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EXPONENTIAL DECAY AND BLOW-UP FOR NONLINEAR HEAT EQUATIONS WITH VISCOELASTIC TERMS AND ROBIN-DIRICHLET CONDITIONS

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Abstract. In this article, we consider a system of nonlinear heat equations with viscoelastic terms and Robin-Dirichlet conditions. First, we prove existence and uniqueness of a weak solution. Next, we prove a blow up result of weak solutions with negative initial energy. Also, we give a sufficient condition that guarantees the existence and exponential decay of global weak solutions. The main tools are the Faedo-Galerkin method, a Lyapunov functional, and a suitable energy functional.

1. Introduction

In this article, we consider the system of nonlinear heat equations containing viscoelastic terms

$$\begin{aligned}&\frac{\partial u_i}{\partial t}-\frac{\partial}{\partial x}\Big(\mu_i(x,t)\frac{\partial u_i}{\partial x}\Big)+\int_0^t g_i(t-s)\frac{\partial}{\partial x}\Big(\bar{\mu}_i(x,s)\frac{\partial u_i}{\partial x}(x,s)\Big)ds\\&=f_i(u_1,\dots,u_N)+F_i(x,t),\end{aligned}\tag{1.1}$$

where $0<x<1$, $t>0$, $1\le i\le N$, with $N\in\mathbb{N}$ and $N\ge 2$, associated with boundary conditions

$$\begin{gathered}\frac{\partial u_1}{\partial x}(0,t)-h_0u_1(0,t)=u_1(1,t)=0,\\ u_2(0,t)=\frac{\partial u_2}{\partial x}(1,t)+h_1u_2(1,t)=0,\\ u_i(0,t)=u_i(1,t)=0,\ 3\le i\le N,\end{gathered}\tag{1.2}$$

and initial conditions

$$u_i(x,0)=\tilde{u}_i(x),\quad 1\le i\le N,\tag{1.3}$$

where $h_0\ge 0$, $h_1\ge 0$ are real numbers and μ_i, g_i, $\overline{\mu}_i$, f_i, F_i, $\tilde{u}_i$ for all $i\in\overline{1,N}$ are given functions satisfying conditions specified later.

System (1.1) arises naturally within frameworks of mathematical models in engineering and physical sciences, which have been studied by many authors and several results concerning existence, nonexistence, regularity, exponential decay, blow-up in finite time and asymptotic behavior have been established, see [4, 5, 6] and references therein.

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Messaoudi [5] considered an initial boundary value problem related to equation

$$u_t - \Delta u - \int_0^t g(t-s)\Delta u(x,s)ds = |u|^{p-2}u,$$

and proved a blow-up result for certain solutions with positive initial energy, under suitable conditions on g and p. In [6], the authors considered a quasilinear parabolic system of the form

$$A(t)|u_t|^{p-2}u_t - \Delta u - \int_0^t g(t-s)\Delta u(x,s)ds = 0,$$

for $m \geq 2$, $p \geq 2$, $A(t)$ a bounded and positive definite matrix, and g a continuously differentiable decaying function, and proved that, under suitable conditions on g and p, a general decay of the energy function for the global solution and a blow-up result for the solution with both positive and negative initial energy.

Long, Y, and Ngoc [4] considered a nonlinear heat equation with a viscoelastic term

$$u_t - \frac{\partial}{\partial x}(\mu_1(x,t)u_x) + \int_0^t g(t-s)\frac{\partial}{\partial x}(\mu_2(x,s)u_x(x,s))ds = f(u) + F(x,t),$$

where $(x,t) \in (0,1) \times (0,T)$, with Robin boundary conditions

$$u_x(0,t) - h_0 u(0,t) = g_0(t), \quad u_x(1,t) + h_1 u(1,t) = g_1(t),$$

and the initial condition

$$u(x,0) = u_0(x),$$

where $h_0 \geq 0$, $h_1 \geq 0$ are real numbers with $h_0 + h_1 > 0$, and μ_1, g, μ_2, f, F, g_0, g_1, u_0 are given functions, under suitable conditions on μ_1, g, μ_2, f, F, g_0, g_1, u_0, a exponential decay of the energy function for the global solution and a blow-up result for the solution have been established.

Motivated by the above mentioned works, we study the blow-up and exponential decay estimates for problem (1.1)-(1.3). This article is organized as follows. In Section 2, we present some preliminaries and notations. In Section 3, by applying the Faedo-Galerkin method and the weak compact method, we establish the existence of a unique weak solution u of (1.1)-(1.3) on $(0,T)$, for every $T > 0$. In Sections 4 and 5, problem (1.1)-(1.3) is considered with $\overline{\mu}_i(x,t) \equiv \overline{\mu}_i(x)$, for all $i \in \overline{1,N}$. In the case of $F_i \equiv 0$, for all $i \in \overline{1,N}$, when some auxiliary conditions are satisfied, we prove that the weak solution u blows up in finite time. In the case of $\|F_i(t)\|$ small enough, for all $i \in \overline{1,N}$, we verify that if the initial energy is also small enough, then the energy of the solution decays exponentially as $t \to +\infty$. For the proof of the blow up result, we divide it into two steps. First, we show that the weak solution obtained here is not a global solution in $\mathbb{R}_+$. Second, we prove that this solution blows up at finite time T_∞, where $[0,T_\infty)$ is a maximal interval on which the solution of (1.1)-(1.3) exists. For the proof of exponential decay result, a Lyapunov functional is constructed via defining a suitable energy functional. The results obtained here is a relative generalization of [4, 7, 8], by improving and developing these previous works, essentially.

2. Preliminary results and notation

First, we put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$, and denote the usual function spaces used throughout the paper by the notation

$$L^p = L^p(\Omega), \quad W^{k,p} = W^{k,p}(\Omega), \quad H^k = W^{k,2}, \quad \forall k \in \mathbb{Z}_+, \quad 1 \le p \le \infty.$$

We denote the usual norm in L^2 by $\|\cdot\|$ and we denote $\|\cdot\|_X$ for the norm in the Banach space X. We will use the notation $\langle \cdot,\cdot\rangle$ for either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \le p \le \infty$, the Banach space of measurable functions $u : (0,T) \to X$ measurable such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \big(\int_0^T \|u(t)\|_X^p dt\big)^{1/p} < +\infty, & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup} \|u(t)\|_X, & \text{if } p = \infty. \end{cases}$$

On H^1, we use the norm

$$\|v\|_{H^1} = \sqrt{\|v\|^2 + \|v_x\|^2}, \quad \forall v \in H^1.$$

We define

$$\begin{gathered} V_1 = \{v \in H^1 : v(1) = 0\}, \\ V_2 = \{v \in H^1 : v(0) = 0\}, \\ V_i = H_0^1 = \{v \in H^1 : v(0) = v(1) = 0\}, \quad i = \overline{3,N}, \end{gathered}$$

it is clear that $V_1, \dots, V_N$ are closed subspaces of H^1. Moreover, we have the following standard lemmas concerning the imbeddings of H^1 into $C^0(\overline{\Omega})$ and of V_i into $C^0(\overline{\Omega})$, and the equivalence between two norms, $v \mapsto \|v_x\|$, $v \mapsto \|v\|_{H^1}$, on V_i for all $i \in \overline{1,N}$.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact, and*

$$\|v\|_{C^0(\overline{\Omega})} \le \sqrt{2}\|v\|_{H^1}, \quad \forall v \in H^1.$$

Lemma 2.2. *For all $i \in \overline{1,N}$, the imbedding $V_i \hookrightarrow C^0(\overline{\Omega})$ is compact. Moreover, we have*

$$\begin{gathered} \|v\|_{C^0(\overline{\Omega})} \le \|v\|_{V_i}, \quad \forall v \in V_i, \\ \frac{1}{\sqrt{2}}\|v\|_{H^1} \le \|v_x\| \le \|v\|_{H^1} \quad \forall v \in V_i. \end{gathered}$$

Let μ_i, $\overline{\mu}_i \in C^0(\overline{\Omega} \times [0,T])$ with $\mu_i(x,t) \ge \mu_{i*} > 0$ and $\overline{\mu}_i(x,t) \ge \overline{\mu}_{i*} > 0$ for all $(x,t) \in \overline{\Omega} \times [0,T]$ and for all $i \in \overline{1,N}$. We consider the families of symmetric bilinear forms $\{a_i(t;\cdot,\cdot)\}_{t\in[0,T]}$, $\{a_i'(t;\cdot,\cdot)\}_{t\in[0,T]}$, $\{\overline{a}_i(t;\cdot,\cdot)\}_{t\in[0,T]}$ defined by

$$
\begin{aligned}
a_1(t;u,v) &= \langle \mu_1(t)u_x, v_x\rangle + h_0\mu_1(0,t)u(0)v(0),\\
a_1'(t;u,v) &= \langle \mu_1'(t)u_x, v_x\rangle + h_0\mu_1'(0,t)u(0)v(0),\\
\overline{a}_1(t;u,v) &= \langle \overline{\mu}_1(t)u_x, v_x\rangle + h_0\overline{\mu}_1(0,t)u(0)v(0), \quad \forall u,v\in V_1,\ t\in[0,T];\\
a_2(t;u,v) &= \langle \mu_2(t)u_x, v_x\rangle + h_1\mu_2(1,t)u(1)v(1),\\
a_2'(t;u,v) &= \langle \mu_2'(t)u_x, v_x\rangle + h_1\mu_2'(1,t)u(1)v(1),\\
\overline{a}_2(t;u,v) &= \langle \overline{\mu}_2(t)u_x, v_x\rangle + h_1\overline{\mu}_2(1,t)u(1)v(1), \quad \forall u,v\in V_2,\ t\in[0,T];\\
a_i(t;u,v) &= \langle \mu_i(t)u_x, v_x\rangle,\\
a_i'(t;u,v) &= \langle \mu_i'(t)u_x, v_x\rangle,\\
\overline{a}_i(t;u,v) &= \langle \overline{\mu}_i(t)u_x, v_x\rangle, \quad \forall u,v\in V_i,\ t\in[0,T],\ i=\overline{3,N}.
\end{aligned}
\tag{2.1}
$$

Then we have the following lemma, whose proof is straightforward so we omit.

Lemma 2.3. *Let μ_i, $\overline{\mu}_i \in C^0(\overline{\Omega}\times[0,T])$ with $\mu_i(x,t)\geq \mu_{i*}>0$ and $\overline{\mu}_i(x,t)\geq \overline{\mu}_{i*}>0$ for all $(x,t)\in\overline{\Omega}\times[0,T]$, $i\in\overline{1,N}$; and $h_0\geq 0$, $h_1\geq 0$. Then, the families of symmetric bilinear forms $\{a_i(t;\cdot,\cdot)\}_{t\in[0,T]}$, $\{\overline{a}_i(t;\cdot,\cdot)\}_{t\in[0,T]}$ defined by (2.1) are continuous on $V_i\times V_i$ and coercive in V_i for all $i\in\overline{1,N}$.*

Moreover, there exist $a_T>0$, $a_0>0$ such that

$$
|a_i(t;u,v)|\leq a_T\|u_x\|\|v_x\|, \quad \overline{a}_i(t;u,v)|\leq a_T\|u_x\|\|v_x\|,
$$

for all $u,v\in V_i$, $t\in[0,T]$, $i\in\overline{1,N}$; and

$$
a_i(t;v,v)\geq a_0\|v_x\|^2, \quad \overline{a}_i(t;v,v)\geq a_0\|v_x\|^2, \quad \forall v\in V_i,\ t\in[0,T],\ i\in\overline{1,N}.
$$

We also have two important lemmas.

