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PYRAMIDAL TRAVELING FRONTS IN THE BELOUSOV-ZHABOTINSKII REACTION-DIFFUSION SYSTEMS IN $\mathbb{R}^3$

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Abstract. In this article, we consider a diffusion system with the Belousov-Zhabotinskii (BZ for short) chemical reaction. The existence and stability of V-shaped traveling fronts for the BZ system in $\mathbb{R}^2$ had been proved in our previous papers [30, 31]. Here we establish the existence and stability of pyramidal traveling fronts for the BZ system in $\mathbb{R}^3$.

1. Introduction

Consider the reaction-diffusion system

$$\begin{aligned} u_t(\mathbf{x},t) &= \Delta u(\mathbf{x},t) + u(\mathbf{x},t)(1-u(\mathbf{x},t)-rv(\mathbf{x},t)),\\ v_t(\mathbf{x},t) &= \Delta v(\mathbf{x},t) - bu(\mathbf{x},t)v(\mathbf{x},t), \end{aligned} \tag{1.1}$$

where r, $b > 0$ are positive parameters and u, v correspond to the concentrations of the bromous acid and bromide ion respectively. (1.1) is called the BZ system, which stems from a typical chemical oscillating reaction. The possible existence of such chemical oscillation was predicted by Turing [39] through the method of mathematical calculation, and the chemical phenomenon was observed by Belousov [1]. When the concentrations of the reactants change orderly along with time and space, chemical waves appear [47]. To investigate the mechanism of the BZ reaction, Field and his coworkers formulated a complex model [8] and then simplified it [9]. Later, the simplified model was nondimensionalized by Murray [25, 26] to be (1.1). It is found that the front solution of (1.1) is an appropriate mathematical tool to describe the planar waves [47].

After that, mathematical studies on system (1.1), a lot of progress has been made, mainly including the existence of 1-D traveling wave fronts, admissible traveling speeds and the asymptotic behavior of traveling wave fronts [12, 20, 21, 22, 37, 38, 41]. In fact, studies of traveling wave solutions on reaction-diffusion equations

$$u_t(\mathbf{x},t) = \Delta u(\mathbf{x},t) + f(u(\mathbf{x},t)), \quad \mathbf{x} \in \mathbb{R}^N,\ t > 0,$$

which can be originated from the pioneer work of Fisher [6], have attracted a lot of attention [4, 10, 11, 23, 24, 42], which mainly focus on 1-D traveling wave solutions and planar traveling wave solutions in $\mathbb{R}^N (N \geq 2)$. The gradually mature theory of

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1-D traveling wave solutions promotes the research on multidimensional traveling wave solutions, which are proper to describe the traveling wave phenomena in multidimensional space, see [2, 3, 13, 14, 15, 28, 29, 34, 35, 44] for the scalar equation and [5, 16, 17, 18, 19, 27, 36, 43, 45, 46] for the reaction diffusion system.

For the BZ reaction, along with the development of research, nonplanar chemical waves were also observed. In 1995, stable V-shaped chemical waves were observed in the BZ reaction [40], for which we have already made rigorous mathematical proofs [30, 31]. However, mathematical studies of multidimensional nonplanar traveling waves on the BZ system (1.1) are still very few.

In this article, we continue to study the pyramidal traveling fronts of (1.1) and expect to give some theoretical implications to the observation of new nonplanar chemical waves in the experiment. Precisely, we go on studying (1.1) under the case $r > 1$, which means that (1.1) is bistable. We still want to emphasize that the case 'bistable' for the BZ system is different from those in Ni and Taniguchi [27] and Wang [43] since it is degenerate at one of its the equilibria.

Now we set $u_1(\mathbf{x}, t) = u(\mathbf{x}, t)$, $u_2(\mathbf{x}, t) = 1 - v(\mathbf{x}, t)$, $\mathbf{u} = (u_1, u_2)$, then system (1.1) can be rewritten as

$$\mathbf{u}_t = \Delta\mathbf{u} + \mathbf{F}(\mathbf{u}), \tag{1.2}$$

where $\mathbf{F}(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u})) = (u_1(1 - r - u_1 + ru_2), bu_1(1 - u_2))$. Under the condition $r > 1$, [37, Theorem 9] tells that system (1.2) admits a unique positive traveling front $\mathbf{U}(\xi) = (U_1(\xi), U_2(\xi))$ satisfying

$$\begin{aligned} &U_1''(\xi) - cU_1'(\xi) + f_1(\mathbf{U}(\xi)) = 0,\\ &U_2''(\xi) - cU_2'(\xi) + f_2(\mathbf{U}(\xi)) = 0 \end{aligned} \tag{1.3}$$

with

$$\mathbf{U}(-\infty) = (0, 0), \quad \mathbf{U}(+\infty) = (1, 1), \quad 0 < U_1(\xi) < U_2(\xi) < 1, \quad \forall \xi \in \mathbb{R},$$

where $c \in \Big(b/(2\sqrt{(r + b)[\min(1, b)(r + b) - 0.5b]}), 2\sqrt{\min(1, b)}\Big)$ is the wave speed, see [37, Theorem 6, Theorem 9 and Proposition 10].

Denote $\mathbf{x}' = (x_1, x_2)$, $\mathbf{x} = (\mathbf{x}', x_3)$. Without loss of generality, we assume that the traveling solutions travel towards the $-x_3$ direction with speed s, then they have the form $\mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}', x_3 + st)$ and satisfy

$$\mathbf{v}_t = \Delta\mathbf{v} - s\mathbf{v}_{x_3} + \mathbf{F}(\mathbf{v}), \quad \mathbf{x} \in \mathbb{R}^3,\ t > 0, \tag{1.4}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{1.5}$$

We aim to find a nontrivial steady state $\mathbf{V}(\mathbf{x})$ of the system

$$-\Delta\mathbf{V} + s\mathbf{V}_{x_3} - \mathbf{F}(\mathbf{V}) = \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{1.6}$$

Since the acceleration effect of the curvature, it is natural to take $s > c$. Fix s and set

$$m_* = \sqrt{s^2 - c^2}/c.$$

Now we introduce the definition of pyramid. Let $n \geq 3$ be a given integer. Assume $\{(A_j, B_j)\}_{j=1}^n$ is a set of unit vectors, i.e. $A_j^2 + B_j^2 = 1$ for all $j \in \{1, 2, \dots, n\}$, and satisfies

$$A_jB_{j+1} - A_{j+1}B_j > 0, \quad 1 \leq j \leq n - 1; \quad A_nB_1 - A_1B_n > 0.$$

We also assume that $(A_j, B_j) \neq (A_i, B_i)$ if $i \neq j$. Now $(-m_* A_j, -m_* B_j, 1)$ is a normal vector of the plane $\{\mathbf{x} \in \mathbb{R}^3 : -x_3 = m_*(A_j x_1 + B_j x_2)\}$. We put

$$h_j(\mathbf{x}') = m_*(A_j x_1 + B_j x_2),$$
$$h(\mathbf{x}') = \max_{1\le j\le n} h_j(\mathbf{x}') = m_* \max_{1\le j\le n}(A_j x_1 + B_j x_2).$$

Then $-x_3 = h(\mathbf{x}')$ represents a pyramid in $\mathbb{R}^3$. Set $\Omega_j = \{\mathbf{x}' \in \mathbb{R}^2 : h(\mathbf{x}') = h_j(\mathbf{x}')\}, j \in \{1, 2, \dots n\}$. Then $\mathbb{R}^2 = \cup_{j=1}^n \Omega_j$. Denote

$$E := \cup_{j=1}^n \partial\Omega_j \subset \mathbb{R}^2.$$

Now the lateral surfaces of a pyramid are $S_j = \{\mathbf{x} \in \mathbb{R}^3 : -x_3 = h_j(\mathbf{x}'), \mathbf{x}' \in \Omega_j\}$ for $j = 1, 2, \dots, n$. We put

$$\Gamma_j = \begin{cases} S_j \cap S_{j+1} & \text{if } 1 \le j \le n-1, \\ S_n \cap S_1 & \text{if } j = n. \end{cases}$$

Then Γ_j represents an edge of the pyramid and $\Gamma = \cup_{j=1}^n \Gamma_j$ represents the set of all edges. Denote

$$\mathbf{v}^-(\mathbf{x}) = \mathbf{U}\left(\frac{c}{s}(x_3 + h(\mathbf{x}'))\right) = \max_{1\le j\le n} \mathbf{U}\left(\frac{c}{s}(x_3 + h_j(\mathbf{x}'))\right). \tag{1.7}$$

Since $\mathbf{U}(x)$ is a planar traveling wave of (1.2), it is easy to see that $\mathbf{v}^-(\mathbf{x})$ is combined with several such planar traveling waves and thus becomes a nonplanar traveling wave with pyramidal level sets. We also define

$$D(\gamma) := \{\mathbf{x} \in \mathbb{R}^3 : \operatorname{dist}(\mathbf{x}, \Gamma) > \gamma\}, \quad \forall \gamma > 0.$$

We now define the a relation of order in $\mathbb{R}^3$. We say that $\mathbf{x} < \mathbf{y}$ (resp. $\mathbf{x} \le \mathbf{y}$) An interval $[\mathbf{x}_1, \mathbf{x}_2] \subset \mathbb{R}^3$ denotes the set of $\mathbf{x} \in \mathbb{R}^3$ with $\mathbf{x}_1 \le \mathbf{x} \le \mathbf{x}_2$. Throughout this paper, we denote $\mathbf{0} = (0, 0)$ and $\mathbf{1} = (1, 1)$. The following theorem is the main assertion in this paper.

Theorem 1.1. *Assume that $r > 1$ and $b > 0$. Then for each $s > c$, there exists a solution $\mathbf{V}(\mathbf{x}) = (V_1(\mathbf{x}), V_2(\mathbf{x}))$ to (1.6) with*

$$\mathbf{v}^-(\mathbf{x}) < \mathbf{V}(\mathbf{x}) < \mathbf{1} \quad \text{in } \mathbb{R}^3$$

and

$$\lim_{\gamma\to\infty} \sup_{\mathbf{x}\in D(\gamma)} \frac{|V_i(\mathbf{x}) - v_i^-(\mathbf{x})|}{\left(v_2^-(\mathbf{x})\right)^{\beta_i}} = 0, \quad i = 1, 2.$$

Furthermore, for any $\mathbf{u}_0(\mathbf{x}) \in C(\mathbb{R}^3, \mathbb{R}^2)$ with $\mathbf{u}_0(\mathbf{x}) \in [\mathbf{0}, \mathbf{1}]$ for $\mathbf{x} \in \mathbb{R}^3$ and

$$\begin{gathered} \lim_{\gamma\to\infty} \sup_{\mathbf{x}\in D(\gamma)} \frac{|V_i(\mathbf{x}) - u_{0,i}(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} = 0, \quad i = 1, 2, \\ \mathbf{v}^-(\mathbf{x}) \le \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \end{gathered} \tag{1.8}$$

the solution $\mathbf{u}(\mathbf{x}, t; \mathbf{u}_0)$ of (1.2) with initial data $\mathbf{u}_0$ satisfies

$$\lim_{t\to\infty} \Big\| \frac{u_i(\cdot,\cdot,\cdot,t;\mathbf{u}_0) - V_i(\cdot,\cdot,\cdot + st)}{(v_2^-(\cdot))^{\beta_i}} \Big\|_{L^\infty(\mathbb{R}^3)} = 0, \quad i = 1, 2.$$

Here, $0 < \beta_2 < \beta_1 < \beta^*$ are arbitrary (see (3.4) for β^*).

To use the comparison argument, we consider a modified system

$$\tilde{\mathbf{v}}_t = \Delta\tilde{\mathbf{v}} - s\tilde{\mathbf{v}}_{x_3} + \tilde{\mathbf{F}}(\tilde{\mathbf{v}}), \quad \mathbf{x} \in \mathbb{R}^3, t > 0, \tag{1.9}$$

$$\tilde{\mathbf{v}}|_{t=0} = \mathbf{v}_0(x, z), \quad \mathbf{x} \in \mathbb{R}^3, \tag{1.10}$$

where $\tilde{\mathbf{F}}(\mathbf{u}) = (\tilde{f}_1(\mathbf{u}), \tilde{f}_2(\mathbf{u})) = \mathbf{F}(\mathbf{u}) + \mathbf{G}(\mathbf{u})$, $\mathbf{G}(\mathbf{u}) = (g_1(\mathbf{u}), g_2(\mathbf{u}))$ with

$$g_1(\mathbf{u}) = 0, \quad g_2(\mathbf{u}) = b(u_1 - 1)\max\{0, u_2 - 1\}.$$

It is clear that both $\mathbf{F}(\mathbf{u})$ and $\tilde{\mathbf{F}}(\mathbf{u})$ are Lipschitz continuous in $\mathbb{R}^2$. Obviously,

$$\partial_{u_2} f_1 \geq 0,\ \partial_{u_1} \tilde{f}_2 \geq 0 \quad \text{if } (u_1, u_2) \in [0, +\infty) \times [0, +\infty).$$

Then the comparison principle (see [32]) gives

$$\tilde{\mathbf{v}}(\mathbf{x}, t; \mathbf{v}_0^1) \leq \tilde{\mathbf{v}}(\mathbf{x}, t; \mathbf{v}_0^2), \quad \forall \mathbf{x} \in \mathbb{R}^3,\ t \geq 0$$

if $\mathbf{0} \leq \mathbf{v}_0^1(\mathbf{x}) \leq \mathbf{v}_0^2(\mathbf{x})$ in $\mathbb{R}^3$, where $\tilde{\mathbf{v}}(\mathbf{x}, t; \mathbf{v}_0)$ denotes the solution of (1.9) and (1.10). In particular, it holds

$$\tilde{\mathbf{v}}(\mathbf{x}, t; \mathbf{v}_0) \in [\mathbf{0}, \mathbf{1}] \quad \text{if } \mathbf{v}_0(\mathbf{x}) \in [\mathbf{0}, \mathbf{1}],\ \forall \mathbf{x} \in \mathbb{R}^3,$$

which implies that the interval $[\mathbf{0}, \mathbf{1}]$ is invariant for the solution of (1.9) and (1.10). Thus, for $\mathbf{v}_0(\mathbf{x}) \in [\mathbf{0}, \mathbf{1}]$, the solution $\tilde{\mathbf{v}}(\mathbf{x}, t; \mathbf{v}_0)$ of (1.9) and (1.10) is also the solution of (1.4) and (1.5), namely, $\tilde{\mathbf{v}}(\mathbf{x}, t; \mathbf{v}_0) \equiv \mathbf{v}(\mathbf{x}, t; \mathbf{v}_0)$, where $\mathbf{v}(\mathbf{x}, t; \mathbf{v}_0)$ denotes the solution of (1.4) and (1.5).

For each unit vector (A_j, B_j), (1.6) admits a solution $\mathbf{U}(\frac{c}{s}(x_3 + h_j(\mathbf{x}')))$, which is called a planar wave. It follows that the function $\mathbf{v}^-(\mathbf{x})$ defined by (1.7) is a subsolution of (1.9), and obviously $\mathbf{v}_z^- = \frac{c}{s}\mathbf{U}'(\frac{c}{s}(x_3 + h(\mathbf{x}'))) > \mathbf{0}$. Throughout this paper, we define the operator $\tilde{\mathcal{L}}$ by

$$\tilde{\mathcal{L}}[\mathbf{v}] := \mathbf{v}_t - \Delta\mathbf{v} + s\mathbf{v}_{x_3} - \tilde{\mathbf{F}}(\mathbf{v}).$$

The remainder of this paper is organized as follows: in Section 2 we give some notation and known results. In Section 3 we prove the existence result of pyramidal fronts by constructing an appropriate supersolution. And in Section 4 we prove the asymptotic stability of the pyramidal traveling fronts constructed in Section 3.

