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GROUND STATE SOLUTIONS FOR QUASILINEAR EQUATIONS OF KIRCHHOFF TYPE

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Abstract. This article concerns quasilinear equations of Kirchhoff type. We prove the existence of ground state signed solutions and sign-changing solutions by using the Nehari method.

1. Introduction

In this article, we consider the quasilinear equation of Kirchhoff type

$$
\begin{gathered}
\begin{aligned}
&a\Delta u + \frac{1}{2} bu\Delta u^2 + \int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x(c\Delta u + \frac{1}{2} du\Delta u^2) \\
&+ f(u) = 0, \quad \text{in } \Omega\,,
\end{aligned} \\
u = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, a, b, c, d are positive constants. If we set $b = 0$, $c = 0, d = 0$, then problem (1.1) reduces to the Dirichlet boundary value problem

$$
\begin{gathered}
a\Delta u + f(u) = 0, \quad \text{in } \Omega\,, \\
u = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.2}
$$

which has been extensively studied and significant results have been made in recent decades. If we set $b = 0$ and $d = 0$, then (1.1) turns to the classical Kirchhoff-type equation

$$
\begin{gathered}
\Big(a + c^2 \int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)\Delta u + f(u) = 0, \quad \text{in } \Omega\,, \\
u = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.3}
$$

which is related to the stationary analogue of the equation

$$
\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\,dx\Big)\frac{\partial^2 u}{\partial^2 x} = 0, \tag{1.4}
$$

proposed by Kirchhoff in [22] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings. In the pioneering work of Lions [29],

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an abstract functional analysis framework was proposed to (1.4)

$$
\begin{gathered}
u_{tt} - \Big(a + c^2 \int_\Omega |\nabla u|^2 \,\mathrm{d}x\Big)\Delta u = f(x,u), \quad \text{in } \Omega\,, \\
u = 0, \quad \text{on } \partial\Omega\,.
\end{gathered}
\tag{1.5}
$$

Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Notice that in (1.5), u denotes the displacement, $f(x,u)$ the external force and c^2 the initial tension while a is related to the intrinsic properties of the string, such as Young's modulus. It is pointed out that the problem (1.5) models several physical and biological systems, where u describes a process which depends on the average of itself, for example, population density. For more physical background of this Kirchhoff problem, we refer the reader to the papers [22, 4, 3, 8, 13]. Mathematically, (1.3) is a nonlocal problem as the appearance of the nonlocal term $\int_\Omega |\nabla u|^2\, dx\Delta u$ implies that (1.3) is not a pointwise identity. This causes some mathematical difficulties, for example, by using the variational method to get the solution, the weak limit of the (PS) sequence to the corresponding functional is not trivially to be the weak solution of the equation. In order to overcome this difficult, several methods have been developed, see [11, 20, 25, 37, 42]. Based on these ideas, the existence of positive solutions, multiple solutions, ground states and semiclassical states, sign-changing solutions for the Kirchhoff type problem have been established by the variational method, see for example [1, 16, 28, 24, 35, 41, 36, 21, 39] and the references therein for the bounded domain and [2, 19, 26, 38, 40, 14] and the references therein for the whole space.

When $c = 0$ and $d = 0$, problem (1.1) does not depend on the nonlocal term any more, that is, it becomes to the following special class of equations

$$
\begin{gathered}
a\Delta u + \frac{1}{2} bu\Delta u^2 + f(u) = 0, \quad \text{in } \Omega\,, \\
u = 0, \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.6}
$$

which is refereed as so called Modified Nonlinear Schrödinger Equation (MNLS) and it appears in many models from mathematical physics, see [5, 6, 18, 23, 33] and the references therein. This class of quasilinear problems has been received considerable attention in the past. When we try to consider the problem (1.6) by using classical critical point theory such as mountain pass theorem and symmetric mountain pass theorem, we find that the quasilinear term make it impossible to find a suitable space in which the corresponding functional I possesses both smoothness and compactness properties. There have been several ideas used to overcome the difficulties such as minimizations with constraints [31], Nehari method [33], a change of variables [12, 34]. In [30], we proposed a new approach, namely the perturbation method. Recently, there are some results about the existence of nontrivial solutions and sign-changing solutions of quasilinear equations, see for example [12, 34, 30, 32] and the references therein.

When $a, b, c, d \neq 0$, problem (1.1) is called a Kirchhoff-type perturbation of the quasilinear Schrödinger equation. To the authors' knowledge, there are a few papers on the existence of the ground state and sign-changing solutions for (1.1). For related work, we can refer to [27, 10], the authors considered the generalized quasilinear Schrödinger equation with a Kirchhoff-type perturbation

$$\begin{aligned}&\Big(1+\lambda\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2\,dx\Big)\big(-\operatorname{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2\big)+V(x)u\\&=K(x)f(u),\quad x\in\mathbb{R}^3\end{aligned}\tag{1.7}$$

where $\lambda > 0$, $g \in C^1(\mathbb{R}, \mathbb{R}^+)$, $V(x)$ and $K(x)$ are both positive continuous functions. Under some suitable assumptions on V and K, by using a change of variables the authors obtained the existence of both the ground state and the ground state sign-changing solutions for (1.7). Moreover, the convergence property and the nodal property for solutions were established in [10]. In fact, their method only can be used to treat the same function g, that is, only there is the integral term $\int_\Omega g(u)|\nabla u|^2\,dx$ in the corresponding functional, and they can make a change of variables as $\nabla v = g^{1/2}(u)\nabla u$ (or $(dv = g^{1/2}(u)\nabla u)$, then $\int_\Omega g(u)|\nabla u|^2\,dx = \int_\Omega |\nabla v|^2\,dx$, thus the quasilinear equation is reduced to the semilinear equation. So essentially they discussed semilinear elliptic equations. If there is the integral term $\int_\Omega \sum_{i,j=1}^N a_{ij}(u)\partial_i u\partial_j u\,dx$ or $\int_\Omega g(u)|\nabla u|^2\,dx$ and $\int_\Omega f(u)|\nabla u|^2\,dx$ in the corresponding functional, then the quasilinear equation can not be reduced to the semilinear equation by applying the idea of change of variables.

