as a perturbation of the equation

$$(-\Delta)^s u - \mu\frac{\Phi(x/|x|)}{|x|^{2s}}u = |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^3. \tag{1.8}$$

We introduce the energy functional associated with equation (1.8) by

$$I_0(u) = \frac{1}{2}\|u\|_\Phi^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x.$$

For λ small enough, we look at the functional I_λ as a perturbation of functional I_0,

$$I_\lambda(u) := I_0(u) + K_\lambda(u),$$

where

$$K_\lambda(u) = \frac{\lambda}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}}\mathrm{d}x\mathrm{d}y.$$

Let $\mathcal{S}$ be the set of ground state critical points of I_0. The perturbation method mainly consists of two aspects:

(1) the mountain pass type critical point of I_0 is a ground state in $D^{s,2}_{\mathrm{rad}}(\mathbb{R}^3)$;
(2) the set $\mathcal{S}$ is compact in $D^{s,2}_{\mathrm{rad}}(\mathbb{R}^3)$.

If conditions (1) and (2) are satisfied, then there exists a Palais-Smale sequence of I_λ near the set $\mathcal{S}$ for sufficiently small λ.

Since the perturbation method is based on the properties of set $\mathcal{S}$, we have to study the existence of ground state solution to equation (1.8) first. The following is our first result.

Theorem 1.1. *Let $s \in (0, 3/4)$ and (Φ_1) hold. Then (1.8) has a ground state solution $v \in D^{s,2}_{\mathrm{rad}}(\mathbb{R}^3)$.*

There is a difficulty in the proof of Theorem 1.1. Because of the lack of compactness of the Sobolev embedding $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$, and the functional I_0 is invariant under the weighted dilation, it is hard to show that the Palais-Smale sequence of I_0 has a convergence subsequence.

For $\Phi \equiv 1$, let us review the method in [45]. Applying their method, one has the existence of ground state solution to the equation

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^3. \tag{1.9}$$

Similar to Section 3 in [45], there exists a Palais-Smale sequence $\{u_n\}$ for the energy functional corresponding to (1.9) such that $u_n \rightharpoonup u$ in $D^{s,2}(\mathbb{R}^3)$ with u solving (1.9). However, it may occur that $u \equiv 0$. The key step of [45] was to rule out the "vanishing" of the Palais-Smale sequence by using the limit equation of (1.9). Taking $v_n(x) = \lambda_n^{\frac{3-2s}{2}} u_n(\lambda_n x + x_n)$ where $\lambda_n > 0$, $x_n \in \mathbb{R}^3$ and $\frac{x_n}{\lambda_n} \to \infty$ as $n \to +\infty$, they derived that $v_n \rightharpoonup v$ in $D^{s,2}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \frac{v_k \phi}{|x + \frac{x_k}{\lambda_k}|^{2s}} \to 0 \quad \text{as } k \to +\infty \tag{1.10}$$

for any $\phi \in D^{s,2}(\mathbb{R}^3)$. Then v weakly solves

$$(-\Delta)^s v = |v|^{2_s^*-2}v, \quad \text{in } \mathbb{R}^3. \tag{1.11}$$

Using the limit equation (1.11), they ruled out the "vanishing" of the Palais-Smale sequence for the energy functional corresponding to (1.9).

Naturally, we would hope to overcome our difficulty by using the method in [45], but unfortunately, for Φ satisfies condition (Φ_1), the behavior of

$$\lim_{n\to\infty} \Phi\Big(\frac{x + \frac{x_n}{\lambda_n}}{|x + \frac{x_n}{\lambda_n}|}\Big), \quad \text{as } \frac{x_n}{\lambda_n} \to \infty,$$

is unknown, so we do not have any information of the expression

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \Phi\Big(\frac{x + \frac{x_n}{\lambda_n}}{|x + \frac{x_n}{\lambda_n}|}\Big) \frac{v_n \varphi}{|x + \frac{x_n}{\lambda_n}|^{2s}} \mathrm{d}x. \tag{1.12}$$

Clearly, their method does not apply to our case, since the expression (1.12) is not equal to zero.

For the above reason, we use another way to prove Theorem 1.1. The crucial point is the utilization of the uniform decay estimate for radial function $v \in D^{s,2}_{\rm rad}(\mathbb{R}^3)$ (see [9])

$$|v(x)| \leqslant \frac{C}{|x|^{\frac{3-2s}{2}}} \|v\|_s \leqslant \frac{C}{|x|^{\frac{3-2s}{2}}}.$$

We consider the function $I_0 : D^{s,2}_{\rm rad}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$J_0 = I_0|_{D^{s,2}_{\rm rad}(\mathbb{R}^3)}.$$

The principle of symmetric criticality implies that the critical points of J_0 are also critical points for I_0. Moreover, in Theorem 1.1 we have the following results:

(1) the mountain pass type critical point of I_0 is a ground state in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$;
(2) the set $\mathcal{S}$ is compact in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$.

Combining the above results and the perturbation method, we give our second result as follows.

Theorem 1.2. *Let $s \in (0,3/4)$, $t \in (0,1)$, $2t+4s=3$ and (Φ_1) hold. Then there exists λ_0 such that system (1.1) has a radially symmetric solution for all $\lambda \in (0,\lambda_0)$.*

This article is organized as follows. In Section 2, we present some notations. In Section 3, we give the proof of Theorem 1.1. In Section 4, we show the proof of Theorem 1.2.

