Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 113, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

CONTINUOUS DEPENDENCE OF RECURRENT SOLUTIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

HAIJING QIU, YAN WANG

Abstract. Existence, uniqueness and asymptotic stability of recurrent solutions have been investigated extensively for semi-linear stochastic differential equations. In this article, we show that the unique recurrent solution depends continuously on the coefficients of the equation in the compact-open topology or uniform topology, which depends on how the coefficients vary with respect to the parameter.

1. Introduction

Recurrent motions are important in dynamical systems since almost all interesting and complicated dynamics happens on the recurrent set. On the other hand, recurrence is also extensively studied for Markov processes since it is closely related to invariant measures (or stationary distributions) of Markov processes. Because of these facts, there are many studies on recurrent solutions for stochastic differential equations (SDEs) so far. Among others, let us mention some works which are closely related to our work: see Khasminskii [13], Morozan [18], Da Prato and Tudor [8], Chen et al [6] and Ji et al [12] for periodic solutions for SDEs, see Halanay [10], Da Prato and Tudor [8], Arnold and Tudor [1], Bezandry and Diagana [2], Wang and Liu [20], Liu and Wang [17], Li et al [14] for almost periodic solutions for SDEs, see Fu and Liu [9], Chen and Lin [7], Wang and Gao [19], Liu and Sun [16], Chang and Tang [4] for almost automorphic solutions for SDEs, and see the very recent work Cheban and Liu [5] (see also Liu and Liu [15] for Lévy noise case) for general recurrent solutions for SDEs.

When the system (or the SDE) depends on a parameter, a natural question is: when the parameter varies, does the SDE still possess recurrent solutions and whether the recurrent solutions vary continuously with respect to the parameter? Answers to this question will enable us to understand deeply the robustness or global bifurcation phenomenon of the system in consideration, which further shed light on our insight into the real problem the system describes. Therefore, to partly investigate this question we consider in present paper continuous dependence of recurrent solutions (including stationary solutions as special case) on the coefficients of stochastic differential equations, in contrast to and based on existence and uniqueness of recurrent solutions that are studied in above mentioned works.

2010 *Mathematics Subject Classification.* 35B30, 37L15, 37B20, 60H10.
Key words and phrases. Stochastic differential equation; continuous dependence; recurrent solution.

Submitted August 2, 2020. Published November 17, 2020.

2. Preliminaries

Let H be a real separable Hilbert space with the norm $|\cdot|$, $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and $L^p(\mathbb{P},H)$ be the space of H-valued random variables with finite p-order moments ($p\geq 1$), i.e. $\|x\|_p := (\mathbb{E}|x|^p)^{1/p} = (\int_\Omega |x|^p d\mathbb{P})^{1/p} < \infty$ for $x\in L^p(\mathbb{P},H)$. We denote by $C_b(\mathbb{R},L^p(\mathbb{P},H))$ the space of bounded continuous mappings from $\mathbb{R}$ to $L^p(\mathbb{P},H)$ with the norm $\|f\|_\infty := \sup_{t\in\mathbb{R}}\|f(t)\|_p$ for $f\in C_b(\mathbb{R},L^p(\mathbb{P},H))$.

Let us consider the stochastic differential equation

$$dx(t) = (Ax(t)+F(t,x(t)))dt + G(t,x(t))dW(t), \tag{2.1}$$

where $F,G\in C(\mathbb{R}\times H,H)$, W is a two-sided standard one-dimensional Brownian motion defined on $(\Omega,\mathcal{F},\mathbb{P})$, and A generates a C_0-semigroup $\{U(t)\}_{t\geq 0}$ on H which is *exponentially stable*, i.e. there are positive constants $\mathcal{N},\nu$ such that

$$\|U(t)\| \leq \mathcal{N}e^{-\nu t} \quad \text{for } t\geq 0. \tag{2.2}$$

Definition 2.1. We say that functions F and G satisfy the condition

(C1) if there exists a number $A_0\geq 0$ such that $|F(t,0)|,|G(t,0)|\leq A_0$ for any $t\in\mathbb{R}$;

(C2) if there exists a number $\mathcal{L}\geq 0$ such that $\mathrm{Lip}(F),\mathrm{Lip}(G)\leq\mathcal{L}$, where

$$\mathrm{Lip}(F) := \sup\Big\{\frac{|F(t,x_1)-F(t,x_2)|}{|x_1-x_2|} : x_1\neq x_2, t\in\mathbb{R}\Big\};$$

(C3) if F and G are continuous in t uniformly w.r.t. x on each bounded subset $Q\subset H$.

