Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 06, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

STATIONARY QUANTUM ZAKHAROV SYSTEMS INVOLVING A HIGHER COMPETING PERTURBATION

SHUAI YAO, JUNTAO SUN, TSUNG-FANG WU

Abstract. We consider the stationary quantum Zakharov system with a higher competing perturbation

$$\Delta^2 u - \Delta u + \lambda V(x)u = K(x)u\phi - \mu|u|^{p-2}u \quad \text{in } \mathbb{R}^3,$$
$$-\Delta\phi + \phi = K(x)u^2 \quad \text{in } \mathbb{R}^3,$$

where $\lambda > 0$, $\mu > 0$, $p > 4$ and functions V and K are both nonnegative. Such problem can not be studied via the common arguments in variational methods, since Palais-Smale sequences may not be bounded. Using a constraint approach proposed by us recently, we prove the existence, multiplicity and concentration of nontrivial solutions for the above problem.

1. Introduction

Our starting point is the quantum Zakharov system

$$\begin{gathered} i\partial_t E + \Delta E - \varepsilon^2\Delta^2 E = nE, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \\ \partial_t^2 n - \Delta n + \varepsilon^2\Delta^2 n = \Delta|E|^2, \end{gathered} \tag{1.1}$$

where $N = 1, 2, 3$, the dimensionless quantum coefficient $0 < \varepsilon \le 1$, the complex valued function $E = E(t,x)$ is the envelope electric field and the real valued function $n = n(t,x)$ is the plasma density fluctuation. Such system has been introduced by Garcia et al. [8] and Haas-Shukla [11] as a model describing the nonlinear interaction between high-frequency quantum Langmuir waves and low-frequency quantum ion-acoustic waves. For more physical meaning, we refer the reader to [10] and the references therein.

In recent years, many researches have studied system(1.1), but they concern mainly the well-posedness of initial value problems, see for example [3, 4, 7, 9, 12]. More precisely, when $N = 1$, Jiang-Lin-Shao [12] proved the local well-posedness of system (1.1) with inital value $(E_0, n_0, \partial_t n_0) \in H^k(\mathbb{R}) \times H^l(\mathbb{R}) \times H^{l-2}(\mathbb{R})$ provided that $|k| - \frac{3}{2} < l < \min\{k + \frac{3}{2}, 2k + \frac{3}{2}\}$ and $k > -\frac{3}{4}$. Chen-Fang-Wang [3] obtained the global well-posedness of system (1.1) with inital value $(E_0, n_0, \partial_t n_0) \in L^2(\mathbb{R}) \times H^l(\mathbb{R}) \times H^{l-2}(\mathbb{R})$ provided that $-3/2 \le l \le 3/2$. When $N = 1, 2, 3$, Guo-Zhang-Guo [9] proved the global well-posedness of system (1.1) with initial value $(E_0, n_0, \partial_t n_0) \in H^k(\mathbb{R}^N) \times H^{k-1}(\mathbb{R}^N) \times H^{k-3}(\mathbb{R}^N)$ with $k \ge 2$. Moreover,

---

2010 *Mathematics Subject Classification.* 35J35, 35B38.
Key words and phrases. Quantum Zakharov system; variational methods; multiple solutions.

Submitted July 21, 2019. Published January 10, 2020.

the classical limit behavior of system (1.1) was studied as the quantum parameter $\varepsilon \to 0$.

If we look for the stationary solution and static solution in the form of

$$E(t,x) = e^{i\omega t}u(x) \quad \text{and} \quad n(t,x) = \phi(x),$$

then system (1.1) is deduced from the elliptic system:

$$\begin{gathered} -\varepsilon^2\Delta^2 u + \Delta u - \omega u = u\phi \quad \text{in } \mathbb{R}^N, \\ \varepsilon^2\Delta\phi - \phi = u^2 \quad \text{in } \mathbb{R}^N. \end{gathered} \tag{1.2}$$

Recently, Fang-Segata-Wu [6] studied the existence of ground state solution for system (1.2) with $0 < \varepsilon \le 1$ and $\omega > 0$. In addition, the existence of bound state radial solution was obtained when $\varepsilon > 0$ is sufficiently small and $\omega > 0$. Later, in [17] the authors considered a class of quantum Zakharov systems with a local perturbation, i.e.

$$\begin{gathered} \Delta^2 u - \Delta u + \lambda V(x)u = u\phi - \mu f(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi + \phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.3}$$

where the parameters $\lambda > 0$, $\mu \in \mathbb{R}$ and the potential $V(x)$ satisfies the following assumptions:

- (A1) $V \in C(\mathbb{R}^3, \mathbb{R})$ with $V(x) \ge 0$ in $\mathbb{R}^3$ and there exists $b > 0$ such that $|\{V < b\}|$ is the finite, where $|\cdot|$ is the Lebesgue measure;
- (A2) $\Omega = \operatorname{int}\{x \in \mathbb{R}^3 \mid V(x) = 0\}$ is nonempty and has smooth boundary with $\overline{\Omega} = \{x \in \mathbb{R}^3 : V(x) = 0\}$.

By using the Nehari manifold method, for λ sufficiently large, in [17] we concluded the following results:

- (i) when $1 < p < 2$ and $-\mu_1 < \mu < 0$, at least two nontrivial solutions exists if $f \in L^{2/(2-p)}(\mathbb{R}^3)$;
- (ii) when $p = 2$ and $-\mu_2 < \mu < 0$, or $p > 2$ and $\mu < 0$, or $\mu = 0$, or $1 < p < 4$ and $\mu > 0$, or $p = 4$ and $0 < \mu < \mu_3$, a nontrivial ground state solution is permitted if $f \in L^{2/(2-p)}(\mathbb{R}^3)$ for $1 < p < 2$ and $f \in L^\infty(\mathbb{R}^3)$ for $p \ge 2$.

We notice that when $\mu > 0$ and $p > 4$ of system (1.3)) has not been studied in [17], since the competing effect of the nonlocal term with the perturbation gives rise to methodological difficulties. Specifically, the common arguments in variational methods, such as mountain pass theorem, can not be applied because Palais-Smale sequences may not be bounded. Moreover, the Nehari manifold method does not work as well, since the energy functional is not bounded below on it.

Motivated by the analysis above, in this paper we are interested in studying the qualitative properties of nontrivial solutions in the case $\mu > 0$ and $p > 4$, including the existence, multiplicity and concentration. Having a little difference with system (1.3), we consider the problem

$$\begin{gathered} \Delta^2 u - \Delta u + \lambda V(x)u = K(x)u\phi - \mu|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi + \phi = K(x)u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.4}$$

where $\lambda > 0$, $\mu > 0$, $p > 4$ and $V(x)$ satisfies conditions (A1) and (A2), and $K \in L^\infty(\mathbb{R}^3) \cup L^{2p/(p-4)}(\mathbb{R}^3)$ with $K(x) \ge 0$ in $\mathbb{R}^3$.

As in [17], system (1.4) can be transformed into the following nonlinear biharmonic equation with a nonlocal term,

$$\Delta^2 u - \Delta u + \lambda V(x)u = K(x)u\phi_{K,u} - \mu|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

where

$$\phi_{K,u}(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|\exp(|x-y|)}dy.$$

Equation (1.5) is variational and its solutions are the critical points of the functional given by

$$I_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx + \frac{\mu}{p}\int_{\mathbb{R}^3}|u|^p dx,$$

where $\|u\|_\lambda = \int_{\mathbb{R}^3}(|\Delta u|^2 + |\nabla u|^2 + \lambda V(x)u^2)dx$. The functional $I_{\lambda,\mu}$ is of class C^1 in X_λ (see Section 2) whose Fréchet derivative is given by

$$\begin{aligned}\langle I'_{\lambda,\mu}(u), v\rangle &= \int_{\mathbb{R}^3}(\Delta u\Delta v + \nabla u\nabla v + \lambda V(x)uv)dx - \int_{\mathbb{R}^3} K(x)\phi_{K,u}uv\,dx\\ &\quad + \mu\int_{\mathbb{R}^3}|u|^{p-2}uv\,dx\end{aligned}$$

for any $v \in H^2(\mathbb{R}^3)$. Hence, if $u \in X_\lambda$ is a critical point of $I_{\lambda,\mu}$, then $(u, \phi_{K,u})$ is a solution of system (1.4).