Lemma 2.4. *Let $f\in C^0(\mathbb{R}^N;\mathbb{R})$, if we set*

$$
\Phi_f(r)=\begin{cases}\sup_{|x|_2\leq r}|f(x)|, & \text{if } r>0,\\ |f(0)|, & \text{if } r=0,\end{cases}
$$

then $\Phi_f\in C^0(\mathbb{R}_+;\mathbb{R}_+)$ is nondecreasing and

$$
|f(x)|\leq \Phi_f(|x|_2), \quad \forall x\in\mathbb{R}^N,
$$

where $|x|_2=\sqrt{x_1^2+\cdots+x_N^2}$ for all $x\in\mathbb{R}^N$.

Proof. With $r>0$, we denote

$$
B_r=\{x\in\mathbb{R}^N: |x|_2<r\}, \quad \bar{B}_r=\{x\in\mathbb{R}^N: |x|_2\leq r\}.
$$

Let $g\in C^0(\mathbb{R}^N;\mathbb{R}_+)$, we set

$$
\varphi_g(r)=\begin{cases}\sup_{|x|_2\leq r} g(x), & \text{if } r>0,\\ g(0), & \text{if } r=0.\end{cases}
$$

We claim that $\varphi_g\in C^0(\mathbb{R}_+;\mathbb{R}_+)$. It is clear that $\varphi_g(r)\geq 0$ for all $r\in\mathbb{R}_+$ and φ_g is nondecreasing in $\mathbb{R}_+$.

(i) We prove that φ_g is continuous from right at 0. For all $\varepsilon>0$, by $g\in C^0(\mathbb{R}^N;\mathbb{R}_+)$, there exists $\delta>0$ such that

$$
|g(x)-g(0)|<\varepsilon, \quad \forall x\in\bar{B}_\delta. \tag{2.2}
$$

From (2.2), we have

$$g(x) < g(0) + \varepsilon = \varphi_g(0) + \varepsilon, \quad \forall x \in \bar{B}_\delta. \tag{2.3}$$

By the definition of φ_g and (2.3), it follows that

$$\varphi_g(0) \le \varphi_g(r) \le \varphi_g(\delta) \le \varphi_g(0) + \varepsilon, \quad \forall r \in [0, \delta].$$

Therefore φ_g is continuous from right at 0.

(ii) For all $r_0 > 0$. We will prove that φ_g is continuous at r_0.

(ii-1) We prove that φ_g is continuous from left at r_0. At first, we define a function $\overline{\varphi}_g$, with $\overline{\varphi}_g(r) = \sup_{|x|_2<r} g(x)$ for all $r > 0$. Easily to see that $\overline{\varphi}_g(r) \le \varphi_g(r)$ for all $r > 0$. We prove that $\overline{\varphi}_g(r) \ge \varphi_g(r)$ for all $r > 0$.

Fixed $r > 0$, by the definition of φ_g, we can assume that

$$\varphi_g(r) = \sup_{|x|_2<r} g(x) = \max_{|x|_2<r} g(x) = g(x_0),$$

where $x_0 \in \bar{B}_r$. We define the sequence $\{x_n\}$ by $x_n = (1 - \frac{1}{n})x_0$. We will have $\{x_n\} \subset B_r$ and $x_n \to x_0$. By the definition of $\overline{\varphi}_g$ and continuity of g, we obtain

$$\overline{\varphi}_g(r) \ge \lim_{n\to+\infty} g(x_n) = g(x_0) = \varphi_g(r).$$

It is clear that $\overline{\varphi}_g$ is nondecreasing in $\mathbb{R}_+$. For all $\varepsilon > 0$, by the definition of $\overline{\varphi}_g$, there exists $x_0 \in B_{r_0}$ such that

$$\overline{\varphi}_g(r_0) - \varepsilon < g(x_0) \le \overline{\varphi}_g(r_0). \tag{2.4}$$

Put $\delta = r_0 - |x_0|_2 > 0$, for all $r \in (r_0 - \delta, r_0]$, we have

$$\overline{\varphi}_g(r_0) - \varepsilon < g(x_0) \le \varphi_g(|x_0|_2) = \overline{\varphi}_g(|x_0|_2) \le \overline{\varphi}_g(r) \le \overline{\varphi}_g(r_0). \tag{2.5}$$

From (2.5), it follows that

$$\varphi_g(r_0) - \varepsilon < \varphi_g(r) \le \varphi_g(r_0), \quad \forall r \in (r_0 - \delta, r_0]. \tag{2.6}$$

Therefore φ_g is continuous from left at r_0.

(ii-2) We prove that φ_g is continuous from right at r_0. By $g \in C^0(\mathbb{R}^N; \mathbb{R}_+)$, we have g is uniform continuous on $\bar{B}_{2r_0}$. For all $\varepsilon > 0$, there exists $\delta \in (0, \frac{r_0}{2})$ such that

$$|g(x) - g(y)| < \varepsilon, \quad \forall x, y \in \bar{B}_{2r_0},\ |x - y|_2 < \delta. \tag{2.7}$$

For all $r \in [r_0, r_0 + \delta)$, by the definition of φ_g, there exists $x_r \in \bar{B}_r$, $y_r = \frac{r_0}{r} x_r \in \bar{B}_{r_0}$ such that $\varphi_g(r) = g(x_r)$ and

$$|g(x_r) - g(y_r)| < \varepsilon. \tag{2.8}$$

From (2.8), we have

$$\varphi_g(r_0) \le \varphi_g(r) = g(x_r) < g(y_r) + \varepsilon \le \varphi_g(|y_r|_2) + \varepsilon \le \varphi_g(r_0) + \varepsilon, \tag{2.9}$$

for all $r \in [r_0, r_0 + \delta)$. Therefore φ_g is continuous from right at r_0. Finally, with $f \in C^0(\mathbb{R}^N; \mathbb{R})$, we have

$$\Phi_f(r) = \varphi_{|f|}(r), \quad \forall r \in \mathbb{R}_+.$$

The fact $|f| \in C^0(\mathbb{R}^N; \mathbb{R}_+)$ leads to $\Phi_f \in C^0(\mathbb{R}_+; \mathbb{R}_+)$. For all $x \in \mathbb{R}^N$, we have

$$|f(x)| \le \varphi_{|f|}(|x|_2) = \Phi_f(|x|_2).$$

Obviously, Φ_f is nondecreasing. The proof is complete. □

Lemma 2.4 is a slight improvement of a result used in [7, Appendix 1, pp. 2734], with $N = 1$ and $f \in C^0(\mathbb{R}; \mathbb{R})$.

Lemma 2.5. *Let $x : [0,T] \to \mathbb{R}_+$ be a continuous function satisfying the inequality*

$$x(t) \le M + \int_0^t k(s)\omega(x(s))ds, \quad \forall t \in [0,T],$$

where $M \ge 0$, $k : [0,T] \to \mathbb{R}_+$ is continuous and $\omega : \mathbb{R}_+ \to (0,+\infty)$ is continuous and nondecreasing. Set

$$\Psi(u) = \int_0^u \frac{dy}{\omega(y)}, \quad u \ge 0.$$

(i) *If $\int_0^{+\infty} \frac{dy}{\omega(y)} = +\infty$, then*

$$x(t) \le \Psi^{-1}\Big(\Psi(M) + \int_0^t k(s)ds\Big), \quad \forall t \in [0,T].$$

(ii) *If $\int_0^{+\infty} \frac{dy}{\omega(y)} < +\infty$, then there exists $T_* \in (0,T]$ such that*

$$\int_0^{T_*} k(s)ds \le \int_0^{+\infty} \frac{dy}{\omega(y)},$$

$$x(t) \le \Psi^{-1}\Big(\Psi(M) + \int_0^t k(s)ds\Big), \quad \forall t \in [0,T_*].$$

For a proof of the above lemma, see [1].

3. Existence and uniqueness of a weak solution to (1.1)-(1.3)

Definition 3.1. A weak solution to (1.1)-(1.3) is a function $\vec{u} = (u_1,\dots,u_N)$ belonging to the functional space

$$W(T) = \{\vec{u} \in L^\infty(0,T;V) : \frac{\partial \vec{u}}{\partial t} \in L^2(0,T;H)\}, \tag{3.1}$$

satisfying the variational problem

$$\begin{aligned} &\langle u_i'(t), v_i\rangle + a_i(t;u_i(t),v_i) - \int_0^t g_i(t-s)\overline{a}_i(s;u_i(s),v_i)ds \\ &= \langle f_i(\vec{u}(t)), v_i\rangle + \langle F_i(t), v_i\rangle, \quad \forall v_i \in V_i,\ i \in \overline{1,N}, \end{aligned} \tag{3.2}$$

and the initial condition

$$u_i(0) = \tilde{u}_i, \quad \forall i \in \overline{1,N}, \tag{3.3}$$

where

$$V = V_1 \times \dots \times V_N, \quad H = (L^2)^N. \tag{3.4}$$

We make the following assumptions:

- (A1) $h_0, h_1 \ge 0$;
- (A2) $\tilde{u}_i \in V_i$ for all $i \in \overline{1,N}$;
- (A3) $\mu_i \in C^1(\overline{\Omega} \times [0,T])$ such that $\mu_i(x,t) \ge \mu_{i*} > 0$ for all $(x,t) \in \overline{\Omega} \times [0,T]$, $i \in \overline{1,N}$;
- (A4) $\overline{\mu}_i \in C^0(\overline{\Omega} \times [0,T])$ such that $\overline{\mu}_i(x,t) \ge \overline{\mu}_{i*} > 0$ for all $(x,t) \in \overline{\Omega} \times [0,T]$, $i \in \overline{1,N}$;
- (A5) $f_i \in C^0(\mathbb{R}^N)$ for all $i \in \overline{1,N}$;
- (A6) $g_i \in H^1(0,T)$ for all $i \in \overline{1,N}$;
- (A7) $F_i \in L^2(Q_T)$ for $i = \overline{1,N}$.

Theorem 3.2. *Let $T > 0$ and* (A1)–(A7) *hold.*

(i) *If*

$$\int_0^{+\infty} \frac{dy}{1+y+\sum_{i=1}^N \Phi_{f_i}^2(\sqrt{y})}, = +\infty$$

then (1.1)-(1.3) *has a global weak solution* $\vec{u} \in W(T)$ *satisfying* (3.2)-(3.3).

(ii) *If*

$$\int_0^{+\infty} \frac{dy}{1+y+\sum_{i=1}^N \Phi_{f_i}^2(\sqrt{y})} < +\infty$$

then (1.1)-(1.3) *has a local weak solution* $\vec{u} \in W(T_*)$ *satisfying* (3.2)*)*-(3.3) *with a certain* T_* *small enough.*

In addition if

(A5*) *For all* $M > 0$*, there exists* $L_M > 0$ *such that*

$$|f_i(x) - f_i(y)| \le L_M |x-y|_2, \quad \forall x, y \in \mathbb{R}^N, \; i \in \overline{1,N},$$

then the solution is unique.

Proof. It consists of four steps.

Step 1: Faedo-Galerkin approximation (introduced by Lions [3]). Let $\{w_i^{(j)}\}_{j\in\mathbb{N}}$ be a denumerable base of V_i for $i = \overline{1,N}$. We find an approximate solution of (1.1)-(1.3) in the form

$$u_i^{(m)}(t) = \sum_{j=1}^m c_i^{(mj)}(t) w_i^{(j)}, \quad \forall i \in \overline{1,N}, \tag{3.5}$$

where the coefficient functions $c_i^{(mj)}$, $1 \le j \le m$, $i \in \overline{1,N}$, satisfy the system of ordinary differential equations

$$\begin{aligned} &\langle \dot{u}_i^{(m)}(t), w_i^{(j)}\rangle + a_i(t; u_i^{(m)}(t), w_i^{(j)}) - \int_0^t g_i(t-s)\overline{a}_i(s; u_i^{(m)}(s), w_i^{(j)}) ds \\ &= \langle f_i(\vec{u}^{(m)}(t)), w_i^{(j)}\rangle + \langle F_i(t), w_i^{(j)}\rangle, \quad j = \overline{1,m}, \; i \in \overline{1,N}, \end{aligned} \tag{3.6}$$

and the initial conditions

$$u_i^{(m)}(0) = \tilde{u}_i^{(0m)}, \quad \forall i \in \overline{1,N}, \tag{3.7}$$

with

$$\tilde{u}_i^{(0m)} = \sum_{j=1}^m \alpha_i^{(mj)} w_i^{(j)} \to \tilde{u}_i \quad \text{strongly in } V_i \text{ for } i \in \overline{1,N}. \tag{3.8}$$

By the above assumptions, we can prove the existence of a solution $\vec{u}^{(m)} = (u_1^{(m)}, \dots, u_N^{(m)})$ for the system (3.6)-(3.8) on the interval $[0, T_m]$, for some $T_m \in (0, T]$. The proofs are straightforward, so we omit the details.