2. Preliminaries

For $s \in (0,1)$, the space $D^{s,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_s^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y.$$

We denote by $D^{s,2}_{\rm rad}(\mathbb{R}^3)$ the space of radial functions in $D^{s,2}(\mathbb{R}^3)$. We could define the best constants

$$S_s := \inf_{u\in D^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_s^2}{\left(\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x\right)^{\frac{2}{2_s^*}}}. \tag{2.1}$$

We know that S_s can be attained in $\mathbb{R}^3$ (see [33]).

For $u \in D^{s,2}(\mathbb{R}^3)$, we have the Hardy inequality (see [22]):

$$\mathcal{C}_{s,2}\int_{\mathbb{R}^3}\frac{|u|^2}{|x|^{2s}}\mathrm{d}x \leqslant \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y, \tag{2.2}$$

with

$$\mathcal{C}_{s,2} := 2\int_0^1 r^{2s-1}|1-r^{\frac{3-2s}{2}}|^2\Phi_{s,2}(r)\mathrm{d}r.$$

We introduce the measure $\mathrm{d}\vartheta$ induced by Lebesgues measure on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. We denote by $\|\cdot\|_{L^q(\mathbb{S}^2)}$ the quantity

$$\|\Phi\|^q_{L^q(\mathbb{S}^2)} = \int_{\mathbb{S}^2}|\Phi(\vartheta)|^q\mathrm{d}\vartheta.$$

Lemma 2.1 ([24]). *Let $s \in (0,1)$ and $0 \leqslant \Phi \in L^{\frac{3}{2s}}(\mathbb{S}^2)$. Then*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y \geqslant \Lambda(\Phi)\int_{\mathbb{R}^3}\frac{\Phi(x/|x|)|u|^2}{|x|^{2s}}\mathrm{d}x,$$

where $u \in D^{s,2}(\mathbb{R}^3)$ and $\Lambda(\Phi) = \mathcal{C}_{s,2}|\mathbb{S}^2|^{\frac{2s}{3}}\|\Phi\|^{-1}_{L^{\frac{3}{2s}}(\mathbb{S}^2)}$.

By using Lemma 2.1 and $\mu \in (0,\Lambda(\Phi))$,

$$\|u\|_\Phi = \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y - \mu\int_{\mathbb{R}^3}\frac{\Phi(x/|x|)|u|^2}{|x|^{2s}}\mathrm{d}x\Big)^{1/2}$$

is an equivalent norm in $D^{s,2}(\mathbb{R}^N)$.

Lemma 2.2 ([29]). *Let $t,r>1$ and $0<\alpha<3$ with $\frac{1}{t}+\frac{1}{r}+\frac{3-\alpha}{3}=2$, $f \in L^t(\mathbb{R}^3)$ and $h \in L^r(\mathbb{R}^3)$. There exists a sharp constant $C(\alpha,t,r)>0$, independent of f,h such that*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|f(x)||h(y)|}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \leqslant C(\alpha,t,r)\|f\|_t\|h\|_r.$$

If $t=r=\frac{6}{3+\alpha}$, then

$$C(\alpha,t,r) = C(\alpha) = \pi^{\frac{3-\alpha}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(3+\frac{\alpha}{2})}\Big\{\frac{\Gamma(\frac{3}{2})}{\Gamma(3)}\Big\}^{-\alpha/3}.$$

A measurable function $u:\mathbb{R}^3\to\mathbb{R}$ belongs to the Morrey space $\mathcal{L}^{q,\varpi}(\mathbb{R}^3)$ with $q\in[1,\infty)$ and $\varpi\in(0,N]$ if

$$\|u\|^q_{\mathcal{L}^{q,\varpi}(\mathbb{R}^3)}=\sup_{R>0,x\in\mathbb{R}^3}R^{\varpi-3}\int_{B(x,R)}|u(y)|^q\mathrm{d}y<\infty.$$

Lemma 2.3 ([35]). *For $s\in(0,1)$, there exists $C>0$ such that for ι and ϑ satisfying $\frac{2}{2^*_s}\leqslant\iota<1$, $1\leqslant\vartheta<2^*_s$, we have*

$$\Big(\int_{\mathbb{R}^3}|u|^{2^*_s}\mathrm{d}x\Big)^{1/(2^*_s)}\leqslant C\|u\|_s^{\iota}\|u\|^{1-\iota}_{\mathcal{L}^{\vartheta,\frac{\vartheta(N-2s)}{2}}(\mathbb{R}^3)},$$

for any $u\in D^{s,2}(\mathbb{R}^3)$.

