Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 02, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LIFESPAN OF SOLUTIONS OF A FRACTIONAL EVOLUTION EQUATION WITH HIGHER ORDER DIFFUSION ON THE HEISENBERG GROUP

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Communicated by Jerome A. Goldstein

Abstract. We consider the higher order diffusion Schrödinger equation with a time nonlocal nonlinearity

$$i\partial_t u - (-\Delta_{\mathbb{H}})^m u = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s)|^p \,\mathrm{d}s,$$

posed in $(\eta, t) \in \mathbb{H} \times (0, +\infty)$, supplemented with an initial data $u(\eta, 0) = f(\eta)$, where $m > 1$, $p > 1$, $< \alpha < 1$, and $\Delta_{\mathbb{H}}$ is the Laplacian operator on the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$. Then, we prove a blow up result for its solutions. Furthermore, we give an upper bound estimate of the life span of blow up solutions.

1. Introduction

In this article, we consider a nonlocal in time higher-order nonlinear Schrödinger equation on the Heisenberg group

$$i\partial_t u - (-\Delta_{\mathbb{H}})^m u = \lambda I^\alpha_{0|t} |u(t)|^p, \quad \eta = (x, y, \tau) \in \mathbb{H},\ t > 0, \tag{1.1}$$

subject to the initial data

$$u(\eta, 0) = f(\eta), \tag{1.2}$$

where $u \equiv u(\eta, t)$ is a complex-valued unknown function, $i^2 = -1$, $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}\backslash\{0\}$, $\lambda_i \in \mathbb{R}$ $(i = 1, 2)$, $f = f(\eta) = f_1(\eta) + if_2(\eta)$, $f_i = f_i(\eta) \in L^1_{\rm loc}(\mathbb{R}^{2N+1})$ $(i = 1, 2)$ are real valued functions, and $I^\alpha_{0|t}\psi$ is the Riemann–Liouville fractional integral of order $(0 < \alpha < 1)$ defined for a continuous function $\psi(t), t > 0$, by

$$\left(I^\alpha_{0|t}\psi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s) \,\mathrm{d}s.$$

Here, $\Gamma(\cdot)$ stands for the gamma function.

First, for the sake of the reader, we give some known facts about the Heisenberg group $\mathbb{H}$ and the operator $\Delta_{\mathbb{H}}$. For their proof and more information, we refer for example to [4, 5, 8, 9, 10]. The Heisenberg group $\mathbb{H}$, whose elements are $\eta =$

2010 *Mathematics Subject Classification.* 35Q55, 35B44, 26A33, 35B30.
Key words and phrases. Schrödinger equation; Heisenberg group; life span;
Riemann-Liouville fractional integrals and derivatives.

Submitted June 8, 2019. Published January 7, 2020.

$(x, y, \tau) \equiv (\tilde{z}, \tau)$ is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the group operation "$\circ$" defined by

$$\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(\langle x, \tilde{y}\rangle - \langle \tilde{x}, y\rangle)),$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^N$. The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vector fields $X_i = \partial_{x_i} + 2y_i\partial_\tau$ and $Y_i = \partial_{y_i} - 2x_i\partial_\tau$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} (X_i^2 + Y_i^2);$$

explicitly, we have

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \Big(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \Big).$$

A natural group of dilitations on $\mathbb{H}$ is given by

$$\delta_\gamma(\eta) = (\gamma x, \gamma y, \gamma^2 \tau), \ \gamma > 0,$$

whose Jacobian determinant is γ^Q, where

$$Q = 2N + 2$$

is the homogeneous dimension of $\mathbb{H}$.

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}$ and homogeneous with respect to the dilatations δ_γ. More precisely, we have

$$\Delta_{\mathbb{H}}(u(\eta \circ \tilde{\eta})) = (\Delta_{\mathbb{H}} u)(\eta \circ \tilde{\eta}), \quad \Delta_{\mathbb{H}}(u \circ \delta_\gamma) = \gamma^2 (\Delta_{\mathbb{H}} u) \circ \delta_\gamma \quad \eta, \tilde{\eta} \in \mathbb{H}.$$

The natural distance from η to the origin is

$$|\eta|_{\mathbb{H}} = \Big(\tau^2 + \Big(\sum_{i=1}^{N} x_i^2 + y_i^2\Big)^2\Big)^{1/4} = \big(\tau^2 + |\tilde{z}|^4\big)^{1/4}.$$