Very recently, we proposed a novel constraint approach to find critical points in the study of Schrödinger-Poisson systems [15, 16] and Kirchhoff type problems [14]. Such approach can effectively solve the difficulties concerned above. In this paper, we shall further develop it to investigate system (1.4) with $\mu > 0$ and $p > 4$. To be specific, by introducing the filtration of the Nehari manifold as follows

$$\mathbf{N}_{\lambda,\mu}(c) = \{u \in \mathbf{N}_{\lambda,\mu} : I_{\lambda,\mu}(u) < c\} \text{ for some } c > 0,$$

where $\mathbf{N}_{\lambda,\mu}$ is the Nehari manifold, we prove that $\mathbf{N}_{\lambda,\mu}(c)$ can be decomposed as

$$\mathbf{N}_{\lambda,\mu}(c) = \mathbf{N}^{(1)}_{\lambda,\mu}(c) \cup \mathbf{N}^{(2)}_{\lambda,\mu}(c),$$

where

$$\mathbf{N}^{(1)}_{\lambda,\mu}(c) = \{u \in \mathbf{N}_{\lambda,\mu}(c) : \|u\|_\lambda < \underline{D}\}, \quad \mathbf{N}^{(2)}_{\lambda,\mu}(c) = \{u \in \mathbf{N}_{\lambda,\mu}(c) : \|u\|_\lambda > \overline{D}\}$$

for $0 < \underline{D} < \overline{D}$, in which each local minimizer of the functional $I_{\lambda,\mu}$ is a critical point of $I_{\lambda,\mu}$ in $H^2(\mathbb{R}^3)$. In consideration of the boundedness of $\mathbf{N}^{(1)}_{\lambda,\mu}(c)$, we can minimize the functional $I_{\lambda,\mu}$ on $\mathbf{N}^{(1)}_{\lambda,\mu}(c)$, where $I_{\lambda,\mu}$ is bounded below, to find a critical point. Furthermore, if we can further prove that $\mathbf{N}^{(2)}_{\lambda}(c)$ is bounded and that $I_{\lambda,\mu}$ is bounded below on $\mathbf{N}^{(2)}_{\lambda}(c)$, then two critical points can be found by minimizing $I_{\lambda,\mu}$ on both $\mathbf{N}^{(1)}_{\lambda,\mu}(c)$ and $\mathbf{N}^{(2)}_{\lambda,\mu}(c)$.

Before stating our results, we introduce some notation. Denote by S_∞ is the best Sobolev constant for the embedding of $H^2(\mathbb{R}^3)$ in $L^\infty(\mathbb{R}^3)$. Let $\overline{A} > 0$ be the sharp constant of Gagliardo-Nirenberg inequality and $\alpha > 0$ be the least energy of the limiting equation (see (2.10) below). Let

$$\mu_* = \frac{2}{(p-4)(1+\bar{A}^{16/3}|\{V<b\}|^{4/3})^{p/2}}\Big[\frac{S_\infty^2(p-4)}{8(p-2)\alpha}\Big]^{(p-2)/2} > 0.$$

We summarize our main results as follows.

Theorem 1.1. *Suppose that $p > 4, K \in L^\infty(\mathbb{R}^3)$ with $K(x) \geq 0$ and conditions* (A1) *and* (A2) *hold. Then there exists a number $\Lambda_* > 0$ such that for every $\lambda \geq \Lambda_*$ and $0 < \mu < \mu_*$, system* (1.4) *admits at least one nontrivial solution $(u^-_{\lambda,\mu}, \phi_{K,u^-_{\lambda,\mu}}) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ which satisfies*

$$\|u^-_{\lambda,\mu}\|_\lambda < 2\Big[\frac{\alpha(p-2)}{p-4}\Big]^{1/2} \quad \text{and} \quad 0 < I_{\lambda,\mu}(u^-_{\lambda,\mu}) < \frac{(p-2)^2}{p(p-4)}\alpha.$$

Theorem 1.2. *Assume that $p > 4, K \in L^{2p/(p-4)}(\mathbb{R}^3)$ with $K(x) \geq 0$ and conditions* (A1) *and* (A2) *hold. Then there exists a number $\Lambda \geq \Lambda_*$ such that for each $\lambda > \Lambda$ and $0 < \mu < \mu_*$, system* (1.4) *admits at least two nontrivial solutions $(u^\pm_{\lambda,\mu}, \phi_{K,u^\pm_{\lambda,\mu}}) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ which satisfy*

$$\|u^-_{\lambda,\mu}\|_\lambda < 2\Big[\frac{\alpha(p-2)}{p-4}\Big]^{1/2} < \|u^+_{\lambda,\mu}\|_\lambda,$$
$$I_{\lambda,\mu}(u^+_{\lambda,\mu}) < 0 < I_{\lambda,\mu}(u^-_{\lambda,\mu}) < \frac{(p-2)^2}{p(p-4)}\alpha.$$

In particular, $(u^+_{\lambda,\mu}, \phi_{K,u^+_{\lambda,\mu}})$ is a ground state solution.

Theorem 1.3. *Suppose that $(u^\pm_{\lambda,\mu}, \phi_{K,u^\pm_{\lambda,\mu}})$ are the nontrivial solutions of* (1.4) *obtained by Theorem 1.2. Then $(u^\pm_{\lambda,\mu}, \phi_{K,u^\pm_{\lambda,\mu}}) \to (u^\pm_\infty, \phi_{K,u^\pm_\infty})$ in $H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ as $\lambda \to \infty$ where $u^\pm_\infty \in H^2_0(\Omega)$ are nontrivial weak solutions of the Dirichlet problem*

$$\begin{gathered} \Delta^2 u - \Delta u = \frac{1}{4\pi} K(x)\Big(\int_\Omega \frac{K(y)u^2(y)}{|x-y|\exp(|x-y|)} dy\Big) u - \mu |u|^{p-2} u \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.6}$$

Remark 1.4. In [17], when $1 < p < 2$ and $\mu < 0$, we obtained the existence of two nontrivial solutions: one is in the neighborhood of the origin whose energy level is negative and the other's energy level is positive. In fact, such case is very similar to the one of concave-convex term. Theorem 1.2 shows that when $p > 4$ and $\mu > 0$, two nontrivial solutions can also be found. However, the solution with negative energy level is away from the origin, which is distinguished from the one in [17].

The remainder of this paper is organized as follows. After presenting some preliminary results in section 2, we prove Theorems 1.1 and 1.2 in sections 3 and 4, respectively. Finally, we explore the concentration of solutions in the section 5.

2. Preliminaries

Let

$$X = \big\{H^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2) dx < \infty\big\}$$

be equipped with the inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\Delta u \Delta v + \nabla u \nabla v + V(x) uv) dx, \quad \|u\| = \langle u, u\rangle^{1/2}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^3} (\Delta u \Delta v + \nabla u \nabla v + \lambda V(x) uv) dx, \quad \|u\|_\lambda = \langle u, u\rangle_\lambda^{1/2}.$$

It is clear that $\|u\| \leq \|u\|_\lambda$ for $\lambda \geq 1$. Now we set $X_\lambda = (X, \|u\|_\lambda)$.

Applying conditions (A1) and (A2), by the Hölder, Young and Gagliardo-Nirenberg inequalities, there exists a sharp constant $\bar{A} > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^3} u^2 dx &\leq \frac{1}{b} \int_{\{V \geq b\}} V(x) u^2 dx + (|\{V < b\}| \int_{\mathbb{R}^3} u^4 dx)^{1/2} \\
&\leq \frac{1}{b} \int_{\mathbb{R}^3} V(x) u^2 dx + \bar{A}^2 |\{V < b\}|^{1/2} (\int_{\mathbb{R}^3} |\Delta u|^2 dx)^{3/8} (\int_{\mathbb{R}^3} u^2 dx)^{5/8} \\
&\leq \frac{1}{b} \int_{\mathbb{R}^3} V(x) u^2 dx + \frac{3\bar{A}^{16/3} |\{V < b\}|^{4/3}}{8} \int_{\mathbb{R}^3} |\Delta u|^2 dx + \frac{5}{8} \int_{\mathbb{R}^3} u^2 dx,
\end{aligned}$$

which shows that

$$\int_{\mathbb{R}^3} u^2 dx \leq \frac{8}{3b} \int_{\mathbb{R}^3} V(x) u^2 dx + \bar{A}^{16/3} |\{V < b\}|^{4/3} \int_{\mathbb{R}^3} |\Delta u|^2 dx.$$

Applying the above inequality leads to

$$\begin{aligned}
&\|u\|_{H^2}^2 \\
&\leq (1 + \bar{A}^{16/3} |\{V < b\}|^{4/3}) \int_{\mathbb{R}^3} |\Delta u|^2 dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{8}{3b} \int_{\mathbb{R}^3} V(x) u^2 dx \\
&\leq \max\big\{1 + \bar{A}^{16/3} |\{V < b\}|^{4/3}, \frac{8}{3b}\big\} \|u\|^2.
\end{aligned} \tag{2.1}$$

This implies that the imbedding $X \hookrightarrow H^2(\mathbb{R}^3)$ is continuous. Similar to the inequality (2.1), we also obtain

$$\|u\|_{H^2}^2 \leq (1 + \bar{A}^{16/3} |\{V < b\}|^{4/3}) \|u\|_\lambda^2 \tag{2.2}$$

for

$$\lambda \geq \lambda_* := \frac{8}{3b} (1 + \bar{A}^{16/3} |\{V < b\}|^{4/3})^{-1}.$$

Since the imbedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ is continuous, by(2.2), for any $r \in [2, +\infty)$ one has

$$\begin{aligned}
\int_{\mathbb{R}^3} |u|^r dx &\leq S_\infty^{-(r-2)} \|u\|_{H^2}^r \\
&\leq S_\infty^{-(r-2)} (1 + \bar{A}^{16/3} |\{V < b\}|^{4/3})^{r/2} \|u\|_\lambda^r
\end{aligned} \tag{2.3}$$

for $\lambda \geq \lambda_*$.