2. Preliminaries

In this section, we give some notation and known results. By [30, Lemma 1.1] or [37, Lemma 13], we have

$$\lim_{\xi \to -\infty} \frac{U_2'(\xi)}{U_2(\xi)} = \lambda_2 = c.$$

Thus we can define

$$N_1 := \sup_{x \in \mathbb{R}} \Big|\frac{U_2'(x)}{U_2(x)}\Big|, \quad N_2 := \sup_{x \in \mathbb{R}} \Big|\frac{U_2''(x)}{U_2(x)}\Big|. \tag{2.1}$$

[30, Lemma 1.1] also implies that there exist two positive constants $L_1 < 1$ and $L_2 > 1$ such that

$$L_1 e^{\max\{\lambda_1, 2\lambda_2\}\xi} < U_1(\xi), U_1'(\xi) < L_2 e^{\min\{\lambda_1, 2\lambda_2\}\xi}, \quad \xi < 0. \tag{2.2}$$

$$L_1 e^{\lambda_1 \xi} < U_2(\xi), U_2'(\xi) < L_2 e^{\lambda_2 \xi}, \quad \xi < 0. \tag{2.3}$$

Using (1.2), the derivative matrix of $\mathbf{F}$ is

$$D\mathbf{F}(\mathbf{u}) = (f_{ij}(\mathbf{u}))_{2\times 2} = \begin{pmatrix} 1-r-2u_1+ru_2 & ru_1 \\ b(1-u_2) & -bu_1 \end{pmatrix}$$

where $f_{ij}(\mathbf{u}) = \frac{\partial f_i(\mathbf{u})}{\partial u_j}$. Because of $r>1$, we can find two positive numbers p_1, p_2 that satisfy $p_1 > p_2 \geq 1$ and $\frac{p_1}{p_2} > r$. Let $\mathbf{p} = (p_1, p_2)^T$, where T means the transpose. We have

$$\mathbf{q} = (q_1, q_2)^T := D\mathbf{F}(\mathbf{1})\cdot \mathbf{p} < \mathbf{0},$$

Fix an appropriate $\varepsilon_1 \in (0,1)$ such that

$$D\mathbf{F}(\mathbf{u})\cdot\mathbf{p} < \frac{1}{2}\mathbf{q}, \quad (1-\varepsilon_1)\mathbf{1} \leq \mathbf{u} \leq (1+\varepsilon_1)\mathbf{1}. \tag{2.4}$$

Now we introduce a mollified pyramid, see [34]. Let $\tilde{\rho}(r) \in C^\infty([0,\infty))$ be a function with the following properties:

$$\begin{gathered} \tilde{\rho}(r) > 0, \quad \tilde{\rho}'(r) \leq 0 \quad \text{for } r \geq 0, \\ \tilde{\rho}(r) \equiv 1 \quad \text{if } 0 \leq r \leq 1, \\ \tilde{\rho}(r) \equiv e^{-r} \quad \text{if } r > 0 \text{ is large enough}, \\ 2\pi \int_0^\infty r\tilde{\rho}(r)dr = 1. \end{gathered}$$

Then $\rho(\mathbf{x}') := \tilde{\rho}(|\mathbf{x}'|)$ belongs to $C^\infty(\mathbb{R}^2)$ and satisfies $\int_{\mathbb{R}^2} \rho(\mathbf{x}')d\mathbf{x}' = 1$. For a pyramid $-x_3 = h(\mathbf{x}')$ we define its corresponding mollified pyramid $-x_3 = \varphi(\mathbf{x}')$, where

$$\varphi(\mathbf{x}') = \int_{\mathbb{R}^2} \rho(\mathbf{x}'-\mathbf{y}')d\mathbf{y}' = \int_{\mathbb{R}^2} \rho(\mathbf{y}')h(\mathbf{x}'-\mathbf{y}')d\mathbf{y}'. \tag{2.5}$$

We set $(a_j, b_j) = m_*(A_j, B_j)$. Then $(a_j, b_j) \in \mathbb{R}^2$ satisfies

$$\frac{s}{\sqrt{1+a_j^2+b_j^2}} = c, \quad \text{for } j = 1, 2, \dots n.$$

We put

$$S(\mathbf{x}') := \frac{s}{\sqrt{1+|\nabla\varphi(\mathbf{x}')|^2}} - c, \tag{2.6}$$

where $\nabla\varphi = (\varphi_{x_1}, \varphi_{x_2})$. The following two lemmas come from Taniguchi [34].

Lemma 2.1. *Let φ and S be as in* (2.5) *and* (2.6)*, respectively. Then*

$$\begin{gathered} h(\mathbf{x}') < \varphi(\mathbf{x}') \leq h(\mathbf{x}') + 2\pi m_* \int_0^\infty r^2 \tilde{\rho}(r)dr, \quad |\nabla\varphi(\mathbf{x}')| < m_*, \\ 0 < S(\mathbf{x}') \leq s - c \end{gathered}$$

for all $\mathbf{x}' \in \mathbb{R}^2$. In particular,

$$\begin{gathered} \lim_{\lambda\to\infty} \sup\{S(\mathbf{x}') | \mathbf{x}' \in \mathbb{R}^2, \operatorname{dist}(\mathbf{x}', E) \geq \lambda\} = 0, \\ \lim_{\lambda\to\infty} \sup\{\varphi(\mathbf{x}') - h(\mathbf{x}') | \mathbf{x}' \in \mathbb{R}^2, \operatorname{dist}(\mathbf{x}', E) \geq \lambda\} = 0 \end{gathered}$$

and there exists positive constants ν_1, ν_2 so that

$$0 < \nu_1 \leq \frac{\varphi(\mathbf{x}') - h(\mathbf{x}')}{S(\mathbf{x}')} \leq \nu_2, \quad \forall \mathbf{x}' \in \mathbb{R}^2.$$

Lemma 2.2. *For all integers $i_1 \geq 0$, $i_2 \geq 0$, one has*

$$C_1 := \sup_{\mathbf{x}' \in \mathbb{R}^2} |D_{x_1}^{i_1} D_{x_2}^{i_2} \varphi(\mathbf{x}')| < +\infty,$$

and furthermore, for $2 \leq i_1 + i_2 \leq 3$ one also has

$$C_2 := \sup_{\mathbf{x}' \in \mathbb{R}^2} \frac{|D_{x_1}^{i_1} D_{x_2}^{i_2} \varphi(\mathbf{x}')|}{S(\mathbf{x}')} < +\infty\,.$$

3. Existence of pyramidal traveling fronts

Set $\mathbf{z}' = \alpha \mathbf{x}'$, $z_3 = \alpha x_3$ and $\mathbf{z} = \alpha \mathbf{x}$. Define

$$\varsigma(\mathbf{x}) = \frac{x_3 + \varphi(\mathbf{z}')/\alpha}{\sqrt{1 + |\nabla \varphi(\mathbf{z}')|^2}}, \quad \eta(\mathbf{x}) = \frac{c}{s}(x_3 + \varphi(\mathbf{z}')/\alpha).$$

Since $1 \leq \sqrt{1 + |\nabla \varphi|^2} < s/c$, we have

$$\frac{s}{c}\eta(\mathbf{x}) < \varsigma(\mathbf{x}) < \eta(\mathbf{x}), \quad \text{if } \varsigma(\mathbf{x}) < 0, \tag{3.1}$$

$$\eta(x, z) < \varsigma(\mathbf{x}) < \frac{s}{c}\eta(\mathbf{x}), \quad \text{if } \varsigma(\mathbf{x}) > 0. \tag{3.2}$$

Now we fix a function $\omega(x) \in C^\infty(\mathbb{R})$ with

$$\begin{gathered} \omega(x) = 1, \quad \text{if } x \leq -1, \\ 0 < \omega(x) < 1, \quad -1 < \omega'(x) < 0, \quad \text{if } -1 < x < 1, \\ \omega(x) = 0, \quad \text{if } x \geq 1. \end{gathered} \tag{3.3}$$

In this section, we denote $\boldsymbol{\beta} := (\beta_1, \beta_2)$ and make it satisfies

$$0 < \beta_2 < \beta_1 < \beta^* := \frac{\lambda_2}{\lambda_1}, \tag{3.4}$$

see [30, Lemmas 1.1 and 1.4] for λ_1 and λ_2. In the proof of the following lemma, we denote

$$\Pi_1(x) := x^2 - cx + 1 - r, \quad \Pi_2(x) := x^2 - cx.$$

Obviously, $\beta^* < 1$ and $\Pi_i(\beta_i \lambda_2) < 0$ for $i = 1, 2$.

Lemma 3.1. *There exist a positive constant $\varepsilon_0^+(\boldsymbol{\beta}) < 1$ and a positive function $\alpha_0^+(\varepsilon, \boldsymbol{\beta})$ such that, for all $0 < \varepsilon < \varepsilon_0^+(\boldsymbol{\beta})$ and $0 < \alpha < \alpha_0^+(\varepsilon, \boldsymbol{\beta})$,*

$$\mathbf{v}^+(\mathbf{x}; \varepsilon, \boldsymbol{\beta}, \alpha) = \mathbf{U}(\varsigma(\mathbf{x})) + \varepsilon S(\mathbf{z}') \big((1 - \omega(\eta(\mathbf{x})))\, \mathbf{p} + \omega(\eta(\mathbf{x})) \mathcal{U}^{\boldsymbol{\beta}}(\eta(\mathbf{x})) \big)$$

is a supersolution to (1.9), where $\mathcal{U}^{\boldsymbol{\beta}}(\xi) := \big(U_2^{\beta_1}(\xi), U_2^{\beta_2}(\xi)\big)$. Furthermore,

$$\lim_{\gamma \to +\infty} \sup_{\mathbf{x} \in D(\gamma)} \frac{|v_i^+(\mathbf{x}; \varepsilon, \boldsymbol{\beta}, \alpha) - v_i^-(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} \leq 2\varepsilon, \quad i = 1, 2, \tag{3.5}$$

$$\mathbf{v}^-(\mathbf{x}) < \mathbf{v}^+(\mathbf{x}; \varepsilon, \boldsymbol{\beta}, \alpha), \quad \mathbf{x} \in \mathbb{R}^3, \tag{3.6}$$

$$\partial_{x_3} \mathbf{v}^+(\mathbf{x}; \varepsilon, \boldsymbol{\beta}, \alpha) > \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{3.7}$$

Proof. For the sake of convenience, we denote $\varsigma(\mathbf{x})$, $\eta(\mathbf{x})$ by ς and η, respectively. We also denote $\mathbf{v}^+(\mathbf{x}; \varepsilon, \boldsymbol{\beta}, \alpha)$ by $\mathbf{v}^+(\mathbf{x})$ for simplicity. By a direct computation, we have

$$\varsigma_{x_1} = -\alpha \frac{\varphi_{z_1}\varphi_{z_1 z_1} + \varphi_{z_2}\varphi_{z_2 z_1}}{1 + |\nabla \varphi(\mathbf{z}')|^2} \varsigma + \frac{\varphi_{z_1}}{\sqrt{1 + |\nabla \varphi(\mathbf{z}')|^2}}, \quad \eta_{x_1} = \frac{c}{s}\varphi_{z_1},$$

$$
\begin{gathered}
\varsigma_{x_2} = -\alpha\frac{\varphi_{z_1}\varphi_{z_1z_2}+\varphi_{z_2}\varphi_{z_2z_2}}{1+|\nabla\varphi(\mathbf{z}')|^2}\varsigma + \frac{\varphi_{z_2}}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}},\\
\eta_{x_2} = \frac{c}{s}\varphi_{z_2}, \quad \varsigma_{x_3} = \frac{1}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}}, \quad \eta_{x_3} = \frac{c}{s},\\
\eta_{x_1x_1} = \alpha\frac{c}{s}\varphi_{z_1z_1}, \quad \eta_{x_2x_2} = \alpha\frac{c}{s}\varphi_{z_2z_2}, \quad \eta_{x_3x_3} = 0, \quad \varsigma_{x_3x_3} = 0,
\end{gathered}
$$

$$
\begin{aligned}
\varsigma_{x_1x_1} &= \alpha^2\Big[\frac{3(\varphi_{z_1}\varphi_{z_1z_1}+\varphi_{z_2}\varphi_{z_2z_1})^2}{(1+|\nabla\varphi(\mathbf{z}')|^2)^2} - \frac{\varphi_{z_1z_1}^2+\varphi_{z_1}\varphi_{z_1z_1z_1}+\varphi_{z_2z_1}^2+\varphi_{z_2}\varphi_{z_2z_1z_1}}{1+|\nabla\varphi(\mathbf{z}')|^2}\Big]\varsigma\\
&\quad + \alpha\frac{\varphi_{z_1z_1}(1-\varphi_{z_1}^2+\varphi_{z_2}^2)-2\varphi_{z_1}\varphi_{z_2}\varphi_{z_1z_2}}{(1+|\nabla\varphi(\mathbf{z}')|^2)^{3/2}},\\
\varsigma_{x_2x_2} &= \alpha^2\Big[\frac{3(\varphi_{z_1}\varphi_{z_1z_2}+\varphi_{z_2}\varphi_{z_2z_2})^2}{(1+|\nabla\varphi(\mathbf{z}')|^2)^2} - \frac{\varphi_{z_1z_2}^2+\varphi_{z_1}\varphi_{z_1z_2z_2}+\varphi_{z_2z_2}^2+\varphi_{z_2}\varphi_{z_2z_2z_2}}{1+|\nabla\varphi(\mathbf{z}')|^2}\Big]\varsigma\\
&\quad + \alpha\frac{\varphi_{z_2z_2}(1+\varphi_{z_1}^2-\varphi_{z_2}^2)-2\varphi_{z_1}\varphi_{z_2}\varphi_{z_1z_2}}{(1+|\nabla\varphi(\mathbf{z}')|^2)^{3/2}}.
\end{aligned}
$$

It is easy to check that $\partial_{x_3}\mathbf{v}^+(\mathbf{x}) > \mathbf{0}$ holds according to the definition of $\mathbf{v}^+(\mathbf{x})$. To prove that $\mathbf{v}^+(\mathbf{x})$ is a supersolution, it suffices to verify that

$$
\tilde{\mathcal{L}}[\mathbf{v}^+(\mathbf{x})] = -\Delta\mathbf{v}^+(\mathbf{x}) + s\mathbf{v}^+_{x_3}(\mathbf{x}) - \tilde{\mathbf{F}}(\mathbf{v}^+(\mathbf{x})) \geq \mathbf{0}.
$$

Throughout the proof, we assume that $\alpha < \varepsilon$. From direct computations and (1.3), we have

$$
\begin{aligned}
\tilde{\mathcal{L}}[\mathbf{v}^+(\mathbf{x})]_i &= \Big(1-\sum_{j=1}^{3}\varsigma_{x_j}^2\Big)U_i''(\varsigma) - \Big(\sum_{j=1}^{3}\varsigma_{x_jx_j}\Big)U_i'(\varsigma)\\
&\quad - \varepsilon\alpha^2\Big(\sum_{j=1}^{2}S_{z_jz_j}\Big)\big[(1-\omega(\eta))p_i + \omega(\eta)U_2^{\beta_i}(\eta)\big]\\
&\quad - 2\varepsilon\alpha\Big(\sum_{j=1}^{2}S_{z_j}\eta_{x_j}\Big)\big[\omega'(\eta)(-p_i+U_2^{\beta_i}(\eta)) + \omega(\eta)\beta_iU_2^{\beta_i-1}(\eta)U_2'(\eta)\big]\\
&\quad - \varepsilon S(\mathbf{z}')\Big\{\Big[\omega''(\eta)\sum_{j=1}^{3}\eta_{x_j}^2 + \omega'(\eta)\sum_{j=1}^{2}\eta_{x_jx_j} - c\omega'(\eta)\Big]\big(-p_i+U_2^{\beta_i}(\eta)\big)\\
&\quad - 2\beta_i\omega'(\eta)U_2'(\eta)U_2^{\beta_i-1}(\eta)\sum_{j=1}^{3}\eta_{x_j}^2\\
&\quad + \omega(\eta)\Big[\beta_i(\beta_i-1)U_2^{\beta_i-2}(\eta)(U_2'(\eta))^2\sum_{j=1}^{3}\eta_{x_j}^2\\
&\quad + \beta_iU_2^{\beta_i-1}(\eta)U_2'(\eta)\sum_{j=1}^{3}\eta_{x_jx_j}\\
&\quad + \beta_iU_2^{\beta_i-1}(\eta)U_2''(\eta)\sum_{j=1}^{3}\eta_{x_j}^2 - c\beta_iU_2^{\beta_i-1}(\eta)U_2'(\eta)\Big]\Big\}\\
&\quad + \Big(\frac{s}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}} - c\Big)U_i'(\varsigma) - \tilde{f}_i(\mathbf{v}^+(\mathbf{x})) + f_i(\mathbf{U}(\varsigma)).
\end{aligned}
$$

Let

$$A_1 := \sup_{\xi\in\mathbb{R}}\Big|\frac{\sum_{j=1}^2 S_{z_jz_j}(\mathbf{z}')}{S(\mathbf{z}')}\Big|, \quad A_2 := \sup_{\xi\in\mathbb{R}}\Big|\frac{\sum_{j=1}^2 S_{z_j}(\mathbf{z}')\eta_{x_j}}{S(\mathbf{z}')}\Big|.$$

By Lemmas 2.1-2.2, we know that $0 \le A_1, A_2 < +\infty$ are well defined. Lemma 2.1 and Lemma 2.2 also imply that there exist positive constants A_3, A_4, A_5 and A_6 such that

$$\big|1-\sum_{j=1}^{3}\varsigma_{x_j}^2\big| \le \alpha(A_3|\varsigma| + A_4\varsigma^2)S(\mathbf{z}') \le \varepsilon\big(A_3|\varsigma| + A_4\varsigma^2\big)S(\mathbf{z}'), \tag{3.8}$$

$$\big|\sum_{j=1}^{2}\varsigma_{x_jx_j}\big| \le \alpha(A_5 + A_6|\varsigma|)S(\mathbf{z}') \le \varepsilon(A_5 + A_6|\varsigma|)S(\mathbf{z}'), \tag{3.9}$$

$$\big|\sum_{j=1}^{2}\eta_{x_jx_j}\big| \le \big|\alpha\frac{c}{s}\sum_{j=1}^{2}\varphi_{z_jz_j}\big| \le 2\alpha C_1. \tag{3.10}$$

Next, we consider three cases.