Before we present our results, let us dwell a while on some existing literature. There are many results about nonexistence of solutions of nonlinear Schrödinger equation (see, e.g. [12, 18, 1, 6] and the references therein). Ikeda and Wakasugi [12] studied the equation

$$i\partial_t u + \Delta u = \lambda |u|^p, \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.3}$$

with $u(x, 0) = f(x)$, and showed that if $1 < p \le 1 + N/2$, $\lambda \in \mathbb{C}\backslash\{0\}$ and $f \in L^2(\mathbb{R}^N)$, then the life span T_m must be finite and

$$\lim_{t \to T_m} \|u(t)\|_{L^2} = +\infty.$$

Later, Kirane and Nabti [13] considered the equation

$$i\partial_t u + \Delta u = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} |u(s)|^p \,\mathrm{d}s, \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.4}$$

with $u(x, 0) = f(x)$, $f \in L^1(\mathbb{R}^N)$ and proved that if $1 < p \le 1 + 2(\alpha+1)/(N - 2\alpha)_+$, $\lambda \in \mathbb{C}\backslash\{0\}$, $\lambda_1 > 0$ and $\int_{\mathbb{R}^N} f_2(x)\, dx < 0$, then equation (1.4) has no global weak solutions.

On the other hand, there are many papers concerning the life span of solutions of various evolution equations (see [11, 14, 19, 13]); we mention in particular that recently Ikeda [11] obtained the upper bound for the life span of solutions for the nonlinear Schrödinger equations (1.3) supplemented with the initial condition $u(x,0)=\varepsilon f(x)$, of the form $T_\varepsilon \le C\varepsilon^{1/\rho}$, $C>0$, $\rho := k/2 - 1/(p-1) < 0$.

Our present work is motivated by [16, 2]. Pohozaev and Véron [16] gave some results about nonexistence of weak solutions of the differential inequality

$$\partial_t u - \Delta_{\mathbb{H}}(au) \ge |\eta|_{\mathbb{H}}^{\gamma}|u|^p, \quad a \in L^\infty,\ \eta \in \mathbb{H},\ t>0, \tag{1.5}$$

subjected to the initial condition $u(x,0)=u_0(x)$, for $\gamma > -2$, $1<p\le (Q+2+\gamma)/Q$ and $\int_{\mathbb{R}^{2N+1}} u_0(x)\,dx \ge 0$. Recently Cazenave and al. [2] studied the global solutions, and blow up solutions for the parabolic equation with nonlocal in time nonlinearity

$$\partial_t u - \Delta u = \int_0^t (t-s)^{-\gamma}|u|^{p-1}u(s)\,\mathrm{d}s, \quad x\in\mathbb{R}^N,\ t>0, \tag{1.6}$$

with $0\le\gamma<1$, $p>1$, $u_0\in C_0(\mathbb{R}^N)$, and proved some results concerning the nonexistence of global weak solutions.

Using the test function method, we study the blow up of weak solutions of problem (1.1)–(1.2). Then we obtain an upper bound of the life span of blow up solutions of equation (1.1) with initial data of the form $u(\eta,0)=\varepsilon f(\eta)$, $\varepsilon>0$.

2. Blow up solutions

In this section, we prove a blow up result for problem (1.1)–(1.2). At first, let us recall some definitions and properties concerning fractional integrals and derivatives (see [17] for more on fractional integrals and derivatives).

We denote by $D^\alpha_{0|t}\psi(t)$ and $D^\alpha_{t|T}\psi(t)$ the left-handed and right-handed Riemann-Liouville fractional derivatives of order $(0<\alpha<1)$ of a continuous function $\psi(t)$, $t>0$ defined by

$$\big(D^\alpha_{0|t}\psi\big)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha}\psi(s)\,\mathrm{d}s,$$
$$\big(D^\alpha_{t|T}\psi\big)(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^T (s-t)^{-\alpha}\psi(s)\,\mathrm{d}s.$$

Let $AC([0,T])$ be the space of absolutely continuous on $[0,T]$ with T finite. We introduce the following lemmas that will be use hereafter.

Lemma 2.1. *Let $\psi,\varphi,D^\alpha_{0|t}\psi,D^\alpha_{t|T}\varphi \in C([0,T])$, we have the formula of integration by parts (see [17, (2.64) p. 46])*

$$\int_0^T \big(D^\alpha_{0|t}\psi\big)(t)\varphi(t)\,\mathrm{d}t = \int_0^T \psi(t)\big(D^\alpha_{t|T}\varphi\big)(t)\,\mathrm{d}t. \tag{2.1}$$

Lemma 2.2. *Let $\psi\in AC^2([0,T]) := \{\psi:[0,T]\to\mathbb{R}$ such that $D\psi\in AC([0,T])\}$. Then, we have*

$$-D\cdot D^\alpha_{t|T}\psi(t) = D^{\alpha+1}_{t|T}\psi(t), \tag{2.2}$$

where $D:=\mathrm{d}/\mathrm{d}t$ is the usual derivative. Moreover, for all $1\le q\le\infty$, the equality

$$D^\alpha_{0|t}I^\alpha_{0|t} = Id_{L^q}(0,T) \tag{2.3}$$

holds almost everywhere on $[0,T]$.