Step 2: A priori estimates. Taking $(w_1^{(j)},\dots,w_N^{(j)}) = (\dot{u}_1^{(m)}(t),\dots,\dot{u}_N^{(m)}(t))$ in (3.6), and summing over i from 1 to N, we obtain

$$\begin{aligned} &\sum_{i=1}^N \|\dot{u}_i^{(m)}(t)\|^2 + \sum_{i=1}^N a_i(t;u_i^{(m)}(t),\dot{u}_i^{(m)}(t)) \\ &- \sum_{i=1}^N \int_0^t g_i(t-s)\overline{a}_i(s;u_i^{(m)}(s),\dot{u}_i^{(m)}(t))ds \\ &= \sum_{i=1}^N \langle f_i(\vec{u}^{(m)}(t)),\dot{u}_i^{(m)}(t)\rangle + \sum_{i=1}^N \langle F_i(t),\dot{u}_i^{(m)}(t)\rangle. \end{aligned} \tag{3.9}$$

First, through a direct calculation, we have

$$\begin{aligned} &\frac{d}{dt} a_i(t,u_i^{(m)}(t),u_i^{(m)}(t)) \\ &= 2a_i(t;u_i^{(m)}(t),u_i^{(m)}(t)) + a_i'(t;u_i^{(m)}(t),u_i^{(m)}(t)), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\frac{d}{dt}\int_0^t g_i(t-s)\overline{a}_i(s;u_i^{(m)}(s),u_i^{(m)}(t))ds \\ &= g_i(0)\overline{a}_i(t;u_i^{(m)}(t),u_i^{(m)}(t)) + \int_0^t g_i'(t-s)\overline{a}_i(s;u_i^{(m)}(s),u_i^{(m)}(t))ds \\ &\quad + \int_0^t g_i(t-s)\overline{a}_i(s;u_i^{(m)}(s),\dot{u}_i^{(m)}(t))ds, \end{aligned} \tag{3.11}$$

$\forall i \in \overline{1,N}$.

Hence, (3.9) can be rewritten as

$$\begin{aligned} &2\sum_{i=1}^N \|\dot{u}_i^{(m)}(t)\|^2 + \frac{d}{dt}\sum_{i=1}^N a_i(t;u_i^{(m)}(t),u_i^{(m)}(t)) \\ &= \sum_{i=1}^N [a_i'(t;u_i^{(m)}(t),u_i^{(m)}(t)) + 2\frac{d}{dt}\int_0^t g_i(t-s)\overline{a}_i(s;u_i^{(m)}(s),u_i^{(m)}(t))ds] \\ &\quad - 2\sum_{i=1}^N g_i(0)\overline{a}_i(t;u_i^{(m)}(t),u_i^{(m)}(t)) \\ &\quad - 2\sum_{i=1}^N \int_0^t g_i'(t-s)\overline{a}_i(s;u_i^{(m)}(s),u_i^{(m)}(t))ds \\ &\quad + 2\sum_{i=1}^N \langle f_i(\vec{u}^{(m)}(t)),\dot{u}_i^{(m)}(t)\rangle + 2\sum_{i=1}^N \langle F_i(t),\dot{u}_i^{(m)}(t)\rangle. \end{aligned} \tag{3.12}$$

Next, integrating (3.12), we obtain

$$\begin{aligned} S_m(t) = S_m(0) &+ \sum_{i=1}^N \int_0^t a_i'(s; u_i^{(m)}(s), u_i^{(m)}(s))ds \\ &+ 2\sum_{i=1}^N \int_0^t g_i(t-s)\overline{a}_i(s; u_i^{(m)}(s), u_i^{(m)}(s))ds \\ &- 2\sum_{i=1}^N g_i(0) \int_0^t \overline{a}_i(s; u_i^{(m)}(s), u_i^{(m)}(s))ds \\ &- 2\sum_{i=1}^N \int_0^t ds \int_0^s g_i'(s-\tau)\overline{a}_i(\tau; u_i^{(m)}(\tau), u_i^{(m)}(s))d\tau \\ &+ 2\sum_{i=1}^N \int_0^t \langle f_i(\vec{u}^{(m)}(s)), \dot{u}_i^{(m)}(s)\rangle ds + 2\sum_{i=1}^N \int_0^t \langle F_i(s), \dot{u}_i^{(m)}(s)\rangle ds \\ = S_m(0) &+ \sum_{k=1}^6 J_k, \end{aligned} \tag{3.13}$$

where

$$S_m(t) = \sum_{i=1}^N \Big(2\int_0^t \|\dot{u}_i^{(m)}(s)\|^2 ds + a_i(t; u_i^{(m)}(t), u_i^{(m)}(t))\Big). \tag{3.14}$$

By (A1)–(A7), and using Lemmas 2.3 and 2.4, we estimate the terms on both sides of (3.13) as follows. At first, we note that

$$S_m(t) \ge \sum_{i=1}^N a_i(t; u_i^{(m)}(t), u_i^{(m)}(t)) \ge a_0 \sum_{i=1}^N \|u_{ix}^{(m)}(t)\|^2. \tag{3.15}$$

Now we estimate the terms J_k on the right-hand side of (3.13) as follows. First term, J_1:

$$\begin{aligned} J_1 &= h_0 \int_0^t \mu_1'(0,s)|u_1^{(m)}(0,s)|^2 ds + h_1 \int_0^t \mu_2'(1,s)|u_2^{(m)}(1,s)|^2 ds \\ &\quad + \sum_{i=1}^N \int_0^t \langle \mu_i'(s) u_{ix}^{(m)}(s), u_{ix}^{(m)}(s)\rangle ds \\ &= J_1^{(1)} + J_1^{(2)} + J_1^{(3)}, \end{aligned} \tag{3.16}$$

in which

$$\begin{aligned} J_1^{(1)} &= h_0 \int_0^t \mu_1'(0,s)|u_1^{(m)}(0,s)|^2 ds \\ &\le h_0 \|\mu_1'\|_{C^0(\overline{\Omega}\times[0,T])} \int_0^t \|u_{1x}^{(m)}(s)\|^2 ds \\ &\le \frac{h_0}{a_0} \|\mu_1'\|_{C^0(\overline{\Omega}\times[0,T])} \int_0^t S_m(s) ds. \end{aligned} \tag{3.17}$$

Using the same techniques, with appropriate modifications, leads to

$$J_1^{(2)} \le \frac{h_1}{a_0} \|\mu_2'\|_{C^0(\overline{\Omega}\times[0,T])} \int_0^t S_m(s) ds. \tag{3.18}$$

Using the Cauchy-Schwarz inequality gives

$$
\begin{aligned}
J_1^{(3)} &= \sum_{i=1}^{N} \int_0^t \langle \mu_i'(s) u_{ix}^{(m)}(s), u_{ix}^{(m)}(s) \rangle ds \\
&\leq \max_{1\leq i\leq N} \|\mu_i'\|_{C^0(\overline{\Omega}\times[0,T])} \int_0^t \sum_{i=1}^{N} \|u_{ix}^{(m)}(s)\|^2 ds \\
&\leq \frac{1}{a_0} \max_{1\leq i\leq N} \|\mu_i'\|_{C^0(\overline{\Omega}\times[0,T])} \int_0^t S_m(s) ds.
\end{aligned} \tag{3.19}
$$

From (3.16)–(3.19), we have

$$
J_1 \leq C_1 \int_0^t S_m(s) ds, \tag{3.20}
$$

where

$$
C_1 = \frac{1}{a_0} \Big(h_0 \|\mu_1'\|_{C^0(\overline{\Omega}\times[0,T])} + h_1 \|\mu_2'\|_{C^0(\overline{\Omega}\times[0,T])} + \max_{1\leq i\leq N} \|\mu_i'\|_{C^0(\overline{\Omega}\times[0,T])} \Big). \tag{3.21}
$$

Second term, J_2. By the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
J_2 &= 2\sum_{i=1}^{N} \int_0^t g_i(t-s) \overline{a}_i(s; u_i^{(m)}(s), u_i^{(m)}(t)) ds \\
&\leq 2\sum_{i=1}^{N} \int_0^t |g_i(t-s)| |\overline{a}_i(s; u_i^{(m)}(s), u_i^{(m)}(t))| ds \\
&\leq 2\sum_{i=1}^{N} \|u_{ix}^{(m)}(t)\| a_T \|g_i\|_{L^\infty(0,T)} \int_0^t \|u_{ix}^{(m)}(s)\| ds \\
&\leq \sum_{i=1}^{N} \Big[\frac{1}{6} a_0 \|u_{ix}^{(m)}(t)\|^2 + \frac{a_T^2 \|g_i\|_{L^\infty(0,T)}^2}{6a_0} \Big(\int_0^t \|u_{ix}^{(m)}(s)\| ds \Big)^2 \Big] \\
&\leq \frac{1}{6} S_m(t) + \frac{1}{6a_0^2} T a_T^2 \max_{1\leq i\leq N} \|g_i\|_{L^\infty(0,T)}^2 \int_0^t S_m(s) ds.
\end{aligned} \tag{3.22}
$$

Third term, J_3. It is clear that

$$
\begin{aligned}
J_3 &= -2 \sum_{i=1}^{N} g_i(0) \int_0^t \overline{a}_i(s; u_i^{(m)}(s), u_i^{(m)}(s)) ds \\
&\leq 2a_T \sum_{i=1}^{N} |g_i(0)| \int_0^t \|u_{ix}^{(m)}(s)\|^2 ds \leq \frac{2a_T}{a_0} \max_{1\leq i\leq N} |g_i(0)| \int_0^t S_m(s) ds;
\end{aligned} \tag{3.23}
$$

Fourth term, J_4.

$$
\begin{aligned}
J_4 &= -2\sum_{i=1}^{N} \int_0^t ds \int_0^s g_i'(s-\tau) \overline{a}_i(\tau; u_i^{(m)}(\tau), u_i^{(m)}(s)) d\tau \\
&\leq 2a_T \sum_{i=1}^{N} \int_0^t \|u_{ix}^{(m)}(s)\| ds \int_0^s |g_i'(s-\tau)| \|u_{ix}^{(m)}(\tau)\| d\tau \\
&= 2a_T \sqrt{T} \sum_{i=1}^{N} \|g_i'\|_{L^2(0,T)} \int_0^t \|u_{ix}^{(m)}(s)\|^2 ds
\end{aligned}
$$

$$\le \frac{2a_T}{a_0}\sqrt{T}\max_{1\le i\le N}\|g_i'\|_{L^2(0,T)}\int_0^t S_m(s)ds.$$

Fifth term, J_5. It is known that

$$|\vec{u}^{(m)}(x,t)|_2 = \sqrt{\sum_{i=1}^N \left|u_i^{(m)}(x,t)\right|^2} \le \sqrt{\sum_{i=1}^N \|u_{ix}^{(m)}(t)\|^2} \le \frac{\sqrt{S_m(t)}}{\sqrt{a_0}}.$$

