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PROPERTIES OF THE RESOLVENT OF SINGULAR q-DIRAC OPERATORS

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Abstract. In this article, we investigate the resolvent operator of a singular q-Dirac system. We obtain an integral representations for the resolvent of this system, in terms of the spectral function. Furthermore, we give a formula for the Titchmarsh-Weyl function of q-Dirac system using the integral representation of the resolvent.

1. Introduction

Quantum analysis is a very interesting subject in mathematics. q-calculus is a type of mathematical analysis in which the concept of limit is not used, thus the functions which are not differentiable can be q-differentiable. The history of this calculus dates back to the beginning of the previous century. First results in q-calculus belong to Euler. q-calculus has important applications in mathematics and physics, such as in the relativity theory, basic hypergeometric functions, orthogonal polynomials, combinatorics and the calculus of variations (see [13]). For a deeper understanding of q-calculus we refer the reader to [1, 12, 13, 15, 16, 17, 21, 24] and the references cited therein.

In the Dirac's relativistic theory of the hydrogen atom, the energy-levels of the atom are the eigenvalues of the one dimensional Dirac operator

$$L := -\frac{l^2}{x} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} c\big(-\frac{ih}{2\pi}\frac{d}{dx}\big) - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} c\frac{jh}{2\pi x} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} mc^2,$$

where l, c, h, m are physical constants and j is an integer, $j \neq 0$ (see [25]). In this work, we discuss the q-analogue of this operator. The authors in [8] introduced a q-analogue of one dimensional Dirac operator. In [8], they investigate the existence and uniqueness of the solution of this problem and discuss some spectral properties of the problem.

In this article, using the spectral function, we construct the integral representation of the resolvent operator of a singular q-Dirac system. In the classical singular Sturm-Liouville equation, the integral representation of the resolvent was first proved by Weyl in 1910. Similar theorems were proved in [2, 3, 4, 5, 6, 7, 20, 22, 26].

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This article is organized as follows. In Section 2, we give some fundamental concepts of quantum analysis. In Section 3, we study the resolvent of the q-Dirac system. In Section 4, we give the integral representation of the resolvent of a q-Dirac operator, in terms of the spectral function. Finally, in Section 5, we give a formula for the Titchmarsh-Weyl function of this problem.

2. Preliminaries

First, we recall some fundamental concepts of quantum analysis. Following the standard notation in [18] and [10], let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R} := (-\infty, \infty)$ and $a \in A$. A *q-difference equation* is an equation that contains q-derivatives of a function defined on A. Let y be a complex-valued function on A. The *q-difference operator* D_q, the *Jackson q-derivative is* defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{(q-1)x} \quad \text{for all } x \in A.$$

Note that there is a connection between the q-deformed Heisenberg uncertainty relation and the Jackson derivative on q-basic numbers (see [23]). In the q-derivative, as $q \to 1$, the q-derivative is reduced to the ordinary derivative. The q-derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty} \frac{y(q^n x) - y(0)}{q^n x} \quad (x \in A),$$

if the limit exists and does not depend on x. A *right-inverse* of D_q, the *Jackson q-integration* is given by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t \ (a, b \in A).$$

The q-integration for a function over $[0, \infty)$ is defined in [14] by the formula

$$\int_0^{\infty} f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

A function f which is defined on A, with $0 \in A$, is said to be q-regular at zero if

$$\lim_{n\to\infty} f(xq^n) = f(0),$$

for every $x \in A$. Throughout the remainder of the paper, we deal only with functions q-regular at zero.

If f and g are q-regular at zero, then we have

$$\int_0^a g(t) D_q f(t) d_q t + \int_0^a f(qt) D_q g(t) d_q t = f(a)g(a) - f(0)g(0).$$

Let $L_q^2(0, \infty)$ be the space of complex-valued functions defined on $[0, \infty)$ such that

$$\|f\| := \Big(\int_0^{\infty} |f(x)|^2 d_q x \Big)^{1/2} < \infty.$$

The space $L_q^2(0,\infty)$ is a separable Hilbert space with the inner product

$$(f,g) := \int_0^\infty f(x)\overline{g(x)}d_qx, \quad f,g \in L_q^2(0,\infty)$$

(see [11]).