Case 1: $\varsigma < -X'$ for some $X' > 0$ large enough. Recalling (3.1), $\varsigma < \eta$ holds in this case. Assume that $\varepsilon \le \frac{1}{2(s-c)}$. And without loss of generality, suppose that $\eta \le -X^* < -1$, where $X^* > 0$ is a positive constant such that $U_2(\eta) \le \frac{1}{2}$ if $\eta \le -X^*$. Under these conditions, we know

$$v_2^+(\mathbf{x}) \le \frac{1}{2} + \varepsilon S(\mathbf{z}') < 1 \quad \text{if } \eta < -X^*,$$

which implies that $\tilde f_2(\mathbf{v}^+(\mathbf{x})) = f_2(\mathbf{v}^+(\mathbf{x}))$. Then by (3.3) we have

$$\begin{aligned}
&\tilde{\mathcal{L}}[\mathbf{v}^+(\mathbf{x})]_i\\
&= \Big(1-\sum_{j=1}^{3}\varsigma_{x_j}^2\Big)U_i''(\varsigma) - \Big(\sum_{j=1}^{3}\varsigma_{x_jx_j}\Big)U_i'(\varsigma)\\
&\quad - \varepsilon S(\mathbf{z}')U_2^{\beta_i}(\eta)\Big\{\frac{\alpha^2\sum_{j=1}^2 S_{z_jz_j}}{S(\mathbf{z}')} + \Big(2\alpha\beta_i\frac{\sum_{j=1}^2 S_{z_j}\eta_{x_j}}{S(\mathbf{z}')} + \beta_i\sum_{j=1}^{2}\eta_{x_jx_j}\Big)\frac{U_2'(\eta)}{U_2(\eta)}\\
&\quad + \beta_i(\beta_i-1)\Big(\frac{U_2'(\eta)}{U_2(\eta)}\Big)^2\sum_{j=1}^{3}\eta_{x_j}^2 + \beta_i\frac{U_2''(\eta)}{U_2(\eta)}\sum_{j=1}^{3}\eta_{x_j}^2 - c\beta_i\frac{U_2'(\eta)}{U_2(\eta)}\Big\}\\
&\quad + \Big(\frac{s}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}} - c\Big)U_i'(\varsigma) - f_i(\mathbf{v}^+(\mathbf{x})) + f_i(\mathbf{U}(\varsigma)).
\end{aligned}$$

Recall (3.1) and the monotonicity of the wave profile $\mathbf{U}$. Then by (3.8)-(3.10) we have

$$\begin{aligned}
&\tilde{\mathcal{L}}[\mathbf{v}^+(\mathbf{x})]_i\\
&\ge -\varepsilon S(\mathbf{z}')U_2^{\beta_i}(\eta)(A_3|\varsigma| + A_4\varsigma^2)\frac{|U_i''(\varsigma)|}{U_2^{\beta_i}(\varsigma)} - \varepsilon S(\mathbf{z}')U_2^{\beta_i}(\eta)(A_5 + A_6|\varsigma|)\frac{U_i'(\varsigma)}{U_2^{\beta_i}(\varsigma)}\\
&\quad - \varepsilon S(\mathbf{z}')U_2^{\beta_i}(\eta)\Big\{\alpha^2A_1 + 2\alpha A_2\frac{U_2'(\eta)}{U_2(\eta)} + 2\alpha C_1\frac{U_2'(\eta)}{U_2(\eta)}\\
&\quad + \beta_i\big(\frac{c}{s}\big)^2\Big| - \big(\frac{U_2'(\eta)}{U_2(\eta)}\big)^2 + \frac{U_2''(\eta)}{U_2(\eta)}\Big|\big(1+|\nabla\varphi(\mathbf{z}')|^2\big)
\end{aligned}$$

$$
\begin{aligned}
&+\beta_i^2\big(\frac{c}{s}\big)^2\Big(\frac{U_2'(\eta)}{U_i(\eta)}\Big)^2\big(1+|\nabla\varphi(\mathbf{z}')|^2\big)-\beta_i^2\Big(\frac{U_2'(\eta)}{U_2(\eta)}\Big)^2\\
&+\beta_i^2\Big(\frac{U_2'(\eta)}{U_2(\eta)}\Big)^2-c\beta_i\frac{U_2'(\eta)}{U_2(\eta)}\Big\}\\
&+\Big(\frac{s}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}}-c\Big)U_i'(\varsigma)+f_i(\mathbf{U}(\varsigma))-f_i(\mathbf{v}^+(\mathbf{x})).
\end{aligned}
$$

Since $\lambda_2=\lim_{x\to-\infty}\frac{U_2'(x)}{U_2(x)}$ and $\lambda_2^2=\lim_{x\to-\infty}\frac{U_2''(x)}{U_2(x)}$, we have

$$
\begin{gathered}
-\Big(\frac{U_2'(x)}{U_2(x)}\Big)^2+\frac{U_2''(x)}{U_2(x)}\to 0,\\
\beta_1^2\Big(\frac{U_2'(x)}{U_2(x)}\Big)^2-c\beta_1\frac{U_2'(x)}{U_2(x)}+1-r\to\Pi_1(\beta_1\lambda_2)<0,\\
\beta_2^2\Big(\frac{U_2'(x)}{U_2(x)}\Big)^2-c\beta_2\frac{U_2'(x)}{U_2(x)}\to\Pi_2(\beta_2\lambda_2)<0
\end{gathered}
$$

as $x\to-\infty$. Thus there exists $X_1>0$ large enough such that

$$
\begin{gathered}
\frac{U_2'(x)}{U_2(x)}<\frac{3}{2}\lambda_2,\quad \Big|-\Big(\frac{U_2'(x)}{U_2(x)}\Big)^2+\frac{U_2''(x)}{U_2(x)}\Big|<-\frac{1}{16}\Pi_i(\beta_i\lambda_2),\\
\beta_1^2\Big(\frac{U_2'(x)}{U_2(x)}\Big)^2-c\beta_1\frac{U_2'(x)}{U_2(x)}+1-r<\frac{1}{2}\Pi_1(\beta_1\lambda_2),\\
\beta_2^2\Big(\frac{U_2'(x)}{U_2(x)}\Big)^2-c\beta_2\frac{U_2'(x)}{U_2(x)}<\frac{1}{2}\Pi_2(\beta_2\lambda_2)
\end{gathered}
$$

for any $x<-X_1$. For the above positive constants A_3,A_4,A_5 and A_6, it follows from [30, Lemma 1.1] that there exists $X_2>0$ large enough such that

$$
\begin{gathered}
(A_3|x|+A_4x^2)\frac{|U_i''(x)|}{U_2^{\beta_i}(x)}<-\frac{1}{16}\Pi_i(\beta_i\lambda_2),\\
(A_5+A_6|x|)\frac{|U_i'(x)|}{U_2^{\beta_i}(x)}<-\frac{1}{16}\Pi_i(\beta_i\lambda_2)
\end{gathered}
$$

for any $x<-X_2$. Also there exists $\alpha_1\in(0,\beta^*)$ small enough such that

$$
\alpha^2A_1+3\alpha A_2\lambda_2+3\alpha C_1\lambda_2<-\frac{1}{16}\Pi_i(\beta_i\lambda_2),\quad \forall\alpha\in(0,\alpha_1),\ i=1,2.
$$

For the reaction term f_i, we have

$$
\begin{aligned}
f_i\big(\mathbf{v}^+(\mathbf{x})\big)-f_i\big(\mathbf{U}(\varsigma)\big)&=\Big(\sum_{j=1}^2 f_{ij}\left(\theta_i\mathbf{v}^+(\mathbf{x})+(1-\theta_i)\mathbf{U}(\varsigma)\right)U_2^{\beta_j}(\eta)\Big)\varepsilon S(\mathbf{z}')\\
&=\Big(\sum_{j=1}^2 f_{ij}\left(\mathbf{U}(\varsigma)+\varepsilon\theta_iS(\mathbf{z}')\mathcal{U}^{\boldsymbol{\beta}}(\eta)\right)U_2^{\beta_j}(\eta)\Big)\varepsilon S(\mathbf{z}'),
\end{aligned}
$$

where $\theta_i\in(0,1)$, $i=1,2$. If $i=1$, then $0<U_1(\xi)<U_2(\xi)<1$ yields

$$
f_{12}\left(\mathbf{U}(\varsigma)+\varepsilon\theta_1S(\mathbf{z}')\mathcal{U}^{\boldsymbol{\beta}}(\eta)\right)=r\left(U_1(\varsigma)+\varepsilon\theta_1S(\mathbf{z}')U_2^{\beta_1}(\eta)\right)\le r(1+\varepsilon s)U_2^{\beta_1}(\eta).
$$

It follows that

$$
f_1(\mathbf{v}^+(\mathbf{x}))-f_1(\mathbf{U}(\varsigma))
$$

$$
\begin{aligned}
&\le \Big(f_{11}\left(\mathbf{U}(\varsigma)+\varepsilon\theta_1 S(\mathbf{z}')\mathcal{U}^{\boldsymbol\beta}(\eta)\right)U_2^{\beta_1}(\eta)+r(1+\varepsilon s)U_2^{\beta_1}(\eta)U_2^{\beta_2}(\eta)\Big)\varepsilon S(\mathbf{z}')\\
&\le \Big(f_{11}\left(\mathbf{U}(\varsigma)+\varepsilon\theta_1 S(\mathbf{z}')\mathcal{U}^{\boldsymbol\beta}(\eta)\right)+r(1+\varepsilon s)U_2^{\beta_2}(\eta)\Big)\varepsilon S(\mathbf{z}')U_2^{\beta_1}(\eta),
\end{aligned}
$$

and $f_{11}\left(\mathbf{U}(x)+\varepsilon\theta_1 S(\mathbf{z}')\mathcal{U}^{\boldsymbol\beta}(y)\right)+r(1+\varepsilon s)U_2^{\beta_2}(y)\to 1-r$ as $x,y\to-\infty$. If $i=2$, then

$$
\begin{aligned}
&f_2\big(\mathbf{v}^+(\mathbf{x})\big)-f_2\big(\mathbf{U}(\varsigma)\big)\\
&=\Big(\sum_{j=1}^2 f_{2j}\left(\mathbf{U}(\varsigma)+\varepsilon\theta_2 S(\mathbf{z}')\mathcal{U}^{\boldsymbol\beta}(\eta)\right)U_2^{\beta_j}(\eta)\Big)\varepsilon S(\mathbf{z}')\\
&=\Big(\sum_{j=1}^2 f_{2j}\left(\mathbf{U}(\varsigma)+\varepsilon\theta_2 S(\mathbf{z}')\mathcal{U}^{\boldsymbol\beta}(\eta)\right)(U_2(\eta))^{\beta_j-\beta_2}\Big)\varepsilon S(\mathbf{z}')U_2^{\beta_2}(\eta).
\end{aligned}
$$

Note that $(U_2(x))^{\beta_1-\beta_2}\to 0$ as $x\to-\infty$, also we have

$$
\sum_{j=1}^2 f_{2j}\left(\mathbf{U}(x)+\varepsilon\theta_2 S(\mathbf{z}')\mathcal{U}^{\boldsymbol\beta}(y)\right)(U_2(y))^{\beta_j-\beta_2}\to 0\quad\text{as } x,y\to-\infty.
$$

Then we know that there exists $X_3>0$ large enough such that

$$
\begin{gathered}
f_{11}\left(\mathbf{U}(x)+\varepsilon\theta_1 S(\mathbf{z}')\mathcal{U}^{\boldsymbol\beta}(y)\right)+r(1+\varepsilon s)U_2^{\beta_2}(y)-(1-r)<-\frac{1}{16}\Pi_1(\beta_1\lambda_2),\\
\sum_{j=1}^2 f_{2j}\left(\mathbf{U}(x)+\varepsilon\theta_2 S(\mathbf{z}')\mathcal{U}^{\boldsymbol\beta}(y)\right)(U_2(y))^{\beta_j-\beta_2}<-\frac{1}{16}\Pi_1(\beta_1\lambda_2)
\end{gathered}
$$

for $x,y<-X_3$.

Let $X'=\max\{\frac{s}{c}X^*,X_1,X_2,X_3\}$, then for $\varsigma<-X'$ we have

$$
\begin{aligned}
\tilde{\mathcal{L}}[\mathbf{v}^+(\mathbf{x})]_i &\ge -U_2^{\beta_i}(\eta)\varepsilon S(\mathbf{z}')\left(A_3|\varsigma|+A_4\varsigma^2\right)\frac{|U_i''(\varsigma)|}{U_2^{\beta_i}(\varsigma)}\\
&\quad -U_i^{\beta_i}(\eta)\varepsilon S(\mathbf{z}')\left(A_5+A_6|\varsigma|\right)\frac{U_i'(\varsigma)}{U_2^{\beta_i}(\varsigma)}\\
&\quad -\varepsilon S(\mathbf{z}')U_2^{\beta_i}(\eta)\Big\{\alpha^2A_1+3\alpha A_2\lambda_2+3\alpha C_1\lambda_2\\
&\quad +\beta_i\Big|-\Big(\frac{U_2'(\eta)}{U_2(\eta)}\Big)^2+\frac{U_2''(\eta)}{U_2(\eta)}\Big|+\beta_i^2\Big(\frac{U_2'(\eta)}{U_2(\eta)}\Big)^2\\
&\quad -\beta_i^2\Big(\frac{U_2'(\eta)}{U_2(\eta)}\Big)^2+\beta_i^2\Big(\frac{U_2'(\eta)}{U_2(\eta)}\Big)^2-c\beta_i\frac{U_2'(\eta)}{U_2(\eta)}\Big\}\\
&\quad -\varepsilon S(\mathbf{z}')\Big(\sum_{j=1}^2 f_{ij}\left(\mathbf{U}(\varsigma)+\varepsilon\theta_i S(\mathbf{z}')\mathcal{U}^{\boldsymbol\beta}(\eta)\right)U_2^{\beta_j}(\eta)\Big)\\
&\ge \varepsilon S(\mathbf{z}')U_2^{\beta_i}(\eta)\Big(\frac{1}{16}\Pi_i(\beta_i\lambda_2)+\frac{1}{16}\Pi_i(\beta_i\lambda_2)+\frac{1}{16}\Pi_i(\beta_i\lambda_2)\\
&\quad +\frac{1}{16}\Pi_i(\beta_i\lambda_2)-\frac{1}{2}\Pi_i(\beta_i\lambda_2)+\frac{1}{16}\Pi_i(\beta_i\lambda_2)\Big)>0.
\end{aligned}
$$

Case 2: $\varsigma > X''$ for some $X'' > 0$ large enough. Without loss of generality, suppose that $\eta > 1$. By [30, Lemma 1.4], we can take $X_1' > 0$ large enough such that

$$(A_3|x| + \alpha A_4 x^2)|U_i''(x)| < -\frac{q_i}{8} \quad \text{and} \quad (A_5 + A_6|x|)|U_i'(x)| < -\frac{q_i}{8}$$

for all $x > X_1'$ and $i = 1, 2$.

Fix a constant $\alpha_2 \in (0, \beta^*)$ such that $\alpha^2 A_1 p_1 < \min_{i=1,2}\{-\frac{q_i}{8}\}$ for any α in $(0, \alpha_2)$. For the reaction term f_i, we have

$$f_i(\mathbf{v}^+(\mathbf{x})) - f_i(\mathbf{U}(\varsigma)) = \Big(\sum_{j=1}^{2} f_{ij}\big(\mathbf{U}(\varsigma) + \theta_i \varepsilon S(\mathbf{z}')\mathbf{p}\big)p_j\Big)\varepsilon S(\mathbf{z}'), \quad i = 1, 2.$$

Since $U_i(x) \to 1$ as $x \to +\infty$ for $i = 1, 2$, there exists $X_2' > 0$ large enough such that for all $\varepsilon \in (0, \frac{\varepsilon_1}{p_1(s-c)})$ (see (2.4) for ε_1), it holds

$$1 - \varepsilon_1 < U_i(x) + \varepsilon\theta_i S(\mathbf{z}')p_i < 1 + \varepsilon_1, \quad x > X_2', \quad i = 1, 2,$$

and then

$$\sum_{j=1}^{2} f_{ij}\big(\mathbf{U}(x) + \varepsilon\theta_i S(\mathbf{z}')\mathbf{p}\big)p_j < \frac{1}{2}q_i, \quad x > X_2', \ i = 1, 2.$$

According to the definition of $\tilde{f}_2$, we know

$$\tilde{f}_2(\mathbf{v}^+(\mathbf{x})) \le f_2(\mathbf{v}^+(\mathbf{x})) + b(\varepsilon S(\mathbf{z}'))^2 p_1 p_2.$$

Take $X'' = \max\{X_1', X_2', 1\}$, then for $\varsigma > X''$, we have

$$\begin{aligned}
&\tilde{\mathcal{L}}[\mathbf{v}^+(\mathbf{x})]_i \\
&= \Big(1 - \sum_{j=1}^{3} \varsigma_{x_j}^2\Big)U_i''(\varsigma) - \Big(\sum_{j=1}^{3} \varsigma_{x_j x_j}\Big)U_i'(\varsigma) - \varepsilon\alpha^2 S(\mathbf{z}')\frac{\sum_{j=1}^{2} S_{z_j}\eta_{x_j}}{S(\mathbf{z}')}p_i \\
&\quad + \Big(\frac{s}{\sqrt{1 + |\nabla\varphi(\mathbf{z}')|^2}} - c\Big)U_i'(\varsigma) - \tilde{f}_i(\mathbf{v}^+(\mathbf{x})) + f_i(\mathbf{U}(\varsigma)) \\
&\ge -\varepsilon S(\mathbf{z}')(A_3|\varsigma| + A_4|\varsigma|^2)|U_i''(\varsigma)| \\
&\quad - \varepsilon S(\mathbf{z}')(A_5 + A_6|\varsigma|)|U_i'(\varsigma)| - \varepsilon S(\mathbf{z}')\alpha^2 A_1 p_1 \\
&\quad - \varepsilon S(\mathbf{z}')\Big(\sum_{j=1}^{2} f_{ij}\left(\mathbf{U}(\varsigma) + \theta_i\varepsilon S(\mathbf{z}')\mathbf{p}\right)p_j\Big) - b(\varepsilon S(\mathbf{z}'))^2 p_1 p_2 \\
&\ge \varepsilon S(\mathbf{z}')\left(\frac{q_i}{8} + \frac{q_i}{8} + \frac{q_i}{8} - \frac{q_i}{2} - b\varepsilon p_1 p_2(s-c)\right) > 0
\end{aligned}$$

provided that $\varepsilon < \min_{i=1,2}\{-\frac{q_i}{8bp_1p_2(s-c)}\}$.