We define the operator $\Phi : X_\lambda \to H^1(\mathbb{R}^3)$ as

$$\Phi[u] = \phi_{K,u}.$$

In the following lemma we state some properties of Φ without any proof. We refer the reader to [6] for more details. These properties are useful to our study of the problem.

Lemma 2.1. *For any $u \in X_\lambda$, we have the following statements:*

(i) $\Phi : X_\lambda \to H^1(\mathbb{R}^3)$ *is continuous;*
(ii) Φ *maps bounded sets in X_λ into bounded sets in $H^1(\mathbb{R}^3)$;*
(iii) $\Phi[tu] = t^2 \Phi[u]$ *for all $t \in \mathbb{R}$;*
(iv) $\Phi[u] > 0$ *when $u \neq 0$.*

Using the arguments in [17], by (2.3), when $K \in L^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx &\le \|K\|_\infty^2 \int_{\mathbb{R}^3} |u|^4dx \\ &\le \|K\|_\infty^2 S_\infty^{-2}(1+\bar{A}^{16/3}|\{V<b\}|^{4/3})^2\|u\|_\lambda^4,\end{aligned} \tag{2.4}$$

and when $K \in L^{2p/(p-4)}(\mathbb{R}^3)$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \\ &\le \Big(\int_{\mathbb{R}^3} |K|^{2p/(p-4)}dx\Big)^{(p-4)/p}\Big(\int_{\mathbb{R}^3} |u|^pdx\Big)^{4/p} \\ &\le \|K\|_{L^{2p/(p-4)}}^2 S_\infty^{-4(p-2)/p}\Big(1+\bar{A}^{16/3}|\{V<b\}|^{4/3}\Big)^2\|u\|_\lambda^4.\end{aligned} \tag{2.5}$$

Set

$$\Theta = \begin{cases} \|K\|_\infty^2 S_\infty^{-2}\big(1+\bar{A}^{16/3}|\{V<b\}|^{4/3}\big)^2 & \text{for } K \in L^\infty(\mathbb{R}^3), \\ \|K\|_{L^{2p/(p-4)}}^2 S_\infty^{-4(p-2)/p}\big(1+\bar{A}^{16/3}|\{V<b\}|^{4/3}\big)^2 & \text{for } K \in L^{2p/(p-4)}(\mathbb{R}^3). \end{cases}$$

Then it follows that

$$\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \le \Theta\|u\|_\lambda^4 \text{ for } \lambda \ge \lambda_*. \tag{2.6}$$

Define the Nehari manifold

$$\mathbf{N}_{\lambda,\mu} = \{u \in X_\lambda\backslash\{0\} : \langle I'_{\lambda,\mu}(u), u\rangle = 0\}.$$

Thus, $u \in \mathbf{N}_{\lambda,\mu}$ if and only if

$$\|u\|_\lambda^2 - \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx + \mu\int_{\mathbb{R}^3} |u|^pdx = 0. \tag{2.7}$$

By this equality and (2.6) one has

$$\begin{aligned}\|u\|_\lambda^2 \le \|u\|_\lambda^2 + \mu\int_{\mathbb{R}^3} |u|^pdx &= \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \\ &\le \Theta\|u\|_\lambda^4 \quad \text{for all } u \in \mathbf{N}_{\lambda,\mu}.\end{aligned}$$

So it leads to

$$\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \ge \|u\|_\lambda^2 \ge \frac{1}{\Theta} \quad \text{for all } u \in \mathbf{N}_{\lambda,\mu}. \tag{2.8}$$

The Nehari manifold $\mathbf{N}_{\lambda,\mu}$ is closely linked to the behavior of the function of the form $h_u : t \to I_{\lambda,\mu}(tu)$ as

$$h_u(t) = \frac{t^2}{2}\|u\|_\lambda^2 - \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx + \frac{\mu t^p}{p}\int_{\mathbb{R}^3} |u|^pdx \quad \text{for } t > 0.$$

For $u \in X$, we find that

$$h'_u(t) = t\|u\|_\lambda^2 - t^3\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx + \mu t^{p-1}\int_{\mathbb{R}^3} |u|^pdx,$$

$$h''_u(t) = \|u\|_\lambda^2 - 3t^2\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx + \mu(p-1)t^{p-2}\int_{\mathbb{R}^3} |u|^pdx.$$

This implies that for $u \in X\backslash\{0\}$ and $t > 0$, $h'_u(t) = 0$ holds if and only if $tu \in \mathbf{N}_{\lambda,\mu}$ by Lemma 2.1. In particular, $h'_u(1) = 0$ holds if and only if $u \in \mathbf{N}_{\lambda,\mu}$. So, $\mathbf{N}_{\lambda,\mu}$

can be split into three parts corresponding to the local minima, local maxima and points of inflection. According to [18], we define

$$\begin{aligned} \mathbf{N}^{+}_{\lambda,\mu} &= \{u \in \mathbf{N}_{\lambda,\mu} : h''_u(1) > 0\}, \\ \mathbf{N}^{0}_{\lambda,\mu} &= \{u \in \mathbf{N}_{\lambda,\mu} : h''_u(1) = 0\}, \\ \mathbf{N}^{-}_{\lambda,\mu} &= \{u \in \mathbf{N}_{\lambda,\mu} : h''_u(1) < 0\}. \end{aligned}$$

Then using the argument in Brown-Zhang [2, Theorem 2.3], we obtain the following result.

Lemma 2.2. *Suppose that u_0 is a local minimizer for $I_{\lambda,\mu}$ on $\mathbf{N}_{\lambda,\mu}$ and that $u_0 \notin \mathbf{N}^0_{\lambda,\mu}$. Then $I'_{\lambda,\mu}(u_0) = 0$ in X^{-1}.*

For each $u \in \mathbf{N}_{\lambda,\mu}$ it holds

$$\begin{aligned} h''_u(1) &= \|u\|^2_\lambda - 3\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx + \mu(p-1)\int_{\mathbb{R}^3}|u|^p dx \\ &= -2\|u\|^2_\lambda + \mu(p-4)\int_{\mathbb{R}^3}|u|^p dx \\ &= (2-p)\|u\|^2_\lambda + (p-4)\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx. \end{aligned} \tag{2.9}$$

Then we have the following result.

Lemma 2.3. *Suppose that $p > 4, K \in L^\infty(\mathbb{R}^3) \cup L^{2p/(p-4)}(\mathbb{R}^3)$ with $K(x) \geq 0$ and conditions* (A1) *and* (A2) *hold. Then $I_{\lambda,\mu}$ is coercive and bounded below on $\mathbf{N}^-_{\lambda,\mu}$ for all $\lambda \geq \lambda_*$ and $\mu > 0$.*

Proof. By (2.7), (2.8) and (2.9) one has

$$I_{\lambda,\mu}(u) = \frac{p-2}{2p}\|u\|^2_\lambda - \frac{p-4}{4p}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \geq \frac{p-2}{4p}\|u\|^2_\lambda \geq \frac{p-2}{4p\Theta},$$

which implies that $I_{\lambda,\mu}$ is coercive and bounded below on $\mathbf{N}^-_{\lambda,\mu}$ for all $\lambda \geq \lambda_*$. □