2.1. **Formulation of system** (1.1)**.** Considering $u\in D^{s,2}(\mathbb{R}^3)$, we define the linear functional $F_u(v):D^{t,2}(\mathbb{R}^3)\to\mathbb{R}$ by

$$F_u(v)=\int_{\mathbb{R}^3}u^2v\mathrm{d}x.$$

Using $2t+4s=3$, Holder's inequality, and (1.9), we obtain

$$\begin{aligned}|F_u(v)|&\leqslant\int_{\mathbb{R}^3}|u|^{\frac{6-4s}{3-2s}}|v|\mathrm{d}x\\&=\int_{\mathbb{R}^3}|u|^{\frac{3+2t}{3-2s}}|v|\mathrm{d}x\\&\leqslant\Big(\int_{\mathbb{R}^3}|u|^{\frac{3+2t}{3-2s}\cdot\frac{6}{3+2t}}\mathrm{d}x\Big)^{\frac{3+2t}{6}}\Big(\int_{\mathbb{R}^3}|v|^{\frac{6}{3-2t}}\mathrm{d}x\Big)^{\frac{3-2t}{6}}\\&\leqslant\Big(\frac{1}{S_s}\|u\|_s^2\Big)^{\frac{3+2t}{6}}\Big(\frac{1}{S_t}\|v\|_t^2\Big)^{\frac{3-2t}{6}},\end{aligned}$$

which implies that F_u is well defined and continuous in $D^{t,2}(\mathbb{R}^3)$.

From the Lax-Milgram theorem it follows that, for every $u\in D^{s,2}(\mathbb{R}^3)$, there exists a unique $\phi_{u,t}\in D^{t,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}(-\Delta)^{t/2}\phi_{u,t}(-\Delta)^{t/2}v\mathrm{d}x=\int_{\mathbb{R}^3}u^2v\mathrm{d}x,$$

which implies that $\phi_{u,t}$ is a weak solution of

$$(-\Delta)^t\phi_{u,t}=u^2,$$

and the representation formula holds,

$$\phi_{u,t}=\int_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|^{3-2t}}\mathrm{d}y. \tag{2.3}$$

It follows from (2.3) that system (1.1) can be rewritten as the equivalent form

$$(-\Delta)^su-\mu\frac{\Phi(x/|x|)}{|x|^{2s}}u+\lambda\big(I_{2t}*|u|^2\big)u=|u|^{2^*_s-2}u\quad\text{in }\mathbb{R}^3.$$

3. Perturbed equation

We look at equation (1.1) as a perturbation of the equation

$$(-\Delta)^s u - \mu \frac{\Phi(x/|x|)}{|x|^{2s}} u = |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^3. \tag{3.1}$$

We introduce the energy functional associated with (1.8) by

$$I_0(u) = \frac{1}{2}\|u\|_\Phi^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} \mathrm{d}x.$$

Define

$$J_0 = I_0|_{D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3)}.$$

3.1. **Mountain pass geometry and Nehari manifold.**

Lemma 3.1. *Under the assumptions of Theorem 1.1, the functional J_0 possesses the mountain pass geometry.*

Proof. We prove that J_0 satisfies all the conditions in Mountain Pass Theorem.
(i) For any $u \in D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3) \setminus \{0\}$, we have

$$J_0(u) \geqslant \frac{1}{2}\|u\|_\Phi^2 - \frac{1}{2_s^* \cdot S_s^{2_s^*}}\|u\|_\Phi^{2_s^*}.$$

Because of $2 < 2_s^*$, there exists a sufficiently small positive number ρ such that

$$\varsigma := \inf_{\|u\|_s = \rho} J_0(u) > 0 = J_0(0).$$

(ii) Given $u \in D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3) \setminus \{0\}$, we have

$$J_0(tu) = \frac{t^2}{2}\|u\|_\Phi^2 - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} \mathrm{d}x < 0,$$

for t large enough. We choose $t_u > 0$ corresponding to u such that $J_0(tu) < 0$ for all $t > t_u$ and $\|t_u u\|_s > \rho$. $\square$

Define

$$c_0 = \inf_{\Upsilon \in \Gamma_0} \max_{t \in [0,1]} J_0(\Upsilon(t)),$$

where

$$\Gamma_0 = \{\Upsilon \in C([0,1], D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3)) | \Upsilon(0) = 0, J_0(\Upsilon(1)) < 0\}.$$

By Lemma 3.1, we obtain that for all $\lambda > 0$, there exists $\{u_n\} \subset D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3)$ such that

$$J_0(u_n) \to c_0 > 0 \text{ and } J_0'(u_n) \to 0 \quad \text{as } n \to \infty.$$

It is also easy to see that $\{u_n\}$ is uniformly bounded in $D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3)$.

The Nehari manifold on $D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3)$ is defined by

$$\mathcal{N} = \{u \in D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3) | \langle J_0'(u), u\rangle = 0, \ u \neq 0\},$$
$$\bar{\bar{c}}_0 = \inf_{u \in \mathcal{N}} J_0(u),$$
$$\bar{c}_0 = \inf_{u \in D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3)} \max_{t \geqslant 0} J_0(tu).$$

With minor changes in the proof of [44, Theorem 4.2], we can show that

$$\bar{\bar{c}}_0 = \bar{c}_0 = c_0.$$

The following result implies the non-vanishing property of $(PS)_{c_0}$ sequence.

Lemma 3.2. *Assume that the assumptions of Theorem 1.1 hold. Let $\{u_n\}$ be a $(PS)_{c_0}$ sequence of J_0 with $c_0 > 0$. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x > 0.$$

Proof. It is easy to see that $\{u_n\}$ is uniformly bounded in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$. Then there exists a constant $0 < C < \infty$ such that $\|u_n\|_\Phi \leqslant C$. Suppose on the contrary that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x = 0. \tag{3.2}$$

According to (3.2) and the definition of $(PS)_{c_0}$ sequence, we obtain

$$c_0 + o(1) = \frac{1}{2}\|u_n\|_\Phi^2 \quad \text{and} \quad \|u_n\|_\Phi^2 = o(1);$$

these imply that $c_0 = 0$, which contradicts with $0 < c_0$. □

3.2. **Ground state solution.**

Proof of Theorem 1.1. We divide the proof of Theorem 1.1 into two Steps.