Proposition 2.2 ([5, Proposition 4.4])**.** *Consider the equation* (2.1)*. Suppose that the functions F and G satisfy conditions* (C1) *and* (C2)*. For $p>2$, denote*

$$c_p := \Big[\frac{p(p-1)}{2}\big(\frac{p}{p-1}\big)^{p-2}\Big]^{p/2}.$$

If

$$\theta_p := 2^{p-1}\mathcal{N}^p\mathcal{L}^p\Big[\Big(\frac{2(p-1)}{\nu p}\Big)^{p-1} + c_p\Big(\frac{p-2}{\nu p}\Big)^{p/2-1}\Big]\frac{2}{\nu p} < 1,$$

then (2.1) *admits a unique bounded solution in $C_b(\mathbb{R},L^p(\mathbb{P},H))$.*

Remark 2.3. Note that the contraction constant θ_p is continuous in p when $p>2$. Furthermore, $c_p=1$ when $p=2$ in Proposition 2.2, so we have

$$\lim_{p\to 2^+}\theta_p = \frac{2\mathcal{N}^2\mathcal{L}^2}{\nu^2} + \frac{2\mathcal{N}^2\mathcal{L}^2}{\nu}.$$

Proposition 2.4 ([5, Theorem 4.6, Corollary 4.7])**.** *Consider the equation* (2.1)*. Suppose that the functions F and G satisfy the conditions* (C1) *and* (C2)*. Then the following statements hold:*

(1) *If $\mathcal{L}<\frac{\nu}{\mathcal{N}\sqrt{2+\nu}}$, then* (2.1) *has a unique solution $\xi\in C(\mathbb{R},B[0,r])$, where*

$$r = \frac{\mathcal{N}A_0\sqrt{2+\nu}}{\nu-\mathcal{N}\mathcal{L}\sqrt{2+\nu}}, \tag{2.3}$$

$$B[0,r] := \{x\in L^2(\mathbb{P},H) : \|x\|_2\leq r\}. \tag{2.4}$$

(2) *If* $\mathcal{L} < \frac{\nu}{2\mathcal{N}\sqrt{1+\nu}}$ *and additionally* F, G *satisfy* (C3), *then the solution* ξ *has the same recurrence property in distribution as the coefficients* (F, G) *with respect to* t*; that is, if* (F, G) *are jointly periodic, quasi-periodic, almost periodic, almost automorphic, Poisson stable etc in* t*, then the distribution* $t \mapsto \mu(t)$ *of* $\xi(\cdot)$ *is periodic, quasi-periodic, almost periodic, almost automorphic, Poisson stable etc.*

By simple change of notation in the proof, we have the following result.

Proposition 2.5 ([11, Theorem 3.2 on page 7])**.** *Let* $\mathcal{A}$ *be a closed subset of a complete metric space* $(\mathcal{X}, d)$ *and* $\mathcal{Y}$ *a metric space. If for each fixed* $x \in \mathcal{A}$ *the mapping* $y \mapsto T_y x$ *is continuous and* $T_y : \mathcal{A} \to \mathcal{A}$ *is a family of uniform contraction on* $\mathcal{A}$*, i.e. there exists a constant* $\theta \in [0, 1)$ *such that*

$$d(T_y x, T_y \bar{x}) \le \theta d(x, \bar{x}) \quad \text{for all } y \in \mathcal{Y} \text{ and } x, \bar{x} \in \mathcal{A}.$$

Then the unique fixed point $g(y)$ *of* T_y *is continuous in* $y \in \mathcal{Y}$.