Now, we consider the biharmonic equation

$$\begin{gathered} \Delta^2 u - \Delta u = \frac{1}{4\pi}K(x)\Big(\int_\Omega \frac{K(y)u^2(y)}{|x-y|\exp(|x-y|)}dy\Big)u \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.10}$$

where Ω is given in condition (A2) and $K \in L^\infty(\mathbb{R}^3) \cup L^{2p/(p-4)}(\mathbb{R}^3)$ with $K(x) \geq 0$. It is easy to verify that (2.10) admits ground state solution with positive energy by using the standard Nehari manifold method. Let ω be the ground state solution of (2.10) and

$$\alpha = \inf_{u \in \mathbf{M}} J(u) = J(\omega) > 0,$$

where J is the energy functional related with (2.10) in $H^2_0(\Omega)$ given by

$$J(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 + |\nabla u|^2)dx - \frac{1}{4}\int_\Omega K(x)\phi_{K,\omega}\omega^2 dx$$

and $\mathbf{M} = \{u \in H^2_0(\Omega)\backslash\{0\} : \langle J'(u), u\rangle = 0\}$. Then it holds

$$\alpha = \frac{1}{2}\int_\Omega (|\Delta \omega|^2 + |\nabla \omega|^2)dx - \frac{1}{4}\int_\Omega K(x)\phi_{K,\omega}\omega^2 dx$$

$$= \frac{1}{4}\int_{\Omega}(|\Delta\omega|^2+|\nabla\omega|^2)dx.$$

For any $u \in \mathbf{N}_{\lambda,\mu}$ with $I_{\lambda,\mu}(u) < \frac{(p-2)^2}{p(p-4)}\alpha$, we have

$$\begin{aligned}
\frac{(p-2)^2}{p(p-4)}\alpha &> \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx + \frac{\mu}{p}\int_{\mathbb{R}^3}|u|^p dx \\
&= \frac{1}{4}\|u\|_\lambda^2 - \frac{\mu(p-4)}{4p}\int_{\mathbb{R}^3}|u|^p dx \\
&\geq \frac{1}{4}\|u\|_\lambda^2 - \frac{\mu(p-4)}{4pS_\infty^{p-2}}\Big(1+\bar{A}^{16/3}|\{V<b\}|^{4/3}\Big)^{p/2}\|u\|_\lambda^p
\end{aligned}$$

for $\lambda \geq \lambda_*$. This indicates that for each $\lambda \geq \lambda_*$ and $0<\mu<2^{(p-2)/2}\mu_*$, there exist two constants $\underline{D}, \overline{D} > 0$ satisfying

$$2\Big[\frac{\alpha(p-2)^2}{p(p-4)}\Big]^{1/2} < \underline{D} < 2\Big[\frac{\alpha(p-2)}{p-4}\Big]^{1/2} < \overline{D} \tag{2.11}$$

such that

$$\|u\|_\lambda < \underline{D} \quad \text{or} \quad \|u\|_\lambda > \overline{D}.$$

Hence, we obtain

$$\begin{aligned}
\mathbf{N}_{\lambda,\mu}\Big(\frac{(p-2)^2}{p(p-4)}\alpha\Big) :=& \big\{u \in \mathbf{N}_{\lambda,\mu} : I_{\lambda,\mu}(u) < \frac{(p-2)^2}{p(p-4)}\alpha\big\} \\
=& \mathbf{N}_{\lambda,\mu}^{(1)} \cup \mathbf{N}_{\lambda,\mu}^{(2)},
\end{aligned}$$

where

$$\mathbf{N}_{\lambda,\mu}^{(1)} = \big\{u \in \mathbf{N}_{\lambda,\mu}\big(\frac{(p-2)^2}{p(p-4)}\alpha\big) : \|u\|_\lambda < \underline{D}\big\}$$

and

$$\mathbf{N}_{\lambda,\mu}^{(2)} = \big\{u \in \mathbf{N}_{\lambda,\mu}\big(\frac{(p-2)^2}{p(p-4)}\alpha\big) : \|u\|_\lambda > \overline{D}\big\}.$$

This shows that

$$\begin{aligned}
\|u\|_\lambda < \underline{D} < 2\Big[\frac{\alpha(p-2)}{p-4}\Big]^{1/2} \quad &\text{for all } u \in \mathbf{N}_{\lambda,\mu}^{(1)}, \\
\|u\|_\lambda > \overline{D} > 2\Big[\frac{\alpha(p-2)}{p-4}\Big]^{1/2} \quad &\text{for all } u \in \mathbf{N}_{\lambda,\mu}^{(2)}.
\end{aligned}$$

It follows from (2.9) and (2.11) that

$$\begin{aligned}
h_u''(1) &= -2\|u\|_\lambda^2 + \mu(p-4)\int_{\mathbb{R}^3}|u|^p dx \\
&\leq -2\|u\|_\lambda^2 + \mu(p-4)S_\infty^{-(p-2)}(1+\bar{A}^{16/3}|\{V<b\}|^{4/3})^{p/2}\|u\|_\lambda^p \\
&< -2\|u\|_\lambda^2 + 2\Big[\frac{(p-4)}{4(p-2)\alpha}\Big]^{(p-2)/2}\|u\|_\lambda^p < 0 \quad \text{for } u \in \mathbf{N}_{\lambda,\mu}^{(1)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{p-2}{2p}\|u\|_\lambda^2 - \frac{p-4}{4p}\int_{\mathbb{R}^3}K(x)\phi_{K,u}u^2dx &= I_{\lambda,\mu}(u) \\
&< \frac{(p-2)^2}{p(p-4)}\alpha
\end{aligned}$$

$$< \frac{p-2}{4p}\|u\|_\lambda^2 \quad \text{for } u \in \mathbf{N}_{\lambda,\mu}^{(2)},$$

and so

$$h_u''(1) = (2-p)\|u\|_\lambda^2 - (4-p)\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx > 0 \quad \text{for } u \in \mathbf{N}_{\lambda,\mu}^{(2)}.$$

Hence, the following statement is true.

Lemma 2.4. *If $p > 4, \lambda \geq \lambda_*$ and $0 < \mu < 2^{(p-2)/2}\mu_*$, then $\mathbf{N}_{\lambda,\mu}^{(1)} \subset \mathbf{N}_{\lambda,\mu}^-$ and $\mathbf{N}_{\lambda,\mu}^{(2)} \subset \mathbf{N}_{\lambda,\mu}^+$ are C^1 sub-manifolds. Furthermore, each local minimizer of the functional $I_{\lambda,\mu}$ on both $\mathbf{N}_{\lambda,\mu}^{(1)}$ and $\mathbf{N}_{\lambda,\mu}^{(2)}$ is a critical point of $I_{\lambda,\mu}$ in X.*

For $u \in X_\lambda\backslash\{0\}$, we define

$$T(u) = \Big(\frac{\|u\|_\lambda^2}{\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx}\Big)^{1/2}.$$

Lemma 2.5. *Suppose that $p > 4$, $K \in L^\infty(\mathbb{R}^3) \cup L^{2p/(p-4)}(\mathbb{R}^3)$ with $K(x) \geq 0$ and conditions* (A1) *and* (A2) *hold. Then for each $\mu > 0$ and $u \in X_\lambda\backslash\{0\}$ satisfying*

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \\
&> \frac{2(p-2)}{p-4}\big(\frac{\mu(p-4)}{2S_\infty^{p-2}}\big)^{2/(p-2)}(1+\bar{A}^{16/3}|\{V<b\}|^{4/3})^{p/(p-2)}\|u\|_\lambda^4,
\end{aligned}$$

there exists a constant $\hat{t}^{(2)} > (\frac{2(p-2)}{p-4})^{1/2}T(u)$ such that

$$\inf_{t\geq 0} I_{\lambda,\mu}(tu) = \inf_{(\frac{2(p-2)}{p-4})^{1/2}T(u)<t<\hat{t}^{(2)}} I_{\lambda,\mu}(tu) < 0.$$

Proof. For any $u \in X_\lambda\backslash\{0\}$ and $t > 0$, we have

$$I_{\lambda,\mu}(tu) = t^p\Big[\frac{t^{2-p}}{2}\|u\|_\lambda^2 - \frac{t^{4-p}}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx + \frac{\mu}{p}\int_{\mathbb{R}^3}|u|^pdx\Big].$$

Let

$$l(t) = \frac{t^{2-p}}{2}\|u\|_\lambda^2 - \frac{t^{4-p}}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx.$$

Clearly, $I_{\lambda,\mu}(tu) = 0$ if and only if

$$l(t) + \frac{\mu}{p}\int_{\mathbb{R}^3}|u|^pdx = 0.$$

It is easily seen that

$$l(t_0) = 0, \lim_{t\to 0^+} l(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} l(t) = 0,$$

where $t_0 = \sqrt{2}T(u)$. Considering the derivative of $l(t)$, we obtain

$$\begin{aligned}
l'(u) &= -\frac{(q-2)t^{1-q}}{2}\|u\|_\lambda^2 + \frac{(p-4)t^{2p-q-1}}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \\
&= t^{1-q}\Big[\frac{(p-4)t^2}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx - \frac{(q-2)}{2}\|u\|_\lambda^2\Big].
\end{aligned}$$