Lemma 2.3 ((See [3])). *Let*

$$\psi(t) = \Big(1 - \frac{t}{T}\Big)_+^{\sigma}$$

with $t \geq 0$, $T > 0$ and $\sigma \gg 1$, then for all $\alpha \in (0,1)$, we have

$$D^{\alpha}_{t|T}\psi(t) = C_1 T^{-\alpha}\Big(1 - \frac{t}{T}\Big)_+^{\sigma-\alpha}, \tag{2.4}$$

$$D^{\alpha+1}_{t|T}\psi(t) = C_2 T^{-\alpha-1}\Big(1 - \frac{t}{T}\Big)_+^{\sigma-\alpha-1}, \tag{2.5}$$

$$\big(D^{\alpha}_{t|T}\psi\big)(T) = 0, \quad \big(D^{\alpha}_{t|T}\psi\big)(0) = C_1 T^{-\alpha}, \tag{2.6}$$

where

$$C_1 = \frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}, \quad C_2 = \frac{(1-\alpha+\sigma)(\sigma-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}.$$

Lemma 2.4 (see [15, Lemma 3.1]). *Let $\chi \in L^1(\mathbb{R}^{2N+1})$ and $\int_{\mathbb{R}^{2N+1}} \chi(\eta)\,\mathrm{d}\eta < 0$. Then there exists a test function $0 \leq \omega \leq 1$ such that*

$$\int_{\mathbb{R}^{2N+1}} \chi(\eta)\omega(\eta)\,\mathrm{d}\eta < 0. \tag{2.7}$$

Definition 2.5. Let $T > 0$. A function u is called a local weak solution of (1.1)–(1.2), if $u \in C([0,T); L^p_{\rm loc}(\mathbb{R}^{2N+1}))$ and satisfies

$$\begin{aligned} &\lambda \int_0^T \int_{\mathbb{R}^{2N+1}} I^{\alpha}_{0|t}|u|^p \phi(\eta,t)\,\mathrm{d}\eta\,\mathrm{d}t + i\int_{\mathbb{R}^{2N+1}} f(\eta)\phi(\eta,0)\,d\eta \\ &= -\int_0^T \int_{\mathbb{R}^{2N+1}} u\,(-\Delta_{\mathbb{H}})^m \phi(\eta,t)\,d\eta dt - i\int_0^T \int_{\mathbb{R}^{2N+1}} u\,\partial_t\phi(\eta,t)\,\mathrm{d}\eta\,\mathrm{d}t \end{aligned} \tag{2.8}$$

for any $\phi \in C_0^{\infty,1}(\mathbb{R}^{2N+1} \times (0,T))$, $\phi \geq 0$, $\phi(\cdot,T) = 0$. If $T = +\infty$, we say that u is a global weak solution of problem (1.1)–(1.2).

Let $f = f_1 + if_2$ satisfy one the the following set of assumptions

$$\begin{gathered} f_1 \in L^1(\mathbb{R}^{2N+1}), \quad \lambda_2 \int_{\mathbb{R}^{2N+1}} f_1(\eta)\,\mathrm{d}\eta > 0, \\ \text{or} \\ f_2 \in L^1(\mathbb{R}^{2N+1}), \quad \lambda_1 \int_{\mathbb{R}^{2N+1}} f_2(\eta)\,\mathrm{d}\eta < 0. \end{gathered} \tag{2.9}$$

Now, we are in a position to announce our results.

Theorem 2.6. *Suppose that $p > 1$ and*

$$p \leq p^* = \frac{Q+2m}{Q-2\alpha m}, \tag{2.10}$$

where if the equality holds, we assume $p > Q/(Q-2m)$ with $Q > 2m\max\{1, 1/\alpha\}$. If the initial data f satisfies (2.9), then problem (1.1)–(1.2) does not admit a global weak solution.