Step 1. Since $\{u_n\}$ is a bounded sequence in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$, up to a subsequence, we assume that $u_n \rightharpoonup u$ in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$, $u_n \to u$ a.e. in $\mathbb{R}^3$, $u_n \to u$ in $L^r_{\rm loc}(\mathbb{R}^3)$ for all $r \in [2, 2_s^*)$.

According to Lemmas 2.3 and 3.2, there exists $C > 0$ such that for any n, we obtain

$$\|u_n\|_{\mathcal{L}^{2,3-2s}(\mathbb{R}^3)} \geqslant C > 0.$$

On the other hand, since $\{u_n\}$ is bounded in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$ and

$$D^{s,2}_{\rm rad}(\mathbb{R}^3) \hookrightarrow D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3) \hookrightarrow \mathcal{L}^{2,3-2s}(\mathbb{R}^3),$$

we have

$$\|u_n\|_{\mathcal{L}^{2,3-2s}(\mathbb{R}^3)} \leqslant C,$$

for some $C > 0$ independent of n. Hence, there exists a positive constant which we denote again by C such that for any n, we obtain

$$C \leqslant \|u_n\|_{\mathcal{L}^{2,3-2s}(\mathbb{R}^3)} \leqslant C^{-1}.$$

Combining the definition of Morrey space and above inequality, we deduce that for any $n \in \mathbb{N}$ there exist $\sigma_n > 0$ and $x_n \in \mathbb{R}^3$ such that

$$\frac{1}{\sigma_n^{2s}}\int_{B(x_n,\sigma_n)}|u_n(y)|^2\mathrm{d}y \geqslant \|u_n\|^2_{\mathcal{L}^{2,3-2s}(\mathbb{R}^3)} - \frac{C}{2n} \geqslant C_1 > 0.$$

Let $v_n(x) = \sigma_n^{\frac{3-2s}{2}}u_n(\sigma_n x)$. We know that

$$J_0(v_n) = J_0(u_n) \to c_0 \text{ and } J_0'(v_n) \to 0 \quad \text{as } n \to \infty.$$

Thus there exists v such that $v_n \rightharpoonup v$ in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$, $v_n \to v$ a.e. in $\mathbb{R}^3$, $v_n \to v$ in $L^r_{\rm loc}(\mathbb{R}^3)$ for all $r \in [2, 2_s^*)$. Then

$$\int_{B(\frac{x_n}{\sigma_n},1)}|v_n(y)|^2\mathrm{d}y = \frac{1}{\sigma_n^{2s}}\int_{B(x_n,\sigma_n)}|u_n(y)|^2\mathrm{d}y \geqslant C_1 > 0. \tag{3.3}$$

Step 2. We will show that $\{\frac{x_n}{\sigma_n}\}$ is bounded. Suppose on the contrary that $\frac{x_n}{\sigma_n}\to\infty$ as $n\to\infty$. By using the boundedness of $\{u_n\}$ in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$, we have $\|v_n\|_s=\|u_n\|_s\leqslant C_2$. Applying the uniform decay estimates of radial functions, we have

$$|v_n(x)|\leqslant \frac{C}{|x|^{\frac{3-2s}{2}}}\|v_n\|_s\leqslant \frac{C_3}{|x|^{\frac{3-2s}{2}}},$$

where $C_3=CC_2$. For any $\sqrt{\frac{C_1}{|B(0,1)|}}>\varepsilon>0$, there exists $M>0$, such that for $n>M$, we have

$$|v_n(x)|\leqslant \frac{C_3}{|\frac{x_n}{\sigma_n}-1|^{\frac{3-2s}{2}}}\leqslant\varepsilon,\quad \text{for all } x\in B^c(0,|\frac{x_n}{\sigma_n}-1|).$$

Since $B(\frac{x_n}{\sigma_n},1)\subset B^c(0,|\frac{x_n}{\sigma_n}-1|)$, we obtain

$$\int_{B(\frac{x_n}{\sigma_n},1)}|v_n(y)|^2\mathrm{d}y\leqslant\varepsilon^2\int_{B(\frac{x_n}{\sigma_n},1)}\mathrm{d}y=\varepsilon^2|B(\frac{x_n}{\sigma_n},1)|=\varepsilon^2|B(0,1)|<C_1.$$

This contradicts (3.3). Hence, $\{\frac{x_n}{\sigma_n}\}$ is bounded, and there exists $R>0$ such that

$$\int_{B(0,R)}|v_n(y)|^2\mathrm{d}y\geqslant\int_{B(\frac{x_n}{\sigma_n},1)}|v_n(y)|^2\mathrm{d}y\geqslant C_1>0.$$

Since the embedding $D^{s,2}_{\rm rad}(\mathbb{R}^3)\hookrightarrow L^r_{\rm loc}(\mathbb{R}^3)$, $r\in[2,2^*_s)$ is compact, we deduce that

$$\int_{B(0,R)}|v(y)|^2\mathrm{d}y\geqslant C_1>0.$$

Hence, $v\not\equiv 0$.