In this article, we consider the case of $a, b, c, d \neq 0$. Because of the two integral terms $\frac{1}{2}\int_\Omega(a+bu^2)|\nabla u|^2\,\mathrm{d}x$ and $\frac{1}{4}\big(\int_\Omega(c+du^2)|\nabla u|^2\,\mathrm{d}x\big)^2$ appear at the same time, we cannot made a change of variables for problem (1.1) to turn into the semilinear equation. However, in the terms $\frac{1}{2}\int_\Omega g(u)|\nabla u|^2\,dx$ and $\frac{1}{4}\left(\int_\Omega g(u)|\nabla u|^2\,dx\right)^2$, $g(u)$ is the same function. Hence, the problem (1.1) is more general than (1.7). So it is rather difficult to obtain the existence of solutions for the problem (1.1). We will utilize the Nehari method to directly treat the quasilinear equation of Kirchhoff-type (1.1), and obtain the existence of ground state signed solutions and sign-changing solutions and compare the critical values, corresponding to signed solutions and sign-changing solutions.

We assume that the nonlinear function f satisfies the following asumptions

- (A1) $\lim_{t\to 0}\frac{f(t)}{t} = 0$;
- (A2) There exist $c > 0$, $8 < p < 12$ such that $|f(t)| \le c(1 + |t|^{p-1})$ for $t \in \mathbb{R}$;
- (A3) $\lim_{|t|\to+\infty}\frac{f(t)}{t^7} = +\infty$;
- (A4) $\frac{f(t\tau)}{(t\tau)^7} \ge \frac{f(\tau)}{\tau^7}$ for $\tau > 1$, $\tau \neq 0$.

We set

$$X = \big\{u : u \in H_0^1(\Omega), \int_\Omega u^2|\nabla u|^2\,\mathrm{d}x < +\infty\big\}.$$

A function $u \in X$ is called a weak solution of (1.1), if for all $\varphi \in C_0^\infty(\Omega)$ it holds that

$$\begin{aligned}&\int_\Omega\big(a\nabla u\nabla\varphi + \frac{1}{2}b\nabla u^2\nabla(u\varphi)\big)\,\mathrm{d}x\\&+\int_\Omega(c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x\int_\Omega\big(c\nabla u\nabla\varphi+\frac{1}{2}d\nabla u^2\nabla(u\varphi)\big)\,\mathrm{d}x\\&=\int_\Omega f(u)\varphi\,\mathrm{d}x\,.\end{aligned}\tag{1.8}$$

Formally problem (1.1) has a variational structure, defined by the functional

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega (a|\nabla u|^2 + bu^2|\nabla u|^2)\,dx + \frac{1}{4}\Big(\int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x\Big)^2 \\ &\quad - \int_\Omega F(u)\,dx, \quad u \in X, \end{aligned}$$

where $F(t) = \int_0^t f(\tau)\,\mathrm{d}\tau$. Given $u, \varphi \in X$ with the property that $\int_\Omega u^2|\nabla\varphi|^2\,\mathrm{d}x < +\infty$ and $\int_\Omega |\nabla u|^2\varphi^2\,\mathrm{d}x < +\infty$, for example $\varphi \in C_0^\infty(\Omega), \varphi = u, u_+$ or u_-, where $u_+ = \max\{u, 0\}$, $u_- = \min\{u, 0\}$, we can define the derivative of I in the direction φ at u, denoted by $\langle DI(u), \varphi\rangle$ as

$$\begin{aligned} \langle DI(u), \varphi\rangle &= \lim_{t\to 0^+} \frac{1}{t}(I(u + t\varphi) - I(u)) \\ &= \int_\Omega (a\nabla u\nabla\varphi + \frac{1}{2}b\nabla u^2\nabla(u\varphi))\,\mathrm{d}x \\ &\quad + \int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x \int_\Omega (c\nabla u\nabla\varphi + \frac{1}{2}d\nabla u^2\nabla(u\varphi))\,\mathrm{d}x \\ &\quad - \int_\Omega f(u)\varphi\,\mathrm{d}x. \end{aligned}$$

Hence u is a weak solution of (1.1), if and only if the derivative $\langle DI(u), \varphi\rangle$ at u is zero in every direction $\varphi \in C_0^\infty(\Omega)$. If $u \in X$ is a (weak) solution of (1.1), we say that u is a critical point of I and $c = I(u)$ is a critical value of I.