Proof. The proof is done by contradiction. Suppose that u is a global bounded weak solution. First we choose the test function. For this aim, we shall use a non-negative smooth function ϕ_1 which was constructed in [7].

$$\phi_1(x)=\phi_1(|x|),\quad \phi_1(0)=1,\quad 0<\phi_1(r)\le 1,\quad \text{for } r\ge 0, \tag{2.11}$$

where $\phi_1(r)$ is decreasing and $\phi_1(r)\to 0$ as $r\to\infty$ sufficiently fast. Moreover, there exists a constant k_m such that

$$|\Delta_{\mathbb{H}}^m\phi_1|\le k_m\phi_1,\quad \eta\in\mathbb{R}^{2N+1}, \tag{2.12}$$

and $\|\phi_1\|_{L^1}=1$. Let

$$\begin{gathered}\phi_2(t)=\big(1-\frac{t}{T}\big)^{\sigma},\quad T>0,\ \sigma\gg 1,\\ \phi(\eta,t):=\phi_1\big(\frac{\eta}{R}\big)\phi_2\big(\frac{t}{R^{2m}}\big),\quad R>0.\end{gathered}$$

Let $\mathcal{Q}:=\mathbb{R}^{2N+1}\times[0,TR^{2m})$. We consider the case $\int_{\mathbb{R}^{2N+1}}f_2(\eta)\,\mathrm{d}\eta<0$ and $\lambda_1>0$ only, since the other cases can be treated similarly (see Remark 2.7).

Using (2.8), we have

$$\begin{aligned}&\lambda\int_{\mathcal{Q}}I_{0|t}^{\alpha}|u|^p\phi(\eta,t)\,d\eta dt+i\int_{\mathbb{R}^{2N+1}}f(\eta)\phi(\eta,0)\,d\eta\\ &=-\int_{\mathcal{Q}}u(-\Delta_{\mathbb{H}})^m\phi(\eta,t)\mathrm{d}\eta dt-i\int_{\mathcal{Q}}u\partial_t\phi(\eta,t)d\eta dt.\end{aligned} \tag{2.13}$$

Replacing $\phi(\eta,t)$ by $D_{t|TR^{2m}}^{\alpha}\phi(\eta,t)$, we arrive at

$$\begin{aligned}&\lambda\int_{\mathcal{Q}}I_{0|t}^{\alpha}|u|^pD_{t|TR^{2m}}^{\alpha}\phi(\eta,t)\,\mathrm{d}\eta\,\mathrm{d}t+i\int_{\mathbb{R}^{2N+1}}f(\eta)D_{t|TR^{2m}}^{\alpha}\phi(\eta,0)\,\mathrm{d}\eta\\ &=-\int_{\mathcal{Q}}u(-\Delta_{\mathbb{H}})^mD_{t|TR^{2m}}^{\alpha}\phi(\eta,t)\,\mathrm{d}\eta dt-i\int_{\mathcal{Q}}uDD_{t|TR^{2m}}^{\alpha}\phi(\eta,t)\,\mathrm{d}\eta\,\mathrm{d}t.\end{aligned} \tag{2.14}$$

Furthermore, by taking the real parts, using (2.1) and (2.3) in the left-hand side of (2.14), and (2.2) in the right-hand side, we obtain

$$\begin{aligned}&\lambda_1\int_{\mathcal{Q}}|u|^p\phi(\eta,t)\,\mathrm{d}\eta\,\mathrm{d}t-D_{t|TR^{2m}}^{\alpha}\phi_2(0)\int_{\mathbb{R}^{2N+1}}f_2(\eta)\phi_1(\eta/R)\mathrm{d}\eta\\ &=-\int_{\mathcal{Q}}(\mathrm{Re}\ u)(-\Delta_{\mathbb{H}})^m\phi_1(\eta/R)D_{t|TR^{2m}}^{\alpha}\phi_2\left(t/R^{2m}\right)\mathrm{d}\eta\,\mathrm{d}t\\ &\quad-\int_{\mathcal{Q}}(\mathrm{Im}\,u)\phi_1(\eta/R)D_{t|TR^{2m}}^{\alpha+1}\phi_2\left(t/R^{2m}\right)\mathrm{d}\eta\,\mathrm{d}t.\end{aligned}$$

By the assumption on f_2 and using the Lemma 2.4, we have

$$D_{t|TR^{2m}}^{\alpha}\phi_2(0)\int_{\mathbb{R}^{2N+1}}f_2(\eta)\phi_1(\eta/R)\,\mathrm{d}\eta=CT^{-\alpha}R^{-2\alpha m}\int_{\mathbb{R}^{2N+1}}f_2(\eta)\phi_1(\eta/R)\,\mathrm{d}\eta\le 0.$$