By Lemma 2.4, we have

$$|f_i(\vec{u}^{(m)}(x,t))| \le \Phi_{f_i}\Big(|\vec{u}^{(m)}(x,t)|_2\Big) \le \Phi_{f_i}(\frac{1}{\sqrt{a_0}}\sqrt{S_m(t)}), \quad \forall i\in\overline{1,N},$$

so

$$\|f_i(\vec{u}^{(m)}(t))\| \le \Phi_{f_i}\Big(\frac{1}{\sqrt{a_0}}\sqrt{S_m(t)}\Big), \quad \forall i\in\overline{1,N};$$

therefore,

$$\begin{aligned} J_5 &= 2\sum_{i=1}^N\int_0^t \langle f_i(\vec{u}^{(m)}(s)), \dot{u}_i^{(m)}(s)\rangle ds \\ &\le \sum_{i=1}^N\int_0^t [3\|f_i(\vec{u}^{(m)}(s))\|^2 + \frac{1}{3}\|\dot{u}_i^{(m)}(s)\|^2]ds \\ &\le 3\sum_{i=1}^N\int_0^t \Phi_{f_i}^2(\frac{1}{\sqrt{a_0}}\sqrt{S_m(s)})ds + \frac{1}{3}\sum_{i=1}^N\int_0^t \|\dot{u}_i^{(m)}(s)\|^2 ds \\ &\le \frac{1}{6}S_m(t) + 3\sum_{i=1}^N\int_0^t \Phi_{f_i}^2(\frac{1}{\sqrt{a_0}}\sqrt{S_m(s)})ds. \end{aligned} \tag{3.24}$$

Sixth term, J_6. We have

$$J_6 = 2\sum_{i=1}^N\int_0^t \langle F_i(s), \dot{u}_i^{(m)}(s)\rangle ds \le \frac{1}{6}S_m(t) + 3\sum_{i=1}^N \|F_i\|^2_{L^2(Q_T)}. \tag{3.25}$$

Now we estimate the term $S_m(0)$. From the convergence in (3.8), we can deduce the existence of a constant $C_0>0$ such that

$$S_m(0) = \sum_{i=1}^N a_i\big(0;\tilde{u}_i^{(0m)},\tilde{u}_i^{(0m)}\big) \le C_0, \quad \forall m\in\mathbb{N}. \tag{3.26}$$

From (3.13), (3.20), (3.22)-(3.26), there exist $M_T>0$, $N_T>0$ constants independent of m such that

$$S_m(t) \le M_T + N_T\int_0^t \omega(S_m(s))ds,\ \forall t\in[0,T], \tag{3.27}$$

with

$$\omega(S) = 1 + S + \sum_{i=1}^N \Phi_{f_i}^2\Big(\frac{1}{\sqrt{a_0}}\sqrt{S}\Big). \tag{3.28}$$

By the same convergence of $\int_0^{+\infty}\frac{dy}{\omega(y)}$ and $\int_0^{+\infty}\frac{dy}{1+y+\sum_{i=1}^N\Phi_{f_i}^2(\sqrt{y})}$, apply Lemma 2.5 with $x(t)\equiv S_m(t)$, $M=M_T$, $k(s)\equiv N_T$, $\omega(S) = 1+S+\sum_{i=1}^N\Phi_{f_i}^2(\frac{1}{\sqrt{a_0}}\sqrt{S})$, we obtain the estimate of $S_m(t)$ in two cases as follows.

Case 1. If

$$\int_0^{+\infty} \frac{dy}{1+y+\sum_{i=1}^N \Phi_{f_i}^2(\sqrt{y})} = +\infty$$

then

$$\begin{aligned} S_m(t) &\le \Psi^{-1}(\Psi(M_T)+N_T t) \\ &\le \Psi^{-1}(\Psi(M_T)+N_T T) \equiv C_T, \quad \forall t \in [0,T],\ m \in \mathbb{N}. \end{aligned} \tag{3.29}$$

Case 2. If

$$\int_0^{+\infty} \frac{dy}{1+y+\sum_{i=1}^N \Phi_{f_i}^2(\sqrt{y})} < +\infty$$

then

$$\begin{aligned} S_m(t) &\le \Psi^{-1}(\Psi(M_T)+N_T t) \\ &\le \Psi^{-1}(\Psi(M_T)+N_T T) \equiv C_T,\ \forall t \in [0,T_*],\ m \in \mathbb{N}, \end{aligned} \tag{3.30}$$

where $T_* \in (0,T]$ chosen such that $T_* N_T \le \int_0^{+\infty} \frac{dy}{\omega(y)}$.

This allows one to take the constant $T_m = T$ or $T_m = T_*$ for all $m \in \mathbb{N}$. In what follows, we will write T_* for both T and T_*.

Step 3: Limiting process. It follows from (3.14), (3.15) and (3.29) (or (3.30)), that

$$\|u_i^{(m)}\|_{L^\infty(0,T_*;V_i)} \le \sqrt{\frac{C_T}{a_0}}, \quad \|\dot{u}_i^{(m)}\|_{L^2(Q_T)} \le \sqrt{C_T}, \quad \forall m \in \mathbb{N},\ \forall i \in \overline{1,N}. \tag{3.31}$$

Applying the Banach-Alaoglu theorem and Kakuntani theorem, the above uniform bounds with respect to m imply that one can extract a subsequence (which we relabel with the index m if necessary) such that

$$\vec{u}^{(m)} \to \vec{u} \quad \text{weak* in } L^\infty(0,T_*;V), \tag{3.32}$$

$$\frac{\partial \vec{u}^{(m)}}{\partial t} \to \frac{\partial \vec{u}}{\partial t} \quad \text{weakly in } L^2(0,T_*;H). \tag{3.33}$$

By Aubin-Lions compactness theorem and Riesz-Fisher theorem, it is straightforward to go on extracting, from weak convergence results (3.32) and (3.33), a subsequence (which we relabel with the index m if necessary) such that

$$\begin{aligned} &\vec{u}^{(m)} \to \vec{u} \quad \text{strongly in } L^2(0,T_*;H), \\ &\vec{u}^{(m)}(x,t) \to \vec{u}(x,t) \quad \text{a.e. } (x,t) \in Q_{T_*}. \end{aligned} \tag{3.34}$$

It remains to show the convergence of the nonlinear terms. Using the continuity argument of f_i for all $i \in \overline{1,N}$ and (3.34), one deduces that

$$f_i(\vec{u}^{(m)}(x,t)) \to f_i(\vec{u}(x,t)) \quad \text{a.e. } (x,t) \in Q_{T_*},\ \forall i \in \overline{1,N}. \tag{3.35}$$

On the other hand,

$$\|f_i(\vec{u}^{(m)})\|_{L^2(Q_{T_*})} \le \sqrt{T} \sup_{|z| \le \sqrt{\frac{C_T}{a_0}}} |f_i(z)|, \quad \forall i \in \overline{1,N}.$$

From [3, Lemma 1.3] we obtain

$$f_i(\vec{u}^{(m)}) \to f_i(\vec{u}) \quad \text{weakly in } L^2(Q_{T_*}),\ \forall i \in \overline{1,N}. \tag{3.36}$$

Combining (3.32), (3.33), (3.36) and (3.8), it is enough to pass to the limit in (3.6) and (3.7) to show that $\vec{u}$ satisfies (3.2) and (3.3). In addition, from (3.32) and (3.33), we have $\vec{u} \in W(T_*)$ and the proof of the existence of a weak solution is complete.

Step 4: Uniqueness of the solution. Suppose $\vec{u}^{(1)}$ and $\vec{u}^{(2)}$ are two solutions of (1.1)-(1.3) on the interval $[0, T_*]$ such that

$$\vec{u}^{(i)} \in W(T_*), \quad i = 1, 2. \tag{3.37}$$

Then $\vec{u} = \vec{u}^{(1)} - \vec{u}^{(2)} = (u_1, \dots, u_N) \in W(T_*)$ satisfies

$$\begin{aligned} &\langle u_i'(t), v_i\rangle + a_i(t; u_i(t), v_i) - \int_0^t g_i(t-s)\overline{a}_i(s; u_i(s), v_i)ds \\ &= \langle f_i(\vec{u}^{(1)}(t)) - f_i(\vec{u}^{(2)}(t)), v_i\rangle, \quad \forall v \in V_i,\ i \in \overline{1,N}, \end{aligned} \tag{3.38}$$

$$u_i(0) = 0, \quad \forall i \in \overline{1,N}. \tag{3.39}$$

Taking $v_i = 2u_i(t)$ in (3.38) and integrating with respect to t, and summing over i from 1 to N, we obtain

$$\begin{aligned} &\sum_{i=1}^N \|u_i(t)\|^2 + 2\int_0^t \sum_{i=1}^N a_i(s; u_i(s), u_i(s))ds \\ &= 2\sum_{i=1}^N \int_0^t ds \int_0^s \overline{a}_i(\tau; u_i(\tau), u_i(s))d\tau \\ &\quad + 2\sum_{i=1}^N \int_0^t \langle f_i(\vec{u}^{(1)}(s)) - f_i(\vec{u}^{(2)}(s)), u_i(s)\rangle ds. \end{aligned} \tag{3.40}$$

Set $\varrho(t) = \sum_{i=1}^N \big(\|u_i(t)\|^2 + \int_0^t \|u_{ix}(s)\|^2 ds\big)$. As in Step 2, we can estimate all terms on the right hand side of (3.40) to obtain

$$\varrho(t) \le D_T \int_0^t \varrho(s)ds, \quad \forall t \in [0, T_*], \tag{3.41}$$

where $D_T > 0$. Applying Gronwall's lemma, (3.41) leads to $\varrho(t) \equiv 0$; i.e., $\vec{u}^{(1)} = \vec{u}^{(2)}$. Theorem 3.2 is proved. □

Lemma 2.4 is a powerful and efficient tool for estimate the nonlinear terms. By Lemma 2.4, we can relax assumptions for $f_i \in C^0(\mathbb{R}^N)$ for all $i \in \overline{1,N}$, that is, f_i can be bounded by the polynomial of $|\vec{u}|_2$ for all $i \in \overline{1,N}$ or not. It is an improvement of the assumptions in [8], here the authors had to suppose that f is bounded by the polynomial of $|u|$ for the initial boundary problem for a nonlinear heat equation $u_t - \frac{\partial}{\partial x}(\mu(x,t)u_x) + f(u) = f_1(x,t)$, $0 < x < 1$, $0 < t < T$, associated with Robin boundary conditions.

