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GENERAL p-CURL SYSTEMS AND DUALITY MAPPINGS ON SOBOLEV SPACES FOR MAXWELL EQUATIONS

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Abstract. We study a general p-curl system arising from a model of type-II superconductors. We show several trace theorems that hold on either a Lipschitz domain with small Lipschitz constant or on a $C^{1,1}$ domain. Certain duality mappings on related Sobolev spaces are computed and used to establish surjectivity results for the p-curl system. We also solve a nonlinear boundary value problem for a general p-curl system on a $C^{1,1}$ domain and provide a variational characterization of the first eigenvalue of the p-curl operator.

1. Introduction

We study the following nonlinear system related to the Maxwell system of electromagnetism in Banach spaces:

$$|\mathbf{u}|^{p-2}\mathbf{u} + \operatorname{curl}_p(\mathbf{u}) + \operatorname{div}_p(\mathbf{u}) = \mathbf{f}(x, \mathbf{u}),$$

where $\mathbf{f} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is a vector-valued Carathéodory function (see Section 7) and the operators curl_p and div_p (see their definitions in (3.5) and (5.2)) act on subspaces of the Sobolev space $W^{1,p}(\operatorname{curl}, \Omega) \cap W^{1,p}(\operatorname{div}, \Omega)$, $1 < p < +\infty$, with Ω a bounded domain in $\mathbb{R}^3$. The operators curl_p and div_p are Banach space generalizations of the classical curl and divergence operators which act on the Hilbert spaces $H(\operatorname{curl}, \Omega)$ and $H(\operatorname{div}, \Omega)$ [24].

The p-curl system we study arises from a model of magnetic induction in a high temperature superconductor [9]. However, the system we study here is more general than the one in Bean's critical state model for type-II superconductors [32], as we allow for vector fields with nonzero divergence.

Recently there has been growing interest in various properties of the p-curl system; see in particular [31] and the references therein. Frequently the roughness of the underlying domain plays a crucial role in the analysis of, for example, well-posedness of the system. Our interest is keeping the domain as rough as possible, i.e. Lipschitz. However, this is not always possible due to various embedding failures and in particular, a lack of simple Poincaré inequality; see Section 4. Frequently, the smoothness of the domain can be relaxed to Lipschitz by restricting the range of p-values for which the corresponding results hold.

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We establish a framework suited for variational methods and calculating duality mappings on various Sobolev spaces associated to the p-curl system; see Sections 3–5. We prove that the p-curl operator can be expressed in terms of a duality mapping. It is worth mentioning that the geometry of Banach spaces is closely related to analytical properties of their duality mappings.

We begin by proving a number of trace results for the Banach spaces under consideration. In many cases, we take the domain to be Lipschitz with small Lipschitz constant. This is needed in order to obtain an L^p-estimate for the gradient of solutions to a certain elliptic boundary value problem.

We generalize the duality mapping procedure to general Banach spaces having dual norm which is uniformly Frechét differentiable on the unit sphere; see Section 6. For further details on duality mappings and their applications to the solvability of nonlinear operator equations in Banach spaces, the reader is referred to [2, 5, 8, 26] and the references therein.

In Section 7, we consider the nonlinear p-curl system on a $C^{1,1}$ or convex domain. Under a particular growth assumption (similar to one commonly employed for the p-Laplace equation), we obtain existence of solutions to the nonlinear boundary value problem (7.1) by using the Nemytskii operator.

Section 8 details the one-dimensional version of the eigenvalue problem considered in Section 7, and we obtain a formula for the first eigenvalue of the p-curl operator explicitly; see equation (8.2). This result closely resembles the result for the first eigenvalue of the p-Laplace operator on $W_0^{1,p}(\Omega)$. This is perhaps not surprising due to the similarities between the p-curl and p-Laplace operator; see in particular Theorem 4.5. Such one-dimensional eigenvalue problems have been studied by Drábek and Manásevich in [15] and Cringanu in [11].

We should mention that we have said "p-curl operator", but the operator we consider in (7.6) also has a divergence term. This is due to the fact that a basic Poincaré inequality does not hold in this setting, and so we must also consider vector fields with well-defined divergence. Equation (4.3) provides the general Friedrichs type inequality for the L^p-norm of the gradient that holds in this setting.

2. Function spaces and trace theorems

In this section, we prove trace theorems with respect to the spaces $W^{1,p}(\mathrm{curl},\Omega)$ and $W^{1,p}(\mathrm{div},\Omega)$ and obtain Green's theorems corresponding to the trace results. We begin with the following definition.