3. Main results

Theorem 3.1. *Consider the equation* (2.1) *and the family of stochastic differential equations*

$$dx(t) = (Ax(t) + F_\lambda(t, x(t)))dt + G_\lambda(t, x(t))dW(t), \tag{3.1}$$

where $\lambda \in \Lambda$*, with* Λ *being a metric space. Assume that the conditions of Proposition 2.4 hold with* (F, G) *replaced by* (F_λ, G_λ)*, and that the constants* $A_0, \mathcal{L}$ *remain the same for all* $\lambda \in \Lambda$*. Then the following statements hold for the unique* L^2*-bounded (also recurrent) solution* ξ_λ *of* (3.1)*:*

(1) *Fix* $\lambda_0 \in \Lambda$*. If for any* $R > 0$*,*

$$\sup_{t\in\mathbb{R}, |x|\le R} |F_\lambda(t, x) - F(t, x)| \to 0 \quad \text{and} \quad \sup_{t\in\mathbb{R}, |x|\le R} |G_\lambda(t, x) - G(t, x)| \to 0$$

as $\lambda \to \lambda_0$*, then* ξ_λ *converges to the unique bounded solution* ξ *of* (2.1) *in the norm* $\|\cdot\|_\infty$ *as* $\lambda \to \lambda_0$.

(2) *If the convergence of coefficients happens in the BUC space, i.e. for any* $R > 0$

$$\sup_{|t|\le R, |x|\le R} |F_\lambda(t, x) - F(t, x)| \to 0, \quad \sup_{|t|\le R, |x|\le R} |G_\lambda(t, x) - G(t, x)| \to 0$$

as $\lambda \to \lambda_0$*, then* ξ_λ *converges to* ξ *in* L^2*-norm uniformly on any compact interval as* $\lambda \to \lambda_0$.

Proof. Since Λ is a metric space, we only need to prove the results for sequences. Let $\lambda_n \to \lambda_0$ as $n \to \infty$. Consider the sequence of equations

$$dx(t) = (Ax(t) + F_n(t, x(t)))dt + G_n(t, x(t))dW(t), \tag{3.2}$$

with $F_n := F_{\lambda_n} \to F$ and $G_n := G_{\lambda_n} \to G$ as $n \to \infty$.

(i) Let the operator $\Phi : C_b(\mathbb{R}, L^2(\mathbb{P}, H)) \to C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ given by

$$(\Phi\phi)(t) := \int_{-\infty}^t U(t - \tau)F(\tau, \phi(\tau))d\tau + \int_{-\infty}^t U(t - \tau)G(\tau, \phi(\tau))dW(\tau)$$

be the contraction mapping which produces the bounded (recurrent) solution for (2.1); see the proof of [5, Theorem 4.6] for details. On the other hand, note that $\lim_{p\to 2^+}\theta_p < 1$ (see Remark 2.3) if and only if

$$\mathcal{L} < \frac{\nu}{\mathcal{N}\sqrt{2(1+\nu)}},$$

which is satisfied by our assumption on $\mathcal{L}$. So it follows from Proposition 2.2 that (2.1) admits a unique L^p-bounded solution for some $p > 2$. This L^p-bounded solution is exactly the unique L^2-bounded solution ξ of (2.1).

Similarly we denote the contraction mapping for the equation (3.2) by Φ_n. Then for given $\phi \in C_b(\mathbb{R}, L^p(\mathbb{P}, H)) \subset C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}
&\mathbb{E}|\Phi_n(\phi)(t) - \Phi(\phi)(t)|^2 \\
&= \mathbb{E}\Big|\int_{-\infty}^t U(t-\tau)F_n(\tau,\phi(\tau))d\tau + \int_{-\infty}^t U(t-\tau)G_n(\tau,\phi(\tau))dW(\tau) \\
&\quad - \int_{-\infty}^t U(t-\tau)F(\tau,\phi(\tau))d\tau - \int_{-\infty}^t U(t-\tau)G(\tau,\phi(\tau))dW(\tau)\Big|^2 \\
&\le 2\mathbb{E}\Big|\int_{-\infty}^t U(t-\tau)[F_n(\tau,\phi(\tau)) - F(\tau,\phi(\tau))]d\tau\Big|^2 \\
&\quad + 2\Big|\int_{-\infty}^t U(t-\tau)[G_n(\tau,\phi(\tau)) - G(\tau,\phi(\tau))]dW(\tau)\Big|^2 \\
&\le 2\mathcal{N}^2 \int_{-\infty}^t e^{-\nu(t-\tau)}d\tau \int_{-\infty}^t e^{-\nu(t-\tau)}\mathbb{E}|F_n(\tau,\phi(\tau)) - F(\tau,\phi(\tau))|^2 d\tau \\
&\quad + 2\mathcal{N}^2 \int_{-\infty}^t e^{-2\nu(t-\tau)}\mathbb{E}|G_n(\tau,\phi(\tau)) - G(\tau,\phi(\tau))|^2 d\tau \\
&\le \frac{2\mathcal{N}^2}{\nu^2}\sup_{\tau\in\mathbb{R}}\mathbb{E}|F_n(\tau,\phi(\tau)) - F(\tau,\phi(\tau))|^2 \\
&\quad + \frac{\mathcal{N}^2}{\nu}\sup_{\tau\in\mathbb{R}}\mathbb{E}|G_n(\tau,\phi(\tau)) - G(\tau,\phi(\tau))|^2.
\end{aligned} \tag{3.3}$$