From $v_n\rightharpoonup v$ weakly in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$ and $\lim_{n\to\infty}\langle J_0'(v_n),\varphi\rangle=o(1)$, we obtain

$$\langle J_\lambda'(v),\varphi\rangle=0.$$

Since $v\not\equiv 0$, we know that $v\in\mathcal{N}$.

Now, we show that $v_n\to v$ strongly in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$. Applying the Brézis-Lieb lemma [10], we obtain

$$\begin{aligned}
c_0&=\lim_{n\to\infty}J_0(v_n)-\lim_{n\to\infty}\frac{1}{2^*_s}\langle J_0'(v_n),v_n\rangle\\
&=\lim_{n\to\infty}\Big(\frac12-\frac{1}{2^*_s}\Big)\|v_n\|_\Phi^2\\
&\geqslant\Big(\frac12-\frac{1}{2^*_s}\Big)\|v\|_\Phi^2\\
&=J_0(v)\geqslant\bar{\bar{c}}_0=c_0.
\end{aligned}\tag{3.4}$$

Thus, the inequalities above have to be equalities. We know that

$$\lim_{n\to\infty}\|v_n\|_\Phi^2=\|v\|_\Phi^2.$$

By using Brézis-Lieb lemma [10] again, we have

$$\lim_{n\to\infty}\|v_n\|_\Phi^2-\lim_{n\to\infty}\|v_n-v\|_\Phi^2=\|v\|_\Phi^2,$$

which implies that

$$\lim_{n\to\infty}\|v_n-v\|_\Phi^2=0.$$

Using (3.4) again, we know that $J_0(v) = c_0$. This implies that v attain the minimum of J_0 at c_0. □

4. Proof of Theorem 1.2

First, we summarize the proof of Theorem 1.1 as follows.

Lemma 4.1. *The energy functional J_0 satisfies the following properties:*

(1) *there exist $\rho, \varsigma > 0$ such that if $\|u\|_s = \rho$, then $J_0(u) \geqslant \varsigma$, and $e \in D^{s,2}_{\rm rad}(\mathbb{R}^3)$ exists such that $\|e\|_s > \rho$ and $J_0(e) < 0$;*

(2) *there exists a critical point v of J_0 such that*
$$J_0(v) = c_0 := \min_{\Upsilon\in\Gamma_0} \max_{t\in[0,1]} J_0(\Upsilon(t)),$$
where $\Gamma_0 = \{\Upsilon \in C([0,1], D^{s,2}_{\rm rad}(\mathbb{R}^3)) | \Upsilon(0) = 0, J_0(\Upsilon(1)) < 0\}$;

(3) *$c_0 := \inf\{J_0(u) | J_0'(u) = 0, u \in D^{s,2}_{\rm rad}(\mathbb{R}^3) \setminus \{0\}\}$;*

(4) *there exists a path $\Upsilon_0(t) \in \Gamma_0$ passing through v at $t = t_0$ and satisfying*
$$J_0(v) > J_0(\Upsilon_0(t)) \text{ for all } t \neq t_0;$$

(5) *the set $\mathcal{S} := \{u \in D^{s,2}_{\rm rad}(\mathbb{R}^3) | J_0'(u) = 0, J_0(u) = c_0\}$ is compact in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$ endowed with the strong topology up to dilations in $\mathbb{R}^3$.*

Proof. It is easy to see that J_0 is invariant by dilations. As a byproduct of the proof of Theorem 1.1, the weak convergence of the dilated subsequence can be upgraded into strong convergence. As a direct consequence, we show that the set $\mathcal{S}$ is compact in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$ endowed with the strong topology up to dilations in $\mathbb{R}^3$. The proof is completed. □

We introduce the energy functional associated with system (1.1) by
$$I_\lambda(u) = \frac{1}{2}\|u\|^2_\Phi + \frac{\lambda}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|^{3-2t}} {\rm d}x{\rm d}y - \frac{1}{2^*_s}\int_{\mathbb{R}^3} |u|^{2^*_s} {\rm d}x.$$
Define
$$J_\lambda = I_\lambda|_{D^{s,2}_{\rm rad}(\mathbb{R}^3)}.$$
As in [12, 25], we define a modified mountain pass energy level of J_λ
$$c_\lambda := \min_{\Upsilon\in\Upsilon_M} \max_{t\in[0,1]} J_\lambda(\Upsilon(t)),$$
where
$$\Upsilon_M = \{\Upsilon \in \Gamma_0 | \sup_{t\in[0,1]} \|\Upsilon(t)\|_s \leqslant M\},$$
$$M = 2\{\sup_{u\in\mathcal{S}} \|u\|_s, \sup_{t\in[0,1]} \|\Upsilon(t)\|_s\}.$$
Clearly, by the choice of M, $\Upsilon_0 \in \Gamma_M$ and thus,
$$c_0 = \min_{\Upsilon\in\Gamma_M} \max_{t\in[0,1]} J_0(\Upsilon(t)).$$
However, since $\Gamma_M \subsetneq \Gamma_0$, the standard mountain pass theorem cannot be applied, so other arguments are needed to prove that c_λ is a critical value.