This indicates that $l(t)$ is decreasing when $0 < t < (\frac{2(p-2)}{p-4})^{1/2}T(u)$ and is increasing when $t > \big(\frac{2(p-2)}{p-4}\big)^{1/2}T(u)$, and hence

$$\inf_{t>0} l(t) = -\frac{1}{p-4}\Big[\frac{2(p-2)\|u\|_\lambda^2}{(p-4)\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx}\Big]^{-(p-2)/2}\|u\|_\lambda^2.$$

For each $u \in X_\lambda\backslash\{0\}$ satisfying

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx\\ &> \frac{2(p-2)}{p-4}\Big(\frac{\mu(p-4)}{2S_\infty^{p-2}}\Big)^{2/(p-2)}\Big(1+\bar{A}^{16/3}|\{V<b\}|^{4/3}\Big)^{p/(p-2)}\|u\|_\lambda^4,\end{aligned}$$

by (2.3) one has

$$\begin{aligned}\inf_{t>0} l(t) &= -\frac{1}{p-4}\Big[\frac{2(p-2)\|u\|_\lambda^2}{(p-4)\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx}\Big]^{-(p-2)/2}\|u\|_\lambda^2\\ &< -\frac{\mu}{pS_\infty^{p-2}}(1+\bar{A}^{16/3}|\{V<b\}|^{4/3})^{p/2}\|u\|_\lambda^p\\ &< -\frac{\mu}{p}\int_{\mathbb{R}^3}|u|^p dx,\end{aligned}$$

which implies that there exist two numbers $\hat{t}^{(i)}$ $(i=1,2)$ satisfying

$$0 < \hat{t}^{(1)} < \Big(\frac{2(p-2)}{p-4}\Big)^{1/2}T(u) < \hat{t}^{(2)}$$

such that

$$I_{\lambda,\mu}(\hat{t}^{(i)}u) = 0 \quad \text{for } i = 1,2.$$

Moreover,

$$I_{\lambda,\mu}\big[\big(\frac{2(p-2)}{p-4}\big)^{1/2}T(u)u\big] < 0,$$

and so $\inf_{t\geq 0} I_{\lambda,\mu}(tu) < 0$. Note that

$$h_u'(t) = pt^{p-1}\Big[l(t) + \frac{\mu}{p}\int_{\mathbb{R}^3}|u|^p dx\Big] + t^p l'(t),$$

leading to

$$h_u'(t) < 0 \quad \text{for all } t \in \big[\hat{t}^{(1)}, big(\frac{2(p-2)}{p-4}\big)^{1/2}T(u)\big] \quad \text{and} \quad h_u'(\hat{t}^{(2)}) > 0.$$

The proof is complete. □

Lemma 2.6. *Suppose that* $p > 4$, $K \in L^\infty(\mathbb{R}^3) \cup L^{2p/(p-4)}(\mathbb{R}^3)$ *with* $K(x) \geq 0$ *and conditions* (A1) *and* (A2) *hold. Then for each* $\mu > 0$ *and* $u \in X_\lambda\backslash\{0\}$ *satisfying*

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx\\ &> \frac{2(p-2)}{p-4}\Big(\frac{\mu(p-4)}{2S_\infty^{p-2}}\Big)^{2/(p-2)}\Big(1+\bar{A}^{16/3}|\{V<b\}|^{4/3}\Big)^{p/(p-2)}\|u\|_\lambda^4,\end{aligned}$$

there are two positive constants $t^+(u)$ *and* $t^-(u)$ *satisfying*

$$T(u) < t^-(u) < \big(\frac{p-2}{p-4}\big)^{1/(2p-2)}T(u) < t^+(u)$$

such that $t^{\pm}(u)u \in \mathbf{N}^{\pm}_{\lambda,\mu}$ *and* $I_{\lambda,\mu}(t^-(u)u) = \sup_{0\le t\le t^+(u)} I_{\lambda,\mu}(tu)$ *and*

$$I_{\lambda,\mu}(t^+(u)u) = \inf_{t\ge t^-(u)} I_{\lambda,\mu}(tu) = \inf_{t\ge 0} I_{\lambda,\mu}(tu) < 0.$$

Proof. Define

$$g(t) = t^{2-p}\|u\|^2_\lambda - t^{4-p}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \quad \text{for} \quad t > 0.$$

Clearly, $tu \in \mathbf{N}_{\lambda,\mu}$ if and only if $g(t)+\mu\int_{\mathbb{R}^3}|u|^pdx = 0$. A straightforward evaluation shows that

$$g(T(u)) = 0, \quad \lim_{t\to 0^+} g(t) = \infty, \quad \lim_{t\to\infty} g(t) = 0.$$

Note that

$$g'(t) = t^{1-p}\Big[-(p-2)\|u\|^2_\lambda + (p-4)t^2\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx\Big].$$

Then we obtain that $g(t)$ is decreasing when $0 < t < (\frac{p-2}{p-4})^{1/2}T(u)$ and is increasing when $t > (\frac{p-2}{p-4})^{1/2}T(u)$, which implies that

$$\inf_{t>0} g(t) = g\Big((\frac{p-2}{p-4})^{1/2}T(u)\Big).$$

For each $u \in X_\lambda\backslash\{0\}$ satisfying

$$\begin{aligned}&\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx\\ &> \frac{2(p-2)}{p-4}\Big(\frac{\mu(p-4)}{2S_\infty^{p-2}}\Big)^{2/(p-2)}(1+\bar{A}^{16/3}|\{V<b\}|^{4/3})^{p/(p-2)}\|u\|^4_\lambda,\end{aligned}$$

it follows from (2.3) that

$$\begin{aligned}g\Big((\frac{p-2}{p-4})^{1/2}T(u)\Big) &= -\frac{2}{p-4}\big[\frac{(p-2)\|u\|^2_\lambda}{(p-4)\int_{\mathbb{R}^3}K(x)\phi_{K,u}u^2dx}\big]^{(2-p)/2}\|u\|^2_\lambda\\ &< -\mu S_\infty^{-(p-2)}\big(1+\bar{A}^{16/3}|\{V<b\}|^{4/3}\big)^{p/2}\|u\|^p_\lambda\\ &\le -\mu\int_{\mathbb{R}^3}|u|^pdx.\end{aligned}$$

Then there exist two constants $t^+(u)$ and $t^-(u)$ such that

$$T(u) < t^-(u) < (\frac{p-2}{p-4})^{1/2}T(u) < t^+(u),$$
$$g(t^{\pm}(u)) + \mu\int_{\mathbb{R}^3}|u|^pdx = 0.$$

Namely, $t^{\pm}(u)u \in \mathbf{N}_{\lambda,\mu}$. By a calculation on the second order derivatives, we find that

$$h''_{t^-(u)u}(1) = (t^-(u))^{p+1}g'(t^-(u)) < 0,$$
$$h''_{t^+(u)u}(1) = (t^+(u))^{p+1}g'(t^+(u)) > 0.$$

These imply that $t^{\pm}(u)u \in \mathbf{N}^{\pm}_{\lambda,\mu}$. It is easily seen that $h'_u(t) > 0$ holds for all $t \in (0,t^-(u)) \cup (t^+(u),\infty)$ and $h'_u(t) < 0$ holds for all $t \in (t^-(u),t^+(u))$, which leads to

$$I_\lambda(t^-(u)u) = \sup_{0\le t\le t^+(u)} I_\lambda(tu) \quad \text{and} \quad I_{\lambda,\mu}(t^+(u)u) = \inf_{t\ge t^-(u)} I_{\lambda,\mu}(tu),$$

and so $I_{\lambda,\mu}(t^+(u)u) < I_{\lambda,\mu}(t^-(u)u)$. By Lemma 2.5, we have

$$I_{\lambda,\mu}(t^+(u)u) = \inf_{t\geq t^-(u)} I_\lambda(tu) = \inf_{t\geq 0} I_\lambda(tu) < 0.$$

This completes the proof. ☐

Since ω is the ground state solution of (2.10) with $J(\omega) = \alpha > 0$, for $0 < \mu < \mu_*$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x)\phi_{K,\omega}\omega^2 dx \\
&= \|\omega\|_\lambda^2 = 4\alpha \\
&> \frac{2(p-2)}{p-4}\Big(\frac{\mu(p-4)}{2S_\infty^{p-2}}\Big)^{2/(p-2)}\Big(1+\bar{A}^{16/3}|\{V<b\}|^{4/3}\Big)^{p/(p-2)}\|\omega\|_\lambda^4.
\end{aligned}$$