Note that X is not even a convex set. It is difficult to find an appropriate space in which the functional I is smooth as well as has necessary compactness property. In this paper we shall utilize the Nehari method. For $u \in X$, define

$$\begin{aligned} \gamma_+(u) &= \langle DI(u), u_+\rangle \\ &= \int_\Omega (a|\nabla u_+|^2 + 2bu_+^2|\nabla u_+|^2)\,\mathrm{d}x \\ &\quad + \int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x \int_\Omega (c|\nabla u_+|^2 + 2du_+^2|\nabla u_+|^2)\,\mathrm{d}x \\ &\quad - \int_\Omega f(u_+)u_+\,\mathrm{d}x\,, \\ \gamma_-(u) &= \langle DI(u), u_-\rangle \\ &= \int_\Omega (a|\nabla u_-|^2 + 2bu_-^2|\nabla u_-|^2)\,\mathrm{d}x \\ &\quad + \int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x \int_\Omega (c|\nabla u_-|^2 + 2du_-^2|\nabla u_-|^2)\,\mathrm{d}x \\ &\quad - \int_\Omega f(u_-)u_-\,\mathrm{d}x \end{aligned} \tag{1.9}$$

and

$$\begin{gathered} S^* = \{u : u \in X, \gamma_+(u) = 0, u_+ \neq 0;\ \gamma_-(u) = 0, u_- \neq 0\}, \\ c^* = \inf_{u\in S^*} I(u)\,. \end{gathered}$$

Theorem 1.1. *Assume* (A1)–(A4) *hold. Then the functional I attains its infimum c^* on S^* at a function u^*, which is a ground state sign-changing weak solution of* (1.1)*, having exactly two nodal domains.*

We also construct ground state signed solutions of the problem (1.1) and compare the critical values, corresponding to signed solutions and sign-changing solutions. For $u \in X$, we define

$$\begin{aligned}\gamma(u) &= \langle DI(u), u\rangle \\ &= \int_\Omega (a|\nabla u|^2 + 2bu^2|\nabla u|^2)\,\mathrm{d}x \\ &\quad + \int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x \int_\Omega (c|\nabla u|^2 + 2du^2|\nabla u|^2)\,\mathrm{d}x \\ &\quad - \int_\Omega f(u)u\,\mathrm{d}x\end{aligned}$$

and

$$\begin{gathered} S = \{u : u \in X, \gamma(u) = 0, u \neq 0\}, \\ c_0 = \inf_{u\in S} I(u). \end{gathered}$$

Theorem 1.2. *Assume* (A1)–(A4) *hold. Then the functional I attains its infimum c_0 on S at a function u, which is a ground state signed weak solution of* (1.1)*. Moreover $c^* > 2c_0$.*

This article is organized as follows. Section 2 and Section 3 are devoted to the proof of Theorem 1.1 and Theorem 1.2, respectively. In Section 4 we indicate some possible extensions.

2. Ground state sign-changing solutions

In this section we prove Theorem 1.1 through a sequence of lemmas.

Lemma 2.1. *The following identities hold for $u \in X, s \geq 0, t \geq 0$: (1)*

$$\begin{aligned}&\frac{1}{2}\int_\Omega (a|\nabla u|^2 + bu^2|\nabla u|^2)\,\mathrm{d}x - \frac{1}{2}\int_\Omega \Big(as^2|\nabla u_+|^2 + at^2|\nabla u_-|^2 + bs^4u_+^2|\nabla u_+|^2 \\ &\quad + bt^4u_-^2|\nabla u_-|^2\Big)\,\mathrm{d}x \\ &= \frac{1}{8}(1-s^8)\int_\Omega (a|\nabla u_+|^2 + 2bu_+^2|\nabla u_+|^2)\,\mathrm{d}x \\ &\quad + \frac{1}{8}(1-t^8)\int_\Omega (a|\nabla u_-|^2 + 2bu_-^2|\nabla u_-|^2)\,\mathrm{d}x \\ &\quad + \frac{1}{8}a(1-s^2)^2(3+2s^2+s^4)\int_\Omega |\nabla u_+|^2\,\mathrm{d}x \\ &\quad + \frac{1}{8}a(1-t^2)^2(3+2t^2+t^4)\int_\Omega |\nabla u_-|^2\,\mathrm{d}x \\ &\quad + \frac{1}{4}b(1-s^4)^2\int_\Omega u_+^2|\nabla u_+|^2\,\mathrm{d}x + \frac{1}{4}b(1-t^4)^2\int_\Omega u_-^2|\nabla u_-|^2\,\mathrm{d}x.\end{aligned}$$

(2)

$$\frac{1}{4}\Big(\int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x\Big)^2$$

$$
\begin{aligned}
&- \frac{1}{4}\Big(\int_\Omega (cs^2|\nabla u_+|^2 + ct^2|\nabla u_-|^2 + ds^4u_+^2|\nabla u_+|^2 + dt^4u_-^2|\nabla u_-|^2)\,\mathrm{d}x\Big)^2 \\
&= \frac{1}{8}(1-s^8)\int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x \int_\Omega (c|\nabla u_+|^2 + 2du_+^2|\nabla u_+|^2)\,\mathrm{d}x \\
&\quad + \frac{1}{8}(1-t^8)\int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x \int_\Omega (c|\nabla u_-|^2 + 2du_-^2|\nabla u_-|^2)\,\mathrm{d}x \\
&+ \frac{1}{8}c^2(1-s^4)^2\Big(\int_\Omega |\nabla u_+|^2\,\mathrm{d}x\Big)^2 + \frac{1}{8}c^2(1-t^4)^2\Big(\int_\Omega |\nabla u_-|^2\,\mathrm{d}x\Big)^2 \\
&\quad + \frac{1}{8}cd(1-s^2)^2(1+2s^2+3s^4)\int_\Omega |\nabla u_+|^2\,\mathrm{d}x \int_\Omega u_+^2|\nabla u_+|^2\,\mathrm{d}x \\
&\quad + \frac{1}{8}cd(1-t^2)^2(1+2t^2+3t^4)\int_\Omega |\nabla u_-|^2\,\mathrm{d}x \int_\Omega u_-^2|\nabla u_-|^2\,\mathrm{d}x \\
&\quad + \frac{1}{8}c^2\Big((s^4-t^4)^2 + 2(1-s^2t^2)^2\Big)\int_\Omega |\nabla u_+|^2\,\mathrm{d}x \int_\Omega |\nabla u_-|^2\,\mathrm{d}x \\
&\quad + \frac{1}{8}cd\big((1-s^4)^2 + 2(s^2-t^4)^2\big)\int_\Omega |\nabla u_+|^2\,\mathrm{d}x \int_\Omega u_-^2|\nabla u_-|^2\,\mathrm{d}x \\
&\quad + \frac{1}{8}cd\big((1-t^4)^2 + 2(t^2-s^4)^2\big)\int_\Omega |\nabla u_-|^2\,\mathrm{d}x \int_\Omega u_+^2|\nabla u_+|^2\,\mathrm{d}x \\
&\quad + \frac{1}{4}d^2(s^4-t^4)^2\int_\Omega u_+^2|\nabla u_+|^2\,\mathrm{d}x \int_\Omega u_-^2|\nabla u_-|^2\,\mathrm{d}x\,.
\end{aligned}
$$