That is,

$$\begin{aligned}
\|\Phi_n(\phi) - \Phi(\phi)\|_\infty^2 \le &\frac{2\mathcal{N}^2}{\nu^2}\sup_{\tau\in\mathbb{R}}\mathbb{E}|F_n(\tau,\phi(\tau)) - F(\tau,\phi(\tau))|^2 \\
&+ \frac{\mathcal{N}^2}{\nu}\sup_{\tau\in\mathbb{R}}\mathbb{E}|G_n(\tau,\phi(\tau)) - G(\tau,\phi(\tau))|^2.
\end{aligned} \tag{3.4}$$

For the above fixed $\phi \in C_b(\mathbb{R}, L^p(\mathbb{P}, H))$, the family $\{|\phi(\tau)|^2 : \tau \in \mathbb{R}\}$ is uniformly integrable (see e.g. [3, page 31] for details), and hence by conditions (C1) and (C2) the family

$$\{|F_n(\tau,\phi(\tau)) - F(\tau,\phi(\tau))|^2 : n \in \mathbb{N}, \tau \in \mathbb{R}\}$$

is uniformly integrable. So it follows that for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\sup_{\tau\in\mathbb{R}, n\in\mathbb{N}} \mathbb{E}[|F_n(\tau,\phi(\tau)) - F(\tau,\phi(\tau))|^2 \mathcal{I}_{|F_n(\tau,\phi(\tau)) - F(\tau,\phi(\tau))|^2 \ge R}] < \varepsilon,$$

where $\mathcal{I}_A$ means the indicator function of the set A. On the other hand, by assumption we have

$$\lim_{n\to\infty} \sup_{\tau\in\mathbb{R},|x|\le R} |F_n(\tau,x)-F(\tau,x)|=0.$$

So we have

$$\lim_{n\to\infty} \sup_{\tau\in\mathbb{R}} \mathbb{E}|F_n(\tau,\phi(\tau))-F(\tau,\phi(\tau))|^2=0.$$

Similarly,

$$\lim_{n\to\infty} \sup_{\tau\in\mathbb{R}} \mathbb{E}|G_n(\tau,\phi(\tau))-G(\tau,\phi(\tau))|^2=0.$$

Therefore, it follows from (3.4) that $\Phi_n(\phi)\to\Phi(\phi)$ in the norm $\|\cdot\|_\infty$ as $n\to\infty$.

Applying Proposition 2.5 by setting $\mathcal{X}=C_b(\mathbb{R},L^2(\mathbb{P},H))$, $\mathcal{A}=C_b(\mathbb{R},L^p(\mathbb{P},H))$ and $\mathcal{Y}=\Lambda$, we obtain the desired result.

(ii) Let now $k>0$, $l>L>0$, $t\in[-L,L]$, and f be a bounded nonnegative function. Then we have

$$\begin{aligned}\int_{-\infty}^t e^{-k(t-\tau)}f(\tau)d\tau &= \int_{-\infty}^{-l} e^{-k(t-\tau)}f(\tau)d\tau + \int_{-l}^t e^{-k(t-\tau)}f(\tau)d\tau\\ &\le \sup_{t\in\mathbb{R}} f(t)\frac{e^{-k(t+l)}}{k} + \sup_{|t|\le l} f(t)\frac{1-e^{-k(t+l)}}{k}.\end{aligned}$$

Consequently,

$$\max_{|t|\le L}\int_{-\infty}^t e^{-k(t-\tau)}f(\tau)d\tau \le \frac{e^{kL}e^{-kl}}{k}\sup_{t\in\mathbb{R}} f(t) + \frac{1-e^{-kL}e^{-kl}}{k}\sup_{|t|\le l} f(t). \tag{3.5}$$