Definition 2.1. A bounded domain $\Omega\subset\mathbb{R}^3$ is called a Lipschitz domain if for each point $p\in\partial\Omega$ there exists an open set $\mathcal{O}\subset\mathbb{R}^3$ such that $p\in\mathcal{O}$, and an orthogonal coordinate system with coordinates $\xi=(\xi_1,\xi_2,\xi_3)$ having the following property: there exists a vector $b\in\mathbb{R}^3$ so that

$$\mathcal{O}=\{\xi:-b_j<\xi_j<b_j,\ 1\le j\le 3\}$$

and a Lipschitz continuous function ϕ defined on the set

$$\mathcal{O}'=\{\xi'\in\mathbb{R}^2:-b_j<\xi_j'<b_j,\ 1\le j\le 2\}$$

such that

$$\begin{aligned}\Omega\cap\mathcal{O}&=\{\xi:\xi_3<\phi(\xi'),\ \xi'\in\mathcal{O}'\},\\ \partial\Omega\cap\mathcal{O}&=\{\xi:\xi_3=\phi(\xi'),\ \xi'\in\mathcal{O}'\}.\end{aligned}$$

The domain is said to be of class $C^{m,1}$ for an integer $m \geq 1$ if the map ϕ can be chosen to be m-times differentiable with Lipschitz continuous partial derivatives of order m.

We also need the notion of a Lipschitz domain with small Lipschitz constant. We say a domain $\Omega \subset \mathbb{R}^3$ is a *Lipschitz domain with small Lipschitz constant* if it is a Lipschitz domain as in Definition 2.1 and there exists $\theta \in (0,1]$ such that

$$\sup_{\xi',\eta' \in \mathcal{O}',\, \xi' \neq \eta'} \frac{|\phi(\xi') - \phi(\eta')|}{|\xi' - \eta'|} \leq \theta.$$

We work with the general spaces

$$W^{k,p}(\mathrm{curl},\Omega) = \big\{\mathbf{u} \in \big(W^{k-1,p}(\Omega)\big)^3 : \nabla \times \mathbf{u} \in \big(W^{k-1,p}(\Omega)\big)^3\big\}, \quad 1 < p < +\infty,$$

with the norm

$$\|\mathbf{u}\|_{W^{k,p}(\mathrm{curl},\Omega)} = \big(\|\mathbf{u}\|^p_{W^{k-1,p}} + \|\nabla \times \mathbf{u}\|^p_{W^{k-1,p}}\big)^{1/p}.$$

Additionally, we define

$$W^{k,p}(\mathrm{div},\Omega) = \big\{\mathbf{u} \in \big(W^{k-1,p}(\Omega)\big)^3 : \nabla \cdot \mathbf{u} \in W^{k-1,p}(\Omega)\big\}$$

with the norm

$$\|\mathbf{u}\|_{W^{k,p}(\mathrm{div},\Omega)} = \big(\|\mathbf{u}\|^p_{W^{k-1,p}} + \|\nabla \cdot \mathbf{u}\|^p_{W^{k-1,p}}\big)^{1/p},$$

where the last norm on the right hand side above is a scalar Sobolev norm.

As in the case of Hilbert spaces, one can prove the denseness of smooth functions $\big(C^\infty(\overline{\Omega})\big)^3$ in these Sobolev spaces. We further define $W_0^{1,p}(\mathrm{curl},\Omega)$ as the completion of $(C_0^\infty(\Omega))^3$ in the $W^{1,p}(\mathrm{curl},\Omega)$ norm, and $W_0^{1,p}(\mathrm{div},\Omega)$ as the completion of $(C_0^\infty(\Omega))^3$ in the $W^{1,p}(\mathrm{div},\Omega)$ norm. We also need the spaces

$$\begin{gathered} W_p = W^{1,p}(\mathrm{curl},\Omega) \cap W^{1,p}(\mathrm{div},\Omega), \\ W_N = \{\mathbf{u} \in W_p : \boldsymbol{\gamma}_t(\mathbf{u}) = \mathbf{0}\}. \end{gathered}$$

We endow these spaces with the obvious graph norm. The map $\boldsymbol{\gamma}_t$ above is the tangential trace map, and it is defined classically for a smooth vector function $\mathbf{u} \in \big(C^\infty(\overline{\Omega})\big)^3$ by

$$\boldsymbol{\gamma}_t(\mathbf{u}) = \boldsymbol{\nu} \times \mathbf{u}\big|_{\partial\Omega},$$

where $\boldsymbol{\nu}$ denotes the outer unit normal on $\partial\Omega$.

Remark 2.2. It is known that W_p does not compactly embed into L^p. However, we do have compact embedding of W_N into L^p; see [3, Lemma 3.3], and this requires that the domain Ω have $C^{1,1}$ regularity.

Furthermore, we need the Besov spaces $B^q_{s,p}$ on the boundary of a Lipschitz domain. In what follows, $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions. Additionally, for $f \in \mathcal{S}$, we denote by $\widehat{f}$ the Fourier transform of f. Moreover, we set $M_j = \big\{\xi \in \mathbb{R}^3 : 2^{j-1} \leq |\xi| \leq 2^{j+1}\big\}$ for $j = 1,2,\dots$ and $M_0 = \{\xi \in \mathbb{R}^3 : |\xi| \leq 2\}$.