4. Blow-up of solutions

In this section we study the blow up in finite time of the solution of (1.1)-(1.3) corresponding to $\overline{\mu}_i(x,t) \equiv \overline{\mu}_i(x)$ and $F_i(x,t) \equiv 0$ for all $i \in \overline{1,N}$,

$$\begin{aligned} &\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial x}\Big(\mu_i(x,t)\frac{\partial u_i}{\partial x}\Big) + \int_0^t g_i(t-s)\frac{\partial}{\partial x}\Big(\overline{\mu}_i(x)\frac{\partial u_i}{\partial x}(x,s)\Big)ds \\ &= f_i(u_1, \dots, u_N), \quad (x,t) \in Q_T,\ \forall i \in \overline{1,N}, \end{aligned} \tag{4.1}$$

with boundary conditions

$$\begin{gathered}\frac{\partial u_1}{\partial x}(0,t)-h_0u_1(0,t)=u_1(1,t)=0,\\ u_2(0,t)=\frac{\partial u_2}{\partial x}(1,t)+h_1u_2(1,t)=0,\\ u_i(0,t)=u_i(1,t)=0,\quad 3\le i\le N,\end{gathered}\tag{4.2}$$

and initial conditions

$$u_i(x,0)=\tilde{u}_i(x),\quad \forall i\in\overline{1,N}.\tag{4.3}$$

We make the following assumptions:

- (A3') $\mu_i\in C^1(\overline{\Omega}\times\mathbb{R}_+)$ such that $\mu_i(x,t)\ge\mu_{i*}>0$, $\frac{\partial\mu_i}{\partial t}(x,t)\le 0$ for all $(x,t)\in\overline{\Omega}\times\mathbb{R}_+$, $i\in\overline{1,N}$;
- (A4') $\overline{\mu}_i\in C^0(\overline{\Omega})$ such that $\overline{\mu}_i(x)\ge\overline{\mu}_{i*}>0$ for all $x\in\overline{\Omega}$, $i=\overline{1,N}$;
- (A5') $f_i\in C^0(\mathbb{R}^N)$ for all $i\in\overline{1,N}$. Furthermore, there exists $\mathcal{F}\in C^1(\mathbb{R}^N)$ such that
  - (i) $\frac{\partial\mathcal{F}}{\partial u_i}=f_i$ for all $i\in\overline{1,N}$,
  - (i) There exists constant $d_1>2$ such that $d_1\mathcal{F}(\vec{u})\le\sum_{i=1}^N u_if_i(\vec{u})$, for all $\vec{u}=(u_1,\dots,u_N)\in\mathbb{R}^N$,
  - (iii) There exist constants $\overline{d}_1>0$, $p_i>2$ for all $i\in\overline{1,N}$, such that $\mathcal{F}(\vec{u})\ge\overline{d}_1\sum_{i=1}^N|u_i|^{p_i}$, for all $\vec{u}=(u_1,\dots,u_N)\in\mathbb{R}^N$;
- (A6') $g_i\in C^1(\mathbb{R}_+;\mathbb{R}_+)\cap L^1(\mathbb{R}_+)$ such that $0<g_i(t)\le g_i(0)$ and $g_i'(t)\le 0$ for all $t\ge 0$, $i\in\overline{1,N}$.

Example 4.1. For $\vec{u}=(u_1,\dots,u_N)\in\mathbb{R}^N$, we define a function that satisfies (A5').

$$\mathcal{F}(\vec{u})=\mathcal{F}(u_1,\dots,u_N)=\sum_{i=1}^N\alpha_i|u_i|^{p_i}+\beta|u_1|^{q_1}\dots|u_N|^{q_N}\ln^k(e+|\vec{u}|_2^2),$$

where $\beta>0$, $k>1$ and $\alpha_i>0$, $p_i>2$, $q_i>2$ for all $i\in\overline{1,N}$ are constants. By direct calculations, we have

$$\begin{aligned}f_i(\vec{u})&=\frac{\partial\mathcal{F}}{\partial u_i}(\vec{u})\\&=p_i\alpha_i|u_i|^{p_i-2}u_i+\beta q_i|u_1|^{q_1}\dots|u_N|^{q_N}u_i^{-1}\ln^k(e+|\vec{u}|_2^2)\\&\quad+2k\beta|u_1|^{q_1}\dots|u_N|^{q_N}\frac{u_i}{e+|\vec{u}|_2^2}\ln^{k-1}(e+|\vec{u}|_2^2),\quad\forall i\in\overline{1,N}.\end{aligned}$$

It is obvious that (A5') holds, since

$$\begin{aligned}\sum_{i=1}^N u_if_i(\vec{u})&=\sum_{i=1}^N p_i\alpha_i|u_i|^{p_i}+\beta(\sum_{i=1}^N q_i)|u_1|^{q_1}\dots|u_N|^{q_N}\ln^k(e+|\vec{u}|_2^2)\\&\quad+2k\beta|u_1|^{q_1}\dots|u_N|^{q_N}\frac{|\vec{u}|_2^2}{e+|\vec{u}|_2^2}\ln^{k-1}(e+|\vec{u}|_2^2)\\&\ge\sum_{i=1}^N p_i\alpha_i|u_i|^{p_i}+\beta(\sum_{i=1}^N q_i)|u_1|^{q_1}\dots|u_N|^{q_N}\ln^k(e+|\vec{u}|_2^2)\\&\ge d_1\mathcal{F}(\vec{u}),\end{aligned}$$

with $d_1 = \min\{p_1, \dots, p_N, \sum_{i=1}^N q_i\}$ and

$$\mathcal{F}(\vec{u}) \geq \sum_{i=1}^N \alpha_i |u_i|^{p_i} \geq \overline{d}_1 \sum_{i=1}^N |u_i|^{p_i}, \quad \overline{d}_1 = \min_{1\leq i\leq N} \alpha_i.$$

Now, on $V_i \times V_i$, we consider the following symmetric bilinear forms:

$$\begin{gathered}
A_1(u,v) = \int_0^1 u_x(x)v_x(x)dx + h_0 u(0)v(0),\\
\overline{a}_1(u,v) = \int_0^1 \overline{\mu}_1(x)u_x(x)v_x(x)dx + h_0\overline{\mu}_1(0)u(0)v(0),\ \forall u,v\in V_1;\\
A_2(u,v) = \int_0^1 u_x(x)v_x(x) + h_1 u(1)v(1),\\
\overline{a}_2(u,v) = \int_0^1 \overline{\mu}_2(x)u_x(x)v_x(x)dx + h_1\overline{\mu}_2(1)u(1)v(1),\ \forall u,v\in V_2;\\
A_i(u,v) = \int_0^1 u_x(v)v_x(x)dx,\\
\overline{a}_i(u,v) = \int_0^1 \overline{\mu}_i(x)u_x(x)v_x(x)dx,\ \forall u,v\in V_i,\ i=\overline{3,N}.
\end{gathered}$$

It is easy to show that the forms $A_i(\cdot,\cdot)$, $\overline{a}_i(\cdot,\cdot)$ are continuous on $V_i \times V_i$ and coercive on V_i for all $i \in \overline{1,N}$. On the other hand, the norm $v \mapsto \|v_x\|$ and the norms $v \mapsto \|v\|_{A_i} = \sqrt{A_i(v,v)}$ and $v \mapsto \|v\|_{\overline{a}_i} = \sqrt{\overline{a}_i(v,v)}$ are equivalent.

Lemma 4.2. *There exist positive constants $\overline{\mu}_*$, $\overline{\mu}^*$, μ_*, μ^* such that:*

(i) $A_i(v,v) \geq \|v_x\|^2$, *for all* $v \in V_i$, $i \in \overline{1,N}$,
(ii) $|A_i(u,v)| \leq (1+\max\{h_0,h_1\})\|u_x\|\|v_x\|$, *for all* $u,v \in V_i$, $i=\overline{1,N}$,
(iii) $\overline{a}_i(v,v) \geq \overline{\mu}_*\|v\|_{A_i}^2$, *for all* $v \in V_i$, $i \in \overline{1,N}$,
(iv) $|\overline{a}_i(u,v)| \leq \overline{\mu}^*\|u\|_{A_i}\|v\|_{A_i}$, *for all* $v \in V_i$, $i=\overline{1,N}$,
(v) $a_i(t;v,v) \geq \mu_*\|v\|_{A_i}^2$, *for all* $v \in V_i$, $i \in \overline{1,N}$,
(vi) $|a_i(t;u,v)| \leq \mu^*\|u\|_{A_i}\|v\|_{A_i}$, *for all* $v \in V_i$, $i \in \overline{1,N}$,
(vii) $a_i'(t,v,v) \leq 0$, *for all* $u,v \in V_i$, $t \geq 0$, $i \in \overline{1,N}$.

Lemma 4.3. *For $i \in \overline{1,N}$, on V_i, the norms $v \mapsto \|v\|_{A_i} = \sqrt{A_i(v,v)}$ and $v \mapsto \|v\|_{\overline{a}_i} = \sqrt{\overline{a}_i(v,v)}$ are equivalent and*

$$\sqrt{\overline{\mu}_*}\|v\|_{A_i} \leq \|v\|_{\overline{a}_i} \leq \sqrt{\overline{\mu}^*}\|v\|_{A_i}, \quad \forall v \in V_i.$$

Now we define the modified energy functional related to (4.1)-(4.3),

$$E(t) = \frac{1}{2}\sum_{i=1}^N [(g_i \star u_i)(t) + a_i(t;u_i(t),u_i(t)) - \tilde{g}_i(t)\|u_i\|_{\overline{a}_i}^2] - \int_0^1 \mathcal{F}(\vec{u}(x,t))dx, \tag{4.4}$$

where

$$(g_i \star u_i)(t) = \int_0^t g_i(t-s)\|u_i(s)-u_i(t)\|_{\overline{a}_i}^2 ds, \quad \tilde{g}_i(t) = \int_0^t g_i(s)ds, \tag{4.5}$$

for all $i \in \overline{1,N}$. By multiplying (4.1) by $u_i'(t)$, and integrating over Ω, and summing over i from 1 to N, we obtain

$$E'(t) = \sum_{i=1}^{N} [-\|u_i'(t)\|^2 + \frac{1}{2} a_i'(t; u_i(t), u_i(t)) - \frac{1}{2} g_i(t)\|u_i\|_{\overline{a}_i}^2 + \frac{1}{2}(g_i' \star u_i)(t)] \le 0, \quad (4.6)$$

for any regular solution. The same result can be established for weak solutions and for almost every t, by a denseness argument.

Theorem 4.4. *Let assumptions* (A1), (A3')–(A6'), (A5*) *hold. If*

$$\max_{1 \le i \le N} \|g_i\|_{L^1(\mathbb{R}_+)} < \frac{\mu_*}{\overline{\mu}^*}\Big(1 - \frac{1}{(d_1 - 1)^2}\Big),$$

then for all $(\tilde{u}_1, \dots, \tilde{u}_N) \in V$ *such that* $E(0) < 0$*, we have:*

- (i) *If* $p_1 = \dots = p_N$*, then the weak solution* u *of* (4.1)-(4.3) *blows up in finite time.*
- (ii) *If there exist* $i,\ j = \overline{1,N}$*,* $i \neq j$ *such that* $p_i \neq p_j$ *and* $\sum_{i=1}^{N} \|\tilde{u}_i\|^2 \ge 4^{1+1/p} N$*, with* $p = \min_{1 \le i \le N} p_i$*, then the weak solution* u *of* (4.1)-(4.3) *blows up in finite time.*

Proof. It consists of two steps.
Step1. First, we prove that

$$\text{Problem (4.1)-(4.3) has no global weak solution.} \quad (4.7)$$

Indeed, by contradiction we assume that

$$\vec{u} \in W(\mathbb{R}_+) = \{\vec{u} \in L^\infty_{\rm loc}(\mathbb{R}_+; V) \cap C(\mathbb{R}_+; H) : \frac{\partial \vec{u}}{\partial t} \in L^2_{\rm loc}(\mathbb{R}_+; H)\},$$

is a global weak solution of (4.1)-(4.3). We define

$$\mathcal{H}(t) = -E(t), \quad t \ge 0. \quad (4.8)$$

Then it follows from (4.6) that $\mathcal{H}'(t) \ge 0$ for all $t \ge 0$. This implies that

$$\mathcal{H}(t) \ge \mathcal{H}(0) = -E(0) > 0, \quad \forall t \ge 0. \quad (4.9)$$

Set

$$\mathcal{L}_1(t) = \frac{1}{2} \sum_{i=1}^{N} \|u_i(t)\|^2. \quad (4.10)$$

By taking the time derivative of (4.10) and using (4.1), we obtain

$$\mathcal{L}_1'(t) = \sum_{i=1}^{N} (\langle f_i(\vec{u}(t)), u_i(t)\rangle - a_i(t; u_i(t), u_i(t)) + \int_0^t g_i(t-s)\overline{a}_i(u_i(s), u_i(t))ds).$$

Hence

$$\begin{aligned} \mathcal{L}_1'(t) \ge & \sum_{i=1}^{N} (\langle f_i(\vec{u}(t)), u_i(t)\rangle - a_i(t; u_i(t), u_i(t))) + \sum_{i=1}^{N} \tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2 \\ & - \sum_{i=1}^{N} \int_0^t g_i(t-s)|\overline{a}_i(u_i(s) - u_i(t), u_i(t))|ds. \end{aligned} \quad (4.11)$$