Let $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$. Then, we define the q-Wronskian of $y(x)$ and $z(x)$ as

$$W_q(y,z)(x) = y_1(x)z_2(q^{-1}x) - z_1(x)y_2(q^{-1}x). \tag{2.1}$$

A convenient Hilbert space $\mathcal{H} = L_q^2((0,\infty);E)$ $(E := \mathbb{C}^2)$ of vector-valued functions is defined by using the inner product

$$(f,g)_{\mathcal{H}} := \int_0^\infty (f(x),g(x))_E d_qx.$$

3. Resolvent of the q-Dirac Operator

In this section, we shall construct the resolvent of the q-Dirac system. Let us consider the q-Dirac system

$$-\frac{1}{q}D_{q^{-1}}y_2 + p(x)y_1 = \lambda y_1 \tag{3.1}$$

$$D_qy_1 + r(x)y_2 = \lambda y_2 \tag{3.2}$$

with the boundary conditions

$$y_1(0,\lambda)\sin\alpha + y_2(0,\lambda)\cos\alpha = 0, \tag{3.3}$$

$$y_1(q^{-n},\lambda)\sin\beta + y_2(q^{-n},\lambda)\cos\beta = 0, \quad \alpha,\beta\in\mathbb{R},\ n\in\mathbb{N}, \tag{3.4}$$

where λ is a complex eigenvalue parameter, p and r are real-valued functions defined on $[0,\infty)$, continuous at zero and $p,r \in L^1_{q,\mathrm{loc}}(0,\infty)$.

We will denote by $\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$ and $\theta(x,\lambda) = \begin{pmatrix} \theta_1(x,\lambda) \\ \theta_2(x,\lambda) \end{pmatrix}$, the solution of the system (3.1)-(3.2) which satisfy the initial conditions

$$\varphi_1(0,\lambda) = \cos\alpha, \quad \varphi_2(0,\lambda) = -\sin\alpha, \quad \theta_1(0,\lambda) = \sin\alpha, \quad \theta_2(0,\lambda) = \cos\alpha. \tag{3.5}$$

Let us define

$$\chi_{q^{-n}}(x,\lambda) = \theta(x,\lambda) + l(\lambda,q^{-n})\varphi(x,\lambda) \in \mathcal{H}, \quad n \in \mathbb{N} = \{1,2,3,\dots\}.$$

Using this notation we now state the result from [9].

Lemma 3.1. *For each non-real number λ, we have $\chi_{q^{-n}}(x,\lambda) \to \chi(x,\lambda)$ and*

$$\int_0^{q^{-n}} \|\chi_{q^{-n}}(x,\lambda)\|_E^2 d_qx \to \int_0^\infty \|\chi(x,\lambda)\|_E^2 d_qx, \quad n\to\infty.$$

Putting

$$\begin{aligned} &G_{q^{-n}}(x,t,\lambda) \\ &= \begin{cases} \chi_{q^{-n}}(x,\lambda)\varphi^T(t,\lambda), & t \le x \\ \varphi(x,\lambda)\chi^T_{q^{-n}}(t,\lambda), & t > x \end{cases} \\ &\begin{cases} \begin{pmatrix} [\chi_{q^{-n}1}(x,\lambda)\varphi_1(t,\lambda) & \chi_{q^{-n}1}(x,\lambda)\varphi_2(t,\lambda) \\ \chi_{q^{-n}2}(x,\lambda)\varphi_1(t,\lambda) & \chi_{q^{-n}2}(x,\lambda)\varphi_2(t,\lambda) \end{pmatrix}, & t \le x \\ \begin{pmatrix} \varphi_1(x,\lambda)\chi_{q^{-n}1}(t,\lambda) & \varphi_1(x,\lambda)\chi_{q^{-n}2}(t,\lambda) \\ \varphi_2(x,\lambda)\chi_{q^{-n}1}(t,\lambda) & \varphi_2(x,\lambda)\chi_{q^{-n}2}(t,\lambda) \end{pmatrix}, & x < t, \end{cases} \end{aligned} \tag{3.6}$$

we have

$$(R_{q^{-n}}f)(x,\lambda) = y(x,\lambda) = \int_0^{q^{-n}} G_{q^{-n}}(x,t,\lambda)f(t)d_qt, \quad \lambda \in \mathbb{C}, \tag{3.7}$$

where $y(x,\lambda) = \begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}$ and $f(\cdot) = \begin{pmatrix} f_1(\cdot) \\ f_2(\cdot) \end{pmatrix} \in \mathcal{H}$. Hence we have

$$\begin{aligned} &G_{q^{-n}}(x,t,\lambda)f(t) \\ &= \begin{cases} \begin{pmatrix} \chi_{q^{-n}1}(x,\lambda)\varphi_1(t,\lambda)f_1(t) + \chi_{q^{-n}1}(x,\lambda)\varphi_2(t,\lambda)f_2(t) \\ \chi_{q^{-n}2}(x,\lambda)\varphi_1(t,\lambda)f_1(t) + \chi_{q^{-n}2}(x,\lambda)\varphi_2(t,\lambda)f_2(t) \end{pmatrix}, & t \le x \\ \begin{pmatrix} \varphi_1(x,\lambda)\chi_{q^{-n}1}(t,\lambda)f_1(t) + \varphi_1(x,\lambda)\chi_{q^{-n}2}(t,\lambda)f_2(t) \\ \varphi_2(x,\lambda)\chi_{q^{-n}1}(t,\lambda)f_1(t) + \varphi_2(x,\lambda)\chi_{q^{-n}2}(t,\lambda)f_2(t) \end{pmatrix}, & x < t. \end{cases} \end{aligned}$$

The function $G_{q^{-n}}(x,t,\lambda)$ is called the *Green function* and the operator $R_{q^{-n}}$ is called the *resolvent operator* of the regular boundary value problem (3.1)-(3.4).