Case 3: $-X' \le \varsigma \le X''$. Define $u_* := \min_{-X' \le x \le X''} \min_{i=1,2} U_i'(x)$ and

$$\begin{gathered}
M_{ij} := \sup_{\mathbf{u} \in [-\varepsilon_1 \mathbf{1}, (1+\varepsilon_1)\mathbf{1}]} f_{ij}(\mathbf{u}), \quad M_0 := \sup_{1 \le i,j \le 2} M_{ij}, \\
M_1 := \sup_{x \in \mathbb{R},\ i=1,2} |U_i'(x)|, \quad M_2 := \sup_{x \in \mathbb{R},\ i=1,2} |x||U_i'(x)|, \\
M_3 := \sup_{x \in \mathbb{R},\ i=1,2} |x||U_i''(x)|, \quad M_4 := \sup_{x \in \mathbb{R},\ i=1,2} x^2|U_i''(x)|.
\end{gathered}$$

We have

$$\tilde{\mathcal{L}}[\mathbf{v}^+(\mathbf{x})]_i \ge -\alpha S(\mathbf{z}')(A_3|\varsigma| + \alpha A_4|\varsigma|^2)|U_i''(\varsigma)| - \varepsilon S(\mathbf{z}')(A_5 + A_6|\varsigma|)|U_i'(\varsigma)|$$

$$
\begin{aligned}
&- \varepsilon S(\mathbf{z}')\alpha^2 A_1 p_i - 2\varepsilon\alpha S(\mathbf{z}')A_2(p_i+N_1)\\
&- \varepsilon S(\mathbf{z}')\left\{(|\omega''(\eta)| + \alpha\,|\Delta_{\mathbf{z}'}\varphi(\mathbf{z}')|)\,p_i + N_1\,|\Delta_{\mathbf{z}'}\varphi(\mathbf{z}')| + N_2\right\}\\
&+ S(\mathbf{z}')u_* - b(\varepsilon S(\mathbf{z}'))^2 p_1 p_2\\
&- \varepsilon S(\mathbf{z}')\Big(\sum_{j=1}^{2} f_{ij}\left(\mathbf{U}(\varsigma) + \varepsilon\theta_i\left(\omega(\eta)\mathbf{p} + (1-\omega(\eta))\mathcal{U}^{\boldsymbol{\beta}}(\eta)\right)\right)\Big)\\
&\times\Big(\omega(\eta)p_j + (1-\omega(\eta))U_2^{\beta_j}(\eta)\Big)\\
\geq{}& S(\mathbf{z}')\Big\{-\alpha A_3M_3 - \alpha A_4M_4 - \alpha A_5M_1 - \alpha A_6M_2\\
&- \alpha A_1p_1 - 2\alpha A_2(p_1+N_1) - \varepsilon\mathcal{A} + u_* - b\varepsilon(s-c)p_1p_2 - 2\varepsilon M_0p_1\Big\},
\end{aligned}
$$

where $\mathcal{A} := \big(\sup_{x\in\mathbb{R}}|\omega''(x)| + \sup_{\mathbf{z}'\in\mathbb{R}^2}|\Delta_{\mathbf{z}'}\varphi(\mathbf{z}')|\big)p_1 + (\sup_{\mathbf{z}'\in\mathbb{R}^2}|\Delta_{\mathbf{z}'}\varphi(\mathbf{z}')|)N_1 + N_2$. See (2.1) for N_1 and N_2. Let

$$
\alpha < \alpha_3 := \frac{u_*}{2[A_3M_3 + A_4M_4 + A_5M_1 + A_6M_2 + A_1p_1 + 2A_2(N_1+p_1)]},
$$
$$
\varepsilon < \varepsilon_2 := \frac{u_*}{2[\mathcal{A} + 2M_0p_1 + b(s-c)p_1p_2]},
$$

then $\tilde{\mathcal{L}}[\mathbf{v}^+(\mathbf{x})]_i > 0$, $i = 1,2$. Combining the three cases above, we have proved that $\mathbf{v}^+(\mathbf{x})$ is a supersolution to (1.9).

Next we prove that $\mathbf{v}^-(\mathbf{x}) < \mathbf{v}^+(\mathbf{x})$. Let

$$
\xi(\mathbf{x}) = \frac{c}{s}(x_3 + h(\mathbf{x}')), \quad \nu(\mathbf{x}) = \frac{1}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}}(x_3 + h(\mathbf{x}')),
$$

and recall that

$$
\eta(\mathbf{x}) = \frac{c}{s}(x_3 + \varphi(\mathbf{z}')/\alpha), \quad \varsigma(\mathbf{x}) = \frac{1}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}}(x_3 + \varphi(\mathbf{z}')/\alpha).
$$

If $\varsigma(\mathbf{x}) \geq \xi(\mathbf{x})$, then it is obvious that $\mathbf{v}^-(\mathbf{x}) < \mathbf{v}^+(\mathbf{x})$ since $U_i(y)(i=1,2)$ are monotone increasing in y. Thus we need only consider the case $\varsigma(\mathbf{x}) < \xi(\mathbf{x})$. It follows from the definitions of $\varsigma(\mathbf{x})$ and $\xi(\mathbf{x})$ that

$$
\begin{aligned}
\varsigma(\mathbf{x}) - \xi(\mathbf{x}) &= \Big(\frac{1}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}} - \frac{c}{s}\Big)(x_3 + h(\mathbf{x}')) + \frac{\varphi(\mathbf{z}')/\alpha - h(\mathbf{x}')}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}}\\
&= \frac{1}{s}S(\mathbf{z}')(x_3 + h(\mathbf{x}')) + \frac{\varphi(\mathbf{z}')/\alpha - h(\mathbf{x}')}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}} < 0.
\end{aligned}
$$

Since $\frac{1}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}} - \frac{c}{s} > 0$ and $\nu_1 \leq \frac{\varphi(\mathbf{z}') - h(\mathbf{z}')}{S(\mathbf{z}')} \leq \nu_2$, we have

$$
x_3 + h(\mathbf{x}') < -s\frac{\varphi(\mathbf{z}')/\alpha - h(\mathbf{x}')}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}}\frac{1}{S(\mathbf{z}')} \leq -\frac{c\nu_1}{\alpha} < 0,
$$

which implies that $\varsigma(\mathbf{x}) < \xi(\mathbf{x}) \leq \frac{c}{s}(-\frac{c\nu_1}{\alpha}) = -\frac{c^2\nu_1}{\alpha s} < 0$ and $\frac{s}{c}\eta(\mathbf{x}) < \varsigma(\mathbf{x}) < \eta(\mathbf{x}) < 0$. Then we have

$$
\begin{aligned}
v_i^+(\mathbf{x}) - v_i^-(\mathbf{x}) &\geq U_i(\nu(\mathbf{x})) + \varepsilon S(\mathbf{z}')U_2^{\beta_i}(\eta(\mathbf{x})) - U_i(\xi(\mathbf{x}))\\
&= \Big(\frac{1}{\sqrt{1+|\nabla\varphi(\mathbf{z}')|^2}} - \frac{c}{s}\Big)(x_3 + h(\mathbf{x}'))U_i'\big(\theta_i\nu(\mathbf{x}) + (1-\theta_i)\xi(\mathbf{x})\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon S(\mathbf{z}')U_2^{\beta_i}(\eta(\mathbf{x}))\\
&=\frac{1}{s}S(\mathbf{z}')\xi(\mathbf{x})U_i'\left(\theta_i\nu(\mathbf{x})+(1-\theta_i)\xi(\mathbf{x})\right)+\varepsilon S(\mathbf{z}')U_2^{\beta_i}(\eta(\mathbf{x}))
\end{aligned}
$$

for some $\theta_i\in(0,1)$, $i=1,2$. Since $x_3+h(\mathbf{x}')<0$, we have $\nu(\mathbf{x})<\varsigma(\mathbf{x})<\xi(\mathbf{x})<\eta(\mathbf{x})<0$, and hence

$$
U_i'(\theta_i\nu(\mathbf{x})+(1-\theta_i)\xi(\mathbf{x}))\le\begin{cases}L_2e^{\min\{\lambda_1,2\lambda_2\}\xi(\mathbf{x})}, & i=1,\\ L_2e^{\lambda_2\xi(\mathbf{x})}, & i=2,\end{cases}
$$

$$
U_2^{\beta_i}(\eta(\mathbf{x}))\ge L_1^{\beta_i}e^{\lambda_1\beta_i\eta(\mathbf{x})}\ge L_1e^{\lambda_1\beta_i\xi(\mathbf{x})},\quad i=1,2.
$$

Then for $i=2$, we have

$$
\begin{aligned}
v_2^+(\mathbf{x})-v_2^-(\mathbf{x})&\ge\frac{1}{s}S(\mathbf{z}')\xi(\mathbf{x})U_2'\left(\theta_2\nu(\mathbf{x})+(1-\theta_2)\xi(\mathbf{x})\right)+\varepsilon S(\mathbf{z}')U_2^{\beta_2}(\eta(\mathbf{x}))\\
&\ge S(\mathbf{z}')\Big(\frac{L_2}{s}\xi(\mathbf{x})e^{\lambda_2\xi(\mathbf{x})}+\varepsilon L_1e^{\lambda_1\beta_2\xi(\mathbf{x})}\Big)\\
&\ge S(\mathbf{z}')e^{\lambda_1\beta_2\xi(\mathbf{x})}\Big(\frac{L_2}{s}\xi(\mathbf{x})e^{(\lambda_2-\lambda_1\beta_2)\xi(\mathbf{x})}+\varepsilon L_1\Big)\\
&\ge S(\mathbf{z}')e^{\lambda_1\beta_2\xi(\mathbf{x})}\Big(\frac{L_2}{s(\lambda_2-\lambda_1\beta_2)^2\xi(\mathbf{x})}\sup_{\omega>0}\omega^2e^{-\omega}+\varepsilon L_1\Big)\\
&\ge S(\mathbf{z}')e^{\lambda_1\beta_2\xi(\mathbf{x})}\Big(-\frac{4L_2\alpha}{c^2e^2(\lambda_2-\lambda_1\beta_2)^2\nu_1}+\varepsilon L_1\Big)>0,
\end{aligned}
$$

provided that

$$
\alpha<\alpha_4:=\min\Big\{\frac{\varepsilon L_1c^2e^2(\lambda_2-\beta_2\lambda_1)^2\nu_1}{4L_2},\quad\frac{\varepsilon L_1c^2e^2\left(\min\{\lambda_1,2\lambda_2\}-\beta_1\lambda_1\right)^2\nu_1}{4L_2}\Big\}.
$$

A similar argument will lead to $v_1^+(\mathbf{x})-v_1^-(\mathbf{x})>0$ for the above α and we omit it. Thus we have that $\mathbf{v}^+(\mathbf{x})>\mathbf{v}^-(\mathbf{x})$ for all $\mathbf{x}\in\mathbb{R}^3$.

Next, we prove (3.5). By Lemma 2.1 we know that for each fixed α, there exits a positive constant $m_\alpha=\frac{1}{\alpha}2\pi m_*\int_0^\infty r^2\tilde\rho(r)dr$ such that

$$
\xi(\mathbf{x})\le\eta(\mathbf{x})\le\xi(\mathbf{x})+m_\alpha,\quad\forall\mathbf{x}\in\mathbb{R}^3. \tag{3.11}
$$

Recall $N_1:=\sup_{x\in\mathbb{R}}\Big|\frac{U_2'(x)}{U_2(x)}\Big|$. Since $U_2(x+y)e^{-N_1y}$ is decreasing in y, we know $U_2(x+y)e^{-N_1y}\le U_2(x)$ for any $y\ge0$. Using this fact and (3.11), we can get

$$
U_2(\xi(\mathbf{x}))\le U_2(\eta(\mathbf{x}))\le U_2(\xi(\mathbf{x}))e^{N_1m_\alpha},\forall\mathbf{x}\in\mathbb{R}^3.
$$

In other words,

$$
1\le\frac{U_2(\eta(\mathbf{x}))}{U_2(\xi(\mathbf{x}))}\le e^{N_1m_\alpha},\quad\forall\mathbf{x}\in\mathbb{R}^3. \tag{3.12}
$$

According to the definition of $\mathbf{v}^+(\mathbf{x})$ and $\mathbf{v}^-(\mathbf{x})$, and using (3.12), it is sufficient to prove that

$$
\lim_{\gamma\to+\infty}\sup_{\mathbf{x}\in D(\gamma)}\frac{|U_i(\varsigma(\mathbf{x}))-U_i(\xi(\mathbf{x}))|}{(U_2(\eta(\mathbf{x})))^{\beta_i}}=0,\quad i=1,2.
$$

We consider three cases.

Case 1: $\xi(\mathbf{x})=\frac{c}{s}(x_3+h(\mathbf{x}'))\to+\infty$. Then we have $0<\xi(\mathbf{x})<\varsigma(\mathbf{x})$ and $\eta(\mathbf{x})<\varsigma(\mathbf{x})<\frac{s}{c}\eta(\mathbf{x})$, which implies that

$$
\frac{|U_i(\varsigma(\mathbf{x}))-U_i(\xi(\mathbf{x}))|}{(U_2(\eta(\mathbf{x})))^{\beta_i}}\to0\quad\text{as }\xi(\mathbf{x})\to+\infty,\ i=1,2.
$$

Case 2: $\xi(\mathbf{x}) = \frac{c}{s}(x_3 + h(\mathbf{x}')) \to -\infty$. We have

$$\varsigma(\mathbf{x}) = \frac{1}{\sqrt{1+|\nabla\varphi(\alpha\mathbf{x}')|^2}}\frac{s}{c}\xi(\mathbf{x}) + \frac{\varphi(\alpha\mathbf{x}')/\alpha - h(\mathbf{x}')}{\sqrt{1+|\nabla\varphi(\alpha\mathbf{x}')|^2}}.$$

Since $0 < \varphi(\alpha\mathbf{x}')/\alpha - h(\mathbf{x}') \le m_\alpha$ and $\frac{c}{s} < \frac{1}{\sqrt{1+|\nabla\varphi(\alpha\mathbf{x}')|^2}} \le 1$, it follows that

$$\frac{s}{c}\xi(\mathbf{x}) \le \varsigma(\mathbf{x}) \le \xi(\mathbf{x}) + m_\alpha, \tag{3.13}$$

which implies $\varsigma(\mathbf{x}) \to -\infty$ as $\xi(\mathbf{x}) \to -\infty$ and vice versa. Thus, we know

$$\xi(\mathbf{x}) \le \eta(\mathbf{x}) \le \frac{c}{s}\varsigma(\mathbf{x}) < \frac{c}{s}\eta(\mathbf{x}) < 0.$$

Using [30, Lemma 1.1], we have

$$\begin{aligned}\frac{U_1'(\xi)}{U_2(\xi)} &= \frac{\lambda_1 e^{\lambda_1\xi} + O(e^{(2\lambda_2-\sigma)\xi})}{A\lambda_2 e^{\lambda_2\xi} + O(e^{(\lambda_1-\sigma)\xi})}\\ &= \frac{\lambda_1 e^{(\lambda_1-\lambda_2)\xi} + O(e^{(\lambda_2-\sigma)\xi})}{A\lambda_2 + O(e^{\lambda_1-\lambda_2-\sigma)\xi})} \to 0 \quad \text{as } \xi \to -\infty.\end{aligned}$$

Thus we know $\mathcal{C} := \sup_{\xi\le 0, i=1,2} \frac{U_i'(\xi)}{U_2(\xi)} < +\infty$. Then by (3.11), (3.13) and (2.3), we have

$$\begin{aligned}&\frac{|U_i(\varsigma(\mathbf{x})) - U_i(\xi(\mathbf{x}))|}{(U_2(\eta(\mathbf{x})))^{\beta_i}}\\ &\le \frac{U_i'(\theta_i\varsigma(\mathbf{x})) + (1-\theta_i)\xi(\mathbf{x}))}{(U_2(\eta(\mathbf{x})))^{\beta_i}}|\varsigma(\mathbf{x}) - \xi(\mathbf{x})|\\ &\le \frac{U_i'(\theta_i\varsigma(\mathbf{x})) + (1-\theta_i)\xi(\mathbf{x}))}{(U_2(\eta(\mathbf{x})))^{\beta_i}}\Big(\frac{s}{c}|\eta(\mathbf{x})| + |\eta(\mathbf{x})| + m_\alpha\Big)\\ &\le \frac{U_i'(\theta_i\varsigma(\mathbf{x})) + (1-\theta_i)\xi(\mathbf{x}))}{U_2(\theta_i\varsigma(\mathbf{x})) + (1-\theta_i)\xi(\mathbf{x}))}\frac{U_2(\eta(\mathbf{x}))}{(U_2(\eta(\mathbf{x})))^{\beta_i}}\Big(\Big(\frac{s}{c}+1\Big)|\eta(\mathbf{x})| + m_\alpha\Big)\\ &\le \mathcal{C}L_2 e^{(1-\beta_i)\lambda_2\eta(\mathbf{x})}\Big(\Big(\frac{s}{c}+1\Big)|\eta(\mathbf{x})| + m_\alpha\Big)\\ &\to 0 \quad \text{as } \eta(\mathbf{x}) \to -\infty.\end{aligned}$$

And notice that $\eta(\mathbf{x}) \to -\infty$ is equivalent to $\xi(\mathbf{x}) \to -\infty$ by (3.11).