Definition 2.3. For $-\infty < q < \infty, 1 < s < \infty, 1 \leq p < \infty$, the Besov space $B^q_{s,p}$ is defined by

$$B^q_{s,p} = \big\{f \in \mathcal{S}' : f = \sum_{j=0}^{\infty} a_j(x), \mathrm{supp}\,(\widehat{a_j}) \subset M_0;\ \|a_j\| < \infty\big\},$$

where the equality of f above is in the sense of tempered distributions, and the norm of a_j is given by

$$\|a_j\| = \Big[\sum_{j=0}^{\infty}\big(2^{qj}\|a_j\|_{L^s}\big)^p\Big]^{1/p},$$

where $\|\cdot\|_{L^s}$ is the usual norm on the Lebesgue space.

For a complete definition of Besov spaces on domains, we refer the reader to [29].

Definition 2.4. We say a distribution u on $\partial\Omega$ belongs to $B^q_{s,p}(\partial\Omega)$ if the composition $u\circ\phi \in B^q_{s,p}\left(\mathcal{O}'\cap\phi^{-1}\left(\partial\Omega\cap\mathcal{O}\right)\right)$ for all possible $\mathcal{O},\phi$ as in Definition 2.1.

We now prove trace results and Green's theorems as their consequences.

Theorem 2.5. *Suppose Ω is a Lipschitz domain with small Lipschitz constant. The mapping $\boldsymbol{\gamma}_t(\mathbf{u}) = \boldsymbol{\nu}\times\mathbf{u}\big|_{\partial\Omega}$ defined on $\left(C^\infty(\overline{\Omega})\right)^3$ can be extended by continuity to a continuous linear map $\boldsymbol{\gamma}_t$ from $W^{1,p}(\mathrm{curl},\Omega)$ to $\left(B^{-1/p'}_{p',p'}(\partial\Omega)\right)^3$. Moreover, the following Green's theorem holds for any $\mathbf{u}\in W^{1,p}(\mathrm{curl},\Omega)$ and $\boldsymbol{\phi}\in W^{1,p'}(\mathrm{curl},\Omega)$:*

$$\langle\boldsymbol{\gamma}_t(\mathbf{u}),\boldsymbol{\phi}\rangle_{\partial\Omega} = \int_\Omega \mathbf{u}\cdot\nabla\times\boldsymbol{\phi}\,dx - \int_\Omega \nabla\times\mathbf{u}\cdot\boldsymbol{\phi}\,dx. \tag{2.1}$$

The angle brackets above denote the duality pairing between $\left(B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)\right)^3$ and $\left(B^{-1/p'}_{p',p'}(\partial\Omega)\right)^3$.

Remark 2.6. Assume $p=2$. Then $W^{1,p}(\mathrm{curl},\Omega)$ is identified with the space $H(\mathrm{curl},\Omega)$. Additionally, the Besov space becomes $B^{-\frac{1}{2}}_{2,2}\approx W^{-\frac{1}{2},2}$ which we can identify with the dual of $H^{-\frac{1}{2}}$. Thus Theorem 2.5 is consistent with the well-known trace theorem for $H(\mathrm{curl},\Omega)$ functions, see [24, Theorem 3.29].

Theorem 2.7. *If Ω is a Lipschitz domain with small Lipschitz constant, then*

$$\begin{aligned} W^{1,p}_0(\mathrm{curl},\Omega) &= \left\{\mathbf{u}\in W^{1,p}(\mathrm{curl},\Omega) : \boldsymbol{\gamma}_t(\mathbf{u}) = \mathbf{0}\right\}\\ &= \Big\{\mathbf{u}\in W^{1,p}(\mathrm{curl},\Omega) : \int_\Omega \mathbf{u}\cdot\nabla\times\boldsymbol{\phi}\,dx = \int_\Omega \nabla\times\mathbf{u}\cdot\boldsymbol{\phi}\,dx \\ &\quad \forall\boldsymbol{\phi}\in\left(C^\infty(\overline{\Omega})\right)^3\Big\}. \end{aligned}\tag{2.2}$$

2.1. **Proof of Theorem 2.5.** We adapt the techniques from the proof of [28, Lemma 6.2]; now to the Banach space setting. Our starting point is the formula

$$\int_\Omega \nabla\times\mathbf{u}\cdot\boldsymbol{\phi}\,dx = \int_\Omega \mathbf{u}\cdot\nabla\times\boldsymbol{\phi}\,dx + \langle\boldsymbol{\gamma}_t(\mathbf{u}),\boldsymbol{\phi}\rangle_{\partial\Omega} \tag{2.3}$$

which holds for any $\mathbf{u},\phi\in\left(C^\infty(\overline{\Omega})\right)^3$. This follows directly from the divergence theorem. By the standard density argument, (2.3) holds for $\boldsymbol{\phi}\in W^{1,p'}(\mathrm{curl},\Omega)$. The Cauchy-Schwarz inequality and Hölder's inequality then yield

$$|\langle\boldsymbol{\gamma}_t(\mathbf{u}),\boldsymbol{\phi}\rangle_{\partial\Omega}| \le \|\mathbf{u}\|_{W^{1,p}(\mathrm{curl},\Omega)}\|\boldsymbol{\phi}\|_{W^{1,p'}(\mathrm{curl},\Omega)}$$

for all $\mathbf{u} \in \left(C^\infty(\overline{\Omega})\right)^3$ and $\boldsymbol{\phi} \in W^{1,p'}(\mathrm{curl}, \Omega)$. Let $\mu \in B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)$. Consider the Neumann problem

$$\begin{gathered}
\Delta v = 0 \quad \text{in } \Omega, \\
\frac{\partial v}{\partial \boldsymbol{\nu}} = \mu \quad \text{on } \partial\Omega, \\
v \in W^{1,p}(\Omega).
\end{gathered} \tag{2.4}$$