The proof of the above lemma is tedious but elementary, so we omit it.

Lemma 2.2. *For $u = u_+ + u_- \in X$, $s \ge 0$, and $t \ge 0$, we have the estimate*

$$
\begin{aligned}
&I(u) - I(su_+ + tu_-) \\
&\ge \frac{1}{8}(1-s^8)\langle DI(u), u_+\rangle + \frac{1}{8}(1-t^4)\langle DI(u), u_-\rangle \\
&\quad + \frac{1}{4}(1-s^2)^2\Big(a\int_\Omega |\nabla u_+|^2\,\mathrm{d}x + b\int_\Omega u_+^2|\nabla u_+|^2\,\mathrm{d}x + \frac{1}{2}c^2\Big(\int_\Omega |\nabla u_+|^2\,\mathrm{d}x\Big)^2\Big) \\
&\quad + \frac{1}{4}(1-t^2)^2\Big(a\int_\Omega |\nabla u_-|^2\,\mathrm{d}x + b\int_\Omega u_-^2|\nabla u_-|^2\,\mathrm{d}x + \frac{1}{2}c^2\Big(\int_\Omega |\nabla u_-|^2\,\mathrm{d}x\Big)^2\Big).
\end{aligned}
\tag{2.1}
$$

Consequently, if $u \in S^*$, then

$$
I(u) > I(su_+ + tu_-) \quad \text{for } s \ge 0,\ t \ge 0,\ (s,t) \ne (1,1). \tag{2.2}
$$

Proof. We have

$$
\begin{aligned}
&I(u) - I(su_+ + tu_-) \\
&= \frac{1}{2}\int_\Omega (a|\nabla u|^2 + bu^2|\nabla u|^2)\,\mathrm{d}x \\
&\quad - \frac{1}{2}\Big(\int_\Omega (as^2|\nabla u_+|^2 + at^2|\nabla u_-|^2 + bs^4u_+^2|\nabla u_+|^2 + bt^4u_-^2|\nabla u_-|^2)\,\mathrm{d}x\Big) \\
&\quad + \frac{1}{4}\Big(\int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x\Big)^2 \\
&\quad - \frac{1}{4}\Big(\int_\Omega (cs^2|\nabla u_+|^2 + ct^2|\nabla u_-|^2 + ds^4u_+^2|\nabla u_+|^2 + dt^4u_-^2|\nabla u_-|^2)\,\mathrm{d}x\Big)^2
\end{aligned}
$$

$$-\int_\Omega (F(u_+)-F(su_+))\,\mathrm{d}x-\int_\Omega (F(u_-)-F(tu_-))\,\mathrm{d}x\,.$$

By the assumption (A4), we obtain the estimate

$$\begin{aligned}
\int_\Omega (F(u_+)-F(su_+))\,\mathrm{d}x &= \int_\Omega dx\int_s^1 \frac{\mathrm{d}}{\mathrm{d}\tau}F(\tau u_+)\,\mathrm{d}\tau\\
&= \int_\Omega \mathrm{d}x\int_s^1 f(\tau u_+)u_+\,\mathrm{d}\tau\\
&\geq \int_\Omega \mathrm{d}x\int_s^1 \tau^7 f(u_+)u_+\,\mathrm{d}t\\
&= \frac{1}{8}(1-s^8)\int_\Omega f(u_+)u_+\,\mathrm{d}x.
\end{aligned}\tag{2.3}$$

Similarly,

$$\int_\Omega (F(u_-)-F(su_-))\,\mathrm{d}x \geq \frac{1}{8}(1-t^8)\int_\Omega f(u_-)u_-\,\mathrm{d}x.\tag{2.4}$$

The estimate (2.1) follows from Lemma 2.1, (2.3), (2.4) and the definition (1.9) of $\gamma_+(u)=\langle DI(u),u_+\rangle$ and $\gamma_-(u)=\langle DI(u),u_-\rangle$. □

Lemma 2.3. *Let $u=u_++u_-\in X, u_+\neq 0$, $u_-\neq 0$. Then there exists a unique pair $(s,t)\in\mathbb{R}_+^2$ such that $su_++tu_-\in S^*$.*

Proof. The uniqueness follows from Lemma 2.2 and formula (2.2). To prove the existence of such a pair, we follow Cerami el al [9] by using a degree theory argument. Denote

$$D_{R,r}=\{(s,t)\in\mathbb{R}^2:0<r\leq s\leq R,0<r\leq t\leq R\}.$$

By the assumption (A3), for R large enough we have

$$\begin{gathered}
\langle DI(Ru_++tu_-),Ru_+\rangle<0,\quad r\leq t\leq R;\\
\langle DI(su_++Ru_-),Ru_-\rangle<0,\quad r\leq s\leq R\,.
\end{gathered}$$