From (3.7), we have

$$\begin{aligned} &y_1(x,\lambda) \\ &= q\chi_{q^{-n}1}(x,\lambda)\int_0^x (\varphi_1(qt,\lambda)f_1(qt) + \varphi_2(qt,\lambda)f_2(qt))d_qt \\ &\quad + q\varphi_1(x,\lambda)\int_x^{q^{-n}} (\chi_{q^{-n}1}(qt,\lambda)f_1(qt) + \chi_{q^{-n}2}(qt,\lambda)f_2(qt))d_qt, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &y_2(x,\lambda) \\ &= q\chi_{q^{-n}2}(x,\lambda)\int_0^x (\varphi_1(qt,\lambda)f_1(qt) + \varphi_2(qt,\lambda)f_2(qt))d_qt \\ &\quad + q\varphi_2(x,\lambda)\int_x^{\infty} (\chi_{q^{-n}1}(qt,\lambda)f_1(qt) + \chi_{q^{-n}2}(qt,\lambda)f_2(qt))d_qt. \end{aligned} \tag{3.9}$$

Now, we shall show that (3.7) satisfies the equation $D_qy_1 + r(x)y_2 = \lambda y_2 + f_2(x)$. From (3.8), it follows that

$$\begin{aligned} D_qy_1(x,\lambda) &= qD_q\chi_{q^{-n}1}(x,\lambda)\int_0^x (\varphi_1(qt,\lambda)f_1(qt) + \varphi_2(qt,\lambda)f_2(qt))d_qt \\ &\quad + qD_q\varphi_1(x,\lambda)\int_x^{q^{-n}} (\chi_{q^{-n}1}(qt,\lambda)f_1(qt) + \chi_{q^{-n}2}(qt,\lambda)f_2(qt))d_qt \\ &\quad + W_q(\varphi,\chi_{q^{-n}})f_2(x) = q\{\lambda - r(x)\}\chi_{q^{-n}2}(x,\lambda) \end{aligned}$$

$$
\begin{aligned}
&\quad \times \int_0^x (\varphi_1(qt,\lambda)f_1(qt)+\varphi_2(qt,\lambda)f_2(qt))d_qt \\
&\quad + q\{\lambda - r(x)\}\varphi_2(x,\lambda) \\
&\quad \times \int_x^{q^{-n}} (\chi_{q^{-n}1}(qt,\lambda)f_1(qt)+\chi_{q^{-n}2}(qt,\lambda)f_2(qt))d_qt + f_2(x) \\
&= \{\lambda - r(x)\}q\chi_{q^{-n}2}(x,\lambda)\int_0^x (\varphi_1(qt,\lambda)f_1(qt)+\varphi_2(qt,\lambda)f_2(qt))d_qt \\
&\quad + \{\lambda - r(x)\}q\varphi_2(x,\lambda) \\
&\quad \times \int_x^{q^{-n}} (\chi_{q^{-n}1}(qt,\lambda)f_1(qt)+\chi_{q^{-n}2}(qt,\lambda)f_2(qt))d_qt + f_2(x) \\
&= \{\lambda - r(x)\}\, y_2(x,\lambda) + f_2(x).
\end{aligned}
$$

The validity of the equality $-\frac{1}{q}D_{q^{-1}}y_2 + p(x)y_1 = \lambda y_1 + f_1(x)$ is proved similarly. We check at once that (3.7) satisfies the boundary conditions (3.3)-(3.4).

4. Integral representation of the resolvent operator

In this section, by using Levitan's technique [20], we will give the integral representation of the resolvent operator.

In [8], the authors prove that the boundary value problem given by (3.1)-(3.2) with the boundary conditions (3.3)-(3.4) has a compact resolvent, thus it has a purely discrete spectrum.

Let $\lambda_{m,q^{-n}}$ ($m \in \mathbb{Z} := \{0, \pm1, \pm2, \dots\}$ and

$$\varphi_{m,q^{-n}}(x) = \begin{pmatrix} \varphi_{m,q^{-n}1}(x) \\ \varphi_{m,q^{-n}2}(x) \end{pmatrix}$$

be the eigenvalues and eigenfunctions, respectively of problem (3.1)-(3.4), and

$$\alpha^2_{m,q^{-n}} = \int_0^{q^{-n}} \|\varphi_{m,q^{-n}}(x)\|_E^2 d_qx.$$

If $f(\cdot) \in L_q^2((0,q^{-n});E)$, then we have

$$\int_0^{q^{-n}} \|f(x)\|_E^2 d_qx = \sum_{m=-\infty}^{\infty} \frac{1}{\alpha^2_{m,q^{-n}}} \Big| \int_0^{q^{-n}} (f(x), \varphi_{m,q^{-n}}(x))_E d_qx \Big|^2$$