For given $\phi\in C(\mathbb{R},B[0,r])$ and $t\in\mathbb{R}$ (recalling that $B[0,r]$ is defined in (2.4)), by (3.3) we have

$$\begin{aligned}&\mathbb{E}|\Phi_n(\phi)(t)-\Phi(\phi)(t)|^2\\ &\le \frac{2\mathcal{N}^2}{\nu}\int_{-\infty}^t e^{-\nu(t-\tau)}\mathbb{E}|F_n(\tau,\phi(\tau))-F(\tau,\phi(\tau))|^2d\tau\\ &\quad + 2\mathcal{N}^2\int_{-\infty}^t e^{-2\nu(t-\tau)}\mathbb{E}|G_n(\tau,\phi(\tau))-G(\tau,\phi(\tau))|^2d\tau.\end{aligned}\tag{3.6}$$

Applying (3.5) to (3.6) with $k=\nu$ and 2ν, $f(\tau)=\mathbb{E}|F_n(\tau,\phi(\tau))-F(\tau,\phi(\tau))|^2$ and $f(\tau)=\mathbb{E}|G_n(\tau,\phi(\tau))-G(\tau,\phi(\tau))|^2$ respectively, we obtain

$$\begin{aligned}&\max_{|t|\le L}\mathbb{E}|\Phi_n(\phi)(t)-\Phi(\phi)(t)|^2\\ &\le \frac{2\mathcal{N}^2}{\nu^2}e^{\nu(L-l)}\sup_{t\in\mathbb{R}}\mathbb{E}I_n^2(t) + \frac{\mathcal{N}^2}{\nu}e^{2\nu(L-l)}\sup_{t\in\mathbb{R}}\mathbb{E}J_n^2(t)\\ &\quad + \frac{2\mathcal{N}^2}{\nu^2}[1-e^{-\nu(L+l)}]\sup_{|t|\le l}\mathbb{E}I_n^2(t) + \frac{\mathcal{N}^2}{\nu}[1-e^{-2\nu(L+l)}]\sup_{|t|\le l}\mathbb{E}J_n^2(t),\end{aligned}\tag{3.7}$$

where $I_n(t):=|F_n(t,\phi(t))-F(t,\phi(t))|$ and $J_n(t):=|G_n(t,\phi(t))-G(t,\phi(t))|$.

By conditions (C1) and (C2), we have

$$\max\Big\{\sup_{t\in\mathbb{R}}\mathbb{E}I_n^1(t),\sup_{t\in\mathbb{R}}\mathbb{E}I_n^2(t)\Big\}\le 4(A_0+\mathcal{L}r)^2.$$

Let now $\{l_n\}$ be a sequence of positive numbers such that $l_n \to +\infty$ as $n \to \infty$. Then by (3.7) we have

$$\begin{aligned} &\max_{|t|\le L} \mathbb{E}|\Phi_n(\phi)(t) - \Phi(\phi)(t)|^2 \\ &\le \Big(\frac{2\mathcal{N}^2}{\nu^2} + \frac{\mathcal{N}^2}{\nu}\Big) e^{\nu(L-l_n)} 4(A_0 + \mathcal{L}r)^2 \\ &\quad + \Big(\frac{2\mathcal{N}^2}{\nu^2} + \frac{\mathcal{N}^2}{\nu}\Big) \max\Big\{ \sup_{|t|\le l_n} \mathbb{E}I_n^2(t), \sup_{|t|\le l_n} \mathbb{E}J_n^2(t)\Big\}. \end{aligned} \tag{3.8}$$

Since $F_n \to F$ and $G_n \to G$ in the BUC space, by [5, Remarks 2.2 and 2.30], passing to limit in (3.8) as $n \to \infty$, for $L > 0$ we obtain

$$\lim_{n\to\infty} \max_{|t|\le L} \mathbb{E}|\Phi_n(\phi)(t) - \Phi(\phi)(t)|^2 = 0$$

by the uniform integrability of the families $\{I_n^2(t) : n \in \mathbb{N}, t \in \mathbb{R}\}$ and $\{J_n^2(t) : n \in \mathbb{N}, t \in \mathbb{R}\}$.