Case 3: $\xi(\mathbf{x}) = \frac{c}{s}(x_3 + h(\mathbf{x}'))$ is bounded. It is obvious by (3.11) that $\eta(\mathbf{x})$ is bounded in this case. Suppose $R_0 > 0$ is a constant such that $|\xi(\mathbf{x})| \le R_0$. For each $\gamma > 2R_0$ and $\mathbf{x} \in D(\gamma)$, there must hold $\operatorname{dist}(\mathbf{x}', E) > \gamma$ and thus

$$\lim_{\gamma\to+\infty}\sup_{\operatorname{dist}(\mathbf{x}',E)>\gamma}(\varphi(\mathbf{x}') - h(\mathbf{x}')) = 0, \quad \lim_{\gamma\to+\infty}\sup_{\operatorname{dist}(\mathbf{x}',E)>\gamma} S(\mathbf{x}') = 0,$$

and hence

$$\lim_{\gamma\to+\infty}\sup_{\mathbf{x}\in D(\gamma)}\frac{|U_i(\varsigma(\mathbf{x})) - U_i(\xi(\mathbf{x}))|}{(U_2(\eta(\mathbf{x})))^{\beta_i}} = 0, \quad i = 1, 2.$$

Finally, let $\alpha_0^+(\varepsilon, \boldsymbol{\beta}) = \min\{\varepsilon, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and

$$\varepsilon_0^+(\boldsymbol{\beta}) = \min\Big\{\frac{\varepsilon_1}{p_1(s-c)}, \min_{i=1,2}\{-\frac{q_i}{8(s-c)bp_1p_2}\}, \varepsilon_2\Big\}.$$

The proof is complete. $\square$

Now we give the existence result for traveling curved fronts.

Theorem 3.2. *For each $s > c$, (1.2) admits a pyramidal traveling front $u(\mathbf{x}, t) = \mathbf{V}(\mathbf{x}', x_3 + st)$. $\mathbf{V}(\mathbf{x})$ satisfies (1.6) with $\partial_{x_3}\mathbf{V}(\mathbf{x}) > \mathbf{0}$ and*

$$\mathbf{v}^-(\mathbf{x}) < \mathbf{V}(\mathbf{x}) < \mathbf{v}^+(\mathbf{x}; \varepsilon, \boldsymbol{\beta}, \alpha), \quad \forall \mathbf{x} \in \mathbb{R}^3,$$

where $0 < \varepsilon < \varepsilon_0^+(\boldsymbol{\beta})$ and $0 < \alpha < \alpha_0^+(\varepsilon, \boldsymbol{\beta})$. Furthermore, we have

$$\lim_{\gamma\to+\infty} \sup_{\mathbf{x}\in D(\gamma)} \frac{|V_i(\mathbf{x}) - v_i^-(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} = 0, \quad i = 1, 2. \tag{3.14}$$

Proof. According to the parabolic estimates, we know that there exists a constant $C > 0$ such that the solution $\mathbf{v}(\mathbf{x}; t)$ of (1.4) and (1.5) with $\mathbf{v}_0(\mathbf{x}) \in [\mathbf{0}, \mathbf{1}]$ satisfies

$$\|\mathbf{v}(\cdot, t; \mathbf{v}_0)\|_{C^3(\mathbb{R}^3)} < C, \quad \forall\ t \geq 1.$$

Since $\mathbf{v}^-$ is a subsolution, we have $\mathbf{v}(\mathbf{x}, t_1; \mathbf{v}^-) < \mathbf{v}(\mathbf{x}, t_2; \mathbf{v}^-)$ for all $\mathbf{x} \in \mathbb{R}^3$ and $0 < t_1 < t_2$, see [33] for more details. Thus,

$$\mathbf{V}(\mathbf{x}) := \lim_{t\to+\infty} \mathbf{v}(\mathbf{x}, t; \mathbf{v}^-), \quad \mathbf{x} \in \mathbb{R}^3, \tag{3.15}$$

is well defined and independent of ε, α, and $\boldsymbol{\beta}$. It follows that $\mathbf{v}(\cdot, t; \mathbf{v}^-)$ converges monotonically to $\mathbf{V}(\cdot)$ under the norm $\|\cdot\|_{C^2_{loc}(\mathbb{R}^3)}$, namely,

$$\lim_{t\to+\infty} \|\mathbf{v}(\cdot, t; \mathbf{v}^-) - \mathbf{V}(\cdot, \cdot)\|_{C^2_{loc}(\mathbb{R}^3)} = 0.$$

By the comparison principle, we know $\mathbf{v}^-(\mathbf{x}) < \mathbf{V}(\mathbf{x}) < \mathbf{v}^+(\mathbf{x}; \varepsilon, \boldsymbol{\beta}, \alpha)$. And the proof of (3.14) is similar to that of [44]. In view of the monotonicity of $\mathbf{v}^-(\mathbf{x})$ on the variable x_3, we come to the conclusion that $\partial_{x_3}\mathbf{V}(\mathbf{x}) \geq \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^3$. Then the strong maximum principle implies the strict inequality. □

4. Stability of traveling curved fronts

This section discusses the stability of the pyramidal traveling fronts constructed in Section 3 by improving the arguments of Taniguchi [35] and Wang [43, 45]. Consider the Cauchy problem

$$\begin{gathered} \frac{\partial}{\partial t}\tilde{\mathbf{w}}(\xi, \eta, t) - \Delta\tilde{\mathbf{w}}(\xi, \eta, t) + \bar{s}\frac{\partial}{\partial \eta}\tilde{\mathbf{w}}(\xi, \eta, t) - \mathbf{F}(\tilde{\mathbf{w}}) = 0, \\ \tilde{\mathbf{w}}(\xi, \eta, 0) = \tilde{\mathbf{w}}_0(\xi, \eta), \end{gathered} \tag{4.1}$$

where $\tilde{\mathbf{w}}(\xi, \eta, t) = (\tilde{w}_1(\xi, \eta, t), \tilde{w}_2(\xi, \eta, t))$ and $(\xi, \eta) \in \mathbb{R}^2, t > 0$. The following theorem is established in [30, 31].

Theorem 4.1. *Assume $b > 0$ and $r > 1$. Then for each $\bar{s} > c$, there exists a steady state $\boldsymbol{\Phi}(\xi, \eta; \bar{s})$ of (4.1) satisfying $\boldsymbol{\Phi}(\xi, \eta; \bar{s}) > \tilde{\mathbf{v}}^-(\xi, \eta)$ and*

$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2>R^2} \Big|\frac{\Phi_i(\xi, \eta) - \tilde{v}_i^-(\xi, \eta)}{(\tilde{v}_2^-(\xi, \eta))^{\beta_i}}\Big| = 0, \quad i = 1, 2,$$

where

$$\tilde{\mathbf{v}}^-(\xi, \eta) = \mathbf{U}\Big(\frac{c}{\bar{s}}\Big(\eta + \frac{\sqrt{\bar{s}^2 - c^2}}{c}|\xi|\Big)\Big),$$

and $\beta_i (i = 1, 2)$ *is defined as in* (3.4)*. Furthermore, for any* $\tilde{\mathbf{w}}_0(\xi, \eta) \in [\mathbf{0}, \mathbf{1}]$ *with* $\tilde{\mathbf{w}}_0 \in C(\mathbb{R}^2, \mathbb{R}^2)$ *and*

$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2>R^2} \Big| \frac{\tilde{v}_i^-(\xi,\eta) - \tilde{w}_{0i}(\xi,\eta)}{(\tilde{v}_2^-(\xi,\eta))^{\beta_i}} \Big| = 0, \quad i = 1, 2,$$

the solution $\tilde{\mathbf{w}}(\xi, \eta, t; \tilde{\mathbf{w}}_0)$ *of* (4.1) *with initial data* $\tilde{\mathbf{w}}_0$ *satisfies*

$$\lim_{t\to\infty} \Big\| \frac{w_i(\cdot,\cdot,t;\tilde{\mathbf{w}}_0) - \Phi_i(\cdot,\cdot+st)}{(\tilde{v}_2^-(\cdot,\cdot))^{\beta_i}} \Big\|_{L^\infty(\mathbb{R}^2)} = 0, \quad i = 1, 2.$$

For any subset $\mathcal{D} \in \mathbb{R}^3$, we denote the characteristic function of $\mathcal{D}$ by $\chi_{\mathcal{D}}$, namely,

$$\chi_{\mathcal{D}}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \mathcal{D}, \\ 0, & \mathbf{x} \in \mathcal{D}^c, \end{cases}$$

where $\mathcal{D}^c$ denotes the complementary set of $\mathcal{D}$. Let $h_{ij}(\mathbf{x}, t) \in C(\mathbb{R}^3 \times \mathbb{R})$ $(i, j = 1, 2)$ be given continuous functions satisfying

$$0 \le h_{ij}(\mathbf{x}, t) \le M_{ij},\ i \ne j; \quad \sup_{\mathbf{x}\in\mathbb{R}^3, t>0} |h_{ii}(\mathbf{x}, t)| \le M_{ii},$$

where $M_{ij} \in \mathbb{R}$ $(i, j = 1, 2)$ are constants. Now consider the linear system

$$\begin{gathered} L_t[w_i(\mathbf{x}, t)] - \sum_{j=1}^{2} h_{ij}(\mathbf{x}, t) w_j(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^3,\ t > 0, \\ w_i(\mathbf{x}, 0) = w_{i,0}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3,\ i = 1, 2, \end{gathered} \tag{4.2}$$

where $L_t := \frac{\partial}{\partial t} - \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2} + s\frac{\partial}{\partial x_3}$ is a linear operator. We have the following result.

Lemma 4.2. *Let* $\mathbf{w}(\mathbf{x}, t) = (w_1(\mathbf{x}, t), w_2(\mathbf{x}, t))$ *be a solution of* (4.2)*. Then there exist positive constants* $\tilde{A}, \tilde{B}$ *and* λ_0 *such that*

$$\begin{aligned} \sup_{i=1,2} \sup_{\mathbf{x}\in D(2\gamma)} \frac{|w_i(\mathbf{x}, t)|}{(v_2^-(\mathbf{x}))^{\beta_i}} &\le 6e^{\lambda_0 t} \frac{\widetilde{A}\pi}{\widetilde{B}} \int_{\frac{\sqrt{3}\gamma}{3t} - \frac{N_1}{2\widetilde{B}}}^{+\infty} e^{-\widetilde{B}r^2} dr \max_{i=1,2} \sup_{\mathbf{x}\in D(\gamma)^c} \frac{|w_{i,0}(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} \\ &\quad + e^{\lambda_0 t} \widetilde{A} \Big(\frac{\pi}{\widetilde{B}}\Big)^{3/2} \max_{i=1,2} \sup_{\mathbf{x}\in D(\gamma)} \frac{|w_{i,0}(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}}, \quad \forall\ t > 0 \end{aligned} \tag{4.3}$$

for any $\gamma > 0$*, where* $D(\gamma)^c$ *is the complementary set of* $D(\gamma)$*. Moreover, we have*

$$\sup_{i=1,2} \sup_{\mathbf{x}\in\mathbb{R}^3} \frac{|w_i(\mathbf{x}, t)|}{(v_2^-(\mathbf{x}))^{\beta_i}} \le e^{\lambda_0 t} \widetilde{A} \Big(\frac{\pi}{\widetilde{B}}\Big)^{3/2} \max_{i=1,2} \sup_{\mathbf{x}\in\mathbb{R}^3} \frac{|w_{i,0}(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}}, \quad \forall t > 0. \tag{4.4}$$

Proof. Let $\widehat{\mathbf{w}}(\mathbf{x}, t) = e^{-\lambda_0' t} \mathbf{w}(\mathbf{x}, t)$, where $\lambda_0' = \sum_{i=1}^{2} M_{ii}$. Then $\widehat{\mathbf{w}}(\mathbf{x}, t)$ satisfies

$$\begin{gathered} L_t \widehat{w}_i(\mathbf{x}, t) + (\lambda_0' - h_{ii}(\mathbf{x}, t)) \widehat{w}_i(\mathbf{x}, t) - h_{ij}(\mathbf{x}, t) \widehat{w}_j(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^3, t > 0, j \ne i, \\ \widehat{w}_{i,0}(\mathbf{x}) = w_{i,0}(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^3. \end{gathered}$$

Now consider a linear system with constant coefficients

$$\begin{gathered} L_t \widetilde{w}_i(\mathbf{x}, t) + (\lambda_0' - M_{ii}) \widetilde{w}_i(\mathbf{x}, t) - M_{ij} \widetilde{w}_j(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^3, t > 0, j \ne i, \\ \widetilde{w}_{i,0}(\mathbf{x}) = |w_{i,0}(\mathbf{x})|, \quad \mathbf{x} \in \mathbb{R}^3. \end{gathered} \tag{4.5}$$

A similar discussion as [45, Lemma 4.2] implies that

$$|\widehat{w}_i(\mathbf{x}, t)| \le \widetilde{w}_i(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^3,\ t > 0,\ i = 1, 2.$$

By [7, Theorems 2 and 3, Chapter 9], we know that there exists a 2×2 matrix function $\mathbf{\Gamma}(\mathbf{x}, \mathbf{y}, t, s) = \mathbf{\Gamma}(\mathbf{x} - \mathbf{y}, t - s)$ such that

$$\widetilde{\mathbf{w}}(\mathbf{x}, t) = \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y}, t) \cdot \widetilde{\mathbf{w}}_0(\mathbf{y}) d\mathbf{y},$$

and there exist positive numbers $\widetilde{A} \geq 1$ and $\widetilde{B} \leq 1$ such that

$$|\mathbf{\Gamma}(\mathbf{x} - \mathbf{y}, t - s)| = \sum_{i,j=1}^{2} |\Gamma_{ij}| \leq \widetilde{A}(t - s)^{-3/2} \exp\{-\widetilde{B} \frac{|\mathbf{x} - \mathbf{y}|^2}{t - s}\}$$

for any $0 \leq s < t \leq 2$. By the uniqueness of solutions, the solution of (4.5) can be written as

$$\begin{aligned}\widetilde{\mathbf{w}}(\mathbf{x}, t) = &\int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y}_1, 1) d\mathbf{y}_1 \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_1 - \mathbf{y}_2, 1) d\mathbf{y}_2 \cdots \\ &\int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_{k-1} - \mathbf{y}_k, 1) d\mathbf{y}_k \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_k - \mathbf{y}, t - k) \cdot \widetilde{\mathbf{w}}_0(\mathbf{y}) d\mathbf{y}\end{aligned}$$

for any $t > 0$, where $k = \max\{[t - 1], 0\}$ and $[t]$ represents the largest integer no more than t. Therefore

$$\begin{aligned}\frac{\widetilde{w}_i(\mathbf{x}, t)}{(v_2^-(\mathbf{x}))^{\beta_i}} &= \int_{\mathbb{R}^3} \mathbf{\Gamma}_i(\mathbf{x} - \mathbf{y}_1, 1) d\mathbf{y}_1 \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_1 - \mathbf{y}_2, 1) d\mathbf{y}_2 \cdots \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_{k-1} - \mathbf{y}_k, 1) d\mathbf{y}_k \\ &\quad \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_k - \mathbf{y}, t - k) \cdot \Big(\chi_{\mathcal{D}(\gamma)^c}(\mathbf{y}) \frac{\widetilde{\mathbf{w}}_0(\mathbf{y})}{(v_2^-(\mathbf{x}))^{\beta_i}} + \chi_{\mathcal{D}(\gamma)}(\mathbf{y}) \frac{\widetilde{\mathbf{w}}_0(\mathbf{y})}{(v_2^-(\mathbf{x}))^{\beta_i}}\Big) d\mathbf{y} \\ &= \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y}_1, 1) d\mathbf{y}_1 \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_1 - \mathbf{y}_2, 1) d\mathbf{y}_2 \cdots \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_{k-1} - \mathbf{y}_k, 1) d\mathbf{y}_k \\ &\quad \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_k - \mathbf{y}, t - k) \cdot \chi_{\mathcal{D}(\gamma)^c}(\mathbf{y}) \frac{\widetilde{\mathbf{w}}_0(\mathbf{y})}{(v_2^-(\mathbf{x}))^{\beta_i}} d\mathbf{y} \\ &\quad + \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y}_1, 1) d\mathbf{y}_1 \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_1 - \mathbf{y}_2, 1) d\mathbf{y}_2 \cdots \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_{k-1} - \mathbf{y}_k, 1) d\mathbf{y}_k \\ &\quad \int_{\mathbb{R}^3} \mathbf{\Gamma}(\mathbf{y}_k - \mathbf{y}, t - k) \cdot \chi_{\mathcal{D}(\gamma)}(\mathbf{y}) \frac{\widetilde{\mathbf{w}}_0(\mathbf{y})}{(v_2^-(\mathbf{x}))^{\beta_i}} d\mathbf{y} \\ &=: I + II\end{aligned}$$

for any $t > 0$ and $i = 1, 2$. Then by a same computation as in [45, Lemma 4.2], we have

$$\begin{aligned}|I| &\leq \widetilde{A}^{k+1}(1 + t)^{3/2} \Big(\int_{\mathbb{R}^3} e^{-\frac{\widetilde{B}|\mathbf{z}|^2}{2}} d\mathbf{z}\Big)^k \int_{\mathbb{R}^3} t^{-3/2} e^{-\widetilde{B}\frac{|\mathbf{x}-\mathbf{y}|^2}{t}} \chi_{\mathcal{D}(\gamma)^c}(\mathbf{y}) \frac{|\widetilde{\mathbf{w}}_0(\mathbf{y})|}{(v_2^-(\mathbf{x}))^{\beta_i}} d\mathbf{y} \\ &\leq \widetilde{A}^{k+1} e^{2t} \Big(\frac{2\pi}{\widetilde{B}}\Big)^{3k/2} \int_{\mathcal{D}(\gamma)^c} t^{-3/2} e^{-\widetilde{B}\frac{|\mathbf{y}|^2}{t}} \frac{(v_2^-(\mathbf{x} - \mathbf{y}))^{\beta_i}}{(v_2^-(\mathbf{x}))^{\beta_i}} \frac{|\widetilde{\mathbf{w}}_0(\mathbf{x} - \mathbf{y})|}{(v_2^-(\mathbf{x} - \mathbf{y}))^{\beta_i}} d\mathbf{y}.\end{aligned}$$