Then by Lemma 2.6, there exist two positive numbers $t^-(\omega)$ and $t^+(\omega)$ such that

$$1 < t^-(\omega) < (\frac{p-2}{p-4})^{1/2} < t^+(\omega)$$

and $t^\pm(\omega)\omega \in \mathbf{N}_{\lambda,\mu}^\pm$. Furthermore, we have

$$I_{\lambda,\mu}(t^-(\omega)\omega) = \sup_{0\leq t\leq t^+(\omega)} I_{\lambda,\mu}(t\omega),$$
$$I_{\lambda,\mu}(t^+(\omega)\omega) = \inf_{t\geq t^-(\omega)} I_{\lambda,\mu}(t\omega) = \inf_{t\geq 0} I_{\lambda,\mu}(t\omega) < 0,$$

which implies that $t^+(\omega)\omega \in \mathbf{N}_{\lambda,\mu}^{(2)}$. A direct calculation shows that

$$\begin{aligned}
I_{\lambda,\mu}(t^-(\omega)\omega) &= \frac{p-2}{2p}\|t^-(\omega)\omega\|_\lambda^2 - \frac{p-4}{4p}\int_{\mathbb{R}^3} K(x)\phi_{K,t^-(\omega)\omega}(t^-(\omega)\omega)^2 dx \\
&= \frac{(t^-(\omega))^2}{4p}\left[2(p-2)-(p-4)(t^-(\omega))^2\right]\|\omega\|_\lambda^2 \\
&< \frac{(p-2)^2}{p(p-4)}\alpha.
\end{aligned}$$

This indicates that $t^-(\omega)\omega \in \mathbf{N}_{\lambda,\mu}^{(1)}$.

We define

$$\gamma_{\lambda,\mu}^- = \inf_{u\in \mathbf{N}_{\lambda,\mu}^{(1)}} I_{\lambda,\mu}(u) = \inf_{u\in \mathbf{N}_{\lambda,\mu}^-} I_{\lambda,\mu}(u).$$

It follows from Lemma 2.3 and the property of ω that

$$\frac{p-2}{4p\Theta} < \gamma_{\lambda,\mu}^- < \frac{(p-2)^2}{p(p-4)}\alpha.$$

We define $\Psi : X_\lambda \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx.$$

We now show that the functional Ψ and its derivative Ψ' have Brezis-Lieb splitting property.

Lemma 2.7. *Assume that $K \in L^\infty(\mathbb{R}^3) \cup L^{2p/(p-4)}(\mathbb{R}^3)$ with $K(x) \geq 0$. Let $u_n \rightharpoonup u$ in X_λ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Then as $n \to \infty$, the following statements hold:*

(i) $\Psi(u_n - u) = \Psi(u_n) - \Psi(u) + o(1)$;

(ii) $\Psi'(u_n - u) = \Psi'(u_n) - \Psi'(u) + o(1)$ *in* X_λ^{-1}.

The proof of the above lemma is similar to that of [19, Lemma 4.2], we omit it here.

3. Proof of Theorem 1.1

First we investigate the compactness condition for the functional $I_{\lambda,\mu}$.

Proposition 3.1. *Suppose that* $p > 4$, $K \in L^\infty(\mathbb{R}^3) \cup L^{2p/(p-4)}(\mathbb{R}^3)$ *with* $K(x) \geq 0$ *and conditions* (A1) *and* (A2) *hold. Then there exists* $\Lambda_* > \lambda_*$ *such that if* $\{u_n\} \subset \mathbf{N}^{(1)}_{\lambda,\mu}$ *is a (PS)*$_\beta$*-sequence for* $I_{\lambda,\mu}$ *with* $\beta < \frac{(p-2)^2}{p(p-4)}\alpha$, *then* $\{u_n\}$ *converges strongly in* X *up to subsequence for all* $\lambda > \Lambda_*$.

Proof. Let $\{u_n\} \subset \mathbf{N}^{(1)}_{\lambda,\mu}$ be a (PS)$_\beta$-sequence for $I_{\lambda,\mu}$ with $\beta < \frac{(p-2)^2}{p(p-4)}\alpha$. It is clear that $\{u_n\}$ is bounded in X_λ. Then there exist a subsequence $\{u_n\}$ and u_0 in X_λ such that

$$u_n \rightharpoonup u_0 \text{ weakly in } X_\lambda;$$
$$u_n \to u_0 \text{ strongly in } L^r_{loc}(\mathbb{R}^3) \text{ for } 2 \leq r < \infty;$$
$$u_n(x) \to u_0(x) \text{ a.e. on } \mathbb{R}^3.$$

Moreover, $I'_{\lambda,\mu}(u_0) = 0$ and $\|u_0\|_\lambda \leq \liminf_{n\to\infty} \|u_n\|_\lambda < \underline{D}$. Let $v_n = u_n - u_0$. Then $v_n \rightharpoonup 0$ in X_λ and

$$\|v_n\|_\lambda \leq 2\underline{D} + o(1). \tag{3.1}$$

It follows from condition (A1) that

$$\int_{\mathbb{R}^3} v_n^2 dx \leq \frac{1}{\lambda b}\int_{\mathbb{R}^3} \lambda V(x) v_n^2 dx + \int_{\{V<b\}} v_n^2 dx \leq \frac{1}{\lambda b}\|v_n\|_\lambda^2 + o(1).$$

From this inequality,(2.2) and the Sobolev inequality, for $r > 2$ we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |v_n|^r dx &\leq |v_n|_\infty^{r-2} \int_{\mathbb{R}^3} v_n^2 dx \\
&\leq S_\infty^{-(r-2)} \|v_n\|_{H^2}^{r-2} \cdot \int_{\mathbb{R}^3} v_n^2 dx \\
&\leq \frac{1}{\lambda b} S_\infty^{-(r-2)} (1 + \bar{A}^{16/3} |\{V < b\}|^{4/3})^{(r-2)/2} \|v_n\|_\lambda^r + o(1).
\end{aligned} \tag{3.2}$$

When $K \in L^\infty(\mathbb{R}^3)$, from (2.4) and (3.2) it follows that

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)\phi_{K,v_n} v_n^2 dx &\leq \|K\|_\infty^2 \int_{\mathbb{R}^3} |v_n|^4 dx \\
&\leq \frac{1}{\lambda b}\|K\|_\infty^2 S_\infty^{-2}\Big(1 + \bar{A}^{16/3}|\{V < b\}|^{4/3}\Big)\|v_n\|_\lambda^4 + o(1).
\end{aligned}$$

When $K \in L^{2p/(p-4)}(\mathbb{R}^3)$, by (2.5) and (3.2) one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x)\phi_{K,v_n} v_n^2 dx \\
&\leq \|K\|_{L^{2p/(p-4)}}^2 \Big(\int_{\mathbb{R}^3} |v_n|^p dx\Big)^{4/p} \\
&\leq (\frac{1}{\lambda b})^{4/p} \|K\|_{L^{2p/(p-4)}}^2 S_\infty^{-4(p-2)/p} \big(1 + \bar{A}^{16/3}|\{V < b\}|^{4/3}\big)^{2(p-2)/p} \|v_n\|_\lambda^4 + o(1).
\end{aligned}$$

Let

$$\Pi_\lambda = \begin{cases} \frac{1}{\lambda b}\|K\|_\infty^2 S_\infty^{-2}(1+\bar{A}^{16/3}|\{V<b\}|^{4/3}) \\ \quad \text{if } K\in L^\infty(\mathbb{R}^3), \\ \big(\frac{1}{\lambda b}\big)^{4/p}\|K\|^2_{L^{2p/(p-4)}} S_\infty^{-4(p-2)/p}\big(1+\bar{A}^{16/3}|\{V<b\}|^{4/3}\big)^{2(p-2)/p} \\ \quad \text{if } K\in L^{2p/(p-4)}(\mathbb{R}^3). \end{cases}$$

Clearly, $\Pi_\lambda \to 0$ as $\lambda\to\infty$. Then

$$\int_{\mathbb{R}^3} K(x)\phi_{K,v_n}v_n^2dx \le \Pi_\lambda\|v_n\|_\lambda^4 + o(1). \tag{3.3}$$

Thus, from Lemma 2.7, (3.1) and (3.3) it follows that

$$\begin{aligned} o(1) &= \|v_n\|_\lambda^2 - \int_{\mathbb{R}^3} K(x)\phi_{K,v_n}v_n^2dx + \mu\int_{\mathbb{R}^3}|v_n|^pdx \\ &\ge \|v_n\|_\lambda^2 - \Pi_\lambda\|v_n\|_\lambda^4 + o(1) \\ &\ge \|v_n\|_\lambda^2(1-\Pi_\lambda \underline{D}^2) + o(1), \end{aligned}$$

which implies that there exists $\Lambda_* > \lambda_*$ such that $v_n \to 0$ strongly in X_λ for $\lambda > \Lambda_*$. This completes the proof. □