By assumption (A1) for r small enough we have

$$\begin{gathered}
\langle DI(ru_++tu_-),ru_+\rangle>0,\quad r\leq t\leq R;\\
\langle DI(su_++ru_-),ru_-\rangle>0,\quad r\leq s\leq R\,.
\end{gathered}$$

By a degree theory argument, we find $(s,t)\in D_{R,r}$ such that $\langle DI(su_++tu_-),tu_-\rangle=0$, and $su_++tu_-\in S^*$. □

Lemma 2.4. *The infimum c^* is attained.*

Proof. There exists $\alpha>0$ such that for $u\in S^*$ it holds that

$$\int_\Omega u_+^p\,\mathrm{d}x\geq\alpha,\quad \int_\Omega u_-^p\,\mathrm{d}x\geq\alpha,\quad \text{for } u\in S^*.\tag{2.5}$$

In fact, by assumptions (A1) and (A2), for $\varepsilon>0$,

$$\begin{aligned}
\varepsilon\int_\Omega u_+^2\,\mathrm{d}x+C_\varepsilon\int_\Omega u_+^p\,\mathrm{d}x &\geq \int_\Omega f(u_+)u_+\,\mathrm{d}x\\
&\geq \int_\Omega a|\nabla u_+|^2\,\mathrm{d}x+2b\int_\Omega u_+^2|\nabla u_+|^2\,\mathrm{d}x\\
&\geq 2\varepsilon\int_\Omega u_+^2\,\mathrm{d}x+c\Big(\int_\Omega u_+^p\,\mathrm{d}x\Big)^{4/p}.
\end{aligned}$$

Hence $\int_\Omega u_+^p \,\mathrm{d}x \geq \alpha$ for some $\alpha > 0$. Similarly $\int_\Omega u_-^p \,\mathrm{d}x \geq \alpha$. For $u \in S^* \subset S$, it holds that

$$
\begin{aligned}
&I(u)\\
&= I(u) - \frac{1}{8}\langle DI(u), u\rangle\\
&= \frac{3}{8}a\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{4}b\int_\Omega u^2|\nabla u|^2 \,\mathrm{d}x\\
&\quad + \frac{1}{8}\int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x \int_\Omega c|\nabla u|^2 \,\mathrm{d}x + \int_\Omega \big(\frac{1}{8}f(u)u - F(u)\big)\,dx\\
&\geq \frac{3}{8}a\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{4}b\int_\Omega u^2|\nabla u|^2 \,\mathrm{d}x.
\end{aligned}
\tag{2.6}
$$

Let $\{u_n\} \subset S^*$ be a minimizing sequence, $I(u_n) \to c^*$ as $n \to \infty$. By (2.6), $\int_\Omega |\nabla u_n|^2 \,\mathrm{d}x \leq c, \int_\Omega u_n^2|\nabla u_n|^2 \,\mathrm{d}x \leq c$. We assume $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $u_n\nabla u_n \rightharpoonup u\nabla u$ in $L^2(\Omega)$, $u_n \to u$ in $L^q(\Omega), 1 \leq q \leq 12$. By (2.5), we have $\int_\Omega u_+^p \,\mathrm{d}x = \lim_{n\to\infty}\int_\Omega (u_n)_+^p \,\mathrm{d}x \geq \alpha > 0$, and $\int_\Omega u_-^p \,\mathrm{d}x \geq \alpha > 0, u_+ \neq 0, u_- \neq 0$. By Lemma 2.3, there exists a pair $(s,t) \in \mathbb{R}_+^2$ such that $su_+ + tu_- \in S^*$. By Lemma 2.2, the formula (2.1) and the lower semicontinuity, we have

$$
\begin{aligned}
c^* &= \lim_{n\to\infty} I(u_n)\\
&\geq \varliminf_{n\to\infty}\{I(s(u_n)_+ + t(u_n)_-) + \frac{1}{4}a(1-s^2)^2\int_\Omega |\nabla (u_n)_+|^2 \,\mathrm{d}x\\
&\quad + \frac{1}{4}a(1-t^2)^2\int_\Omega |\nabla (u_n)_-|^2 \,\mathrm{d}x\}\\
&\geq I(su_+ + tu_-) + \frac{1}{4}a(1-s^2)^2\int_\Omega |\nabla u_+|^2 \,\mathrm{d}x + \frac{1}{4}a(1-t^2)^2\int_\Omega |\nabla u_-|^2 \,\mathrm{d}x\\
&\geq c^* + \frac{1}{4}a(1-s^2)^2\int_\Omega |\nabla u_+|^2 \,\mathrm{d}x + \frac{1}{4}a(1-t^2)^2\int_\Omega |\nabla u_-|^2 \,\mathrm{d}x\,.
\end{aligned}
$$

Hence $s = 1$, $t = 1$, $u_+ + u_- = u^* \in S^*$, and $I(u^*) = c^*$. □

Lemma 2.5. *The minimizer u^* is a ground state sign-changing weak solution of* (1.1).