Setting

$$I_R:=\int_{\mathcal{Q}}|u|^p\phi(\eta,t)\,\mathrm{d}\eta\,\mathrm{d}t,$$

we may write the estimate

$$\begin{aligned}\lambda_1 I_R &\le -\int_{\mathcal{Q}} (\mathrm{Re}\ u)(-\Delta_{\mathbb{H}})^m \phi_1(\eta/R) D^{\alpha}_{t|TR^{2m}} \phi_2\left(t/R^{2m}\right)\,\mathrm{d}\eta\mathrm{d}t \\ &\quad -\int_{\mathcal{Q}} (\mathrm{Im}\,u)\phi_1(\eta/R) D^{\alpha+1}_{t|TR^{2m}} \phi_2\left(t/R^{2m}\right)\,\mathrm{d}\eta\mathrm{d}t \\ &\le \int_{\mathcal{Q}} |u|\,|\Delta^m_{\mathbb{H}}\phi_1(\eta/R)||D^{\alpha}_{t|TR^{2m}}\phi_2\left(t/R^{2m}\right)|\,\mathrm{d}\eta\mathrm{d}t \\ &\quad + \int_{\mathcal{Q}} |u|\phi_1(\eta/R)|D^{\alpha+1}_{t|TR^{2m}}\phi_2\left(t/R^{2m}\right)|\,\mathrm{d}\eta\mathrm{d}t \equiv \mathcal{A}_1 + \mathcal{A}_2.\end{aligned} \tag{2.15}$$

Now, applying ε-Young's inequality,

$$XY \le \varepsilon X^p + C(\varepsilon)Y^q, \quad X \ge 0,\ Y \ge 0,\ p+q = pq,$$

with $0 < \varepsilon \ll 1$, $C(\varepsilon) = (1/q)(p\varepsilon)^{-q/p})$ in

$$\mathcal{A}_1 \quad \text{with } X = |u|\phi(\eta,t)^{1/p},\ Y = \phi(\eta,t)^{-1/p}|\Delta^m_{\mathbb{H}}\phi_1(\eta/R)|\,|D^{\alpha}_{t|TR^{2m}}\phi_2\left(t/R^{2m}\right)|,$$
$$\mathcal{A}_2 \quad \text{with } X = |u|\phi(\eta,t)^{1/p},\ Y = \phi(\eta,t)^{-1/p}\phi_1(\eta/R)|D^{\alpha+1}_{t|TR^{2m}}\phi_2\left(t/R^{2m}\right)|,$$

we obtain

$$\begin{aligned}&(\lambda_1 - 2\varepsilon)I_R \\ &\le C(\varepsilon)\int_{\mathcal{Q}} \phi_1(\eta/R)^{-\frac{1}{p-1}}|\Delta^m_{\mathbb{H}}\phi_1(\eta/R)|^{\frac{p}{p-1}}\phi_2\left(t/R^{2m}\right)^{-\frac{1}{p-1}} \\ &\quad \times |D^{\alpha}_{t|TR^{2m}}\phi_2\left(t/R^{2m}\right)|^{\frac{p}{p-1}}\,\mathrm{d}\eta\mathrm{d}t \\ &\quad + C(\varepsilon)\int_{\mathcal{Q}} \phi_1(\eta/R)\phi_2\left(t/R^{2m}\right)^{-\frac{1}{p-1}}|D^{\alpha+1}_{t|TR^{2m}}\phi_2\left(t/R^{2m}\right)|^{\frac{p}{p-1}}\,\mathrm{d}\eta\,\mathrm{d}t \\ &\equiv \mathcal{A}_3 + \mathcal{A}_4.\end{aligned} \tag{2.16}$$

At this stage, we pass to the scaled variables $s = t/R^{2m}$, $\tilde{\eta} = (\tilde{x}, \tilde{y}, \tilde{\tau})$ such that $\tilde{\tau} = \tau/R^2$, $\tilde{x} = x/R$, $\tilde{y} = y/R$, we obtain

$$\mathcal{A}_3 \le CR^{\beta}\int_0^T \int_{\mathbb{R}^{2N+1}} \phi_1(\tilde{\eta})\phi_2^{\alpha_1}(s)\,\mathrm{d}\tilde{\eta}\mathrm{d}s,$$
$$\mathcal{A}_4 \le CR^{\beta}\int_0^T \int_{\mathbb{R}^{2N+1}} \phi_1(\tilde{\eta})\phi_2^{\alpha_2}(s)\,\mathrm{d}\tilde{\eta}\mathrm{d}s,$$

where

$$\alpha_1 = \frac{p(\sigma-\alpha)-\sigma}{\sigma(p-1)}, \quad \alpha_2 = \frac{p(\sigma-\alpha-1)-\sigma}{\sigma(p-1)}, \quad \beta = Q + 2m - \frac{2mp(\alpha+1)}{p-1}.$$

Finally, we arrive at

$$(\lambda_1 - 2\varepsilon)I_R \le CR^{\beta}. \tag{2.17}$$

Note that inequality (2.10) is equivalent to $\beta \le 0$. So, we have to consider two cases:

- Case $\beta < 0$: we pass to the limit in (2.17) as R goes to $+\infty$; we obtain

$$\int_0^{\infty}\int_{\mathbb{R}^{2N+1}} |u|^p\,\mathrm{d}\eta\mathrm{d}t = 0 \quad \Longrightarrow \quad u \equiv 0,$$

this is a contradiction.