Now, we are ready to prove Theorem 1.1. By Lemma 2.3 and the Ekeland variational principle [5], there exists a minimizing sequence $\{u_n\}\subset \mathbf{N}^{(1)}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n) = \gamma^-_{\lambda,\mu} + o(1) \quad\text{and}\quad I'_{\lambda,\mu}(u_n) = o(1) \text{ in } X.$$

It follows from Proposition 3.1 and $0<\gamma^-_{\lambda,\mu}<\frac{(p-2)^2}{p(p-4)}\alpha$ that there exist a subsequence $\{u_n\}$ and $u^-_{\lambda,\mu}\in X\backslash\{0\}$ such that $u_n\to u^-_{\lambda,\mu}$ strongly in X_λ for all $\lambda>\Lambda_*$ and $0<\mu<\mu_*$. Thus, $u^-_{\lambda,\mu}$ is a minimizer for $I_{\lambda,\mu}$ on $\mathbf{N}^{(1)}_{\lambda,\mu}$. This indicates that $u^-_{\lambda,\mu}$ is a critical point of $I_{\lambda,\mu}$ by Lemma 2.4. Hence, $(u^-_{\lambda,\mu},\phi_{K,u^-_{\lambda,\mu}})\in H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ is a nontrivial solution of system $(Z_{\lambda,\mu})$.

4. Proof of Theorem 1.2

We define

$$\gamma^+_{\lambda,\mu} = \inf_{u\in\mathbf{N}^{(2)}_{\lambda,\mu}} I_{\lambda,\mu}(u) = \inf_{u\in\mathbf{N}^{+}_{\lambda,\mu}} I_{\lambda,\mu}(u).$$

Lemma 4.1. *Suppose that $p>4, K\in L^{2p/(p-4)}(\mathbb{R}^3)$ with $K(x)\ge 0$ and conditions(A1) and (A2) hold. Then for $\lambda\ge\lambda_*$ and $\mu>0$, the following statements are true:*

- (i) $\mathbf{N}^+_{\lambda,\mu}$ *is a bounded set;*
- (ii) *there exists a positive constant D_0 such that $0>\gamma^+_\lambda>-D_0$.*

Proof. (i) Let $u\in\mathbf{N}^+_{\lambda,\mu}$. By condition (A1) and (2.5), we obtain

$$\begin{aligned} 1 &= \frac{\int_{\mathbb{R}^3}K(x)\phi_{K,u}u^2dx}{\|u\|_\lambda^2+\mu\int_{\mathbb{R}^3}|u|^pdx} \\ &< \frac{(\int_{\mathbb{R}^3}|K|^{2p/(p-4)}dx)^{(p-4)/p}(\int_{\mathbb{R}^3}|u|^pdx)^{4/p}}{\mu\int_{\mathbb{R}^3}|u|^pdx} \end{aligned}$$

$$= \frac{(\int_{\mathbb{R}^3} |K|^{2p/(p-4)}dx)^{(p-4)/p}}{\mu(\int_{\mathbb{R}^3} |u|^p dx)^{(p-4)/p}},$$

which implies that there exists a constant $d_1 > 0$, depending on μ such that

$$\int_{\mathbb{R}^3} |u|^p dx \le d_1 \quad \text{for } u \in \mathbf{N}^+_{\lambda,\mu}. \tag{4.1}$$

Thus, according to (2.9) one has

$$\|u\|^2_\lambda < \frac{\mu(p-4)}{2}\int_{\mathbb{R}^3} |u|^p dx \le \frac{\mu(p-4)}{2} d_1 \quad \text{for } u \in \mathbf{N}^+_{\lambda,\mu}.$$

This indicates that $\mathbf{N}^+_{\lambda,\mu}$ is a bounded set.

(ii) Let $u \in \mathbf{N}^+_{\lambda,\mu}$. From Lemma 2.6, we have $\gamma^+_{\lambda,\mu} < 0$. Using (4.1) gives

$$\begin{aligned} I_{\lambda,\mu}(u) &= \frac{1}{4}\|u\|^2_\lambda - \frac{\mu(p-4)}{4p}\int_{\mathbb{R}^3} |u|^p dx \\ &> -\frac{\mu(p-4)}{4p}\int_{\mathbb{R}^3} |u|^p dx \\ &\ge -\frac{\mu(p-4)}{4p} d_1, \end{aligned}$$

which shows that there exists a constant $D_0 > 0$ such that $\gamma^+_{\lambda,\mu} > -D_0$ for all $\lambda \ge \lambda_*$. This completes the proof. □

Similar to Proposition 3.1, we can establish a compactness result for the functional $J_{\lambda,a}$ in $\mathbf{N}^{(2)}_{\lambda,\mu}$.

Proposition 4.2. *Suppose that $p > 4$, $K \in L^{2p/(p-4)}(\mathbb{R}^3)$ with $K(x) \ge 0$ and conditions* (A1) *and* (A2) *hold. Then there exists a number $\Lambda_{**} \ge \lambda_*$ such that $I_{\lambda,\mu}$ satisfies (PS)$_\beta$-condition in $\mathbf{N}^{(2)}_{\lambda,\mu}$ with $\beta < \frac{(p-2)^2}{p(p-4)}\alpha$ for all $\lambda \ge \Lambda_{**}$ and $0 < \mu < \mu_*$.*

Now, we are ready to proof Theorem 1.2. Similar to the argument of Theorem 1.1, we obtain that $u^-_{\lambda,\mu}$ is a critical point of $I_{\lambda,\mu}$ satisfying $I_{\lambda,\mu}(u^-_{\lambda,\mu}) = \gamma^-_{\lambda,\mu} = \inf_{u\in\mathbf{N}^{(1)}_{\lambda,\mu}} I_{\lambda,\mu}(u) > 0$ for all $\lambda > \Lambda_*$ and $0 < \mu < \mu_*$.

By Lemma 4.1 and the Ekeland variational principle [5], there exists a minimizing sequence $\{u_n\} \subset \mathbf{N}^{(2)}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n) = \gamma^-_{\lambda,\mu} + o(1) \quad \text{and} \quad I'_{\lambda,\mu}(u_n) = o(1) \text{ in } X.$$

From Proposition 4.2 there exist a subsequence $\{u_n\}$ and $u^+_{\lambda,\mu} \in X\backslash\{0\}$ such that $u_n \to u^+_{\lambda,\mu}$ strongly in X_λ for all $\lambda > \Lambda_{**}$. Thus, $u^+_{\lambda,\mu}$ is a minimizer for $I_{\lambda,\mu}$ on $\mathbf{N}^{(2)}_{\lambda,\mu}$. Hence, $u^+_{\lambda,\mu}$ is a critical point of $I_{\lambda,\mu}$ by Lemma 2.4. Note that

$$\gamma^+_{\lambda,\mu} = I_{\lambda,\mu}(u^+_{\lambda,\mu}) \le I_{\lambda,\mu}(t^+\omega) < 0,$$

implying $u^+_{\lambda,\mu} \in \mathbf{N}^{(2)}_{\lambda}$. Therefore, we conclude that for $\lambda > \Lambda := \max\{\Lambda_*, \Lambda_{**}\}$, system $(Z_{\lambda,\mu})$ admits at least two nontrivial solutions $(u^\pm_{\lambda,\mu}, \phi_{K,u^\pm_{\lambda,\mu}}) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ satisfying

$$0 < \|u^-_{\lambda,\mu}\|_\lambda < 2\Big[\frac{\alpha(p-2)}{p-4}\Big]^{1/2} < \|u^+_{\lambda,\mu}\|_\lambda$$

and

$$I_{\lambda,\mu}(u^+_{\lambda,\mu}) < 0 < I_{\lambda,\mu}(u^-_{\lambda,\mu}) < \frac{(p-2)^2}{p(p-4)}\alpha.$$

In particular, $u^+_{\lambda,\mu}$ is a ground state solution.