Proof. We prove that the minimizer $u^* = u_+ + u_-$ solves the equation (1.8). Otherwise there exists $\varphi \in C_0^\infty(\Omega)$ and $m > 0$ such that

$$\langle DI(u^*), \varphi\rangle = -2m < 0.$$

By the continuity, there exist $\delta > 0, \varepsilon_0 > 0$ such that

$$\langle DI(su_+ + tu_- + \varepsilon\varphi, \varphi\rangle \leq -m \quad \text{if } |s-1| \leq \delta, |t-1| \leq \delta, 0 \leq \varepsilon \leq \varepsilon_0. \tag{2.7}$$

By the assumption (A4), $f(t\tau)t\tau \geq sf(\tau)\tau$ for $s \geq 1$, $\tau \neq 0$. For $1-\delta \leq t \leq 1+\delta$, we have

$$
\begin{aligned}
\gamma_+((1+\delta)u_+ + tu_-) &= \langle DI((1+\delta)u_+ + tu_-), (1+\delta)u_+\rangle\\
&\leq \langle DI((1+\delta)u), (1+\delta)u_+\rangle\\
&< (1+\delta)^8\langle DI(u), u_+\rangle = 0.
\end{aligned}
$$

Similarly, for $1-\delta \leq t \leq 1+\delta$,

$$\gamma_+((1-\delta)u_+ + tu_-) = \langle DI((1-\delta)u_+ + tu_-), (1-\delta)u_+\rangle$$

$$\begin{aligned}&\ge \langle DI((1-\delta)u),(1-\delta)u_+\rangle\\&> (1-\delta)^8\langle DI(u),u_+\rangle = 0.\end{aligned}$$

And for $1-\delta\le s\le 1+\delta$,

$$\begin{gathered}\gamma_-(su_+ + (1+\delta)u_-) = \langle DI(su_+ + (1+\delta)u_-),(1+\delta)u_-\rangle < 0,\\ \gamma_+(su_+ + (1-\delta)u_-) = \langle DI(su_+ + (1-\delta)u_-),(1+\delta)u_-\rangle > 0\,.\end{gathered}$$

Take ε sufficiently small so that

$$\gamma_+((1+\delta)u_+ + tu_- + \varepsilon\varphi) < 0,\quad \gamma_+((1-\delta)u_+ + tu_- + \varepsilon\varphi) > 0,$$

for $1-\delta\le t\le 1+\delta$; and

$$\gamma_-(su_+ + (1+\delta)u_- + \varepsilon\varphi) < 0,\quad \gamma_-(su_+ + (1-\delta)u_- + \varepsilon\varphi) > 0,$$

for $1-\delta\le s\le 1+\delta$.

Again by a degree theory argument there exists a pair (s,t) such that $|s-1|\le\delta, |t-1|\le\delta$ and $\gamma_+(su_+ + tu_- + \varepsilon\varphi)=0$, $\gamma_-(su_+ + tu_- + \varepsilon\varphi)=0$; that is, $su_+ + tu_- + \varepsilon\varphi\in S^*$. Now by Lemma 2.4 and (2.7),

$$\begin{aligned}c^* &\le I(su_+ + tu_- + \varepsilon\varphi)\\&\le I(u^*) + I(su_+ + tu_- + \varepsilon\varphi) - I(su_+ + tu_-)\\&= c^* + \int_0^1 \langle DI(su_+ + tu_- + \tau\varepsilon\varphi),\varepsilon\varphi\rangle\,\mathrm{d}\tau\\&\le c^* - \varepsilon m,\end{aligned}$$

which is a contradiction. □

Proof of Theorem 1.1. We only need to prove that the minimizer u^* has exactly two nodal domains. We follow the argument in [39]. If u^* has more than two nodal domains, say, D_1, D_2 positive nodal domains, and D_3 a negative nodal domain. Set $v_+ = u^*\chi_{D_1}$, $v_- = u^*\chi_{D_3}$, $v = v_1 + v_2$, $w = u^*\chi_{D_2}$, $v + w = u^*$, where χ_D denotes the eigenfunction of D, that is if $x\in D$, $\chi_D(x)=1$, or $\chi_D(x)=0$. We have

$$\begin{aligned}c^* &= I(u^*) = I(v+w) - \frac{1}{8}\langle DI(v+w),v+w\rangle\\&= \Big\{I(v)+I(w)+\frac12\int_\Omega (c|\nabla v|^2 + dv^2|\nabla v|^2)\,\mathrm{d}x\int_\Omega (c|\nabla w|^2 + dw^2|\nabla w|^2)\,\mathrm{d}x\Big\}\\&\quad - \frac18\Big\{\langle DI(v),v\rangle + \langle DI(w),w\rangle\\&\quad + \int_\Omega (c|\nabla v|^2 + dv^2|\nabla v|^2)\,\mathrm{d}x\int_\Omega (c|\nabla w|^2 + 2dw^2|\nabla w|^2)\,\mathrm{d}x\\&\quad + \int_\Omega (c|\nabla w|^2 + dv^2|\nabla w|^2)\,\mathrm{d}x\int_\Omega (c|\nabla v|^2 + 2dv^2|\nabla v|^2)\,\mathrm{d}x\Big\}\\&> I(v) - \frac18\langle DI(v),v\rangle.\end{aligned}$$

In the above we have used the fact that $I(w)-\frac18\langle DI(w),w\rangle\ge 0$ (see (2.6)). Notice that $0=\langle DI(u^*),v_+\rangle\ge\langle DI(v),v_+\rangle$ and $0=\langle DI(u^*),v_-\rangle\ge\langle DI(v),v_-\rangle$. Let $s>0$, $t>0$ be such that $sv_+ + tv_-\in S^*$. By Lemma 2.2,

$$c^* > I(v) - \frac18\langle DI(v),v\rangle$$

$$
\begin{aligned}
&\geq I(sv_+ + tv_-) + \frac{1}{8}(1-s^8)\langle DI(v), v_+\rangle + \frac{1}{8}(1-t^8)\langle DI(v), v_+\rangle - \frac{1}{8}\langle DI(v), v\rangle \\
&= I(sv_+ + tv_-) - \frac{1}{8}s^8\langle DI(v), v_+\rangle - \frac{1}{8}t^8\langle DI(v), v_-\rangle \\
&\geq I(sv_+ + tv_-) \geq c^*,
\end{aligned}
$$

which is a contradiction. □

3. Ground state signed solutions

In this section, we prove Theorem 1.2. Since the proof is analogous to, and easier than that of Theorem 1.1, we will omit some details.