By using the Schwarz inequality and Young inequality, for all $\delta_1 > 0$, we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_0^t g_i(t-s)|\overline{a}_i(u_i(s)-u_i(t),u_i(t))|ds \\ &\le \sum_{i=1}^N [\frac{1}{2\delta_1}\tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2 + \frac{\delta_1}{2}(g_i \star u_i)(t)]. \end{aligned} \tag{4.12}$$

From (4.4) and (4.9), we obtain

$$\int_0^1 \mathcal{F}(\vec{u}(x,t))dx \ge \frac{1}{2}\sum_{i=1}^N [(g_i \star u_i)(t) + a_i(t;u_i(t),u_i(t)) - \tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2]. \tag{4.13}$$

Since $\varphi(x) = \frac{\mu_*}{\mu^*}(1 - \frac{1}{(x-1)^2})$ is continuous and nondecreasing on $(2, d_1)$, it follows that

$$0 = \varphi(2) < \max_{1\le i\le N}\|g_i\|_{L^1(\mathbb{R}_+)} < \frac{\mu_*}{\mu^*}\Big(1 - \frac{1}{(d_1-1)^2}\Big) = \varphi(d_1).$$

Then there exists a unique constant $\hat{p} \in (2, d_1)$ such that

$$\max_{1\le i\le N}\|g_i\|_{L^1(\mathbb{R}_+)} = \varphi(\hat{p}). \tag{4.14}$$

Set $\delta_1 = \hat{p}$ and $\delta_2 = \frac{\delta_1}{d_1}$. From (4.11)-(4.14) we deduce that

$$\begin{aligned} \mathcal{L}_1'(t) &\ge (1-\delta_2)\sum_{i=1}^N \langle f_i(\vec{u}(t)), u_i(t)\rangle + \delta_2 \sum_{i=1}^N \langle f_i(\vec{u}(t)), u_i(t)\rangle \\ &\quad - \sum_{i=1}^N a_i(t;u_i(t),u_i(t)) + \big(1 - \frac{1}{2\delta_1}\big)\sum_{i=1}^N \tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2 - \frac{\delta_1}{2}\sum_{i=1}^N (g_i \star u_i)(t) \\ &\ge (1-\delta_2)\sum_{i=1}^N \langle f_i(\vec{u}(t)), u_i(t)\rangle + \frac{\delta_2 d_1}{2}\sum_{i=1}^N (g_i \star u_i)(t) \\ &\quad + \frac{\delta_2 d_1}{2}\sum_{i=1}^N a_i(t;u_i(t),u_i(t)) \\ &\quad - \frac{\delta_2 d_1}{2}\sum_{i=1}^N \tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2 - \sum_{i=1}^N a_i(t;u_i(t),u_i(t)) \\ &\quad + \big(1 - \frac{1}{2\delta_1}\big)\sum_{i=1}^N \tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2 - \frac{\delta_1}{2}\sum_{i=1}^N (g_i \star u_i)(t) \\ &= \big(1 - \frac{\hat{p}}{d_1}\big)\sum_{i=1}^N \langle f_i(\vec{u}(t)), u_i(t)\rangle + \big(\frac{\hat{p}}{2} - 1\big)\sum_{i=1}^N a_i(t;u_i(t),u_i(t)) \\ &\quad - \frac{(\hat{p}-1)^2}{2\hat{p}}\sum_{i=1}^N \tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2 \\ &\ge \big(1 - \frac{\hat{p}}{d_1}\big)\sum_{i=1}^N \langle f_i(\vec{u}(t)), u_i(t)\rangle + \big(\frac{\hat{p}}{2} - 1\big)\frac{\mu_*}{\mu^*}\sum_{i=1}^N \|u_i(t)\|_{\overline{a}_i}^2 \end{aligned}$$

$$
\begin{aligned}
&- \frac{(\hat{p}-1)^2}{2\hat{p}} \max_{1\le i\le N} \|g_i\|_{L^1(\mathbb{R}_+)} \sum_{i=1}^N \|u_i(t)\|_{\overline{a}_i}^2 \\
&= \big(1-\frac{\hat{p}}{d_1}\big) \sum_{i=1}^N \langle f_i(\vec{u}(t)), u_i(t)\rangle \\
&\quad + \frac{(\hat{p}-1)^2}{2\hat{p}} (\varphi(\hat{p}) - \max_{1\le i\le N} \|g_i\|_{L^1(\mathbb{R}_+)}) \sum_{i=1}^N \|u_i(t)\|_{\overline{a}_i}^2 \\
&= \big(1-\frac{\hat{p}}{d_1}\big) \sum_{i=1}^N \langle f_i(\vec{u}(t)), u_i(t)\rangle \\
&\ge (1-\frac{\delta_1}{d_1})\overline{d}_1 \sum_{i=1}^N \|u_i(t)\|_{L^{p_i}}^{p_i} \\
&\ge (1-\frac{\delta_1}{d_1})\overline{d}_1 \sum_{i=1}^N \|u_i(t)\|^{p_i} \equiv \theta \sum_{i=1}^N \|u_i(t)\|^{p_i}.
\end{aligned}
$$

We consider the following cases:

(i) If $p_1 = \dots = p_N = p$, then from the inequality $(\sum_{i=1}^N y_i)^\alpha \le N^{\alpha-1}\sum_{i=1}^N y_i^\alpha$, for all $\alpha \ge 1$, $y_1, \dots, y_N \ge 0$, we obtain

$$
\begin{aligned}
\mathcal{L}_1'(t) &\ge \theta \sum_{i=1}^N \|u_i(t)\|^p \ge N^{1-\frac{p}{2}}\theta\Big(\sum_{i=1}^N \|u_i(t)\|^2\Big)^{p/2} \\
&\ge \frac{N\theta}{(\sqrt{2N})^p}\mathcal{L}_1^{p/2}(t) \equiv \theta_1 \mathcal{L}_1^{p/2}(t).
\end{aligned} \tag{4.15}
$$

A direct integration of (4.15) yields

$$
\mathcal{L}_1^{\frac{p}{2}-1}(t) \ge \frac{2}{(p-2)\theta_1(T_*-t)}, \quad \forall t \in [0, T_*),
$$

with $T_* = \frac{2}{(p-2)\theta_1}\mathcal{L}_1^{1-\frac{p}{2}}(0)$. Therefore, $\lim_{t\to T_*^-}\mathcal{L}_1(t) = +\infty$. This is a contradiction with $\vec{u} \in C([0,T_*]; H)$. Thus, (4.7) holds.

(ii) If there exist $i, j \in \overline{1,N}$, $i \ne j$ such that $p_i \ne p_j$. We put $p = \min_{1\le i\le N} p_i$, using the inequality $x^p \le x^{p_i} + 1$, for all $x \ge 0$, $i \in \overline{1,N}$, we obtain

$$
\begin{aligned}
\mathcal{L}_1'(t) &\ge \theta \sum_{i=1}^N \|u_i(t)\|^{p_i} \ge \theta(\sum_{i=1}^N \|u_i(t)\|^p - N) \\
&\ge \theta[N^{1-\frac{p}{2}}(\sum_{i=1}^N \|u_i(t)\|^2)^{\frac{p}{2}} - N] = \frac{N\theta}{(\sqrt{2N})^p}(\mathcal{L}_1^{p/2}(t) - (\sqrt{2N})^p).
\end{aligned} \tag{4.16}
$$

From $\mathcal{L}_1'(t) \ge 0$, for all $t \ge 0$, we have $\mathcal{L}_1(t) \ge \mathcal{L}_1(0) = \frac{1}{2}\sum_{i=1}^N \|\tilde{u}_i\|^2$, for all $t \ge 0$. It follows from $\sum_{i=1}^N \|\tilde{u}_i\|^2 \ge 4^{1+1/p}N$, that $\frac{1}{2}\mathcal{L}_1^{p/2}(t) \ge \frac{1}{2}\mathcal{L}_1^{p/2}(0) \ge (\sqrt{2N})^p$, for all $t \ge 0$. Therefore, from (4.16) we deduce that

$$
\begin{aligned}
\mathcal{L}_1'(t) &\ge \frac{N\theta}{(\sqrt{2N})^p}(\frac{1}{2}\mathcal{L}_1^{p/2}(t) + \frac{1}{2}\mathcal{L}_1^{p/2}(t) - (\sqrt{2N})^p) \\
&\ge \frac{N\theta}{2(\sqrt{2N})^p}\mathcal{L}_1^{\frac{p}{2}}(t) \equiv \theta_2 \mathcal{L}_1^{\frac{p}{2}}(t), \ \forall t \ge 0.
\end{aligned} \tag{4.17}
$$

A direct integration of (4.17) gives
$$\mathcal{L}_1^{\frac{p}{2}-1}(t) \geq \frac{2}{(p-2)\theta_2(T_*-t)}, \quad \forall t \in [0,T_*),$$
with $T_* = \frac{2}{(p-2)\theta_2}\mathcal{L}_1^{1-\frac{p}{2}}(0)$.

Therefore $\lim_{t\to T_*^-} \mathcal{L}_1(t) = +\infty$. This is a contradiction with $\vec{u} \in C([0,T_*];H)$. Thus, (4.7) holds.

Step 2. Next, we put
$$T_\infty = \sup\{T > 0 : (4.1)\text{-}(4.3) \text{ has a unique solution } \vec{u} \in W(T)\}.$$
By (4.7), we have $T_\infty < +\infty$. We now prove that
$$\lim_{t\to T_\infty^-} \|\vec{u}_x(t)\| = +\infty. \tag{4.18}$$
Indeed, assume that (4.18) is not true, then there exists a constant $M > 0$ and a sequence $\{t_m\}$ with $\{t_m\} \subset (0,T_\infty)$, $t_m \to T_\infty$ such that
$$\|\vec{u}_x(t_m)\|^2 \leq M, \quad \forall m \in \mathbb{N}.$$
Following the argument as above, for each $m \in \mathbb{N}$, there exists a unique weak solution
$$\vec{u}_* \in \{\vec{u} \in L^\infty(t_m, t_m+\eta; V) \cap C([t_m,t_m+\eta];H) : \frac{\partial \vec{u}}{\partial t} \in L^2(t_m,t_m+\eta;H)\}$$
of (4.1)-(4.3) with the initial data
$$\vec{u}_*(t_m) = \vec{u}(t_m),$$
with $\eta > 0$ independent of $m \in \mathbb{N}$. By $t_m \to T_\infty$, we can get $t_m + \eta > T_\infty$ for $m \in \mathbb{N}$ sufficiently large. It is clear that the function
$$\vec{U}(t) = \begin{cases} \vec{u}(t), & 0 \leq t \leq t_m, \\ \vec{u}_*(t), & t_m \leq t \leq t_m+\eta, \end{cases}$$
is a weak solution of (4.1)-(4.3) on $[0, t_m+\eta]$, $t_m+\eta > T_\infty$, we obtain a contradiction to the maximality of T_∞. Thus, (4.18) holds. Theorem 4.4 is proved . □

5. Exponential decay of solutions

In this section, we study the global solution of (1.1)-(1.3), corresponding to $\overline{\mu}_i(x,t) \equiv \overline{\mu}_i(x)$ as in Section 4. We shall make suitable and necessary assumptions, for which the solution obtained here decays exponentially, these assumptions are as follows.