Take $\boldsymbol{\phi} = \nabla v$ where v solves (2.4). Then

$$\|\boldsymbol{\phi}\|_{W^{1,p'}(\mathrm{curl},\Omega)} = \|\nabla v\|_{L^{p'}(\Omega)}$$

since any gradient is in the kernel of the curl (as viewed as operators on L^q). Thus, we need to estimate the $L^{p'}$-norm of the gradient of the solution of (2.4). This highly depends on the geometry of the domain, which is why we restrict the domain to be Lipschitz with small Lipschitz constant. (Note that any C^1 domain satisfies this assumption.) Then, by [14, Theorem 5], we can find a constant $C > 0$, depending on the Lipschitz nature of $\partial\Omega$, such that

$$\|\nabla v\|_{L^{p'}(\Omega)} \le C\|\mu\|_{B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)}.$$

For a general Lipschitz domain, the entire range of $p's$ for such an estimate to hold is not expected; thus, if we wish to weaken the smoothness of the boundary we also have to decrease the range of allowed p's. This is due to restrictions on solvability of the Neumann problem (2.4). Indeed, there is a (sharp) range of p values for solvability together with an L^p estimate for the gradient, see in particular [16], [19], or [33].

Note also that the result in [14] does not characterize the trace estimates using Besov spaces, but by adapting the ideas of [16] one can easily obtain the above estimate. Indeed, this can be done by using the fact that the trace of $W^{1,p}(\Omega)$ is the Besov space $B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)$, see [19]. We then have that

$$\begin{aligned}
\|\boldsymbol{\gamma}_t(\mathbf{u})\|_{\left(B_{p',p'}^{-1/p'}(\partial\Omega)\right)^3} &= \sup_{\boldsymbol{\phi} \in \left(B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)\right)^3,\ \|\boldsymbol{\phi}\|=1} |\langle \boldsymbol{\gamma}_t(\mathbf{u}), \boldsymbol{\phi}\rangle| \\
&\le \|\mathbf{u}\|_{W^{1,p}(\mathrm{curl},\Omega)} \|\boldsymbol{\phi}\|_{W^{1,p'}(\mathrm{curl},\Omega)} \\
&\le C\|\mathbf{u}\|_{W^{1,p}(\mathrm{curl},\Omega)} \|\mu\|_{B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)} \\
&= C\|\mathbf{u}\|_{W^{1,p}(\mathrm{curl},\Omega)},
\end{aligned}$$

where $C = C(\theta)$, i.e. the constant depends on the Lipschitz character of the domain. For more on the dual of Besov spaces, see [25]. Additionally, for this characterization of the Besov space norm on the boundary, see [21].

2.2. **Proof of Theorem 2.7.** We need the following lemma to prove the theorem.

Lemma 2.8. *Suppose that* $\mathbf{u} \in W^{1,p}(\mathrm{curl}, \Omega)$ *is such that for each* $\boldsymbol{\phi} \in \left(C^\infty(\overline{\Omega})\right)^3$, *it holds*

$$\int_\Omega \nabla \times \mathbf{u} \cdot \boldsymbol{\phi}\, dx - \int_\Omega \mathbf{u} \cdot \nabla \times \boldsymbol{\phi}\, dx = 0.$$

Then $\mathbf{u} \in W_0^{1,p}(\mathrm{curl}, \Omega)$.

Assuming Lemma 2.8 for now; it in particular implies that the set

$$\big\{\mathbf{u} \in W^{1,p}(\mathrm{curl},\Omega) : \int_\Omega \nabla\times\mathbf{u}\cdot\boldsymbol{\phi}\,dx - \int_\Omega \mathbf{u}\cdot\nabla\times\boldsymbol{\phi}\,dx = 0,\ \forall\boldsymbol{\phi}\in\big(C^\infty(\overline{\Omega})\big)^3\big\}$$

is a subset of $W_0^{1,p}(\mathrm{curl},\Omega)$. Then we apply Theorem 2.5 to $\mathbf{u}$ such that $\boldsymbol{\gamma}_t(\mathbf{u}) = \mathbf{0}$ to obtain

$$\begin{aligned}&\big\{\mathbf{u}\in W^{1,p}(\mathrm{curl},\Omega) : \boldsymbol{\gamma}_t(\mathbf{u}) = \mathbf{0}\big\}\\ &\subset \big\{\mathbf{u} \in W^{1,p}(\mathrm{curl},\Omega) : \int_\Omega \nabla\times\mathbf{u}\cdot\boldsymbol{\phi}\,dx - \int_\Omega \mathbf{u}\cdot\nabla\times\boldsymbol{\phi}\,dx = 0,\ \forall\boldsymbol{\phi}\in\big(C^\infty(\overline{\Omega})\big)^3\big\}.\end{aligned}$$