which is called the *Parseval equality.*

Now let us define the non-decreasing step function $\varrho_{q^{-n}}$ on $(-\infty,\infty)$ by

$$\varrho_{q^{-n}}(\lambda) = \begin{cases} -\sum_{\lambda<\lambda_{m,q^{-n}}<0} \frac{1}{\alpha^2_{m,q^{-n}}}, & \text{for } \lambda \le 0 \\ \sum_{0\le\lambda_{m,q^{-n}}<\lambda} \frac{1}{\alpha^2_{m,q^{-n}}} & \text{for } \lambda \ge 0. \end{cases} \tag{4.1}$$

The function $\varrho_{q^{-n}}$ is called the *spectral function* of the regular boundary value problem (3.1)-(3.4). Then the Parseval equality can be written as

$$\int_0^{q^{-n}} \|f(x)\|_E^2 d_qx = \int_{-\infty}^{\infty} F^2(\lambda) d\varrho_{q^{-n}}(\lambda),$$

where

$$F(\lambda) = \int_0^{q^{-n}} (f(x), \varphi(x,\lambda))_E d_qx.$$

Next we will prove a lemma, but first, we recall some definitions. A function f defined on an interval $[a,b]$ is said to be of *bounded variation* if there is a constant $C>0$ such that

$$\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\leq C$$

for every partition of $[a,b]$ by points of subdivision $x_0,x_1,\dots,x_n$ with

$$a=x_0<x_1<\dots<x_n=b\,. \tag{4.2}$$

Let f be a function of bounded variation. Then the *total variation* of f on $[a,b]$ is denoted by

$$V_a^b(f):=\sup\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|,$$

where the least upper bound is taken over all (finite) partitions (4.2) of the interval $[a,b]$ (see [19]).

Lemma 4.1. *For any positive κ, there is a positive constant $\Upsilon=\Upsilon(\kappa)$ not depending on q^{-n} such that*

$$V_{-\kappa}^{\kappa}\{\varrho_{q^{-n}}(\lambda)\}=\sum_{-\kappa\leq\lambda_{m,q^{-n}}<\kappa}\frac{1}{\alpha_{m,q^{-n}}^{2}}=\varrho_{q^{-n}}(\kappa)-\varrho_{q^{-n}}(-\kappa)<\Upsilon. \tag{4.3}$$

Proof. Let $\sin\alpha\neq 0$. Since $\varphi_2(x,\lambda)$ is continuous at zero, by condition $\varphi_2(0,\lambda)=-\sin\alpha$, there is a positive number δ nearby 0 such that

$$\Big(\frac{1}{\delta}\int_0^{\delta}\varphi_2(x,\lambda)d_qx\Big)^2>\frac{1}{2}\sin^2\alpha. \tag{4.4}$$

Let us define $f_\delta(x)=\begin{pmatrix}f_{1\delta}(x)\\ f_{2\delta}(x)\end{pmatrix}$, where $f_{1\delta}(x)=0$ and

$$f_{2\delta}(x)=\begin{cases}1/\delta, & 0\leq x\leq\delta\\ 0, & x>\delta.\end{cases}$$

From the Parseval equality and (4.4), we obtain

$$\begin{aligned}\int_0^{\delta}(f_{1\delta}^2(x)+f_{2\delta}^2(x))d_qx=\frac{1}{\delta}&=\int_{-\infty}^{\infty}\Big(\frac{1}{\delta}\int_0^{\delta}\phi_2(x,\lambda)d_qx\Big)^2d\varrho_{q^{-n}}(\lambda)\\ &\geq\int_{-\kappa}^{\kappa}\Big(\frac{1}{\delta}\int_0^{\delta}\phi_2(x,\lambda)d_qx\Big)^2d\varrho_{q^{-n}}(\lambda)\\ &>\frac{1}{2}\sin^2\alpha\{\varrho_{q^{-n}}(\kappa)-\varrho_{q^{-n}}(-\kappa)\},\end{aligned}$$

which proves (4.3).