- (A5") $f_i \in C^0(\mathbb{R}^N)$ for all $i \in \overline{1,N}$. Furthermore, there exists $\mathcal{F} \in C^1(\mathbb{R}^N;\mathbb{R})$ such that
  - (i) $\frac{\partial \mathcal{F}}{\partial u_i} = f_i$ for all $i \in \overline{1,N}$,
  - (ii) There exists a nondecreasing function $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that
  $$\lim_{z\to 0^+} G(z) = 0, \quad \mathcal{F}(\vec{u}) \leq G(|\vec{u}|_2)|\vec{u}|_2^2, \quad \forall \vec{u} \in \mathbb{R}^N,$$
  - (iii) There exists a constant $d_2 > 2$ such that $d_2\mathcal{F}(\vec{u}) \geq \sum_{i=1}^N u_i f_i(\vec{u})$, for all $\vec{u} \in \mathbb{R}^N$;
- (A6") $g_i \in C^1(\mathbb{R}_+,\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ such that
  - (i) $0 < g_i(t) \leq g_i(0)$, $g_i'(t) \leq 0$ for all $t \geq 0$, $i \in \overline{1,N}$,

(ii) $L \equiv \mu_* - \overline{\mu}^* \max_{1 \leq i \leq N} \|g_i\|_{L^1(\mathbb{R}_+)} > 0$,

(iii) There exist constants $\xi_i > 0$ for all $i \in \overline{1,N}$ such that

$$g_i'(t) \leq -\xi_i g_i(t), \quad \forall t \geq 0,\ i \in \overline{1,N};$$

(A7") $F_i \in L^2(\mathbb{R}_+; L^2)$ and there exist constants $C_i > 0$, $\gamma_i > 0$ for all $i \in \overline{1,N}$ such that

$$\|F_i(t)\| \leq C_i \exp(-\gamma_i t),\ \forall t \geq 0,\ i \in \overline{1,N}.$$

Example 5.1. We note that the function $\mathcal{F}$ given in Example 4.1 also satisfies (A5"). Indeed, we have

$$\mathcal{F}(\vec{u}) \leq \sum_{i=1}^{N} \alpha_i |\vec{u}|_2^{p_i} + \beta |\vec{u}|_2^{q_1+\dots+q_N} \ln^k(e+|\vec{u}|_2^2) \leq G(|\vec{u}|_2)|\vec{u}|_2^2, \quad \forall \vec{u} \in \mathbb{R}^N,$$

where $G(z) = \sum_{i=1}^{N} \alpha_i z^{p_i-2} + \beta z^{q_1+\dots+q_N-2} \ln^k(e+z^2) \to 0$ as $z \to 0_+$. From Example 4.1, we have

$$\begin{aligned}
\sum_{i=1}^{N} u_i f_i(\vec{u}) &= \sum_{i=1}^{N} p_i \alpha_i |u_i|^{p_i} + \beta\Big(\sum_{i=1}^{N} q_i\Big)|u_1|^{q_1} \dots |u_N|^{q_N} \ln^k(e+|\vec{u}|_2^2) \\
&\quad + 2k\beta |u_1|^{q_1} \dots |u_N|^{q_N} \frac{|\vec{u}|_2^2}{e+|\vec{u}|_2^2} \ln^{k-1}(e+|\vec{u}|_2^2) \\
&\leq \sum_{i=1}^{N} p_i \alpha_i |u_i|^{p_i} + \beta\Big(\sum_{i=1}^{N} q_i + 2k\Big)|u_1|^{q_1} \dots |u_N|^{q_N} \ln^k(e+|\vec{u}|_2^2) \\
&\leq d_2 \mathcal{F}(\vec{u}),
\end{aligned}$$

where $d_2 = \max\{p_1, \dots, p_N, 2k + \sum_{i=1}^{N} q_i\} > 2$. Hence, (A5") holds.

Now, for $\delta > 0$ to be chosen later, we define

$$\mathcal{L}(t) = E(t) + \frac{\delta}{2} \sum_{i=1}^{N} \|u_i(t)\|^2 = E(t) + \delta \mathcal{L}_1(t), \tag{5.1}$$

where

$$E(t) = \frac{1}{2} \sum_{i=1}^{N} ((g_i \star u_i)(t) + a_i(t; u_i(t), u_i(t)) - \tilde{g}_i(t)\|u_i(t)\|_{\tilde{a}_i}^2) - \int_0^1 \mathcal{F}(\vec{u}(x,t))dx.$$

With $p \in (2, d_2)$, we can rewrite the energy functional $E(t)$ as follows

$$E(t) = (\frac{1}{2} - \frac{1}{p}) \sum_{i=1}^{N} [(g_i \star u_i)(t) + a_i(t; u_i(t), u_i(t)) - \tilde{g}_i(t)\|u_i(t)\|_{\tilde{a}_i}^2] + \frac{1}{p} I(t),$$

where

$$\begin{aligned}
I(t) &= I(\vec{u}(t)) \\
&= \sum_{i=1}^{N} \big[(g_i \star u_i)(t) + a_i(t; u_i(t), u_i(t)) - \tilde{g}_i(t)\|u_i(t)\|_{\tilde{a}_i}^2\big] \\
&\quad - p \int_0^1 \mathcal{F}(\vec{u}(x,t))dx.
\end{aligned}$$

Lemma 5.2. *Assume that* (A1), (A3'), (A4'), (A5")–(A7") *hold. Then*

$$E'(t) \le -(1-\frac{\varepsilon_1}{2})\sum_{i=1}^N \|u_i'(t)\|^2 - \frac{1}{2}\sum_{i=1}^N \xi_i (g_i \star u_i)(t) + \frac{1}{2\varepsilon_1}\sum_{i=1}^N \|F_i(t)\|^2, \tag{5.2}$$

for all $\varepsilon_1 > 0$.

.

Proof. Multiplying i^{th} equation in (1.1) by u_i', and integrating over Ω, and summing over i from 1 to N; we obtain

$$\begin{aligned} E'(t) &= -\sum_{i=1}^N \|u_i'(t)\|^2 + \frac{1}{2}\sum_{i=1}^N a_i'(t;u_i(t),u_i(t)) - \frac{1}{2}\sum_{i=1}^N g_i(t)\|u_i(t)\|_{\bar{a}_i}^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^N (g_i' \star u_i)(t) + \sum_{i=1}^N \langle F_i(t), u_i'(t)\rangle, \end{aligned} \tag{5.3}$$

for any regular solution u. We can extend (5.3) to weak solutions by using denseness arguments.

On the other hand, we have

$$\begin{gathered} \sum_{i=1}^N \langle F_i(t), u_i'(t)\rangle \le \frac{\varepsilon_1}{2}\sum_{i=1}^N \|u_i'(t)\|^2 + \frac{1}{2\varepsilon_1}\sum_{i=1}^N \|F_i(t)\|^2, \\ \frac{1}{2}\sum_{i=1}^N (g_i' \star u_i)(t) \le -\frac{1}{2}\sum_{i=1}^N \xi_i (g_i \star u_i)(t), \\ \frac{1}{2}\sum_{i=1}^N a_i'(t;u_i(t),u_i(t)) \le 0. \end{gathered} \tag{5.4}$$

From (5.3) and (5.4), we obtain (5.2). Lemma 5.2 is proved. □

Lemma 5.3. *Assume that* (A1), (A3'), (A4'), (A5")–(A7") *hold. Suppose* $I(0) > 0$ *and*

$$R_* = \sqrt{\frac{2p}{(p-2)L}\Big(E(0) + \frac{1}{2}\sum_{i=1}^N \|F_i\|_{L^2(\mathbb{R}_+;L^2)}^2\Big)} \quad \textit{is small enough} \tag{5.5}$$

such that

$$\eta_* = L - pG(R_*) > \frac{d_2 - p}{d_2}\mu^*. \tag{5.6}$$

Then $I(t) > 0$ *for all* $t \ge 0$.

Proof. By the continuity of $I(t)$ and $I(0) > 0$, there exists $T_1 > 0$ such that

$$I(t) = I(\vec{u}(t)) > 0, \quad \forall t \in [0, T_1]. \tag{5.7}$$

This gives

$$\begin{aligned} E(t) &\ge \Big(\frac{1}{2} - \frac{1}{p}\Big)\sum_{i=1}^N \big(a_i(t;u_i(t),u_i(t)) - \tilde{g}_i(t)\|u_i(t)\|_{\bar{a}_i}^2\big) \\ &\ge \Big(\frac{1}{2} - \frac{1}{p}\Big)\big(\mu_* - \overline{\mu}^* \max_{1\le i\le N}\|g_i\|_{L^1(\mathbb{R}_+)}\big)\sum_{i=1}^N \|u_i(t)\|_{A_i}^2 \end{aligned}$$

$$= \frac{(p-2)L}{2p} \sum_{i=1}^{N} \|u_i(t)\|_{A_i}^2, \quad \forall t \in [0, T_1].$$

Hence

$$\sum_{i=1}^{N} \|u_i(t)\|_{A_i}^2 \leq \frac{2p}{(p-2)L} E(t), \quad \forall t \in [0, T_1]. \tag{5.8}$$

From (5.2) with $\varepsilon_1 = 1$ and (5.8), we obtain

$$\begin{aligned} \sum_{i=1}^{N} \|u_i(t)\|_{A_i}^2 &\leq \frac{2p}{(p-2)L} E(t) \\ &\leq \frac{2p}{(p-2)L} \Big(E(0) + \frac{1}{2} \sum_{i=1}^{N} \|F_i\|_{L^2(\mathbb{R}_+; L^2)}^2 \Big) \equiv R_*^2, \end{aligned} \tag{5.9}$$

for all $t \in [0, T_1]$; so

$$|\vec{u}(x,t)|_2 = \sqrt{\sum_{i=1}^{N} |u_i(x,t)|^2} \leq \sqrt{\sum_{i=1}^{N} \|u_{ix}(t)\|^2} \leq \sqrt{\sum_{i=1}^{N} \|u_i(t)\|_{A_i}^2} \leq R_*.$$

By (A5”), we have

$$\int_0^1 \mathcal{F}(\vec{u}(x,t))dx \leq \int_0^1 G(|\vec{u}(x,t)|_2)|\vec{u}(x,t)|_2^2 dx \leq G(R_*) \sum_{i=1}^{N} \|u_i(t)\|_{A_i}^2. \tag{5.10}$$

Consequently

$$\begin{aligned} I(\vec{u}(t)) &= \sum_{i=1}^{N} [(g_i \star u_i)(t) + a_i(t; u_i(t), u_i(t)) - \tilde{g}_i(t)\|u_i(t)\|_{\bar{a}_i}^2] - p \int_0^1 \mathcal{F}(\vec{u}(x,t))dx \\ &\geq \sum_{i=1}^{N} (g_i \star u_i)(t) + (L - pG(R_*)) \sum_{i=1}^{N} \|u_i(t)\|_{A_i}^2 \\ &\equiv \sum_{i=1}^{N} ((g_i \star u_i)(t) + \eta_* \|u_i(t)\|_{A_i}^2) > 0, \quad \forall t \in [0, T_1]. \end{aligned}$$

Now, we set $T_\infty = \sup\{T > 0 : I(t) > 0, \forall t \in [0, T]\}$. If $T_\infty < +\infty$, then by the continuity of $I(t)$, we have $I(T_\infty) \geq 0$.

In the case $I(T_\infty) > 0$, by the same arguments as above, we can deduce that there exists $T'_\infty > T_\infty$ such that $I(t) > 0$, for all $t \in [0, T'_\infty]$. We obtain a contradiction to the definition of T_∞.