Since $\big(C_0^\infty(\overline{\Omega})\big)^3 \subset \{\mathbf{u}\in W^{1,p}(\mathrm{curl},\Omega) : \boldsymbol{\gamma}_t(\mathbf{u}) = \mathbf{0}\}$ and the set $\{\mathbf{u}\in W^{1,p}(\mathrm{curl},\Omega) : \boldsymbol{\gamma}_t(\mathbf{u}) = \mathbf{0}\}$ is closed due to continuity of the trace map, we conclude that

$$W_0^{1,p}(\mathrm{curl},\Omega) \subset \big\{\mathbf{u}\in W^{1,p}(\mathrm{curl},\Omega) : \boldsymbol{\gamma}_t(\mathbf{u}) = \mathbf{0}\big\}\,.$$

Proof of Lemma 2.8. The proof is similar to the proof of [24, Lemma 3.27] with a few adjustments, and so we shall provide a sketch of the proof with the necessary adjustments. Since Ω is a bounded Lipschitz domain, we can find a collection of open sets U_j such that $\Omega\subset\cup_{j=1}^M U_j$ and such that each $\Omega_j := U_j\cap\Omega$ is a bounded and starlike Lipschitz domain. Then there is a partition of unity subordinate to this open cover; that is, there exist functions $\{\alpha_j\}_{j=1}^M$ such that each $\alpha_j\in C_0^\infty(U_j)$, as well as $0\le\alpha_j(x)\le 1$ and $\sum_{j=1}^M \alpha_j = 1$ for all $x\in\Omega$. Let $\widetilde{\mathbf{u}}$ denote the extension of $\mathbf{u}$ by zero outside of Ω. Clearly, $\widetilde{\mathbf{u}}\in W^{1,p}(\mathrm{curl},\mathbb{R}^3)$. By the construction of α_j, we have

$$\widetilde{\mathbf{u}}(x) = \sum_{j=1}^M \alpha_j\widetilde{\mathbf{u}}(x),\quad x\in\Omega,$$

and $\widetilde{\mathbf{u}}_j := \alpha_j\widetilde{\mathbf{u}} \in W^{1,p}(\mathrm{curl},\mathbb{R}^3)$. Let $\widetilde{\mathbf{u}}_{j,t}(x) := \widetilde{\mathbf{u}}_j(x/t)$ for $0<t<1$. Then $\widetilde{\mathbf{u}}_{j,t}\to\widetilde{\mathbf{u}}_j$ in $W^{1,p}(\mathrm{curl},\mathbb{R}^3)$ as $t\to 1$.

Let $\mathbf{M}_\epsilon = \boldsymbol{\rho}_\epsilon * \mathbf{v}$ for $\mathbf{v}\in(L^p(\mathbb{R}^3))^3$ denote the mollification of $\mathbf{v}$ for a mollifier $\boldsymbol{\rho}_\epsilon$. Then $\mathbf{M}_\epsilon\to\mathbf{v}$ in $(L^p(\mathbb{R})^3)^3$ as $\epsilon\to 0$, and by differentiability properties of the convolution, we have $\nabla\times\mathbf{M}_\epsilon = \boldsymbol{\rho}_\epsilon * (\nabla\times\mathbf{v})$. Thus, $\boldsymbol{\rho}_\epsilon * \widetilde{\mathbf{u}}_{j,t}\to\widetilde{\mathbf{u}}_{j,t}$ as $\epsilon\to 0$ in $W^{1,p}(\mathrm{curl},\mathbb{R}^3)$. We can then find sequences $\{t_k\},\{\epsilon_k\}$, with $0<t_k,\epsilon_k<1$, such that $\epsilon_k\to 0$, $t_k\to 1$ and

$$\boldsymbol{\rho}_{\epsilon_k} * \widetilde{\mathbf{u}}_{j,t_k}\to\widetilde{\mathbf{u}}_j\quad\text{in } W^{1,p}(\mathrm{curl},\Omega_j).$$

The function

$$\widetilde{\mathbf{u}}^{(k)} := \sum_{j=1}^M \boldsymbol{\rho}_{\epsilon_k} * \widetilde{\mathbf{u}}_{j,t_k}\to\mathbf{u}\quad\text{in } W^{1,p}(\mathrm{curl},\Omega).$$

Thus, we conclude that $\mathbf{u}\in W_0^{1,p}(\mathrm{curl},\Omega)$ (note that $\widetilde{\mathbf{u}}^{(k)}\in(C_0^\infty(\Omega))^3$ for each k). $\square$

2.3. **Traces of $W^{1,p}(\mathrm{div},\Omega)$ functions.** We can similarly analyze traces of functions belonging to $W^{1,p}(\mathrm{div},\Omega)$. First, we define for a smooth vector $\mathbf{u}$ the normal trace

$$\gamma_n(\mathbf{u}) = \mathbf{u}\big|_{\partial\Omega}\cdot\boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ is the outer unit normal vector on $\partial\Omega$.