Lemma 4.2. *Let $\lambda > 0$. Then $\lim_{\lambda\to 0} c_\lambda = c_0$.*

Proof. Since $\lambda > 0$, it is easy to see that $c_\lambda \geqslant c_0$. On the other hand, we can take $e = Tv$ in property (1), where

$$T > \Big(\frac{2_s^*}{2}\Big)^{\frac{1}{2_s^*-2}}.$$

Then $\Upsilon_0(t) \in C([0,1], D^{s,2}(\mathbb{R}^3))$ is defined as $\Upsilon_0(t) = te = tTv$, and $t_0 = \frac{1}{T}$ in property (4). We know that

$$\begin{aligned}
\lim_{\lambda\to 0} c_\lambda &= \lim_{\lambda\to 0} J_\lambda(\Upsilon_0(t)) \\
&\leqslant J_0(\Upsilon_0(t)) + \lim_{\lambda\to 0}\frac{\lambda}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|Tv(y)|^2|Tv(x)|^2}{|x-y|^{3-2t}}\mathrm{d}x\mathrm{d}y \\
&= J_0(v) = c_0.
\end{aligned}$$

The proof is complete. □

For any positive number d, we denote

$$B_d(u) := \{v \in D^{s,2}_{\rm rad}(\mathbb{R}^3) | \|u - v\|_s \leqslant d\},$$

and for any subset A of $D^{s,2}_{\rm rad}(\mathbb{R}^3)$, we set

$$A^d := \cup_{u\in A} B_d(u).$$

Lemma 4.3. *Set $d_1 = \sqrt{2\cdot 2_s^* \cdot c_0/(2_s^*-2)}$. Let $d \in (0, d_1)$. For any $u \subset \mathcal{S}^d$, we have $u \not\equiv 0$.*

Proof. For any $v \subset \mathcal{S}^d$, we have

$$\|v\|_s^2 \geqslant \|v\|_\Phi^2 = \frac{2\cdot 2_s^*}{2_s^*-2}\Big(J_0(v) - \frac{1}{2_s^*}\langle J_0'(v), v\rangle\Big) = \frac{2\cdot 2_s^*\cdot c_0}{2_s^*-2},$$

which gives

$$\|v\|_s > d_1. \tag{4.1}$$

By the definition of $\mathcal{S}^d$ and $u \subset \mathcal{S}^d$, there exists some $v \subset \mathcal{S}^d$ such that

$$\|u - v\|_s \leqslant d < d_1. \tag{4.2}$$

Applying the triangle inequality, we have

$$\|v\|_s = \|v - u + u\|_s \leqslant \|v - u\|_s + \|u\|_s.$$

which gives

$$\|v\|_s - \|v - u\|_s \leqslant \|u\|_s. \tag{4.3}$$

Putting (4.1) and (4.2) into (4.3), we obtain

$$\|u\|_s > d_1 - \|v - u\|_s > 0.$$

The proof is complete. □

Lemma 4.4. *Let $d > 0$ be a fixed number and let $\{u_j\} \subset \mathcal{S}^d$. Then there exists $\{\sigma_j\}$, such that*

$$\|\bar{u}_j\|_s = \|u_j\|_s,$$

where $\bar{u}_j(x) = \sigma_j^{\frac{3-2s}{2}} u_j(\sigma_j x)$. And, up to a subsequence, $\bar{u}_j \rightharpoonup \bar{u} \in \mathcal{S}^{2d}$.

Proof. Let $\{u_j\} \subset \mathcal{S}^d$. By the definition of $\mathcal{S}^d$ and Lemma 4.1(5), there exists $w_j \in \mathcal{S}$ such that

$$\|u_j - w_j\|_s \leqslant d.$$

By property (5) again, there exists $\{\sigma_j\}$, such that $\bar{w}_j \in \mathcal{S}$, one can prove that $\bar{w}_j \to \bar{w} \in \mathcal{S}$, where $\bar{w}_j(x) = \sigma_j^{\frac{3-2s}{2}} w_j(\sigma_j x)$ and $\bar{u}_j(x) = \sigma_j^{\frac{3-2s}{2}} u_j(\sigma_j x)$. Also

$$\|\bar{u}_j\|_s = \|u_j\|_s, \quad \|\bar{u}_j - \bar{w}_j\|_s = \|u_j - w_j\|_s \leqslant d.$$

Hence, for j large enough, we obtain

$$\|\bar{u}_j - \bar{w}\|_s = \|\bar{u}_j - \bar{w}_j + \bar{w}_j - \bar{w}\|_s \leqslant \|\bar{u}_j - \bar{w}_j\|_s + \|\bar{w}_j - \bar{w}\|_s \leqslant 2d.$$

Thus, $\{\bar{u}_j\}$ is bounded and, up to a subsequence, $\bar{u}_j \rightharpoonup \bar{u}$ in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$. Since $B_{2d}(\bar{w})$ is weakly closed in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$, we obtain $\bar{u} \in B_{2d}(\bar{w}) \subset \mathcal{S}^{2d}$. The proof is complete. □

Lemma 4.5. *Set $d \in (0, d_1)$. Suppose that there exist sequences $\lambda_j > 0$, $\lambda_j \to 0$, and $\{u_j\} \subset \mathcal{S}^d$ satisfying*

$$\lim_{j\to\infty} J_{\lambda_j}(u_j) \leqslant c_0 \quad \textit{and} \quad \lim_{j\to\infty} J'_{\lambda_j}(u_j) = 0.$$

Then there is sequence $\{\sigma_j\}$ such that $\|\bar{u}_j\|_s = \|u_j\|_s$, where $\bar{u}_j(x) = \sigma_j^{\frac{3-2s}{2}} u_j(\sigma_j x)$. And, up to a subsequence, $\{\bar{u}_j\}$ converges to some $\bar{u} \in \mathcal{S}$.