Lemma 3.1. *The following identities hold for $u \in X$ and $s \geq 0$: (1)*

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega (a|\nabla u|^2 + bu^2|\nabla u|^2)\,\mathrm{d}x - \frac{1}{2}\int_\Omega (as^2|\nabla u|^2 + bs^4u^2|\nabla u|^2)\,\mathrm{d}x \\
&= \frac{1}{8}a(1-s^8)\int_\Omega (a|\nabla u|^2 + 2b|\nabla u|^2)\,\mathrm{d}x + \frac{1}{8}a(1-s^2)^2(3+2s^2+s^4)\int_\Omega |\nabla u|^2\,\mathrm{d}x \\
&\quad + \frac{1}{4}b(1-s^4)^2\int_\Omega u^2|\nabla u|^2\,\mathrm{d}x
\end{aligned}
$$

and (2)

$$
\begin{aligned}
&\Big(\frac{1}{4}\int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x\Big)^2 - \frac{1}{4}\Big(\int_\Omega (cs^2|\nabla u|^2 + ds^4u^2|\nabla u|^2)\,\mathrm{d}x\Big)^2 \\
&= \frac{1}{8}(1-s^8)\int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x\int_\Omega (c|\nabla u|^2 + 2du^2|\nabla u|^2)\,\mathrm{d}x \\
&\quad + \frac{1}{8}c^2(1-s^4)^2\int_\Omega |\nabla u|^2\,\mathrm{d}x \\
&\quad + \frac{1}{8}cd(1-s^2)(1+2s^2+3s^4)\int_\Omega |\nabla u|^2\,\mathrm{d}x\int_\Omega u^2|\nabla u|^2\,\mathrm{d}x.
\end{aligned}
$$

Lemma 3.2. *For $u \in X, s \geq 0$, we have the estimate*

$$
\begin{aligned}
I(u) - I(su) \geq \frac{1}{8}(1-s^8)\langle DI(u), u\rangle + \frac{1}{4}(1-s^2)^2\Big(a\int_\Omega |\nabla u|^2\,\mathrm{d}x \\
+ b\int_\Omega u^2|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}c^2\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)^2\Big).
\end{aligned}
\tag{3.1}
$$

Consequently, if $u \in S$, then

$$
I(u) > I(su) \quad \text{for } s \geq 0,\ s \neq 1.
$$

Lemma 3.3. *Let $u \in X, u \neq 0$. Then there exists a unique positive number s such that $su \in S$.*

Proof. The uniqueness follows from Lemma 3.2. To prove the existence, just notice that $\langle DI(Ru), Ru\rangle < 0$ for R large enough and $\langle DI(ru), ru\rangle > 0$ for r small enough, hence there exists $s \in (r, R)$ such that $\langle DI(su), su\rangle = 0$, that is, $su \in S$. □

Lemma 3.4. *The infimum c_0 is attained.*

Proof. Let $\{u_n\} \subset S$ be a minimizing sequence, $I(u_n) \to c_0$ as $n \to \infty$. By (2.6),

$$
\int_\Omega |\nabla u_n|^2\mathrm{d}x \leq c, \quad \int_\Omega u_n^2|\nabla u_n|^2\mathrm{d}x \leq c.
$$

Assume $u_n \rightharpoonup u$ in $H_0^1(\Omega), u_n\nabla u_n \rightharpoonup u\nabla u$ in $L^2(\Omega)$, $u_n \to u$ in $L^q(\Omega)$, $1 \le q < 12$. Again there exists $\alpha > 0$ such that $\int_\Omega |u_n|^p \,\mathrm{d}x \ge \alpha > 0$, hence

$$\int_\Omega |u|^p \,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega |u_n|^p \,\mathrm{d}x \ge \alpha > 0, \quad u \ne 0.$$

By Lemma 2.3, there exists a positive number s such that $su \in S$. By Lemma 3.2, formula (3.1),

$$\begin{aligned} c_0 &= \lim_{n\to\infty} I(u_n) \\ &\ge \varliminf_{n\to\infty} \Big\{ I(su_n) + \frac{1}{4}(1-s^2)a \int_\Omega |\nabla u_n|^2 \,\mathrm{d}x \Big\} \\ &\ge I(su) + \frac{1}{4}(1-s^2)^2 a \int_\Omega |\nabla u|^2 \,\mathrm{d}x \\ &\ge c_0 + \frac{1}{4}(1-s^2) a \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \end{aligned}$$

hence $s = 1$ and $u \in S$, $I(u) = c_0$, and u is a minimizer. □

Lemma 3.5. *The minimizer u is a ground state solution of* (1.1).