Theorem 2.9. *Suppose Ω is a $C^{1,1}$ domain. The mapping $\gamma_n(\mathbf{u})$ defined on $\left(C^\infty(\overline{\Omega})\right)^3$ can be extended by continuity to a continuous linear map γ_n from $W^{1,p}(\operatorname{div},\Omega)$ to $B^{-1/p}_{p,p}(\partial\Omega)$. Moreover, the following Green's theorem holds for any $\mathbf{u} \in W^{1,p}(\operatorname{div},\Omega)$ and $\phi \in W^{1,p'}(\Omega)$:*

$$\langle \gamma_n(\mathbf{u}), \phi\rangle_{\partial\Omega} = (\mathbf{u}, \nabla\phi)_{L^2} + (\nabla\cdot\mathbf{u}, \phi)_{L^2}. \tag{2.5}$$

Note that Ω is $C^{1,1}$ is needed here to obtain the estimate (2.6) below. Again, if Ω were merely Lipschitz, then further restrictions of p would need to be imposed; see [19].

Proof. Similar to the proof of Theorem 2.5, we start with the Green's formula for smooth functions $\phi \in C^\infty(\Omega)$

$$(\mathbf{v}, \nabla\phi)_{L^2} + (\nabla\cdot\mathbf{v}, \phi)_{L^2} = \langle \phi, \gamma_n(\mathbf{v})\rangle_{\partial\Omega}$$

which by density argument can be extended to hold for $\phi \in W^{1,p'}(\Omega)$. By the Cauchy-Schwarz inequality, we have

$$|\langle \gamma_n(\mathbf{v}), \phi\rangle| \le \|\mathbf{v}\|_{W^{1,p}(\operatorname{div},\Omega)}\|\phi\|_{W^{1,p'}(\Omega)} \quad \forall \phi \in W^{1,p'}(\Omega),\ \mathbf{v} \in (C^\infty(\overline{\Omega}))^3.$$

Let $g \in B^{1-\frac{1}{p'}}_{p',p'}(\partial\Omega)$. Take $\phi = u$ where u solves $\Delta u = 0$ in Ω, with boundary condition $u\big|_{\partial\Omega} = g$. Such a solution exists, and one can find a constant $c > 0$ such that

$$\|u\|_{W^{1,p'}(\Omega)} \le c\|g\|_{B^{1-\frac{1}{p'}}_{p',p'}(\partial\Omega)}, \tag{2.6}$$

see [19]. Then, as in the proof of Theorem 2.5, we see that

$$\|\gamma_n(\mathbf{v})\|_{B^{-1/p}_{p,p}(\partial\Omega)} \le c\|\mathbf{v}\|_{W^{1,p}(\operatorname{div},\Omega)}\ \|g\|_{B^{1-\frac{1}{p'}}_{p',p'}(\partial\Omega)},$$

and therefore the continuity is established. □

Remark 2.10. Just as in Remark 2.6, when $p = 2$, the space $W^{1,p}(\operatorname{div},\Omega)$ is identified with the space $H(\operatorname{div},\Omega)$. We recover the normal trace result [24, Theorem 3.24] in this case as well.

Later on, we will need the space

$$W_N^0 := \{\mathbf{u} \in W_N : \gamma_n(\mathbf{u}) = 0\}.$$

Similar to Theorem 2.7, it is straightforward to see that the following result holds.

Theorem 2.11. *If Ω is a Lipschitz domain with small Lipschitz constant, then*

$$W_0^{1,p}(\operatorname{div},\Omega) = \left\{\mathbf{u} \in W^{1,p}(\operatorname{div},\Omega) : \gamma_n(\mathbf{u}) = 0\right\}. \tag{2.7}$$

Thus, we see that

$$W_N^0 = W_0^{1,p}(\operatorname{curl},\Omega) \cap W_0^{1,p}(\operatorname{div},\Omega). \tag{2.8}$$

3. Duality mapping on $W^{1,p}(\mathrm{curl},\Omega)$

Let X be a real Banach space and X^* its dual. Let $\langle\cdot,\cdot\rangle$ denote the duality pairing. Given an operator $T : X \to 2^{X^*}$, define the range of T by

$$\mathcal{R}(T) = \cup_{x\in\mathcal{D}(T)} Tx,$$

where as usual $\mathcal{D}(T) := \{x \in X : Tx \neq \emptyset\}$ is the effective domain of T. The graph of T is the set $G(T) := \{(x, y) \in X \times X^* :\ y \in Tx,\ x \in \mathcal{D}(T)\}$. The operator T is said to be monotone if

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq 0 \tag{3.1}$$

for all $(x_1, x_1^*), (x_2, x_2^*) \in G(T)$. The operator T is strictly monotone if it is monotone and the equality in (3.1) implies $x_1 = x_2$.