If $\sin\alpha=0$, then we define $f_\delta(x)=\begin{pmatrix}f_{1\delta}(x)\\ f_{2\delta}(x)\end{pmatrix}$ where $f_{2\delta}(x)=0$ and

$$f_{1\delta}(x)=\begin{cases}\frac{1}{\delta^2}, & 0\leq x\leq\delta\\ 0, & x>\delta\,.\end{cases}$$

Thus we obtain the inequality (4.3) by applying the Parseval equality. □

Now we obtain an expansion into a Fourier series of the resolvent if one knows the expansion of the function $f(\cdot)$. By q-integration by parts, we find

$$\begin{aligned}
&\int_0^{q^{-n}} \Big[\begin{pmatrix} 0 & -\frac{1}{q}D_{q^{-1}} \\ D_q & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\Big]^T \\
&\times \varphi_{m,q^{-n}}(x)d_qx \\
&= \int_0^{q^{-n}} \big[-\frac{1}{q}D_{q^{-1}}y_2 + p(x)y_1\big]\varphi_{m,q^{-n}1}(x)d_qx \\
&\quad + \int_0^{q^{-n}} \big[D_qy_1 + p(x)y_2\big]\varphi_{m,q^{-n}2}(x)d_qx \\
&= \int_0^{q^{-n}} \big[-\frac{1}{q}D_{q^{-1}}\varphi_{m,q^{-n}2}(x) + p(x)y_1\varphi_{m,q^{-n}1}(x)\big]y_2d_qx \\
&\quad + \int_0^{q^{-n}} \big[D_q\varphi_{m,q^{-n}1}(x) + p(x)\varphi_{m,q^{-n}2}(x)\big]y_1d_qx \\
&= \lambda_{m,q^{-n}}\int_0^{q^{-n}} \varphi^T_{m,q^{-n}}(x)y(x,\lambda)d_qx = \lambda_{m,q^{-n}}t_m(\lambda).
\end{aligned} \tag{4.5}$$

For $m \in \mathbb{Z}$, we set

$$y(x,\lambda) = \sum_{m=-\infty}^{\infty} t_m(\lambda)\varphi_{m,q^{-n}}(x), \quad a_m = \int_0^{q^{-n}} f^T(x)\varphi_{m,q^{-n}}(x)d_qx\,.$$

Then we have

$$\begin{aligned}
a_m &= \int_0^{q^{-n}} f^T(x)\varphi_{m,q^{-n}}(x)d_qx \\
&= \int_0^{q^{-n}} \Big[\begin{pmatrix} 0 & -\frac{1}{q}D_{q^{-1}} \\ D_q & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\Big]^T \\
&\quad \times \varphi_{m,q^{-n}}(x)d_qx - \lambda\int_0^{q^{-n}} y^T(x,\lambda)\varphi_{m,q^{-n}}(x)d_qx \\
&= (\lambda_{m,q^{-n}} - \lambda)t_m(\lambda),\ m \in \mathbb{Z}.
\end{aligned}$$

Then, we obtain

$$\begin{gathered}
t_m(\lambda) = \frac{a_m}{\lambda_{m,q^{-n}} - \lambda}, \\
y(x,\lambda) = \int_0^{q^{-n}} G_{q^{-n}}(x,t,\lambda)f(t)d_qt = \sum_{m=-\infty}^{\infty} \frac{a_m}{\lambda_{m,q^{-n}} - \lambda}\varphi_{m,q^{-n}}(x).
\end{gathered}$$

Hence the expansion of the resolvent is

$$\begin{aligned}
(R_{q^{-n}}f)(x,z) &= \sum_{m=-\infty}^{\infty} \frac{\varphi_{m,q^{-n}}(x)\int_0^{q^{-n}} f^T(t)\varphi_{m,q^{-n}}(t)d_qt}{\alpha^2_{m,q^{-n}}(\lambda_{m,q^{-n}} - z)} \\
&= \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z}\Big\{\int_0^{q^{-n}} f^T(t)\varphi_{m,q^{-n}}(t,\lambda)d_qt\Big\}d\varrho_{q^{-n}}(\lambda).
\end{aligned} \tag{4.6}$$

Lemma 4.2. *Let z be a non-real number and x be a fixed number. Then*
$$\int_{-\infty}^{\infty} \frac{\|\varphi(x,\lambda)\|_E^2}{|z-\lambda|^2} d\varrho_{q^{-n}}(\lambda) < K. \tag{4.7}$$

Proof. Putting $f(t) = \varphi_{m,q^{-n}}(t)$ in (4.6), we obtain
$$\frac{1}{\alpha_{m,q^{-n}}} \int_0^{q^{-n}} G_{q^{-n}}(x,t,z)\varphi_{m,q^{-n}}(t) d_q t = \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}(\lambda_{m,q^{-n}} - z)}, \tag{4.8}$$
since the eigenfunctions $\varphi_{m,q^{-n}}(x)$ are orthogonal. Using (4.8), if we apply the Parseval equality to $G_{q^{-n}}(x,t,z)$, we obtain
$$\begin{aligned}\int_0^{q^{-n}} \|G_{q^{-n}}(x,t,z)\|_E^2 d_q t &= \sum_{m=-\infty}^{\infty} \frac{\|\varphi_{m,q^{-n}}(x)\|_E^2}{\alpha_{m,q^{-n}}^2 |\lambda_{m,q^{-n}} - z|^2} \\ &= \int_{-\infty}^{\infty} \frac{\|\varphi(x,\lambda)\|_E^2}{|z-\lambda|^2} d\varrho_{q^{-n}}(\lambda).\end{aligned}$$
Since the last integral is convergent by Lemma 3.1, the proof is complete. □

Now we recall a well-known theorems by Helly.