• Case $\beta = 0$: using inequality (2.17) with $R \to +\infty$, and taking into account the fact that $p = p^*$, we obtain

$$u \in L^p((0, +\infty) \times \mathbb{R}^{2N+1}).$$

On the other hand, repeating the same calculations as above, with

$$\phi(x,t) = \phi_1\Big(\frac{\eta}{RL^{-1}}\Big)\phi_2\Big(\frac{t}{R^{2m}}\Big),$$

where $1 \le L < R$ is large enough such that when $R \to +\infty$ we do not have $L \to +\infty$ at the same time, we arrive at

$$\lambda I_R \le CL^{-Q} + CL^{\frac{2pm}{p-1}-Q}, \tag{2.18}$$

thanks to the change of variables $\tilde{\tau} = \tau/(RL^{-1})^2$, $\tilde{x} = x/RL^{-1}$, $\tilde{y} = y/RL^{-1}$ and $s = t/R^{2m}$. Thus, using $p > Q/(Q-2m)$ and passing to the limit when $R \to +\infty$, and then when $L \to +\infty$ in (2.18), we obtain

$$\int_0^\infty \int_{\mathbb{R}^{2N+1}} |u|^p \, \mathrm{d}\eta \mathrm{d}t = 0 \quad \Longrightarrow \quad u \equiv 0,$$

which is also a contradiction. □

Remark 2.7. For the other cases, setting

$$I_R \equiv \begin{cases} -\int_{\mathcal{Q}} \lambda_1 |u|^p \phi(\eta,t)\, \mathrm{d}\eta \mathrm{d}t & \text{if } \lambda_1 < 0,\ \lambda_1 \int_{\mathbb{R}^{2N+1}} f_2(\eta)\, \mathrm{d}\eta < 0, \\ \int_{\mathcal{Q}} \lambda_2 |u|^p \phi(\eta,t)\, \mathrm{d}\eta \mathrm{d}t & \text{if } \lambda_2 > 0,\ \lambda_2 \int_{\mathbb{R}^{2N+1}} f_1(\eta)\, \mathrm{d}\eta > 0, \\ -\int_{\mathcal{Q}} \lambda_2 |u|^p \phi(\eta,t)\, \mathrm{d}\eta \mathrm{d}t & \text{if } \lambda_2 < 0,\ \lambda_2 \int_{\mathbb{R}^{2N+1}} f_1(\eta)\, \mathrm{d}\eta > 0, \end{cases}$$

we can prove the same conclusion in the same manner as above.

3. Life span of blow up solutions

To estimate the life span of blow up solutions, we assume that f satisfies one of the two sets of conditions

$$\begin{gathered} f_1 \in L^1_{\rm loc}(\mathbb{R}^{2N+1}), \quad \lambda_2 f_1(\eta) \ge |\eta|_{\mathbb{H}}^{-k},\ |\eta|_{\mathbb{H}} > 1, \\ \text{or} \\ f_2 \in L^1_{\rm loc}(\mathbb{R}^{2N+1}), \quad -\lambda_1 f_2(\eta) \ge |\eta|_{\mathbb{H}}^{-k},\ |\eta|_{\mathbb{H}} > 1, \end{gathered} \tag{3.1}$$

where

$$Q - 2\alpha m < k < \frac{2m(\alpha+1)}{p-1}. \tag{3.2}$$

We also consider the case when $\lambda_1 > 0$ only; the other cases can be treated in a similar manner.

Theorem 3.1. *Suppose that conditions* (3.1), (2.10) *and* (3.2) *are satisfied, and let u be the solution of* (1.1) *with the initial data $u(\eta,0) = \varepsilon f(\eta)$, where $\varepsilon > 0$. Denote by $[0, T_\varepsilon)$ the life span of u. Then there exists a positive constant C such that*

$$T_\varepsilon \le C\varepsilon^{1/\rho},$$

where $\rho = \frac{k}{2m} - \frac{\alpha+1}{p-1} < 0$.