Finally, applying Proposition 2.5 with $\mathcal{X} = C(\mathbb{R}, L^2(\mathbb{P}, H))$, $\mathcal{A} = C(\mathbb{R}, L^p(\mathbb{P}, H))$, both endowed with the Bebutov metric (see the metric d in [5, §2.1] for details), and $\mathcal{Y} = \Lambda$, we obtain the desired result. The proof is complete. □

Remark 3.2. To simplify the notation and highlight the idea, we consider only the most simple noise: one-dimensional Brownian motion. Indeed, the main results of this paper still hold for more general noise case, i.e. when the noise W in (2.1) is replaced by a Q-Wiener process, which brings no essential but just notational difference; see e.g. [15, 16, 20] for details.

As a simple but important case, we have the following result on the continuous dependence of recurrent solutions for asymptotically autonomous stochastic systems.

Corollary 3.3. *Consider the equation* (2.1) *and the family of stochastic differential equations* (3.1)*. Assume that the coefficients F, G of* (2.1) *are independent of t and that the conditions of Theorem 3.1 hold. Then the unique L^2-bounded (also recurrent) solution of* (3.1) *converges to the unique stationary solution of* (2.1) *as $\lambda \to \lambda_0$, and the mode of convergence is the same as their coefficients.*

A much simpler case is the continuous dependence of stationary distributions.

Corollary 3.4. *Consider the family of stochastic differential equations* (3.1) *and its limit equation* (2.1)*. Assume that the coefficients F_λ, G_λ of* (3.1) *and F, G of* (2.1) *are independent of t and that the conditions of Theorem 3.1 hold. Then the unique stationary solution (hence stationary distribution) of* (3.1) *converges to that of* (2.1) *as $\lambda \to \lambda_0$.*

4. Applications

In this section we illustrate our results with two examples.

Example 4.1. Consider the stochastic differential equation

$$\begin{aligned} dy &= \Big(-4y + \frac{\lambda \sin t}{3 + \cos\sqrt{3}t} \frac{y}{y^2+1}\Big) dt + \Big(\frac{1}{2} y + \frac{1+\lambda}{2 + \cos t + \cos\sqrt{2}\,t}\Big) dW \\ &=: (Ay + f_\lambda(t,y))dt + g_\lambda(t,y) dW, \end{aligned} \tag{4.1}$$

where $\lambda \in [0,1]$ is a parameter and W is a one-dimensional two-sided Brownian motion. Clearly A generates an exponentially stable semigroup on $\mathbb{R}$ with $\mathcal{N} = 1$ and $\nu = 4$. Note that for each λ, f_λ is quasi-periodic in t and g_λ is Levitan almost periodic in t, uniformly with respect to y on any bounded subset of $\mathbb{R}$, so f_λ, g_λ are jointly Levitan almost periodic. The Lipschitz constants of f_λ, g_λ satisfy $\max\{\mathrm{Lip}(f_\lambda), \mathrm{Lip}(g_\lambda)\} \le 1/2$ for all $\lambda \in [0,1]$, so the conditions of Proposition 2.4 are met. Hence (4.1) admits a unique L^2-bounded mild solution ξ_λ and this unique L^2-bounded solution is Levitan almost periodic in distribution.

For fixed $\lambda_0 \in [0,1]$, we note that for any $R > 0$,

$$\begin{aligned}
&\lim_{\lambda\to\lambda_0} \sup_{t\in\mathbb{R}, |y|\le R} |f_\lambda(t,y) - f_{\lambda_0}(t,y)| = 0,\\
&\lim_{\lambda\to\lambda_0} \sup_{t\in\mathbb{R}, |y|\le R} |g_\lambda(t,y) - g_{\lambda_0}(t,y)| = \infty,\\
&\lim_{\lambda\to\lambda_0} \sup_{|t|\le R, |y|\le R} |g_\lambda(t,y) - g_{\lambda_0}(t,y)| = 0.
\end{aligned}$$

Therefore, by Theorem 3.1 the unique Levitan almost periodic solution ξ_λ converges to ξ_{λ_0} in the norm $\|\cdot\|_2$ uniformly on any compact interval but cannot be uniformly on $\mathbb{R}$.