Theorem 4.3 ([19])**.** *Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real non-decreasing functions on a finite interval $a \le \lambda \le b$. Then there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a non-decreasing function w such that*
$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda), \quad a \le \lambda \le b.$$

Theorem 4.4 ([19])**.** *Assume that $(w_n)_{n\in\mathbb{N}}$ is a real, uniformly bounded, sequence of non-decreasing functions on a finite interval $a \le \lambda \le b$, and suppose that*
$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda), \ a \le \lambda \le b.$$
If f is any continuous function on $a \le \lambda \le b$, then
$$\lim_{n\to\infty} \int_a^b f(\lambda) dw_n(\lambda) = \int_a^b f(\lambda) dw(\lambda).$$

By Lemma 4.1, the set $\{\varrho_{q^{-n}}(\lambda)\}$ is bounded. Using Lemma 4.2 and Theorem 4.3, we can find a sequence $\{q^{-n_k}\}$ such that the functions $\varrho_{q^{-n_k}}(\lambda)$ $(n_k \to \infty)$ converge to a monotone function $\varrho(\lambda)$. The function $\varrho(\lambda)$ is called the *spectral function* of the singular boundary value problem (3.1)-(3.3) on $[0,\infty)$.

Lemma 4.5. *Let z be a non-real number and x be a fixed number. Then*
$$\int_{-\infty}^{\infty} \frac{\|\varphi(x,\lambda)\|_E^2}{|z-\lambda|^2} d\varrho(\lambda) \le K. \tag{4.9}$$

Proof. By inequality (4.7), for arbitrary $\eta > 0$, we have
$$\int_{-\eta}^{\eta} \frac{\|\varphi(x,\lambda)\|_E^2}{|z-\lambda|^2} d\varrho_{q^{-n}}(\lambda) < K.$$
Letting $\eta \to \infty$ and $n \to \infty$, we obtain the desired result. □

Lemma 4.6. *For arbitrary $\eta > 0$, we have the inequalities*
$$\int_{-\infty}^{-\eta} \frac{d\varrho(\lambda)}{\lambda^2} < \infty, \quad \int_{\eta}^{\infty} \frac{d\varrho(\lambda)}{\lambda^2} < \infty. \tag{4.10}$$

Proof. Since $\|\varphi(0,\lambda)\|_E^2 \neq 0$, putting $x=0$ in (4.9), we obtain

$$\int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty,$$

which completes the proof. □

Lemma 4.7. *Let $f(\cdot) \in \mathcal{H}$ and*

$$(Rf)(x,z) = \int_0^{\infty} G(x,t,z) f(t) d_q t,$$

where

$$G(x,t,z) = \begin{cases} \chi(x,z)\varphi^T(t,z), & t \le x \\ \varphi(x,z)\chi^T(t,z), & t > x. \end{cases}$$

Then

$$\int_0^{\infty} \|(Rf)(x,z)\|_E^2 d_q x \le \frac{1}{v^2} \int_0^{\infty} \|f(x)\|_E^2 d_q x, \ \ z = u + iv.$$

Proof. From (4.6) and the Parseval equality, we obtain

$$\begin{aligned}
&\int_0^{q^{-n}} \|(Rf)(x,z)\|_E^2 d_q x \\
&= \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m,q^{-n}}^2 |\lambda_{m,q^{-n}} - z|^2} \Big| \int_0^{q^{-n}} f^T(t)\varphi_{m,q^{-n}}(t) d_q t \Big|^2 \\
&\le \frac{1}{v^2} \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_{m,q^{-n}}^2} \Big| \int_0^{q^{-n}} f^T(t)\varphi_{m,q^{-n}}(t) d_q t \Big|^2 \\
&= \frac{1}{v^2} \int_0^{q^{-n}} \|f(t)\|_E^2 d_q t.
\end{aligned}$$

Letting $n \to \infty$, we obtain the desired result. □

The function $G(x,t,z)$ is called the *Green function* and the operator R is called the *resolvent operator* of the singular boundary value problem (3.1)-(3.3) on $[0,\infty)$. Now we obtain the integral representations for the resolvent.