Remark 3.2. When $p = \frac{Q+2m}{Q-2\alpha m}$, we have $\rho = \frac{k-Q+2\alpha m}{2m}$.

Proof of Theorem 3.1. First, repeating the same calculations as in Theorem 2.6, we obtain

$$
\begin{aligned}
&\lambda_1 I_R - CT^{-\alpha}R^{-2\alpha m}\int_{\mathbb{R}^{2N+1}} \varepsilon f_2(\eta)\phi_1(\eta/R)\,\mathrm{d}\eta \\
&\leq \int_{\mathcal{Q}} |u||\Delta_{\mathbb{H}}^m \phi_1(\eta/R)||D^{\alpha}_{t|TR^{2m}}\phi_2\left(t/R^{2m}\right)|\,\mathrm{d}\eta\mathrm{d}t \\
&\quad + \int_{\mathcal{Q}} |u|\phi_1(\eta/R)|D^{\alpha+1}_{t|TR^{2m}}\phi_2\left(t/R^{2m}\right)|\,\mathrm{d}\eta\mathrm{d}t \equiv \mathcal{A}_1 + \mathcal{A}_2.
\end{aligned} \tag{3.3}
$$

By Hölder's inequality applied to $\mathcal{A}_1$ and $\mathcal{A}_2$, we have

$$
\begin{aligned}
&\lambda_1 I_R - CT^{-\alpha}R^{-2\alpha m}\int_{\mathbb{R}^{2N+1}} \varepsilon f_2(\eta)\phi_1(\eta/R)\,\mathrm{d}\eta \\
&\leq I_R^{1/p}\Big(\int_{\mathcal{Q}} \phi_1(\eta/R)\phi_2\left(t/R^{2m}\right)^{-\frac{1}{p-1}}|D^{\alpha+1}_{t|TR^{2m}}\phi_2\left(t/R^{2m}\right)|^{\frac{p}{p-1}}\,\mathrm{d}\eta\mathrm{d}t\Big)^{\frac{p-1}{p}} \\
&\quad + I_R^{1/p}\Big(\int_{\mathcal{Q}} \phi_1(\eta/R)^{-\frac{1}{p-1}}|\Delta_{\mathbb{H}}^m\phi_1(\eta/R)|^{\frac{p}{p-1}}\phi_2\left(t/R^{2m}\right)^{-\frac{1}{p-1}} \\
&\quad \times |D^{\alpha}_{t|TR^{2m}}\phi_2\left(t/R^{2m}\right)|^{\frac{p}{p-1}}\,\mathrm{d}\eta\mathrm{d}t\Big)^{\frac{p-1}{p}}.
\end{aligned} \tag{3.4}
$$

Using (2.4), (2.5), and passing to the scaled variables $s = t/TR^{2m}$, $\tilde{\eta} = (\tilde{x}, \tilde{y}, \tilde{\tau})$ such that $\tilde{\tau} = \tau/R^2$, $\tilde{x} = x/R$, $\tilde{y} = y/R$, we arrive at

$$
\lambda_1 I_R + CT^{-\alpha}V_R \leq R^{\frac{\beta}{q}} I_R^{1/p}(A(T) + B(T)), \tag{3.5}
$$

where

$$
\begin{gathered}
V_R := \varepsilon R^{-2\alpha m}\int_{\mathbb{R}^{2N+1}} -f_2(\eta)\phi_1(\eta/R)\,\mathrm{d}\eta, \\
\mathcal{A}(T) := CT^{-\alpha}\Big(\int_0^T\int_{\mathbb{R}^{2N+1}} \phi_1(\tilde{\eta})\phi_2(s)^{\alpha_1}\,\mathrm{d}\tilde{\eta}\mathrm{d}s\Big)^{\frac{p-1}{p}}, \\
\mathcal{B}(T) := CT^{-(\alpha+1)}\Big(\int_0^T\int_{\mathbb{R}^{2N+1}} \phi_1(\tilde{\eta})\phi_2(s)^{\alpha_2}\,\mathrm{d}\tilde{\eta}\mathrm{d}s\Big)^{\frac{p-1}{p}}.
\end{gathered}
$$

Thus

$$
V_R \leq C\lambda_1 T^{\alpha}\Big(\frac{R^{\frac{\beta}{q}}}{\lambda_1}\left(\mathcal{A}(T) + \mathcal{B}(T)\right) I_R^{1/p} - I_R\Big).
$$