We say that a continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a *normalization function* if it is strictly increasing, $\phi(0) = 0$, and $\phi(t) \to \infty$ as $t \to \infty$. The *duality mapping corresponding to* ϕ is the set valued mapping $J_\phi : X \to 2^{X^*}$ defined by

$$J_\phi x = \{x^* \in X^* : \langle x^*, x\rangle = \phi(\|x\|)\|x\|,\ \|x^*\| = \phi(\|x\|)\}, \quad x \in X.$$

We note that $\mathcal{D}(J_\phi) = X$ by the Hahn-Banach theorem. Some main properties of J_ϕ are collected in the following theorem (cf. [20]).

Theorem 3.1. *If ϕ is as above, then*

(1) *for all $x \in X$, $J_\phi x$ is a bounded, closed, and convex subset of X^*;*
(2) *J_ϕ is monotone, i.e.*
$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq (\phi(\|x_1\|) - \phi(\|x_2\|))\,(\|x_1\| - \|x_2\|) \geq 0$$
for all $(x_1, x_1^*), (x_2, x_2^*) \in G(J_\phi)$; and
(3) *for every $x \in X$, there holds $J_\phi x = \partial\psi(x)$, where*
$$\psi(x) = \int_0^{\|x\|} \phi(t)\, dt \tag{3.2}$$
and $\partial\psi : X \to 2^{X^*}$ is the subdifferential of ψ defined by
$$\partial\psi(x) = \{x^* \in X^* : \psi(y) - \psi(x) \geq \langle x^*, y - x\rangle\ \forall y \in X\}.$$

Further, recall that a functional $f : X \to \mathbb{R}$ is called Gâteaux differentiable at $x \in X$ if there exists $f'(x) \in X^*$ such that

$$\lim_{t\to 0} \frac{f(x + th) - f(x)}{t} = \langle f'(x), h\rangle$$

for all $h \in X$. Additionally, we need the following definition.

Definition 3.2. A real Banach space X is said to be

(1) uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta = \delta(\epsilon) > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, then $\|x + y\| \leq 2(1 - \delta)$;
(2) locally uniformly convex if for $\|x\| = \|x_n\| = 1$ and $\|x_n + x\| \to 2$ as $n \to \infty$, then $x_n \to x$ strongly in X; and
(3) strictly convex if for every $x, y \in X$ with $\|x\| = \|y\| = 1$, $x \neq y$ and $\lambda \in (0, 1)$, there holds $\|\lambda x + (1 - \lambda)y\| < 1$.

Remark 3.3. It is well-known that if X is reflexive with both X and X^* locally uniformly convex, the duality mapping J_ϕ is a single-valued homeomorphism of X onto X^*. For these and further properties of duality mappings, the reader is referred to [8, 10].

Example 3.4. For $X = W_0^{1,p}(\Omega)$ with $1 < p < \infty$, and $\phi(t) = t^{p-1}$, it is shown in [13] by applying the Poincaré inequality that J_ϕ in this context is precisely the negative of the p-Laplacian Δ_p:

$$\begin{gathered} J_\phi : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega), \\ J_\phi(u) = -\Delta_p u := -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad u \in W_0^{1,p}(\Omega). \end{gathered}$$

When $X = W^{1,p}(\Omega)$, it is shown in [11] that

$$\begin{gathered} J_\phi : W^{1,p}(\Omega) \to \left(W^{1,p}(\Omega)\right)^*, \\ J_\phi u = -\Delta_p u + |u|^{p-2}u, \quad u \in W^{1,p}(\Omega). \end{gathered}$$

Throughout this section, unless otherwise noted, we assume that $p \geq 2$ and that Ω is a bounded Lipschitz domain. We next compute the duality mapping on $W^{1,p}(\operatorname{curl},\Omega)$ with respect to its norm given by

$$\|\mathbf{u}\|^p_{W^{1,p}(\operatorname{curl},\Omega)} = \|\mathbf{u}\|^p_{L^p} + \|\nabla\times\mathbf{u}\|^p_{L^p}$$

and corresponding to the normalization function $\phi(t) = t^{p-1}$. Recall now that if a convex functional $f : W^{1,p}(\operatorname{curl},\Omega) \to \mathbb{R}$ is Gâteaux differentiable at $\mathbf{u}$, then $\partial f(\mathbf{u}) = f'(\mathbf{u})$, where ∂f is the subdifferential of f. By Theorem 3.1, part (3), we know that $J_\phi = \partial\psi$, where

$$\psi(\mathbf{u}) = \frac{1}{p}\left(\|\mathbf{u}\|^p_{L^p} + \|\nabla\times\mathbf{u}\|^p_{L^p}\right) := \psi_1(\mathbf{u}) + \psi_2(\mathbf{u}).$$

It is well-known that the functional $F : (L^p(\Omega))^3 \to \mathbb{R}$ given by $\mathbf{u} \mapsto p^{-1}\|\mathbf{u}\|^p_{L^p}$ is Gâteaux differentiable and

$$\langle F'(\mathbf{v}), \mathbf{h}\rangle = \sum_{i=1}^3 \int_\Omega |v_i|^{p-2} v_i h_i \, dx \quad \forall \mathbf{v}, \mathbf{h} \in (L^p(\Omega))^3. \tag{3.3}$$