5. Concentration of solutions

Proof of Theorem 1.3. We follow the arguments in [1, 13]. Choosing a positive sequence $\{\lambda_n\}$ such that $\Lambda < \lambda_1 \le \lambda_2 \le \dots \le \lambda_n \to \infty$ as $n \to \infty$. Let $u_n^\pm := u^\pm_{\lambda_n,\mu}$ be the solutions obtained in Theorem 1.2 with $u^-_{\lambda_n,\mu} \in \mathbf{N}^{(1)}_{\lambda_n,\mu}$ and $u^+_{\lambda_n,\mu} \in \mathbf{N}^{(2)}_{\lambda_n,\mu}$. By Lemma 4.1 (i) and the definition of $\mathbf{N}^{(1)}_{\lambda_n,\mu}$, there exists a constant $M > 0$, independent of λ_n such that $\|u_n^\pm\|_{\lambda_n} \le M$, leading to $\|u_n^\pm\|_{\lambda_1} \le M$. Thus, there exist $u^\pm_\infty \in X$ such that

$$u_n^\pm \rightharpoonup u^\pm_\infty \quad \text{weakly in } X_{\lambda_1};$$
$$u_n^\pm \to u^\pm_\infty \quad \text{strongly in } L^r_{loc}(\mathbb{R}^3) \text{ for } 2 \le r < \infty;$$
$$u_n^\pm(x) \to u^\pm_\infty(x) \quad \text{a.e. on } \mathbb{R}^3.$$

Similar to the proof of Proposition 3.1, we conclude that

$$u_n^\pm \to u^\pm_\infty \quad \text{strongly in } X_{\lambda_1}.$$

This shows that $u_n^\pm \to u^\pm_\infty$ strongly in $H^2(\mathbb{R}^3)$ by (2.2). By Fatou's Lemma, we obtain

$$\int_{\mathbb{R}^3} V(x)(u^\pm_\infty)^2 dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^3} V(x)(u_n^\pm)^2 dx \le \liminf_{n\to\infty} \frac{\|u_n^\pm\|^2_{\lambda_n}}{\lambda_n} = 0,$$

which implies that $u^\pm_\infty = 0$ a.e. in $\mathbb{R}^3\backslash\Omega$. Moreover, fixing $\varphi \in C_0^\infty(\mathbb{R}^3\backslash\overline{\Omega})$, we have

$$\int_{\mathbb{R}^3\backslash\overline{\Omega}} \nabla u^\pm_\infty(x)\varphi(x)dx = -\int_{\mathbb{R}^3\backslash\overline{\Omega}} u^\pm_\infty(x)\nabla\varphi(x)dx = 0,$$

which indicates that $\nabla u^\pm_\infty(x) = 0$ a.e. in $\mathbb{R}^3\backslash\overline{\Omega}$. Since $\partial\Omega$ is smooth, $u^\pm_\infty \in H^2(\mathbb{R}^3\backslash\overline{\Omega})$ and $\nabla u^\pm_\infty \in H^1(\mathbb{R}^3\backslash\overline{\Omega})$, it follows from Trace Theorem that there are constants $\overline{C}, \widetilde{C} > 0$ such that

$$\|u^\pm_\infty\|_{L^2(\partial\Omega)} \le \overline{C}\|u^\pm_\infty\|_{H^2(\mathbb{R}^3\backslash\overline{\Omega})} = 0,$$
$$\|\nabla u^\pm_\infty\|_{L^2(\partial\Omega)} \le \widetilde{C}\|\nabla u^\pm_\infty\|_{H^1(\mathbb{R}^3\backslash\overline{\Omega})} = 0.$$

These show that $u^\pm_\infty \in H_0^2(\Omega)$.

Since $\langle I_{\lambda_n,\mu}(u_n^\pm), \varphi\rangle = 0$ for any $\varphi \in C_0^\infty(\Omega)$, it is easy to verify that

$$\int_\Omega (\Delta u^\pm_\infty \Delta\varphi + \nabla u^\pm_\infty \nabla\varphi)dx + \mu\int_\Omega |u^\pm_\infty|^{p-2}u^\pm_\infty\varphi dx = \int_\Omega K(x)\phi_{K,u^\pm_\infty} u^\pm_\infty \varphi dx.$$

This tells us that $u^\pm_\infty$ are weak solutions of (1.6) by the denseness of $C_0^\infty(\Omega)$ in $H_0^2(\Omega)$. By (2.8) and the facts of $u_n^\pm \to u^\pm_\infty$ strongly in X_{λ_1} and $u^\pm_\infty \in H_0^2(\Omega)$, we have

$$\int_\Omega (|\Delta u^\pm_\infty|^2 + |\nabla u^\pm_\infty|^2)dx \ge \frac{1}{\Theta} > 0.$$

This implies that $u^\pm_\infty \ne 0$. Furthermore,

$$I_{\lambda_n,\mu}(u_n^+) < 0 < \frac{p-2}{4p\Theta} < I_{\lambda_n,\mu}(u_n^-) < \frac{(p-2)^2}{p(p-4)}\alpha.$$

Then $u_\infty^+ \neq u_\infty^-$. This completes the proof. □

Acknowledgments. J. Sun was supported by the National Natural Science Foundation of China (Grant No. 11671236). T. F. Wu was supported by the Ministry of Science and Technology, Taiwan (Grant No. 108-2115-M-390-007-MY2), and by the National Center for Theoretical Sciences, Taiwan.

References

[1] T. Bartsch, A. Pankov, Z. Q. Wang; *Nonlinear Schrödinger equations with steep potential well*, Commun. Contemp. Math., 3 (2001), 549–569.

[2] K. J. Brown, Y. Zhang; *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*, J. Differential Equations, 193 (2003), 481–499.

[3] T. J. Chen , Y. F. Fang, K. H. Wang; *Low regularity global well-posedness for the quantum Zakharov system in 1D*, Taiwanese J. Math., 21 (2017), 341–361.

[4] I. Chueshov; *Quantum Zakharov model in a bounded domain*, Z. Angew. Math. Phys., 64 (2013), 967–989.

[5] I. Ekeland; *On the variational principle*, J. Math. Anal. Appl. 17 (1974) 324–353.

[6] Y. F. Fang, J. Segata, T. F. Wu; *On the standing waves of quantum Zakharov system*, J. Math. Anal. Appl,. 458 (2018), 1427–1448.

[7] Y. F. Fang, H. W. Shih. K. H. Wang; *Local well-posedness for the quantum Zakharov system in one spatial dimension*, J. Hyperbolic Differential Equations, 14 (2017), 157–192.

[8] L. G. Garcia, F. Haas, L. P. L. de Oliveira, J. Goedert; *Modified Zakharov equations for plasmas with a quantum correction*, Phys. Plasmas 12 (2005) 012302.

[9] Y. Guo, J. Zhang, B. Guo; *Global well-posedness and the classical limit of the solution for the quantum Zakharov system*, Z. Angew. Math. Phys., 64 (2013), 53–68.

[10] F. Haas, L. G. Garcia, J. Goedert; *Quantum Zakharov equations*, in Fourth International Winter Conference on Mathematical Methods in Physics, CBPF, Rio de Janeiro (SISSA, WC2004), J. High Energy Phys., (2005) 015.

[11] F. Haas, P. K. Shukla; *Quantum and classical dynamics of Langmuir wave packets*, Phys. Rev., E 79 (2009), 066402.

[12] J. C. Jiang, C. K. Lin, S. Shao; *On one dimensional quantum Zakharov system*, Discrete Contin. Dyn. Syst., 36 (2016), 5445–5475.

[13] J. Sun, T. F. Wu; *Multiplicity and concentration of nontrivial solutions for generalized extensible beam equations in* $\mathbb{R}^N$, Electron. J. Differential Equations, 2019 (2019), No. 41, pp. 1–23.

[14] J. Sun, T. F. Wu; *Steep potential well may help Kirchhoff type equations to generate multiple solutions*, Nonlinear Anal., 190 (2020), 111609.

[15] J. Sun, T. F. Wu, Z. Feng; *Multiplicity of positive solutions for a nonlinear Schrödinger-Poisson system*, J. Differential Equations, 260 (2016), 586–627.

[16] J. Sun, T. F. Wu, Z. Feng; *Non-autonomous Schrödinger-Poisson problem in* $\mathbb{R}^3$, Discrete Contin. Dyn. Syst., 38 (2018) 1889–1933.

[17] J. Sun, S. Yao, T. F. Wu; *The stationary quantum Zakharov system perturbed by a local nonlinearity*, Applied Mathematics Letters, 95 (2019) 172–178.

[18] G. Tarantello; *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 9 (1992), 281–304.

[19] L. Zhao, F. Zhao; *On the existence of solutions for the Schrödinger-Poisson equations*, J. Math. Anal. Appl., 346 (2008), 155–169.

Shuai Yao
School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, China
Email address: shyao2019@163.com

Juntao Sun
School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, China.
School of Mathematical Sciences, Qufu Normal University, Shandong 273165, China
Email address: jtsun@sdut.edu.cn

Tsung-Fang Wu
Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan
Email address: tfwu@nuk.edu.tw