Theorem 4.8. *For every non-real z and for each $f(\cdot) \in \mathcal{H}$, one has*

$$(Rf)(x,z) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} F(\lambda) d\varrho(\lambda), \tag{4.11}$$

where

$$F(\lambda) = \lim_{\xi\to\infty} \int_0^{q^{-\xi}} f^T(x)\varphi(x,\lambda) d_q x.$$

Proof. Let the function $f_\xi(x)$ vanish outside the interval $[0, q^{-\xi}]$ (where $q^{-\xi} < q^{-n}$) and satisfy the boundary condition (3.3). Let a be an arbitrary positive number. Set

$$F_\xi(\lambda) = \int_0^{q^{-\xi}} f_\xi^T(x)\varphi(x,\lambda) d_q x.$$

From (4.6), we obtain

$$\begin{aligned}(R_{q^{-n}}f_\xi)(x,z) &= \int_{-\infty}^{\infty}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)d\varrho_{q^{-n}}(\lambda)\\ &= \int_{-\infty}^{-a}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)d\varrho_{q^{-n}}(\lambda)+\int_{-a}^{a}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)d\varrho_{q^{-n}}(\lambda)\\ &\quad+\int_{a}^{\infty}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)d\varrho_{q^{-n}}(\lambda)\\ &= I_1+I_2+I_3.\end{aligned}\tag{4.12}$$

Now we estimate I_1. By (4.6), we obtain

$$\begin{aligned}\|I_1\|_E &= \|\int_{-\infty}^{-a}\frac{\varphi(x,\lambda)}{\lambda-z}F_\xi(\lambda)d\varrho_{q^{-n}}(\lambda)\|_E\\ &= \|\sum_{\lambda_{k,q^{-n}}<-a}\frac{\varphi_{k,q^{-n}}(x)\int_0^{q^{-\xi}}f_\xi^T(x)\varphi_{k,q^{-n}}(x)d_qx}{\alpha_{k,q^{-n}}^2(\lambda_{k,q^{-n}}-z)}\|_E\\ &\le \Big(\sum_{\lambda_{k,q^{-n}}<-a}\frac{\|\varphi_{k,q^{-n}}(x)\|_E^2}{\alpha_{k,q^{-n}}^2|\lambda_{k,q^{-n}}-z|^2}\Big)^{1/2}\\ &\quad\times\Big(\sum_{\lambda_{k,q^{-n}}<-a}\frac{1}{\alpha_{k,q^{-n}}^2}\Big|\int_0^{q^{-\xi}}f_\xi^T(x)\varphi_{k,q^{-n}}(x)d_qx\Big|^2\Big)^{1/2}.\end{aligned}\tag{4.13}$$

By q-integration by parts, we have

$$\begin{aligned}&\int_0^{q^{-\xi}}f_\xi^T(x)\varphi_{k,q^{-n}}(x)d_qx\\ &=\frac{1}{\lambda_{k,q^{-n}}}\int_0^{q^{-\xi}}f_{\xi1}(x)\Big\{-\frac{1}{q}D_{q^{-1}}\varphi_{k,q^{-n}2}(x)+p(x)\varphi_{k,q^{-n}1}(x)\Big\}d_qx\\ &\quad+\frac{1}{\lambda_{k,q^{-n}}}\int_0^{q^{-\xi}}f_{\xi2}(x)\{D_q\varphi_{k,q^{-n}1}(x)+r(x)\varphi_{k,q^{-n}2}(x)\}d_qx\\ &=\frac{1}{\lambda_{k,q^{-n}}}\int_0^{q^{-\xi}}\varphi_{k,q^{-n}1}(x)\{-\frac{1}{q}D_{q^{-1}}f_{\xi2}(x)+p(x)f_{\xi1}(x)\}d_qx\\ &\quad+\frac{1}{\lambda_{k,q^{-n}}}\int_0^{q^{-\xi}}\varphi_{k,q^{-n}2}(x)\{D_qf_{\xi1}(x)+r(x)f_{\xi2}(x)\}d_qx.\end{aligned}\tag{4.14}$$

By Lemma 4.2, we have

$$\|I_1\|_E\le\frac{K^{1/2}}{a}\Big(\sum_{\lambda_{k,q^{-n}}<-a}\frac{1}{\alpha_{k,q^{-n}}^2}|\int_0^{q^{-\xi}}h_\xi^T(x)\varphi_{k,q^{-n}}(x)d_qx|^2\Big)^{1/2},$$

where

$$h_\xi(x)=\begin{pmatrix}-\frac{1}{q}D_{q^{-1}}f_{\xi2}(x)+p(x)f_{\xi1}(x)\\ D_qf_{\xi1}(x)+r(x)f_{\xi2}(x)\end{pmatrix}.$$

By using Bessel's inequality, we obtain

$$\|I_1\|_E\le\frac{K^{1/2}}{a}\Big[\int_0^{q^{-\xi}}\|h_\xi^T(x)\|_E^2d_qx\Big]^{1/2}=\frac{C_1}{a}.$$

By a similar method, one can prove that $\|I_3\|_E \le \frac{C_2}{a}$. Then, I_1 and I_3 tend to zero as $a \to \infty$, uniformly in q^{-n}. By using Theorems 4.3 and 4.4 in (4.12), we obtain

$$(Rf_\xi)(x,z) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} F_\xi(\lambda) d\varrho(\lambda). \tag{4.15}$$