In the case $I(T_\infty) = 0$, it follows that

$$0 = I(T_\infty) \geq \sum_{i=1}^{N} \Big((g_i \star u_i)(T_\infty) + \eta_* \|u_i(T_\infty)\|_{A_i}^2 \Big) \geq 0.$$

Therefore

$$u_i(T_\infty) = (g_i \star u_i)(T_\infty) = 0, \quad \forall i \in \overline{1, N}.$$

From the fact that $g(T_\infty - s) > 0$, for all $s \in [0, T_\infty]$, we have

$$(g_i \star u_i)(T_\infty) = \int_0^{T_\infty} g_i(T_\infty - s)\|u_i(s)\|_{\bar{a}_i}^2 ds = 0,$$

it follows that $\|u_i(s)\| \leq \|u_i(s)\|_{\overline{a}_i} = 0$, a.e. $s \in [0, T_\infty]$. By $u_i \in C([0, T_\infty]; L^2)$, we deduce that $u_i(s) = 0$, for all $s \in [0, T_\infty]$, i.e. $u_i(0) = 0$. This leads to $I(0) = 0$. We get in contradiction with $I(0) > 0$. Consequently, $T_\infty = +\infty$, i.e. $I(t) > 0$, for all $t \geq 0$. This completes the proof. □

Lemma 5.4. *Let $I(0) > 0$ and (5.5), (5.6) hold. Set*

$$E_1(t) = \sum_{i=1}^{N} \Big((g_i \star u_i)(t) + \|u_i(t)\|_{A_i}^2 \Big) + I(t). \tag{5.11}$$

Then there exist positive constants β_1, β_2 such that

$$\beta_1 E_1(t) \leq \mathcal{L}(t) \leq \beta_2 E_1(t), \quad \forall t \geq 0. \tag{5.12}$$

Proof. It is not difficult to see that

$$\begin{aligned}
\mathcal{L}(t) &= \big(\frac{1}{2} - \frac{1}{p}\big) \sum_{i=1}^{N} \big((g_i \star u_i)(t) + a_i(t; u_i(t), u_i(t)) - \tilde{g}_i(t) \|u_i(t)\|_{\overline{a}_i}^2 \big) \\
&\quad + \frac{I(t)}{p} + \delta \mathcal{L}_1(t) \\
&\geq \frac{p-2}{2p} \sum_{i=1}^{N} \big((g_i \star u_i)(t) + L \|u_i(t)\|_{A_i}^2 \big) + \frac{I(t)}{p} \geq \beta_1 E_1(t),
\end{aligned}$$

where $\beta_1 = \min\{\frac{p-2}{2p}, \frac{(p-2)L}{2p}, \frac{1}{p}\}$. Similarly,

$$\begin{aligned}
\mathcal{L}(t) &\leq \frac{p-2}{2p} \sum_{i=1}^{N} ((g_i \star u_i)(t) + \mu^* \|u_i(t)\|_{A_i}^2) + \frac{I(t)}{p} + \frac{\delta}{2} \sum_{i=1}^{N} \|u_i(t)\|_{A_i}^2 \\
&= \frac{p-2}{2p} \sum_{i=1}^{N} (g_i \star u_i)(t) + \big(\frac{(p-2)\mu^*}{2p} + \frac{\delta}{2} \big) \sum_{i=1}^{N} \|u_i(t)\|_{A_i}^2 + \frac{I(t)}{p} \leq \beta_2 E_1(t),
\end{aligned}$$

where $\beta_2 = \max\{\frac{p-2}{2p}, \frac{(p-2)\mu^*}{2p} + \frac{\delta}{2}, \frac{1}{p}\}$. The proof is complete. □

Lemma 5.5. *Suppose $I(0) > 0$ (5.5), (5.6) hold. Then*

$$\begin{aligned}
\mathcal{L}_1'(t) &\leq \Big(\frac{1}{2\varepsilon_2} + \frac{d_2}{p} \Big) \sum_{i=1}^{N} (g_i \star u_i)(t) - \frac{\varepsilon_3 d_2}{p} I(t) + \frac{1}{2\varepsilon_2} \sum_{i=1}^{N} \|F_i(t)\|^2 \\
&\quad - \Big(\frac{d_2}{p} - 1 - \frac{\varepsilon_2}{2} \Big) \sum_{i=1}^{N} \tilde{g}_i(t) \|u_i(t)\|_{\overline{a}_i}^2 \\
&\quad - \big[\frac{(1-\varepsilon_3) d_2 \eta_*}{p} - (\frac{d_2}{p} - 1)\mu^* - \frac{\varepsilon_2}{2} \big] \sum_{i=1}^{N} \|u_i(t)\|_{A_i}^2,
\end{aligned} \tag{5.13}$$

for all $\varepsilon_2 > 0$, $\varepsilon_3 \in (0, 1)$.

Proof. Multiplying the i^{th} equation in (1.1) by u_i', and integrating over Ω, and summing over i from 1 to N, we obtain

$$\begin{aligned} \mathcal{L}_1'(t) &= -\sum_{i=1}^N a_i(t;u_i(t),u_i(t)) + \sum_{i=1}^N \tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2 + \sum_{i=1}^N \langle f_i(\vec{u}(t)),u_i(t)\rangle \\ &\quad + \sum_{i=1}^N \langle F_i(t),u_i(t)\rangle + \sum_{i=1}^N \int_0^t g_i(t-s)\overline{a}_i(u_i(s)-u_i(t),u_i(t))ds. \end{aligned} \tag{5.14}$$

For all $\varepsilon_2 > 0$, we have

$$\int_0^t g_i(t-s)\overline{a}_i(u_i(s)-u_i(t),u_i(t))ds \le \frac{\varepsilon_2}{2}\tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2 + \frac{1}{2\varepsilon_2}(g_i\star u_i)(t), \tag{5.15}$$

$$\sum_{i=1}^N \langle F_i(t),u_i(t)\rangle \le \frac{1}{2\varepsilon_2}\sum_{i=1}^N \|F_i(t)\|^2 + \frac{\varepsilon_2}{2}\sum_{i=1}^N \|u_i(t)\|_{A_i}^2. \tag{5.16}$$

For each $\varepsilon_3 \in (0,1)$, we have

$$\begin{aligned} &\sum_{i=1}^N \langle f_i(\vec{u}(t)),u_i(t)\rangle \\ &\le d_2\int_0^1 \mathcal{F}(\vec{u}(x,t))dx \\ &= \frac{d_2}{p}\Big[\sum_{i=1}^N ((g_i\star u_i)(t) + a_i(t;u_i(t),u_i(t)) - \tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2) - I(t)\Big] \\ &= \frac{d_2}{p}\sum_{i=1}^N \big[(g_i\star u_i)(t) + a_i(t;u_i(t),u_i(t)) - \tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2\big] \\ &\quad - \frac{\varepsilon_3 d_2}{p}I(t) - \frac{(1-\varepsilon_3)d_2}{p}I(t) \\ &\le \frac{d_2}{p}\sum_{i=1}^N \big[(g_i\star u_i)(t) + a_i(t;u_i(t),u_i(t)) - \tilde{g}_i(t)\|u_i(t)\|_{\overline{a}_i}^2\big] \\ &\quad - \frac{\varepsilon_3 d_2}{p}I(t) - \frac{(1-\varepsilon_3)d_2\eta_*}{p}\sum_{i=1}^N \|u_i(t)\|_{A_i}^2. \end{aligned} \tag{5.17}$$

Since $\frac{d_2}{p} - 1 > 0$, we have

$$\big(\frac{d_2}{p}-1\big)\sum_{i=1}^N a_i(t;u_i(t),u_i(t)) \le \big(\frac{d_2}{p}-1\big)\mu^*\sum_{i=1}^N \|u_i(t)\|_{A_i}^2. \tag{5.18}$$

From (5.14)–(5.18), we obtain (5.13). The proof is complete. □

Theorem 5.6. *Assume* (A1), (A3'), (A4'), (A5")–(A7") *hold, and* $(\tilde{u}_1,\dots,\tilde{u}_N)\in V$. *If* $I(0) > 0$ *and the initial energy* $E(0)$ *satisfies* (5.5) *and* (5.6), *Then there exist positive constants* C, γ *such that*

$$E_1(t) \le C\exp(-\gamma t), \quad \forall t \ge 0. \tag{5.19}$$

Proof. From (5.1), (5.2) and (5.13) it follows that

$$
\begin{aligned}
\mathcal{L}'(t) \le& -\big(1-\frac{\varepsilon_1}{2}\big)\sum_{i=1}^N \|u_i'(t)\|^2 - \frac{1}{2}\sum_{i=1}^N \big[\xi_i - \delta\big(\frac{1}{\varepsilon_2}+\frac{2d_2}{p}\big)\big](g_i\star u_i)(t) \\
&- \frac{\delta\varepsilon_3 d_2}{p} I(t) - \delta\big[\frac{(1-\varepsilon_3)d_2\eta_*}{p} - \big(\frac{d_2}{p}-1\big)\mu^* - \frac{\varepsilon_2}{2}\big]\sum_{i=1}^N \|u_i(t)\|_{A_i}^2 \\
&- \delta\big(\frac{d_2}{p}-1-\frac{\varepsilon_2}{2}\big)\sum_{i=1}^N \tilde{g}_i(t)\|u_i(t)\|_{a_i}^2 + \rho(t),
\end{aligned}
\tag{5.20}
$$

for all δ, ε_1, $\varepsilon_2 > 0$ and $\varepsilon_3 \in (0,1)$, where

$$
\rho(t) = \frac{1}{2}\big(\frac{1}{\varepsilon_1}+\frac{\delta}{\varepsilon_2}\big)\sum_{i=1}^N \|F_i(t)\|^2. \tag{5.21}
$$

As $\eta_* > \frac{d_2-p}{d_2}\mu^*$, we can choose $\varepsilon_2 > 0$ and $\varepsilon_3 \in (0,1)$ such that

$$
\sigma_1 = \frac{d_2(1-}{\varepsilon}{}_3)\eta_* p - \big(\frac{d_2}{p}-1\big)\mu^* - \frac{\varepsilon_2}{2} > 0, \quad \sigma_2 = \frac{d_2}{p} - 1 - \frac{\varepsilon_2}{2} > 0. \tag{5.22}
$$

We continue by choosing δ and $\varepsilon_1 > 0$ such that

$$
1-\frac{\varepsilon_1}{2} > 0, \quad \sigma_3 = \frac{1}{2}\big[\min_{1\le i\le N}\xi_i - \delta\big(\frac{1}{\varepsilon_2}+\frac{2d_2}{p}\big)\big] > 0. \tag{5.23}
$$

From (5.21), we have

$$
\rho(t) = \frac{1}{2}\big(\frac{1}{\varepsilon_1}+\frac{\delta}{\varepsilon_2}\big)\sum_{i=1}^N \|F_i(t)\|^2 \le C_0\exp(-2\gamma_0 t), \quad \forall t \ge 0, \tag{5.24}
$$

where $C_0 = \frac{1}{2}\big(\frac{1}{\varepsilon_1}+\frac{\delta}{\varepsilon_2}\big)\min_{1\le i\le N} C_i^2$, $\gamma_0 = \max_{1\le i\le N}\gamma_i$. Then, we deduce from (5.20) -(5.24), that there exists $\gamma_* > 0$ such that

$$
\begin{aligned}
\mathcal{L}'(t) &\le -\gamma_*\Big[\sum_{i=1}^N((g_i\star u_i)(t) + \|u_i(t)\|_{A_i}^2) + I(t)\Big] + C_0\exp(-2\gamma_0 t) \\
&\le -\frac{\gamma_*}{\beta_2}\mathcal{L}(t) + C_0\exp(-2\gamma_0 t),
\end{aligned}
\tag{5.25}
$$

where $\gamma_* = \min\{\frac{\varepsilon_3 d_2\delta}{p}, \delta\sigma_1, \sigma_3\}$, $0 < \gamma < \min\{\frac{\gamma_*}{\beta_2}, 2\gamma_0\}$. By integrating (5.25), we deduce

$$
E_1(t) \le \frac{1}{\beta_1}\mathcal{L}(t) \le \frac{1}{\beta_1}\Big(\mathcal{L}(0) + \frac{C_0}{2\gamma_0-\gamma}\Big)\exp(-\gamma t) \equiv C\exp(-\gamma t), \quad \forall t \ge 0.
$$

This implies (5.19), and completes the proof. □

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