Moreover, since $U_2(x+y)e^{-N_1 y}$ is decreasing in y, we know $U_2(x+y)e^{-N_1 y} \leq U_2(x)$ for any $y \geq 0$. With the help of this fact, we have

$$\begin{aligned}\frac{v_2^-(\mathbf{x} - \mathbf{y})}{v_2^-(\mathbf{x})} &= \frac{U_2\left(\frac{c}{s}(x_3 - y_3 + h(\mathbf{x}' - \mathbf{y}'))\right)}{U_2\left(\frac{c}{s}(x_3 + h(\mathbf{x}'))\right)} \\ &\leq \frac{U_2\left(\frac{c}{s}\left(x_3 + h(\mathbf{x}') + \max\{1, m_*\}\Sigma_{i=1}^3 |y_i|\right)\right)}{U_2\left(\frac{c}{s}(x_3 + h(\mathbf{x}'))\right)} \\ &\leq e^{N_1 \frac{c}{s} \max\{1, m_*\}\Sigma_{i=1}^3 |y_i|}\end{aligned}$$

$$\le e^{N_1\Sigma_{i=1}^3|y_i|}.$$

Note that if $\mathbf{x}\in\mathcal{D}(2\gamma)$, then $\gamma\le|\operatorname{dist}(\mathbf{x},\Gamma)-\operatorname{dist}(\mathbf{y},\Gamma)|\le|\mathbf{x}-\mathbf{y}|$ for all $\mathbf{y}\in\mathcal{D}(\gamma)^c$. Therefore,

$$\begin{aligned}
|I| &\le C_{k,t}t^{-3/2}\int_{B(\mathbf{x},\gamma)^c} e^{-\widetilde{B}\frac{|\mathbf{x}-\mathbf{y}|^2}{t}}e^{N_1\sum_{i=1}^3|x_i-y_i|}d\mathbf{y}\max_{i=1,2}\sup_{\mathbf{y}\in\mathcal{D}(\gamma)^c}\frac{|\widetilde{w}_{i,0}(\mathbf{y})|}{(v_2^-(\mathbf{y}))^{\beta_i}}\\
&= C_{k,t}t^{-3/2}e^{\frac{3tN_1^2}{4\widetilde{B}}}\int_{B(\mathbf{x},\gamma)^c} e^{-\frac{\widetilde{B}}{t}\sum_{i=1}^3\left(|y_i-x_i|-\frac{tN_1}{2\widetilde{B}}\right)^2}d\mathbf{y}\max_{i=1,2}\sup_{\mathbf{y}\in\mathcal{D}(\gamma)^c}\frac{|\widetilde{w}_{i,0}(\mathbf{y})|}{(v_2^-(\mathbf{y}))^{\beta_i}}\\
&\le C_{k,t}t^{-3/2}e^{\frac{3tN_1^2}{4\widetilde{B}}}3\int_{\mathbf{y}\in\mathbb{R}^3,|y_1|\ge\frac{\sqrt{3}}{3}\gamma-\frac{tN_1}{2\widetilde{B}}} e^{-\frac{\widetilde{B}}{t}|\mathbf{y}|^2}d\mathbf{y}\max_{i=1,2}\sup_{\mathbf{y}\in\mathcal{D}(\gamma)^c}\frac{|\widetilde{w}_{i,0}(\mathbf{y})|}{(v_2^-(\mathbf{y}))^{\beta_i}}\\
&= 6C_{k,t}e^{\frac{3tN_1^2}{4\widetilde{B}}}\frac{\widetilde{A}\pi}{\widetilde{B}}\int_{\frac{\sqrt{3}\gamma}{3t}-\frac{N_1}{2\widetilde{B}}}^{+\infty} e^{-\widetilde{B}r^2}dr\cdot\max_{i=1,2}\sup_{\mathbf{y}\in\mathcal{D}(\gamma)^c}\frac{|\widetilde{w}_{i,0}(\mathbf{y})|}{(v_2^-(\mathbf{y}))^{\beta_i}},
\end{aligned}$$

where $C_{k,t}=\widetilde{A}^{k+1}e^{2t}\left(\frac{2\pi}{\widetilde{B}}\right)^{3k/2}$. Similarly,

$$\begin{aligned}
|II| &\le \widetilde{A}^{k+1}(1+t)^{3/2}\Big(\int_{\mathbb{R}^3}e^{-\frac{\widetilde{B}|\mathbf{z}|^2}{2}}d\mathbf{z}\Big)^k\\
&\quad\times\int_{\mathbb{R}^3}t^{-3/2}e^{-\widetilde{B}\frac{|\mathbf{x}-\mathbf{y}|^2}{t}}e^{N_1\sum_{i=1}^3|x_i-y_i|}\chi_{\mathcal{D}(\gamma)}(\mathbf{y})\frac{|\widetilde{\mathbf{w}}_0(\mathbf{y})|}{(v_2^-(\mathbf{x}))^{\beta_i}}d\mathbf{y}\\
&\le C_{k,t}t^{-3/2}e^{\frac{3tN_1^2}{4\widetilde{B}}}\int_{\mathbb{R}^3}e^{-\widetilde{B}\frac{|\mathbf{x}-\mathbf{y}|^2}{t}}\chi_{\mathcal{D}(\gamma)}(\mathbf{y})d\mathbf{y}\max_{i=1,2}\sup_{\mathbf{y}\in\mathcal{D}(\gamma)}\frac{|\widetilde{w}_{i,0}(\mathbf{y})|}{(v_2^-(\mathbf{y}))^{\beta_i}}\\
&\le C_{k,t}\left(\frac{\pi}{\widetilde{B}}\right)^{3/2}e^{\frac{3tN_1^2}{4\widetilde{B}}}\max_{i=1,2}\sup_{\mathbf{y}\in\mathcal{D}(\gamma)}\frac{|\widetilde{w}_{i,0}(\mathbf{y})|}{(v_2^-(\mathbf{y}))^{\beta_i}}.
\end{aligned}$$

Note that $0<t-k<2$ and let $\lambda_0:=\lambda_0'+2+\ln\left(\frac{2\sqrt{2}\pi\sqrt{\pi}\widetilde{A}}{\widetilde{B}\sqrt{\widetilde{B}}}\right)+\frac{3N_1^2}{4\widetilde{B}}$, then (4.3) follows. To prove the inequality (4.4), we just need to replace $\mathcal{D}(\gamma)$ by $\mathbb{R}^3$ in the term II. Then the proof is complete. □

By a proper coordinate change, we show that the pyramidal traveling front $\mathbf{V}(\mathbf{x})$ converges to a two dimensional V-form front on edges of the pyramid at infinity. For each positive integer $j\in\{1,2,\dots,n\}$, we consider a plane perpendicular to an edge $\Gamma_j=S_j\cap S_{j+1}$. Then the cross section of $-x_3=\max\{h_j(\mathbf{x}'),h_{j+1}(\mathbf{x}')\}$ in this plane is V-shaped. Let $\mathbf{V}^j$ be the two dimensional V-form front as in Theorem 4.1 corresponding to the cross section of $-x_3=\max\{h_j(\mathbf{x}'),h_{j+1}(\mathbf{x}')\}$. We now make some preparations before giving the formulation of $\mathbf{V}^j$.

It is easy to see that the expression of Γ_j is

$$\frac{x_1}{B_j-B_{j+1}}=\frac{x_2}{A_{j+1}-A_j}=\frac{x_3}{m_*(A_jB_{j+1}-A_{j+1}B_j)},\quad x_3<0.$$

Define

$$\begin{gathered}
p_j:=A_jB_{j+1}-A_{j+1}B_j>0,\quad 1\le j\le n;\\
q_j:=\sqrt{(A_j-A_{j+1})^2+(B_j-B_{j+1})^2}>0,\quad 1\le j\le n.
\end{gathered}$$

Then the direction of Γ_j is given by

$$\frac{1}{\sqrt{m_*^2 p_j^2 + q_j^2}}\big(B_{j+1} - B_j, A_j - A_{j+1}, m_*(A_{j+1}B_j - A_j B_{j+1})\big)$$

and the traveling direction of the two dimensional V-form wave $\mathbf{V}^j$ is perpendicular to the direction of Γ_j and given by

$$\frac{1}{q_j\sqrt{m_*^2 p_j^2 + q_j^2}}\left(m_*(B_{j+1} - B_j)p_j, m_*(A_j - A_{j+1})p_j, q_j^2\right).$$

Let s_j be the speed of $\mathbf{V}^j$ and $2\theta_j \in (0, \pi)$ be the angle between S_j and S_{j+1}. It is not difficult to see that

$$s_j \sin\theta_j = c, \quad \sin\theta_j = \frac{\sqrt{m_*^2 p_j^2 + q_j^2}}{q_j\sqrt{1 + m_*^2}}, \quad s_j = \frac{s q_j}{\sqrt{m_*^2 p_j^2 + q_j^2}}.$$

The speed of $\mathbf{V}^j$ toward the x_3-axis equals

$$s_j\sqrt{m_*^2 p_j^2 + q_j^2}/q_j = c\sqrt{1 + m_*^2} = s,$$

which coincides with the speed of $\mathbf{V}$. Now we define a matrix

$$\mathbf{R}_j = \begin{pmatrix} \frac{A_{j+1}-A_j}{q_j} & \frac{m_*(B_{j+1}-B_j)p_j}{q_j\sqrt{m_*^2p_j^2+q_j^2}} & \frac{B_{j+1}-B_j}{\sqrt{m_*^2p_j^2+q_j^2}} \\ \frac{B_{j+1}-B_j}{q_j} & \frac{m_*(A_j-A_{j+1})p_j}{q_j\sqrt{m_*^2p_j^2+q_j^2}} & \frac{A_j-A_{j+1}}{\sqrt{m_*^2p_j^2+q_j^2}} \\ 0 & \frac{q_j}{\sqrt{m_*^2p_j^2+q_j^2}} & -\frac{m_*p_j}{\sqrt{m_*^2p_j^2+q_j^2}} \end{pmatrix}$$

and make the following coordinate transformation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{R}_j \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \mathbf{R}_j^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Define $\mathbf{V}^j(\mathbf{x}) := \mathbf{\Phi}(\xi, \eta; s_j)$. Since $\mathbf{R}_j$ is an orthogonal matrix, the graph of $\mathbf{V}^j(\mathbf{x})$ is the same as that of $\mathbf{\Phi}(\xi, \eta; s_j)$ except the position in the space. Direct calculations show that $\mathbf{V}^j(\mathbf{x})$ satisfies (1.6) with speed s_j for each $j \in \{1, 2, \dots, n\}$. Thus we call $\mathbf{V}^j$ a planar V-form front corresponding to the edge Γ_j. Set

$$Q_j := \{\mathbf{x} \in \mathbb{R}^3 : \operatorname{dist}(\mathbf{x}, \Gamma) = \operatorname{dist}(\mathbf{x}, \Gamma_j)\}.$$

Then we have $\mathbb{R}^3 = \cup_{j=1}^n Q_j$. Define $\hat{\mathbf{V}}(\mathbf{x}) = \max_{1\le j\le n} \mathbf{V}^j(\mathbf{x})$. From the monotonicity of $\mathbf{V}^j$ in x_3, $\hat{\mathbf{V}}(\mathbf{x})$ is strictly monotone in x_3. In addition, $\hat{\mathbf{V}}(\mathbf{x})$ has the following properties.

Lemma 4.3. *$\hat{\mathbf{V}}(\mathbf{x})$ satisfies $\mathbf{v}^-(\mathbf{x}) < \hat{\mathbf{V}}(\mathbf{x}) < \mathbf{V}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$ and*

$$\lim_{\gamma\to\infty} \sup_{\mathbf{x}\in D(\gamma)} \frac{|\hat{V}_i(\mathbf{x}) - v_i^-(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} = 0. \tag{4.6}$$

Proof. By Theorem 4.1 we have

$$\max\big\{\mathbf{U}\big(\frac{c}{s}(x_3 + h_j(\mathbf{x}'))\big), \mathbf{U}\big(\frac{c}{s}(x_3 + h_{j+1}(\mathbf{x}'))\big)\big\} < \mathbf{V}^j(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3.$$

It follows that $\mathbf{v}^-(\mathbf{x}) = \mathbf{U}\left(\frac{c}{s}(x_3 + h(\mathbf{x}'))\right) < \hat{\mathbf{V}}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$. Moreover, taking the left and right sides of the inequality

$$\max\left\{\mathbf{U}\left(\frac{c}{s}(x_3 + h_j(\mathbf{x}'))\right), \mathbf{U}\left(\frac{c}{s}(x_3 + h_{j+1}(\mathbf{x}'))\right)\right\} < \mathbf{v}^-(\mathbf{x})$$

as initial values of (1.6) respectively, we obtain $\mathbf{V}^j(\mathbf{x}) \le \mathbf{V}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$. Then we have $\hat{\mathbf{V}}(\mathbf{x}) \le \mathbf{V}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$. Finally, (4.6) follows from (3.14). □

Let $\mathbf{v}(\mathbf{x}, t; \mathbf{v}_0)$ be the solution of (1.4) with initial value $\mathbf{v}_0 \in [\mathbf{0}, \mathbf{1}]$ which satisfies (1.8). By Lemma 4.2 we have

$$\begin{aligned}
&\max_{i=1,2} \sup_{\mathbf{x}\in D(\gamma)} \frac{|v_i(\mathbf{x}, t; \mathbf{v}_0) - V_i(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} \\
&\le 6e^{\lambda_0 t} \frac{\pi \tilde{A}}{\tilde{B}} \int_{\frac{\sqrt{3}\gamma}{3t} - \frac{N_1}{2\tilde{B}}}^{+\infty} e^{-\tilde{B}r^2} dr \max_{i=1,2} \sup_{\mathbf{x}\in D(\gamma)^c} \frac{|V_i(\mathbf{x}) - v_{i,0}(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} \\
&\quad + e^{\lambda_0 t} \tilde{A} \Big(\frac{\pi}{\tilde{B}}\Big)^{3/2} \max_{i=1,2} \sup_{\mathbf{x}\in D(\gamma)} \frac{|V_i(\mathbf{x}) - v_{i,0}(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}}
\end{aligned}$$

for any $\gamma > 0$ and $t > 0$. It follows that

$$\lim_{\gamma\to+\infty} \sup_{\mathbf{x}\in D(\gamma)} \frac{|v_i(\mathbf{x}, t; \mathbf{v}_0) - V_i(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} = 0,$$

which implies that

$$\lim_{\gamma\to+\infty} \sup_{\mathbf{x}\in D(\gamma), \mathbf{x}\in Q_j} \frac{|v_i(\mathbf{x}, t; \mathbf{v}_0) - V_i^j(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} = 0, \tag{4.7}$$

$$\lim_{\gamma\to+\infty} \sup_{\mathbf{x}\in D(\gamma)} \frac{|v_i(\mathbf{x}, t; \mathbf{v}_0) - \hat{V}_i(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} = 0 \tag{4.8}$$

for any fixed $t > 0$. Using Theorem 4.1 and (4.7), we obtain the following result through a similar discussion as Wang et al. [45, Proposition 4.5] or a slight modification of the proof in [3, Proposition 1].

Proposition 4.4. *Assume that $\mathbf{v}_0 \in [\mathbf{0}, \mathbf{1}]$ satisfies (1.8). For any given $\epsilon > 0$, one can choose a $T^* > 0$ large enough such that*

$$\lim_{R\to\infty} \max_{1\le j\le n} \sup_{|\mathbf{x}|\ge R, \mathbf{x}\in Q_j} \frac{|v_i(\mathbf{x}, t; \mathbf{v}_0) - V_i^j(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} < \epsilon \tag{4.9}$$

for any fixed $t \ge T^*$.