As is known, if $f(\cdot) \in \mathcal{H}$, then one can find a sequence $\{f_\xi(x)\}_{\xi=1}^{\infty}$ which satisfies the previous conditions and tends to $f(x)$ as $\xi \to \infty$. From the Parseval's equality, the sequence of Fourier transforms converges to the transform of $f(\cdot)$. By Lemmas 4.5 and 4.7, we can pass to the limit as $\xi \to \infty$ in (4.15). Hence the proof is complete. □

5. Titchmarsh-Weyl function

In this section, we will derive formulas for the Titchmarsh-Weyl function $m(z)$ and the spectral function $\varrho(\lambda)$, with the help of the integral representation of the resolvent.

First, we recall the Stieltjes inversion formula. Let $\sigma(\lambda) = \sigma_1(\lambda) + i\sigma_2(\lambda)$ be a complex function of bounded variation on the entire line. We put

$$\varphi(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{z - \lambda},$$

$$\psi(\sigma,\tau) = \frac{sgn\tau}{\pi} \frac{\varphi(z) - \varphi(\overline{z})}{2i} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\tau| d\sigma(\lambda)}{(\lambda-\sigma)^2 + \tau^2}, \quad z = \sigma + i\tau.$$

Theorem 5.1 ([20])**.** *If a and b are the points of continuity of $\sigma(\lambda)$, then we have*

$$\sigma(b) - \sigma(a) = -\lim_{\tau \to 0} \int_a^b \psi(\sigma,\tau) d\sigma.$$

Theorem 5.2. *(i) For any non-real z, one has*

$$m(z) - m(z_0) = \int_{-\infty}^{\infty} \Big[\frac{1}{\lambda - z} - \frac{1}{\lambda - z_0}\Big] d\varrho(\lambda), \quad \operatorname{Im} z_0 \neq 0. \tag{5.1}$$

(ii) If λ and μ are points of continuity of $\varrho(\lambda)$, then one has

$$\varrho(\lambda) - \varrho(\mu) = -\frac{1}{\pi} \lim_{\tau \to 0} \int_{\mu}^{\lambda} \operatorname{Im}\{m(\sigma + i\tau)\} d\sigma, \quad z = \sigma + i\tau,\ \tau > 0. \tag{5.2}$$

Proof. (i) Since $f(x)$ is arbitrary, from (4.11) it follows that

$$G(x,t,z) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)\varphi^T(t,\lambda)}{\lambda - z} d\varrho(\lambda).$$

Hence

$$G(x,t,z) - G(x,t,z_0) = \int_{-\infty}^{\infty} \varphi(x,\lambda)\varphi^T(t,\lambda) \Big[\frac{1}{\lambda - z} - \frac{1}{\lambda - z_0}\Big] d\varrho(\lambda). \tag{5.3}$$

Since both sides in (5.3) are matrices, their corresponding elements are equal. Thus, by using (3.6) and the definition of the product $\varphi(x,\lambda)\varphi^T(t,\lambda)$, putting $x = t = 0$, and then taking the initial conditions (3.5), we obtain

$$\begin{aligned} &\{\sin\alpha + m(z)\cos\alpha\}\cos\alpha - \{\sin\alpha + m(z_0)\cos\alpha\}\cos\alpha \\ &= \int_{-\infty}^{\infty} \cos^2\alpha \Big[\frac{1}{\lambda - z} - \frac{1}{\lambda - z_0}\Big] d\varrho(\lambda), \quad \operatorname{Im} z \neq 0,\ \operatorname{Im} z_0 \neq 0, \end{aligned}$$

i.e.,

$$m(z)-m(z_0)=\int_{-\infty}^{\infty}\big[\frac{1}{\lambda-z}-\frac{1}{\lambda-z_0}\big]d\varrho(\lambda).$$

(ii) From (5.1), we obtain

$$\psi(\sigma,\tau)=\frac{sgn\tau}{\pi}\frac{m(z)-m(\overline{z})}{2i}=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\tau|d\sigma(\lambda)}{(\lambda-\sigma)^2+\tau^2}.$$

By Theorem 5.1, we have

$$\varrho(\lambda)-\varrho(\mu)=-\lim_{\tau\to 0}\int_{\mu}^{\lambda}\psi(\sigma,\tau)d\sigma. \tag{5.4}$$

Since $m(\overline{z})=\overline{m(z)}$, it follows that

$$\psi(\sigma,\tau)=\frac{sgn\tau}{\pi}\frac{m(z)-m(\overline{z})}{2i}=\frac{sgn\tau}{\pi}\operatorname{Im}\{m(z)\}. \tag{5.5}$$

For $\tau>0$, we obtain (5.2) by using (5.4) and (5.5). Thus the theorem is proved. □

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