Proof. Considering that $\lim_{j\to\infty} J'_{\lambda_j}(u_j) = 0$ and that $\{u_j\}$ is bounded. By Lemma 4.4, up to a subsequence, $\bar{u}_j \rightharpoonup \bar{u} \in \mathcal{S}^{2d}$, and by the choice of d_1, $\bar{u} \neq 0$. We may readily verify that

$$\lim_{j\to\infty} J_{\lambda_j}(\bar{u}_j) = \lim_{j\to\infty} J_{\lambda_j}(u_j) \leqslant c_0 \quad \text{and} \quad \lim_{j\to\infty} J'_{\lambda_j}(\bar{u}_j) = 0.$$

For all $\varphi \in D^{s,2}_{\rm rad}(\mathbb{R}^3)$, we obtain

$$|\langle J'_{\lambda_j}(\bar{u}_j), \varphi\rangle| = |\langle J'_{\lambda_j}(u_j), \bar{\varphi}\rangle| \leqslant \|J'_{\lambda_j}(u_j)\|_{s^{-1}} \|\bar{\varphi}\|_s = o(1)\|\bar{\varphi}\|_s,$$

where $\bar{\varphi} = \sigma_j^{-\frac{3-2s}{2}} \varphi(\frac{x}{\sigma_j})$. Since $\|\bar{\varphi}\|_s = \|\varphi\|_s$, we obtain

$$J'_{\lambda_j}(\bar{u}_j) \to 0 \quad \text{as } j \to \infty.$$

Now, we have

$$\langle J'_0(\bar{u}), \varphi\rangle = \lim_{j\to\infty} \langle J'_{\lambda_j}(\bar{u}_j), \varphi\rangle - \lambda_j \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\bar{u}_j(y)|^2 \bar{u}_j(x)\varphi(x)}{|x-y|^{3-2t}} \mathrm{d}x\mathrm{d}y = 0.$$

Hence, $J'_0(\bar{u}) = 0$. Furthermore, applying Lemma 2.2 and $\bar{u}_j \in \mathcal{S}^{2d}$ for all j, we obtain

$$\begin{aligned}
\lim_{j\to\infty} \langle J'_0(\bar{u}_j), \varphi\rangle &= \lim_{j\to\infty} \langle J'_{\lambda_j}(\bar{u}_j), \varphi\rangle - \lambda_j \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\bar{u}_j(y)|^2 \bar{u}_j(x)\varphi(x)}{|x-y|^{3-2t}} \mathrm{d}x\mathrm{d}y \\
&= o(1)\|\varphi\|_s.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
c_0 &\geqslant \lim_{j\to\infty} J_{\lambda_j}(\bar{u}_j) \\
&= \lim_{j\to\infty} J_0(\bar{u}_j) + \frac{\lambda_j}{4} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\bar{u}_j(y)|^2 |\bar{u}_j(x)|^2}{|x-y|^{3-2t}} \mathrm{d}x\mathrm{d}y \qquad (4.4) \\
&= \lim_{j\to\infty} J_0(\bar{u}_j).
\end{aligned}$$

So $\{\bar{u}_j\}$ is a $(PS)_m$ sequence for J_0 with $m := \lim_{j\to\infty} J_0(\bar{u}_j)$. Thus, up to a subsequence, $\bar{u}_j \rightharpoonup \bar{u}$ and

$$\begin{aligned} J_0(\bar{u}) =& \frac{1}{2}\|\bar{u}\|_\Phi^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |\bar{u}|^{2_s^*} \mathrm{d}x \\ =& \big(\frac{1}{2} - \frac{1}{2_s^*}\big)\|\bar{u}\|_\Phi^2 \\ \leqslant& \big(\frac{1}{2} - \frac{1}{2_s^*}\big)\liminf_{j\to\infty} \|\bar{u}_j\|_\Phi^2 \\ =& \liminf_{j\to\infty}\Big(J_0(\bar{u}_j) - \frac{1}{2_s^*}\langle J_0'(\bar{u}_j), \bar{u}_j\rangle\Big) = m. \end{aligned}$$

Then, property (3) implies $m \geqslant J_0(\bar{u}) \geqslant c_0$, which, combined with (4.4), yields $m = J_0(\bar{u}) = c_0$, which implies $\bar{u} \in \mathcal{S}$. $\square$

Now, we set

$$m_\lambda := \max_{t\in[0,1]} J_\lambda(\Upsilon_0(t)).$$

Then $c_\lambda \leqslant m_\lambda$. It is easy to see that $\lim_{\lambda\to 0} m_\lambda \leqslant c_0$, this inequality together with Lemma 4.2, provides $\lim_{\lambda\to 0} c_\lambda = \lim_{\lambda\to 0} m_\lambda = c_0$. We also define

$$J_\lambda^{m_\lambda} = \{u \in D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3) | J_\lambda(u) \leqslant m_\lambda\}.$$