Proof. We prove that the minimizer u solves equation (1.8). Otherwise there exists $\varphi \in C_0^\infty(\Omega)$ and $m > 0$ such that

$$\langle DI(u), \varphi\rangle = -2m < 0.$$

Choose $\delta > 0$, $\varepsilon_0 > 0$ such that

$$\langle DI(su + \varepsilon\varphi), \varphi\rangle \le -m, \quad \text{if } |s-1| \le \delta,\ 0 \le \varepsilon \le \varepsilon_0. \tag{3.2}$$

We have $\gamma((1+\delta)u) < 0$, $\gamma((1-\delta)u) > 0$. Choose ε so small that $\gamma((1+\delta)u+\varepsilon\varphi) < 0$, $\gamma((1-\delta)u+\varepsilon\varphi) > 0$. Then there exists $s \in (1-\delta, 1+\delta)$ such that $\gamma(su+\varepsilon\varphi) = 0$; that is $su + \varepsilon\varphi \in S$. By (3.2)

$$\begin{aligned} c_0 &\le I(su+\varepsilon\varphi) \\ &\le I(u) + I(su+\varepsilon\varphi) - I(su) \\ &= c_0 + \int_0^1 \langle DI(su+\tau\varepsilon\varphi), \varepsilon\varphi\rangle \,\mathrm{d}\tau \\ &\le c_0 - m\varepsilon, \end{aligned}$$

which is a contradiction. □

Proof of Theorem 1.2. We prove $c^* > 2c_0$. Let $u^* = u_+ + u_- \in S^*$ be a minimizer, $I(u^*) = c^*$. Choose $s > 0, t > 0$ such that $su_+ \in S, tu_- \in S$. Then we have

$$\begin{aligned} c^* &= I(u^*) = I(u_+ + u_-) \\ &\ge I(su_+ + tu_-) \\ &= I(su_+) + I(tu_-) \\ &\quad + \frac{1}{2}\int_\Omega (cs^2|\nabla u_+|^2 + ds^4u_+^2|\nabla u_+|^2)\,\mathrm{d}x \int_\Omega (ct^2|\nabla u_-|^2 + dt^4u_-^2|\nabla u_-|^2)\,\mathrm{d}x \\ &> I(su_+) + I(tu_-) \ge 2c_0. \end{aligned}$$

Finally we prove that the minimizer $u \in S$ is signed. Otherwise $u = u_+ + u_-, u_+ \neq 0, u_- \neq 0$. Since u is a solution of (1.1), $\langle DI(u), u_+\rangle = 0, \langle DI(u), u_-\rangle = 0$; that is, $u \in S^*$. Now we have

$$c_0 = I(u) \geq c^* > 2c_0,$$

which is a contradiction, since we know $c_0 > 0$. □

4. Final remarks

Assumption (A4) can be slightly weakened, with

(A4') There exists $\lambda \in (0, \lambda_1)$ such that

$$\frac{f(t\tau) - a\lambda t\tau}{(t\tau)^7} \geq \frac{f(\tau) - a\lambda\tau}{\tau^7}, \quad \text{for } t \geq 1,\ \tau \neq 0\,,$$

where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian operator $(-\Delta)$ with the zero Dirichlet boundary condition. We rewrite the functional I as

$$\begin{aligned} I(u) &= \frac{1}{2}a\int_\Omega (|\nabla u|^2 - \lambda u^2)\,\mathrm{d}x + \frac{1}{2}b\int_\Omega u^2|\nabla u|^2\,\mathrm{d}x \\ &\quad + \frac{1}{4}\Big(\int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x\Big)^2 - \int_\Omega \tilde{F}(u)\,\mathrm{d}x, \end{aligned}$$

where $\tilde{F}(s) = F(s) - \frac{1}{2}a\lambda s^2$, $\tilde{f}(s) = \frac{\mathrm{d}F(x)}{\mathrm{d}s} = f(s) - a\lambda s$. Since

$$\|u\| = \Big(\int_\Omega (|\nabla u|^2 - \lambda u^2)\,\mathrm{d}x\Big)^{1/2}$$

is an equivalent norm of $H_0^1(\Omega)$, everything we obtain under the assumption (A4) remains true under the new, weakened assumption (A4').

The authors in [39] considered semilinear equations of Kirchhoff type

$$\begin{gathered} (a + b\int_\Omega |\nabla u|^2\,\mathrm{d}x)\Delta u + f(u) = 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

They introduced the following condition (in a less explicit form)

$$\frac{f(t\tau) - a\lambda t\tau}{(t\tau)^3} \geq \frac{f(\tau)}{\tau^3}, \quad \text{for } t \geq 1,\ \tau \neq 0.$$

We can consider the problems on unbounded domains, for example, the whole space $\mathbb{R}^3$,

$$\begin{gathered} a\Delta u - V(x)u + \frac{1}{2}bu\Delta u^2 + \int_\Omega (c|\nabla u|^2 + du^2|\nabla u|^2)\,\mathrm{d}x \cdot (c\Delta u \\ + \frac{1}{2}du\Delta u^2) + f(u) = 0, \quad \text{in } \mathbb{R}^3 \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{gathered}$$

where V is the potential function, for example, $V \equiv$ a positive number or $V = V(x)$ satisfies suitable decay assumptions as $|x| \to \infty$ (see[9]).

A typical example for the nonlinear term f in (1.1) is the monomial $f(s) = |s|^{q-2}s$, $8 < q < 12$. If $f(s) = s^{11}$; that is, $\frac{1}{2}q = 6$ is the critical Sobolev exponent, then by Pohožaev identity, problem (1.1) may have not any nontrivial solutions. Instead, we can consider $f(s) = s^{11} + \lambda|s|^{q-2}s$, where $|s|^{q-2}s$ is a lower term.

Following the idea in [15] (see also [7] for p-Laplacian equations, [17] for quasilinear equations), we can prove the existence of infinitely many solutions.

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