If f_λ remains unchanged but $g_\lambda(t,y) = \frac{1}{2}y + (1+\lambda)(2 + \cos t + \cos\sqrt{2}\,t)$, then f_λ and g_λ are quasi-periodic. So it follows from Proposition 2.4 again that (4.1) admits a unique L^2-bounded mild solution ξ_λ which is quasi-periodic in distribution. Furthermore, by Theorem 3.1 the unique quasi-periodic solution ξ_λ converges to ξ_{λ_0} in the norm $\|\cdot\|_2$ uniformly on $\mathbb{R}$, because in this case we have

$$\lim_{\lambda\to\lambda_0} \sup_{t\in\mathbb{R}, |y|\le R} |g_\lambda(t,y) - g_{\lambda_0}(t,y)| = 0.$$

Example 4.2. Consider the following stochastic heat equation on the interval [0,1] with Dirichlet boundary condition,

$$\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2} + u\sin t + (1-\lambda)(2 - \sin t - \sin\sqrt{3}t) + \lambda\frac{\partial W}{\partial t}\\
=: \frac{\partial^2 u}{\partial \xi^2} + f_\lambda(t,u) + \lambda\frac{\partial W}{\partial t},\\
u(t,0) = u(t,1) = 0, \quad t > 0.
\end{gathered} \tag{4.2}$$

Here W is a one-dimensional two-sided Brownian motion and $\lambda \in [0,1]$ is a parameter. Let A be the Laplace operator, then $A : D(A) = H^2(0,1)\cap H_0^1(0,1) \to L^2(0,1)$. Denote $H := L^2(0,1)$ and the norm on H by $|\cdot|$. Then the above equation can be written as the evolution equation

$$dY(t) = (AY(t) + F_\lambda(t, Y(t)))dt + \lambda dW(t) \tag{4.3}$$

on the Hilbert space H with

$$Y(t) := u(t,\cdot), \quad F_\lambda(t, Y(t)) := f_\lambda(t, u(t,\cdot)).$$

Note that, operator A has eigenvalues $\{-n^2\pi^2\}_{n=1}^\infty$ and generates a C_0-semigroup $\{U(t)\}_{t\ge0}$ on H satisfying $\|U(t)\| \le e^{-\pi^2 t}$ for $t \ge 0$, i.e. $\mathcal{N} = 1$ and $\nu = \pi^2$. Note that $\mathrm{Lip}(F) \le 1$ for all $\lambda \in [0,1]$, so the conditions of Proposition 2.4 hold. Thus (4.3), hence (4.2), admits a unique L^2-bounded mild solution ξ_λ which is quasi-periodic in distribution.

Note that as $\lambda \to 0$, for any $R > 0$ we have

$$\lim_{\lambda\to 0} \sup_{t\in\mathbb{R},|Y|\le R} |F_\lambda(t,Y) - F_0(t,Y)| = 0,$$

so it follows from Theorem 3.1 that $\xi_\lambda \to \xi_0$ in the norm $\|\cdot\|_2$ uniformly on $\mathbb{R}$, with ξ_0 being the deterministic quasi-periodic solution for the equation (4.2) with parameter $\lambda = 0$. Similarly, as $\lambda \to 1$, we have $\xi_\lambda \to \xi_1$ uniformly on $\mathbb{R}$, with ξ_1 being the periodic solution of the equation (4.2) with parameter $\lambda = 1$.

If

$$f_\lambda(t,u) = u\sin t + \frac{1-\lambda}{2-\sin t - \sin\sqrt{3}t}$$

in (4.2), then for any $R > 0$ we have

$$\lim_{\lambda\to 0} \sup_{t\in\mathbb{R},|Y|\le R} |F_\lambda(t,Y) - F_0(t,Y)| = \infty, \quad \lim_{\lambda\to 0} \sup_{|t|\le R,|Y|\le R} |F_\lambda(t,Y) - F_0(t,Y)| = 0.$$

So it follows from Theorem 3.1 that $\xi_\lambda \to \xi_0$ in the norm $\|\cdot\|_2$ uniformly on any compact interval but cannot be uniformly on $\mathbb{R}$, with ξ_0 being the deterministic Levitan almost periodic solution for the equation (4.2) with parameter $\lambda = 0$. Similarly, as $\lambda \to 1$, we have $\xi_\lambda \to \xi_1$ uniformly on any compact interval but cannot be uniformly on $\mathbb{R}$, with ξ_1 being the periodic solution of the equation (4.2) with parameter $\lambda = 1$.