Lemma 4.6. *For any $d_2, d_3 > 0$ satisfying $d_3 < d_2 < d_1$, there are constants $\iota > 0$ and $\tilde{\lambda} > 0$ depending on d_2, d_3 such that for $\lambda \in (0, \tilde{\lambda})$, we have*

$$\|J_\lambda'(u)\|_{s^{-1}} \geqslant \iota, \quad \text{for all } u \in J_\lambda^{m_\lambda} \cap (\mathcal{S}^{d_2} \backslash \mathcal{S}^{d_3}).$$

Proof. To the contrary, suppose that for some $d_2, d_3 > 0$ satisfying $d_3 < d_2 < d_1$, there exist sequences $\{\lambda_j\}$ with $\lim_{j\to\infty} \lambda_j = 0$ and $\{u_j\} \in J_{\lambda_j}^{m_{\lambda_j}} \cap (\mathcal{S}^{d_2} \setminus \mathcal{S}^{d_3})$, such that

$$\lim_{j\to\infty} J_{\lambda_j}(u_j) \leqslant c_0 \quad \text{and} \quad \lim_{j\to\infty} J_{\lambda_j}'(u_j) = 0.$$

By using property (5), there exists a sequence $\{\sigma_j\}$ such that

$$\{\bar{u}_j\} \in J_{\lambda_j}^{m_{\lambda_j}} \cap (\mathcal{S}^{d_2} \setminus \mathcal{S}^{d_3}), \quad \lim_{j\to\infty} J_{\lambda_j}(\bar{u}_j) \leqslant c_0, \quad \lim_{j\to\infty} J_{\lambda_j}'(\bar{u}_j) = 0,$$

where $\bar{u}_j(x) = \sigma_j^{\frac{3-2s}{2}} u_j(\sigma_j x)$. Hence, we can apply Lemma 4.4 and assert the existence of some $\bar{u} \in \mathcal{S}$ such that $\bar{u}_j \to \bar{u}$ in $D_{\mathrm{rad}}^{s,2}(\mathbb{R}^3)$. As a consequence, $\mathrm{dist}(\bar{u}_j, \mathcal{S}) \to 0$ as $j \to \infty$, contradicting the relation $\bar{u}_j \notin \mathcal{S}^{d_3}$. $\square$

Lemma 4.7. *For $d > 0$, there exists $\delta > 0$, such that if $\lambda > 0$ with λ small enough, $t \in [0,1]$, $J_\lambda(\Upsilon_0(t)) \geqslant c_\lambda - \delta$ implies $\Upsilon_0(t) \in \mathcal{S}^d$.*

Since the proof of the above lemma is quite similar to that of [25, Propositions 4], we omit it here.

Lemma 4.8. *To each $0 < d < d_1$ corresponds a number $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, there exists a sequence $\{u_j\} \in J_\lambda^{m_\lambda} \cap \mathcal{S}^d$ such that $J_\lambda'(u_j) \to 0$ as $j \to \infty$.*

Since the proof of the above lemma is quite similar to that of [12, Propositions 5.3], we omit it here.

Proof of Theorem 1.2. Taking $d \in (0, d_1)$, by Lemma 4.8, there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, there exists a Palais Smale sequence $\{u_j^\lambda\} \subset \mathcal{S}^{\frac{d}{2}}$. By using property (5), there exists sequence $\{\sigma_j\}$ such that $\{\bar{u}_j^\lambda\} \subset \mathcal{S}^{\frac{d}{2}}$ where $\bar{u}_j^\lambda(x) = \sigma_j^{\frac{3-2s}{2}} u_j^\lambda(\sigma_j x)$. It is easy to see that $\{\bar{u}_j^\lambda\}$ is bounded in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$. Then by Lemma 4.4, up to a subsequence, there exists some $\bar{u}^\lambda \in \mathcal{S}^{\frac{d}{2}\cdot 2} = \mathcal{S}^d$ such that $\bar{u}_j^\lambda \rightharpoonup \bar{u}^\lambda$. Then we obtain $J'_\lambda(\bar{u}^\lambda) = 0$, and by the choice of d, $\bar{u}^\lambda \not\equiv 0$. Hence $\bar{u}^\lambda$ is a nontrivial critical point of J_λ. The principle of symmetric criticality implies that the critical points of J_λ are also critical points for I_λ. □

5. Open problem

In Theorem 1.1, we obtain a ground state solution $v \in D^{s,2}_{\rm rad}(\mathbb{R}^3)$ to equation (1.8). Since $D^{s,2}_{\rm rad}(\mathbb{R}^3) \subset D^{s,2}(\mathbb{R}^3)$, so v just is a ground state solution in $D^{s,2}_{\rm rad}(\mathbb{R}^3)$, not a ground state solution in $D^{s,2}(\mathbb{R}^3)$. It is natural to ask: Does there exist a ground state solution to equation (1.8) in $D^{s,2}(\mathbb{R}^3)$?

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Yu Su (corresponding author)
School of Mathematics and Big Data, Anhui University of Science and Technology, Huainan, 232001 Anhui, China.
School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, China
Email address: yusumath@qq.com

Haibo Chen
School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, China
Email address: math_chb@163.com

Senli Liu
School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, China
Email address: jasonliu0615@163.com

Xianwen Fang
School of Mathematics and Big Data, Anhui University of Science and Technology, Huainan, 232001 Anhui, China
Email address: xwfang@aust.edu.cn