Lemma 4.5. *Assume that $\mathbf{v}_0 \in [\mathbf{0}, \mathbf{1}]$ satisfies (1.8). Let $\mathbf{V}$ be defined by (3.15), then for any given $\epsilon > 0$ one can choose a $T^* > 0$ large enough such that*

$$\lim_{R\to\infty} \max_{i=1,2} \sup_{|\mathbf{x}|\ge R} \frac{|v_i(\mathbf{x}, t; \mathbf{v}_0) - V_i(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} < \epsilon \tag{4.10}$$

for any fixed $t \ge T^*$. In particular, one has

$$\lim_{R\to\infty} \max_{i=1,2} \sup_{|\mathbf{x}|\ge R} \frac{|\hat{V}_i(\mathbf{x}) - V_i(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} = 0. \tag{4.11}$$

Proof. For any $\epsilon > 0$, by taking $\mathbf{v}_0 = \mathbf{V}$ in Proposition 4.4 we have

$$\lim_{R\to\infty} \max_{1\le j\le n} \sup_{|\mathbf{x}|\ge R, \mathbf{x}\in Q_j} \frac{|V_i(\mathbf{x}) - V_i^j(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} < \epsilon.$$

Thus because of the arbitrariness of ϵ, we have

$$\lim_{R\to\infty} \max_{1\le j\le n} \sup_{|\mathbf{x}|\ge R, \mathbf{x}\in Q_j} \frac{|V_i(\mathbf{x}) - V_i^j(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}} = 0. \tag{4.12}$$

Then (4.11) follows from (4.12). And (4.10) follows from (4.12) and (4.9). The proof is complete. □

Equality (4.11) shows that a pyramidal traveling front $\mathbf{V}$ converges to two dimensional V-form fronts $\mathbf{\Phi}$ near edges. By a similar proof to that of Wang [45], we can get the following lemma.

Lemma 4.6. *Let* $\mathbf{V}$ *be defined by* (3.15)*. Then for any* $\delta \in (0, 1)$ *we have*

$$\min_{i=1,2} \inf_{V_i^j(\mathbf{x})\in[\delta,1-\delta]} \frac{\partial}{\partial x_3} V_i^j(\mathbf{x}) > 0, \quad j = 1, 2, 3, \dots, n;$$

$$\min_{i=1,2} \inf_{V_i(\mathbf{x})\in[\delta,1-\delta]} \frac{\partial}{\partial x_3} V_i(\mathbf{x}) > 0.$$

Inequality (3.7) and Lemma 4.6 yield that for any $M > 0$, there exists a positive C_* such that

$$\min_{i=1,2} \inf_{|\frac{c}{s}(x_3+h(\mathbf{x}'))|\le M} \frac{\partial}{\partial x_3} V_i(\mathbf{x}) \ge C_*, \quad \min_{i=1,2} \inf_{|\varsigma(\mathbf{x})|\le M} \frac{\partial}{\partial x_3} v_i^+(\mathbf{x}) \ge C_*.$$

Then we can construct the following two supersolutions.

Lemma 4.7. *Let* $\mathbf{V}$ *be as in* (3.15)*. For any* $0 < \vartheta < \min\big\{\min_{i=1,2}\big(-\frac{\prod_i(\beta_i\lambda_2)}{24\lambda_2 C_1}\big), 1\big\}$, *there exists a positive constant* κ *small enough and a positive constant* $\rho = \rho(\kappa)$ *sufficiently large such that, for any* $\delta \in (0, \delta_0)$ *where*

$$\delta_0 := \min\Big\{\frac{1}{2(s-c+1)}, \frac{\varepsilon_1}{p_1}, \min_{i=1,2}\Big\{-\frac{q_i}{8bp_1p_2}\Big\}\Big\},$$

the function

$$\mathbf{W}^+(\mathbf{x}, t; \delta) = \mathbf{V}(\mathbf{x}', x_3 + \xi + \rho\delta(1 - e^{-\kappa t})) + \delta e^{-\kappa t}\big(\omega(\theta)\mathbf{p} + (1 - \omega(\theta))\,\mathcal{U}^{\boldsymbol{\beta}}(\theta)\big)$$

is a supersolution to (1.9)*, where* $\theta = \frac{c}{s}(x_3+\xi+\rho\delta(1-e^{-\kappa t})+\varphi(\vartheta\mathbf{x}')/\vartheta)$ *and* $\xi \in \mathbb{R}$ *is a constant. See* (2.4) *for* ε_1.

Proof. By a direct computation, we have

$$\begin{aligned}
&\tilde{\mathcal{L}}[\mathbf{W}^+]_i \\
&= \rho\delta\kappa e^{-\kappa t}\partial_{x_3} V_i - \delta\kappa e^{-\kappa t}[\omega(\theta)p_i + (1 - \omega(\theta))U_2^{\beta_i}(\theta)] \\
&\quad + \frac{c}{s}\rho\delta^2\kappa e^{-2\kappa t}\big[\big(p_i - U_2^{\beta_i}(\theta)\big)\omega'(\theta) + \beta_i(1 - \omega(\theta))U_2^{\beta_i-1}(\theta)U_2'(\theta)\big] \\
&\quad - \delta e^{-\kappa t}\Big\{\omega''(\theta)\big(p_i - U_2^{\beta_i}(\theta)\big)\sum_{j=1}^3 \theta_{x_j}^2 + \omega'(\theta)\big(p_i - U_2^{\beta_i}(\theta)\big)\sum_{j=1}^3 \theta_{x_jx_j} \\
&\quad - c\omega'(\theta)\big(p_i - U_2^{\beta_i}(\theta)\big) - 2\beta_i\omega'(\theta)U_2^{\beta_i-1}(\theta)U_2'(\theta)\sum_{j=1}^3 \theta_{x_j}^2
\end{aligned}$$

$$
\begin{aligned}
&+(1-\omega(\theta))\Big[\beta_i(\beta_i-1)U_2^{\beta_i-2}(\theta)(U_2'(\theta))^2\sum_{j=1}^3\theta_{x_j}^2+\beta_iU_2^{\beta_i-1}(\theta)U_2''(\theta)\sum_{j=1}^3\theta_{x_j}^2\\
&+\beta_iU_2^{\beta_i-1}(\theta)U_2'(\theta)\sum_{j=1}^3\theta_{x_jx_j}-c\beta_iU_2^{\beta_i-1}(\theta)U_2'(\theta)\Big]\Big\}\\
&-f_i(\mathbf{W}^+)-g_i(\mathbf{W}^+)+f_i(\mathbf{V}).
\end{aligned}
$$

Then the rest of proof is almost the same as that of [30, Lemma 4.2], we omit it. □

By a similar argument to that of Lemma 4.7, we obtain the following lemma.

Lemma 4.8. *There exists a positive κ constant small enough and a positive constant $\rho=\rho(\kappa)$ sufficiently large such that, for any $\delta\in(0,\delta_0)$ (δ_0 is given in Lemma 4.7), $\xi\in\mathbb{R}$ and $0<\alpha<\min\{\alpha_0^+(\varepsilon,\boldsymbol{\beta}),\min_{i=1,2}\big(-\frac{\prod_i(\beta_i\lambda_2)}{24C_1\lambda_2}\big)\}$, the function*

$$
\mathbf{w}^+(\mathbf{x},t;\delta)=\mathbf{v}^+(\mathbf{x}',x_3+\xi+\rho\delta(1-e^{-\kappa t});\varepsilon,\boldsymbol{\beta},\alpha)+\delta e^{-\kappa t}\Big(\omega(\hat\theta)\mathbf{p}+\big(1-\omega(\hat\theta)\big)\mathcal{U}^{\boldsymbol{\beta}}(\hat\theta)\Big)
$$

is a supersolution to (1.9)*, where $\hat\theta=\frac{c}{s}(x_3+\xi+\rho\delta(1-e^{-\kappa t})+\varphi(\alpha\mathbf{x}')/\alpha)$ and $\xi\in\mathbb{R}$ is a constant.*

Lemma 4.9. *Let* $\mathbf{V}$ *be defined by* (3.15)*. Then it satisfies*

$$
\lim_{R\to\infty}\sup_{|x_3+h(\mathbf{x}')|\ge R}\frac{\partial_{x_3}V_i(\mathbf{x})}{(v_2^-(\mathbf{x}))^{\beta_i}}=0,\quad i=1,2. \tag{4.13}
$$

Proof. We split up the proof into two steps.

Step 1. We prove that

$$
\lim_{R\to\infty}\sup_{x_3+h(\mathbf{x}')\ge R}\frac{\partial_{x_3}V_i(\mathbf{x})}{(v_2^-(\mathbf{x}))^{\beta_i}}=0,\quad i=1,2.
$$

It is sufficient to prove that $\lim_{R\to\infty}\sup_{x_3+h(\mathbf{x}')\ge R}\partial_{x_3}V_i(\mathbf{x})=0$, since

$$
\lim_{R\to\infty}\sup_{x_3+h(\mathbf{x}')\ge R}v_2^-(\mathbf{x})=\lim_{R\to\infty}\sup_{x_3+h(\mathbf{x}')\ge R}U_2\Big(\frac{c}{s}(x_3+h(\mathbf{x}'))\Big)=1.
$$

Obviously, the assumption $x_3+h(\mathbf{x}')\to\infty$ implies that $\operatorname{dist}(\mathbf{x},\Gamma)\to\infty$. It follows that

$$
\lim_{R\to\infty}\sup_{x_3+h(\mathbf{x}')\ge R}|\mathbf{V}(\mathbf{x})-\mathbf{1}|=0,
$$

and thus $\lim_{R\to\infty}\sup_{x_3+h(\mathbf{x}')\ge R}|\mathbf{F}(\mathbf{V}(\mathbf{x}))|=0$. Applying the interior Schauder's estimate to

$$
-\Delta V_i+s\partial_{x_3}V_i=f_i(\mathbf{V})\quad\text{in } B(\mathbf{x}_0,2),\quad\forall\mathbf{x}_0\in\mathbb{R}^3,\ i=1,2,
$$

we have

$$
\lim_{R\to\infty}\sup_{i=1,2}\{\|V_i\|_{W^{2,p}(B(\mathbf{x}_0,1))}\big|\mathbf{x}_0\in\mathbb{R}^3,|x_3^0+h(\mathbf{x}_0')|\ge R\}=0
$$

for $p>3$. Therefore

$$
\lim_{R\to\infty}\sup_{x_3+h(\mathbf{x}')\ge R}\partial_{x_3}V_i(\mathbf{x})=0,\quad i=1,2.
$$

Step 2. We prove that

$$\lim_{R\to\infty}\sup_{x_3+h(\mathbf{x}')\le -R}\frac{\partial_{x_3}V_i(\mathbf{x})}{(v_2^-(\mathbf{x}))^{\beta_i}}=0,\quad i=1,2.$$

Note that the set $\{\mathbf{x}\in\mathbb{R}^3|x_3+h(\mathbf{x}')\le -R\}\subseteq D(R)=\{\mathbf{x}\in\mathbb{R}^3|\operatorname{dist}(\mathbf{x},\Gamma)\ge R\}$. It follows from (3.14) that for $i=1,2$,

$$\lim_{R\to\infty}\sup_{x_3+h(\mathbf{x}')\le -R}\frac{\left|V_i(\mathbf{x})-v_i^-(\mathbf{x})\right|}{(v_2^-(\mathbf{x}))^{\beta_i}}\le\lim_{R\to+\infty}\sup_{\mathbf{x}\in D(R)}\frac{|V_i(\mathbf{x})-v_i^-(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}}=0.$$

Then a similar discussion as that in [30, Lemma 4.6] shows that (4.13) holds for the case $x_3+h(\mathbf{x}')\to-\infty$. The proof is complete. □

To prove the stability result, we also need the following lemma, which can be referred to [45, Lemma 4.8].

Lemma 4.10. *Assume that $\delta\in(0,\varepsilon_1)$, where ε_1 is in (2.4). For any $\mathbf{x}\in\mathbb{R}^3$ with*

$$\delta\le\hat{V}_i(\mathbf{x})=\max_{1\le j\le n}V_i^j(\mathbf{x})\le 1-\delta,$$

we have

$$\inf_{0<\varrho<\varrho_0}\frac{\hat{V}_i(\mathbf{x}',x_3+\varrho)-\hat{V}_i(\mathbf{x})}{\varrho}\ge\min_{1\le j\le n}\min_{i=1,2}\inf_{\frac{\delta}{2}\le V_i^j(\mathbf{x})\le 1-\frac{\delta}{2}}\frac{\partial}{\partial x_3}V_i^j(\mathbf{x})>0,$$

where ϱ_0 is a positive constant depending only on δ.

Let $\mathbf{v}^+(\mathbf{x};\varepsilon,\boldsymbol{\beta},\alpha)$ be as in Lemma 3.1. Define

$$\mathbf{V}^*(\mathbf{x}):=\lim_{t\to\infty}\tilde{\mathbf{v}}(\mathbf{x},t;\mathbf{v}^+),\quad \mathbf{x}\in\mathbb{R}^3.$$

Since $\mathbf{v}^+$ is a supersolution, $\mathbf{V}^*(\mathbf{x})$ is well defined and satisfies

$$-\Delta\mathbf{V}^*+s\mathbf{V}^*_{x_3}-\tilde{\mathbf{F}}(\mathbf{V}^*)=\mathbf{0},$$

and it may depend on ε,α and $\boldsymbol{\beta}$. By the comparison principle, we have

$$\mathbf{v}^-(\mathbf{x})<\hat{\mathbf{V}}(\mathbf{x})<\mathbf{V}(\mathbf{x})\le\mathbf{V}^*(\mathbf{x})<\mathbf{v}^+(\mathbf{x};\varepsilon,\boldsymbol{\beta},\alpha),\quad \mathbf{x}\in\mathbb{R}^3.$$

From (3.5), we have

$$\lim_{\gamma\to+\infty}\sup_{\mathbf{x}\in D(\gamma)}\frac{|V_i^*(\mathbf{x})-v_i^-(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}}=0,\quad i=1,2.$$

Then applying Proposition 4.4 to $\mathbf{V}^*$ we obtain

$$\lim_{R\to\infty}\max_{i=1,2}\sup_{|\mathbf{x}|\ge R}\frac{|\hat{V}_i(\mathbf{x})-V_i^*(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}}=0. \tag{4.14}$$

It follows immediately that

$$\lim_{R\to\infty}\max_{i=1,2}\sup_{|\mathbf{x}|\ge R}\frac{|V_i(\mathbf{x})-V_i^*(\mathbf{x})|}{(v_2^-(\mathbf{x}))^{\beta_i}}=0. \tag{4.15}$$

The next lemma says that $\mathbf{V}^*(\mathbf{x})$ is independent of ε,α and $\boldsymbol{\beta}$.

Lemma 4.11. *One has $\mathbf{V}(\mathbf{x})=\mathbf{V}^*(\mathbf{x})$ in $\mathbb{R}^3$.*

Proof. Assume that $\mathbf{V}(\mathbf{x}) \not\equiv \mathbf{V}^*(\mathbf{x})$. We take $\delta \in (\frac{\delta_0}{2}, \delta_0)$, where δ_0 is given in Lemma 4.7. By the definition of $\mathbf{V}^*$ and (4.15), there exists $\lambda > 0$ sufficiently large such that

$$\begin{aligned}\mathbf{V}^*(\mathbf{x}) &\le \mathbf{V}(\mathbf{x}', x_3 + \lambda) + \delta(v_2^-(\theta))^{\beta_i} \\ &\le \mathbf{V}(\mathbf{x}', x_3 + \lambda) + \delta\big(\omega(\theta)\mathbf{p} + (1-\omega(\theta))\mathcal{U}^{\boldsymbol{\beta}}(\theta)\big),\end{aligned}$$

where $\theta = \frac{c}{s}(x_3 + \lambda + \varphi(\vartheta \mathbf{x}')/\vartheta)$. Then the comparison principle implies that

$$\mathbf{V}^*(\mathbf{x}) \le \mathbf{W}^+(\mathbf{x}', x_3 + \lambda, t; \delta), \quad \mathbf{x} \in \mathbb{R}^3,\ t > 0.$$

Letting $t \to \infty$, we obtain

$$\mathbf{V}^*(\mathbf{x}) \le \mathbf{V}(\mathbf{x}', x_3 + \lambda + \rho\delta), \quad \mathbf{x} \in \mathbb{R}^3.$$

Define

$$\Lambda := \inf\{\lambda > 0 : \mathbf{V}^*(\mathbf{x}) \le \mathbf{V}(\mathbf{x}', x_3 + \lambda), \quad \mathbf{x} \in \mathbb{R}^3\}.$$