We clearly have

$$
\mathcal{A}(T) = \frac{C}{(\sigma + 1 - q\alpha)^{1/q}} T^{\frac{p-1}{p}-\alpha} = a_p T^{\frac{p-1}{p}-\alpha}, \tag{3.6}
$$

$$
\mathcal{B}(T) = \frac{C}{(\sigma + 1 - q(\alpha+1))^{1/q}} T^{\frac{p-1}{p}-(\alpha+1)} = b_p T^{\frac{p-1}{p}-(\alpha+1)}. \tag{3.7}
$$

Note that

$$
\max_{x>0}(\gamma x^w - x) = (1-w)w^{w/(1-w)}\gamma^{1/(1-w)},
$$

for $\gamma > 0$ and $0 < w < 1$. Whereupon

$$
V_R \leq CT^{\alpha}R^{\beta}\mathcal{E}(T)^q, \tag{3.8}
$$

for any $T > 0$ and $R > 0$, where

$$
C = \lambda_1^{-1/(p-1)}(p-1)(1/p)^q,
$$

$$\mathcal{E}(T) = \mathcal{A}(T) + \mathcal{B}(T) = a_p T^{1-\frac{p\alpha+1}{p}} + b_p T^{-\frac{p\alpha+1}{p}}.$$

On the other hand, by the definition of V_R and the assumption on the initial data f, we have

$$\begin{aligned} V_R &= \varepsilon R^{-2\alpha m} \int_{\mathbb{R}^{2N+1}} -f_2(\eta)\phi_1(\eta/R)\,\mathrm{d}\eta \\ &\geq \varepsilon R^{-2\alpha m} \int_{|\eta|_{\mathbb{H}}\geq 1} -f_2(\eta)\phi_1(\eta/R)\,\mathrm{d}\eta \\ &\geq \varepsilon \lambda_1^{-1} R^{-2\alpha m} \int_{|\eta|_{\mathbb{H}}\geq 1} |\eta|_{\mathbb{H}}^{-k}\phi_1(\eta/R)\,\mathrm{d}\eta; \end{aligned}$$

passing to the scaled variables $\tilde{\eta} = (\tilde{x}, \tilde{y}, \tilde{\tau})$ such that $\tilde{x} = x/R$, $\tilde{y} = y/R$, $\tilde{\tau} = \tau/R^2$, we obtain

$$\begin{aligned} V_R &\geq \varepsilon R^{Q-k-2\alpha m}\lambda_1^{-1} \int_{|\tilde{\eta}|_{\mathbb{H}}\geq \frac{1}{R}} |\tilde{\eta}|_{\mathbb{H}}^{-k}\phi_1(\tilde{\eta})\,\mathrm{d}\tilde{\eta} \\ &\geq \varepsilon R^{Q-k-2\alpha m}\lambda_1^{-1} \int_{|\tilde{\eta}|_{\mathbb{H}}\geq \frac{1}{R_0}} |\tilde{\eta}|_{\mathbb{H}}^{-k}\phi_1(\tilde{\eta})\,\mathrm{d}\tilde{\eta} \\ &= C_k \varepsilon R^{Q-k-2\alpha m}, \end{aligned}$$

for any $R > R_0$, where R_0 is a constant independent of R and ε.

Now, let $t_0 \in (0, T_\varepsilon)$ and $R > R_0$. By using (3.8) with $T = t_0 R^{-2m}$, we obtain

$$\varepsilon \leq C R^{2\alpha m+k-Q} \Big(T^{\frac{\alpha}{q}} R^{\frac{\beta}{q}} E(t_0 R^{-2m})\Big)^q \equiv C H(t_0, R). \tag{3.9}$$

Furthermore,

$$\begin{aligned} H(t_0, R) &= \Big(a_p\, t_0^{1-\frac{p\alpha+1}{p}} R^{\frac{k(p-1)}{p}-2m} + b_p\, t_0^{-\frac{p\alpha+1}{p}} R^{\frac{k(p-1)}{p}}\Big)^{\frac{p}{p-1}} \\ &= t_0^{-\frac{\alpha+1}{p-1}} \Big(a_p\, t_0 R^{\frac{k(p-1)}{p}-2m} + b_p\, R^{\frac{k(p-1)}{p}}\Big)^{\frac{p}{p-1}}. \end{aligned} \tag{3.10}$$

Substituting $R = t_0^{1/2m}$ in (3.10), we can restate inequality (3.9) as

$$\varepsilon \leq C H(t_0, t_0^{1/2m}) \leq C t_0^{\frac{k}{2m}-\frac{\alpha+1}{p-1}},$$

with some $C > 0$. Consequently, the inequality

$$t_0 \leq C \varepsilon^{1/\rho}$$

holds for any $t_0 \in (0, T_\varepsilon)$. This completes the proof of the theorem. □

Acknowledgements. This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, under grant no. (RG-36-130-40). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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