Acknowledgements. The authors are grateful to Professor Zhenxin Liu for helpful discussions. The authors sincerely thank the anonymous referees for their careful reading and helpful suggestions. This work is partially supported by the NSFC, Grant 11871132, and by the startup fund from Dalian University of Technology.

References

[1] L. Arnold, C. Tudor; *Stationary and almost periodic solutions of almost periodic affine stochastic differential equations*, Stochastics Stochastics Rep., 64 (1998), 177–193.

[2] P. H. Bezandry, T. Diagana; *Existence of almost periodic solutions to some stochastic differential equations*, Appl. Anal., 86 (2007), 819–827.

[3] P. Billingsley; *Convergence of Probability Measures.* Second edition. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999. x+277 pp.

[4] Y.-K. Chang, C. Tang; *Asymptotically almost automorphic solutions to stochastic differential equations driven by a Lévy process*, Stochastics, 88 (2016), no. 7, 980–1011.

[5] D. Cheban, Z. Liu; *Periodic, quasi-periodic, almost periodic, almost automorphic, Birkhoff recurrent and Poisson stable solutions for stochastic differential equations*, J. Differential Equations, 269 (2020), no. 4, 3652–3685.

[6] F. Chen, Y. Han, Y. Li, X. Yang; *Periodic solutions of Fokker-Planck equations*, J. Differential Equations, 263 (2017), 285–298.

[7] Z. Chen, W. Lin; *Square-mean weighted pseudo almost automorphic solutions for non-autonomous stochastic evolution equations*, J. Math. Pures Appl., 100 (2013), 476–504.

[8] G. Da Prato, C. Tudor; *Periodic and almost periodic solutions for semilinear stochastic equations*, Stoch. Anal. Appl., 13 (1995), 13–33.

[9] M. Fu, Z. Liu; *Square-mean almost automorphic solutions for some stochastic differential equations*, Proc. Amer. Math. Soc., 138 (2010), 3689–3701.

[10] A. Halanay; *Periodic and almost periodic solutions to affine stochastic systems*, Proceedings of the Eleventh International Conference on Nonlinear Oscillations (Budapest, 1987), 94–101, János Bolyai Math. Soc., Budapest, 1987.

[11] J. Hale; *Ordinary Differential Equations.* Second edition. Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980. xvi+361 pp.

[12] M. Ji, W. Qi, Z. Shen, Y. Yi; *Existence of periodic probability solutions to Fokker-Planck equations with applications*, J. Funct. Anal., 277 (2019), no. 11, 108281, 41 pp.

[13] R. Khasminskii; *Stochastic Stability of Differential Equations.* Translated from the Russian by D. Louvish. Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics and Analysis, 7. Sijthoff & Noordhoff, Alphen aan den Rijn-Germantown, Md., 1980. xvi+344 pp.

[14] Y. Li, Z. Liu, W. Wang; *Almost periodic solutions and stable solutions for stochastic differential equations*, Discrete Contin. Dyn. Syst. Ser. B, 24 (2019), 5927–5944.

[15] X. Liu, Z. Liu; *Poisson stable solutions for stochastic differential equations with Lévy noise*, Acta Math. Sin. (Engl. Ser.), In Press. (also available at arXiv:2002.00395)

[16] Z. Liu, K. Sun; *Almost automorphic solutions for stochastic differential equations driven by Lévy noise*, J. Funct. Anal., 226 (2014), 1115–1149.

[17] Z. Liu, W. Wang; *Favard separation method for almost periodic stochastic differential equations*, J. Differential Equtions, 260 (2016), 8109–8136.

[18] T. Morozan; *Periodic solutions of affine stochastic differential equations*, Stochastic Anal. Appl., 4 (1986), 87–110.

[19] B. Wang, H. Gao, *Neutral stochastic partial functional integro-differential equations driven by G-Brownian motion,* Electron. J. Differential Equations, 2019 (2019), no. 120, 1–15.

[20] Y. Wang, Z. Liu; *Almost periodic solutions for stochastic differential equations with Lévy noise*, Nonlinearity, 25 (2012), 2803–2821.

Haijing Qiu
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
Email address: 18233150561@163.com

Yan Wang (corresponding author)
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
Email address: wangyan84@dlut.edu.cn