Then $\Lambda \ge 0$, and $\mathbf{V}^*(\mathbf{x}) \le \mathbf{V}(\mathbf{x}', x_3 + \Lambda)$ for all $\mathbf{x} \in \mathbb{R}^3$. The assumption $\mathbf{V}(\mathbf{x}) \not\equiv \mathbf{V}^*(\mathbf{x})$ implies that $\Lambda \ne 0$. Thus the strong maximum principle yields that either $\mathbf{V}^*(\mathbf{x}) \equiv \mathbf{V}(\mathbf{x}', x_3 + \Lambda)$ or $\mathbf{V}^*(\mathbf{x}) < \mathbf{V}(\mathbf{x}', x_3 + \Lambda)$. We assert that the former case is impossible. To see this, we choose a sequence $\{\mathbf{x}'_m\}_{m\in\mathbb{N}} \subseteq \mathbb{R}^2$ satisfying $h(\mathbf{x}'_m) \to \infty$ and $\operatorname{dist}(\mathbf{x}'_m, E) \to \infty$. Then by the fact that $\mathbf{v}^- < \mathbf{V} \le \mathbf{V}^* < \mathbf{v}^+$, we have

$$\lim_{m\to\infty} \mathbf{V}^*(\mathbf{x}'_m, -h(\mathbf{x}'_m)) = \mathbf{U}(0), \quad \liminf_{m\to\infty} \mathbf{V}(\mathbf{x}'_m, -h(\mathbf{x}'_m) + \Lambda) \ge \mathbf{U}\big(\frac{c}{s}\Lambda\big),$$

which contradicts $\mathbf{V}^*(\mathbf{x}) \equiv \mathbf{V}(\mathbf{x}', x_3 + \Lambda)$. Now we assume that

$$\mathbf{V}^*(\mathbf{x}) < \mathbf{V}(\mathbf{x}', x_3 + \Lambda), \quad \mathbf{x} \in \mathbb{R}^3.$$

By Lemma 4.9, for any fixed $\rho > 0$ defined in Lemma 4.7, we can take a $R_* = R_*(\rho) > 0$ large enough such that

$$\sup_{|x_3 + h(\mathbf{x}')| \ge R_* - \frac{\Lambda}{2}} \frac{\partial_{x_3} V_i(\mathbf{x})}{(v_2^-(\mathbf{x}))^{\beta_i}} \le \frac{1}{3\rho}, \quad i = 1, 2.$$

Define

$$\mathcal{D} := \{\mathbf{x} \in \mathbb{R}^3 : |x_3 + h(\mathbf{x}')| \le R^*\}.$$

Choose a constant σ satisfying $0 < \sigma < \{\frac{\delta_0}{2}, \frac{\Lambda}{4\rho}, \frac{\ln\frac{3}{2}}{2N_1\rho}\}$, where $N_1 = \sup_{\mathbf{x}\in\mathbb{R}^3} \frac{U_2'(\mathbf{x})}{U_2(\mathbf{x})}$. Using Lemma 4.10 for $\mathbf{x} \in \mathcal{D}$, we have

$$\hat{V}_i(\mathbf{x}', x_3 + \Lambda) - \hat{V}_i(\mathbf{x}) \ge \min\{\varrho_0, \Lambda\} \min_{1\le j\le n} \min_{i=1,2} \inf_{\frac{\delta_*}{2} \le V_i^j(\mathbf{x}) \le 1 - \frac{\delta_*}{2}} \frac{\partial}{\partial x_3} V_i^j(\mathbf{x}) > 0,$$

where

$$\delta_* = \min_{i=1,2} \min\big\{\frac{\delta_0}{2}, 1 - \max_{1\le j\le n} \sup_{\mathbf{x}\in\mathcal{D}} V_i^j(\mathbf{x}', x_3 + \Lambda), \min_{1\le j\le n} \inf_{\mathbf{x}\in\mathcal{D}} V_i^j(\mathbf{x})\big)\big\},$$

and ϱ_0 is defined in Lemma 4.10 associated with δ_*. Thus, for $\mathbf{x} \in \mathcal{D}$, it follows that

$$\begin{aligned}&\lim_{R\to\infty} \inf_{|\mathbf{x}|>R, \mathbf{x}\in\mathcal{D}} (V_i(\mathbf{x}', x_3 + \Lambda) - V_i^*(\mathbf{x})) \\ &\ge \lim_{R\to\infty} \inf_{|\mathbf{x}|>R, \mathbf{x}\in\mathcal{D}} \Big(\hat{V}_i(\mathbf{x}', x_3 + \Lambda) - V_i^*(\mathbf{x})\Big) \\ &\ge \lim_{R\to\infty} \inf_{|\mathbf{x}|>R, \mathbf{x}\in\mathcal{D}} \Big(\hat{V}_i(\mathbf{x}', x_3 + \Lambda) - \hat{V}_i(\mathbf{x})\Big) + \lim_{R\to\infty} \inf_{|\mathbf{x}|>R, \mathbf{x}\in\mathcal{D}} \Big(\hat{V}_i(\mathbf{x}) - V_i^*(\mathbf{x})\Big)\end{aligned}$$

$$> 0,$$

since $\lim_{R\to\infty}\sup_{|\mathbf{x}|>R,\mathbf{x}\in\mathcal{D}}\big(V_i^*(\mathbf{x})-\hat V_i(\mathbf{x})\big)=0$ by (4.14). Thus we can choose a small σ such that

$$V_i(\mathbf{x}', x_3+\Lambda-2\rho\sigma) > V_i^*(\mathbf{x}) \quad \text{in } \mathcal{D}.$$

In the domain $\mathbb{R}^3\setminus\mathcal{D}$, we have

$$\begin{aligned}
&\frac{V_i(\mathbf{x}', x_3+\Lambda-2\rho\sigma)-V_i(\mathbf{x}', x_3+\Lambda)}{U_2^{\beta_i}\big(\frac{c}{s}(x_3+\Lambda-2\rho\sigma+\varphi(\alpha\mathbf{x}')/\alpha)\big)}\\
&\ge \frac{V_i(\mathbf{x}', x_3+\Lambda-2\rho\sigma)-V_i(\mathbf{x}', x_3+\Lambda)}{\big(v_2^-(\mathbf{x}', x_3+\Lambda-2\rho\sigma)\big)^{\beta_i}}\\
&\ge \frac{-2\rho\sigma\int_0^1\partial_{x_3}V_i(\mathbf{x}', x_3+\Lambda-2\rho\sigma\tau)d\tau}{\big(v_2^-(\mathbf{x}', x_3+\Lambda-2\rho\sigma)\big)^{\beta_i}}\\
&= -2\rho\sigma\Big(\frac{v_2^-(\mathbf{x}', x_3+\Lambda)}{v_2^-(\mathbf{x}', x_3+\Lambda-2\rho\sigma)}\Big)^{\beta_i}\int_0^1\frac{\partial_{x_3}V_i(\mathbf{x}', x_3+\Lambda-2\rho\sigma\tau)}{(v_2^-(\mathbf{x}', x_3+\Lambda))^{\beta_i}}d\tau\\
&\ge -2\rho\sigma e^{2N_1\rho\sigma\beta_i}\int_0^1\frac{\partial_{x_3}V_i(\mathbf{x}', x_3+\Lambda-2\rho\sigma\tau)}{\big(v_2^-(\mathbf{x}', x_3+\Lambda-2\rho\sigma\tau)\big)^{\beta_i}}d\tau\\
&\ge -2\rho\sigma\frac{3}{2}\frac{1}{3\rho}=-\sigma.
\end{aligned}$$

In other other words,

$$V_i(\mathbf{x}', x_3+\Lambda)\le V_i(\mathbf{x}', x_3+\Lambda-2\rho\sigma)+\sigma U_2^{\beta_i}\Big(\frac{c}{s}\,(x_3+\Lambda-2\rho\sigma+\varphi(\alpha\mathbf{x}')/\alpha)\Big).$$

Combining the above two cases, we obtain

$$V_i^*(\mathbf{x})\le V_i(\mathbf{x}', x_3+\Lambda-2\rho\sigma)+\sigma U_2^{\beta_i}\Big(\frac{c}{s}\,(x_3+\Lambda-2\rho\sigma+\varphi(\alpha\mathbf{x}')/\alpha)\Big),$$

for all $\mathbf{x}\in\mathbb{R}^3$. Then Lemma 4.7 and the comparison principle yield

$$V_i^*(\mathbf{x})\le W_i^+(\mathbf{x}', x_3+\Lambda-2\rho\sigma, t;\sigma),\quad \mathbf{x}\in\mathbb{R}^3,\ t>0.$$

Letting $t\to\infty$, we have

$$V_i^*(\mathbf{x})\le V_i(\mathbf{x}', x_3+\Lambda-2\rho\sigma),\quad \mathbf{x}\in\mathbb{R}^3.$$

This contradicts the definition of Λ. Thus $\Lambda\equiv 0$. The proof is complete. □

Theorem 4.12. *Assume $b>0$ and $r>1$. Fix a couple of $\beta_1,\beta_2\in(0,\beta^*)$ with $\beta_2<\beta_1$. Assume that $\mathbf{v}_0\in C(\mathbb{R}^3,\mathbb{R}^2)$ with $\mathbf{v}_0\in[\mathbf{0},\mathbf{1}]$, $\mathbf{v}_0(\mathbf{x})\ge\mathbf{v}^-(\mathbf{x})$ for $\mathbf{x}\in\mathbb{R}^3$ and*

$$\lim_{\gamma\to\infty}\sup_{\mathbf{x}\in D(\gamma)}\frac{|v_{i,0}(\mathbf{x})-V_{i,0}(\mathbf{x})|}{(v_2(\mathbf{x}))^{\beta_i}}=0,\quad i=1,2.$$

Then the solution of (1.2) *with initial value $\mathbf{v}_0$ satisfies*

$$\lim_{t\to\infty}\Big\|\frac{v_i(\cdot,t;\mathbf{v}_0)-V_i(\cdot)}{(v_2(\mathbf{x}))^{\beta_i}}\Big\|_{L^\infty(\mathbb{R}^3)}=0,\quad i=1,2. \tag{4.16}$$

Proof. Under the condition $\mathbf{v}_0\in[\mathbf{0},\mathbf{1}]$, the solution $\tilde{\mathbf{v}}(\mathbf{x},t;\mathbf{v}_0)$ of (1.9) and (1.10) is also the solution of (1.4) and (1.5), namely, $\tilde{\mathbf{v}}(\mathbf{x},t;\mathbf{v}_0)\equiv\mathbf{v}(\mathbf{x},t;\mathbf{v}_0)$. For a random

$m > 1$ and any given $\epsilon > 0$, by Proposition 4.4, Lemma 4.5 and Lemma 4.11, we can choose a positive constant R_0 such that

$$\begin{aligned}
&V_i(\mathbf{x}) - \frac{\epsilon}{m}(v_2^-(\mathbf{x}))^{\beta_i} \le \tilde{v}_i(\mathbf{x},t;\mathbf{v}^-) \le V_i(\mathbf{x}), \quad |\mathbf{x}| \ge R_0,\ i=1,2;\\
&V_i(\mathbf{x}) \le \tilde{v}_i(\mathbf{x},t;\mathbf{v}^+) \le V_i(\mathbf{x}) + \frac{\epsilon}{m}(v_2^-(\mathbf{x}))^{\beta_i}, \quad |\mathbf{x}| \ge R_0,\ i=1,2,
\end{aligned}$$

for any fixed $t \ge T^*$. On the domain $B_{R_0}(\mathbf{0}) = \{\mathbf{x}||\mathbf{x}| \le R_0\}$, applying the interior Shauder estimate to $\tilde{v}_i(\mathbf{x},t;\mathbf{v}^-) - V_i(\mathbf{x})$ and $\tilde{v}_i(\mathbf{x},t;\mathbf{v}^+) - V_i(\mathbf{x})$, and noticing that $\mathbf{v}^-$ is bounded on $B_{R_0}(\mathbf{0})$, we have

$$\lim_{t\to\infty} \sup_{\mathbf{x}\in B_{R_0}(\mathbf{0})} \Big|\frac{\tilde{v}_i(\cdot,t;\mathbf{v}^-) - V_i(\cdot)}{(v_2(\mathbf{x}))^{\beta_i}}\Big| = 0, \quad \lim_{t\to\infty} \sup_{\mathbf{x}\in B_{R_0}(\mathbf{0})} \Big|\frac{\tilde{v}_i(\cdot,t;\mathbf{v}^+) - V_i(\cdot)}{(v_2(\mathbf{x}))^{\beta_i}}\Big| = 0$$

for $i = 1, 2$. Since $m > 1$ can be taken arbitrarily large, we have

$$\lim_{t\to\infty} \Big\|\frac{\tilde{v}_i(\cdot,t;\mathbf{v}^-) - V_i(\cdot)}{(v_2(\mathbf{x}))^{\beta_i}}\Big\|_{L^\infty(\mathbb{R}^3)} = 0, \quad \lim_{t\to\infty} \Big\|\frac{\tilde{v}_i(\cdot,t;\mathbf{v}^+) - V_i(\cdot)}{(v_2(\mathbf{x}))^{\beta_i}}\Big\|_{L^\infty(\mathbb{R}^3)} = 0,$$

for $i = 1, 2$. Let $\delta \in (0, \frac{\delta_0}{2})$ and $\varepsilon < \min\{\varepsilon_0^+(\boldsymbol{\beta}), \frac{\delta_0}{4s}\}$. Taking $\alpha \in (0, \alpha^+(\varepsilon, \boldsymbol{\beta}))$ small enough and using Lemma 4.5, we have

$$v_i(\mathbf{x}, T^*; \mathbf{v}_0) \le V_i(\mathbf{x}) + \delta(v_2^-(\mathbf{x}))^{\beta_i} \le v_i^+(\mathbf{x}) + \delta(v_2^-(\mathbf{x}))^{\beta_i}.$$

A similar discussion as in [35] shows that

$$\begin{aligned}
&\lim_{t\to\infty} \Big\|\frac{v_i(\mathbf{x},t;\mathbf{v}^-) - V_i(\mathbf{x})}{(v_2(\mathbf{x}))^{\beta_i}}\Big\|_{L^\infty(\mathbb{R}^3)} = 0,\\
&\lim_{t\to\infty} \Big\|\frac{v_i(\mathbf{x},t;\mathbf{v}_0) - V_i(\mathbf{x})}{(v_2(\mathbf{x}))^{\beta_i}}\Big\|_{L^\infty(\mathbb{R}^3)} = 0
\end{aligned}$$

for $i = 1, 2$. Take $\hat{t}$ large enough such that

$$v_i(\mathbf{x}, \hat{t}; \mathbf{v}^-) \le v_i(\mathbf{x}, \hat{t}; \mathbf{v}^+) < V_i(\mathbf{x}) + \delta(v_2^-(\mathbf{x}))^{\beta_i}, \quad t \ge \hat{t},\ i = 1, 2. \tag{4.17}$$

Since $\mathbf{w}^+(\mathbf{x}, t; \delta)$ is a supersolution, there exists a $\tilde{t} > 0$ such that

$$v_i(\mathbf{x}, t + T^* + 1; \mathbf{v}_0) \le V_i(\mathbf{x}', x_3 + \rho\delta) + \delta e^{-\lambda_0 \hat{t}}(v_2^-(\mathbf{x}))^{\beta_i}, \quad t \ge \tilde{t},\ i = 1, 2.$$

Denote $\mathbf{v}^{+,\delta}(\mathbf{x}) = \mathbf{v}^+(\mathbf{x}', x_3 + \rho\delta)$. Then Lemma 4.2 implies that

$$v_i(\mathbf{x}, \tilde{t} + T^* + \hat{t} + 1; \mathbf{v}_0) - \mathbf{v}_i^{+,\delta}(\mathbf{x}) \le \delta(v_2^-(\mathbf{x}))^{\beta_i}, \quad i = 1, 2.$$

Then by (4.17), we have

$$v_i(\mathbf{x}, \tilde{t} + T^* + \hat{t} + 1; \mathbf{v}_0) \le V_i(\mathbf{x}', x_3 + \rho\delta) + 2\delta(v_2^-(\mathbf{x}))^{\beta_i}.$$

Thus it follows from Lemma 4.7 that

$$\mathbf{v}(\mathbf{x}, t + \tilde{t} + T^* + \hat{t} + 1; \mathbf{v}_0) \le \mathbf{W}^+(\mathbf{x}', x_3 + \rho\delta, t; 2\delta).$$

Therefore, by letting $t \to \infty$ we have

$$\begin{aligned}
&V_i(\mathbf{x}) \le v_i(\mathbf{x}, t; \mathbf{v}_0) \le V_i(\mathbf{x}', x_3 + \rho\delta + 2\rho\delta) + 2\delta(v_2^-(\mathbf{x}))^{\beta_i},\\
&0 \le v_i(\mathbf{x}, t; \mathbf{v}_0) - V_i(\mathbf{x}) \le V_i(\mathbf{x}', x_3 + \rho\delta + 2\rho\delta) - V_i(\mathbf{x}) + 2\delta(v_2^-(\mathbf{x}))^{\beta_i}\\
&\le (v_2^-(\mathbf{x}))^{\beta_i}\Big(\frac{\partial_{x_3} V_i(\mathbf{x}', x_3 + 3\rho\delta\tau)}{(v_2^-(\mathbf{x}', x_3 + 3\rho\delta\tau))^{\beta_i}} \frac{(v_2^-(\mathbf{x}', x_3 + 3\rho\delta\tau))^{\beta_i}}{(v_2^-(\mathbf{x}))^{\beta_i}} + 2\delta\Big)\\
&\le (v_2^-(\mathbf{x}))^{\beta_i}\left(M_* e^{\frac{c}{s} N_1 3\rho\delta} 3\rho + 2\right)\delta,
\end{aligned}$$

where $\tau \in (0,1)$ and

$$M_* = \max_{\mathbf{x}\in\mathbb{R}^3} \frac{\partial_{x_3} V_i(\mathbf{x})}{(v_2^-(\mathbf{x}))^{\beta_i}}.$$

Because of the arbitrariness of δ, (4.16) follows. Thus the proof is complete. □

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