Thus, it remains to compute the Gâteaux derivative of ψ_2. Now, we write $\psi_2 = FG$, where F is the functional above and $G : W^{1,p}(\operatorname{curl},\Omega) \to (L^p(\Omega))^3$ is defined by $G(v) = |\nabla\times\mathbf{v}|$. We need to check differentiability of the functional G. But, the derivative is easily computed to be

$$G'(\mathbf{u})\cdot\mathbf{v} = \frac{\nabla\times\mathbf{u}\cdot\nabla\times\mathbf{v}}{|\nabla\times\mathbf{u}|}.$$

We obtain

$$\langle\psi'(\mathbf{u}), \mathbf{v}\rangle = \langle\psi_1'(\mathbf{u}), \mathbf{v}\rangle + \langle\psi_2'(\mathbf{u}), \mathbf{v}\rangle = \langle|\mathbf{u}|^{p-2}\mathbf{u} + \operatorname{curl}_p(\mathbf{u}), \mathbf{v}\rangle, \tag{3.4}$$

where we define $\operatorname{curl}_p : W^{1,p}(\operatorname{curl},\Omega) \to \left(W^{1,p}(\operatorname{curl},\Omega)\right)^*$ by

$$\langle\operatorname{curl}_p(\mathbf{u}), \mathbf{v}\rangle = \int_\Omega |\nabla\times\mathbf{u}|^{p-2}\nabla\times\mathbf{u}\cdot\nabla\times\mathbf{v}\, dx \quad \forall \mathbf{u}, \mathbf{v} \in W^{1,p}(\operatorname{curl},\Omega).$$

In view of Theorem 2.7, we see that $\operatorname{curl}_p : W_0^{1,p}(\operatorname{curl},\Omega) \to (W_0^{1,p}(\operatorname{curl},\Omega))^*$ is given by

$$\operatorname{curl}_p(\mathbf{u}) := \nabla\times\left(|\nabla\times\mathbf{u}|^{p-2}\nabla\times\mathbf{u}\right) \quad \forall \mathbf{u}, \mathbf{v} \in W_0^{1,p}(\operatorname{curl},\Omega). \tag{3.5}$$

Thus, we have shown the following result.

Theorem 3.5. *The duality mapping* $J_\phi : W^{1,p}(\mathrm{curl},\Omega) \to \big(W^{1,p}(\mathrm{curl},\Omega)\big)^*$ *corresponding to the normalization function* $\phi(t) = t^{p-1}$ *is given by*

$$J_\phi \mathbf{u} = |\mathbf{u}|^{p-2}\mathbf{u} + \mathrm{curl}_p(\mathbf{u})$$

for each $\mathbf{u} \in W^{1,p}(\mathrm{curl},\Omega)$*. In particular, it is coercive and a strictly monotone homeomorphism.*

As a result of the surjectivity of the duality mapping, we obtain the following results.

Corollary 3.6. *For each* $\mathbf{f} \in \big(W^{1,p}(\mathrm{curl},\Omega)\big)^*$*, the equation* $|\mathbf{u}|^{p-2}\mathbf{u}+\mathrm{curl}_p(\mathbf{u}) = \mathbf{f}$ *has a unique solution in* $W^{1,p}(\mathrm{curl},\Omega)$*.*

Theorem 3.7. *The operator* $\nabla_p \times \mathbf{u} := |\mathbf{u}|^{p-2}\mathbf{u} + \mathrm{curl}_p(\mathbf{u})$ *satisfies the* (S_+)*-condition; i.e., if* $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ *weakly in* $W^{1,p}(\mathrm{curl},\Omega)$ *and*

$$\limsup_{n\to\infty} \langle -\nabla_p \times \mathbf{u}_n, \mathbf{u}_n - \mathbf{u}_0\rangle \le 0,$$

then $\mathbf{u}_n \to \mathbf{u}_0$ *strongly in* $W^{1,p}(\mathrm{curl},\Omega)$*.*

This follows immediately from the previous results coupled with [13, Prop. 2]. Our next goal is to show that the functional $\psi(\mathbf{u}) = \frac{1}{p}\left(\|\mathbf{u}\|_{L^p}^p + \|\nabla\times\mathbf{u}\|_{L^p}^p\right)$ is continuously Fréchet differentiable on $W^{1,p}(\mathrm{curl},\Omega)$. To do so, we need the following lemma from [17].

Lemma 3.8. *If* $p \ge 2$*, then for all* $x, y, z \in \mathbb{R}^n$*, there exists a constant* $C_1 > 0$ *such that*

$$\big||z|^{p-2}z - |y|^{p-2}y\big| \le C_1|z-y|\,(|z|+|y|)^{p-2}.$$

Using the above lemma we prove the next theorem.

Theorem 3.9. *The functional* $\psi(\mathbf{u}) = \frac{1}{p}\left(\|\mathbf{u}\|_{L^p}^p + \|\nabla\times\mathbf{u}\|_{L^p}^p\right)$ *is continuously Fréchet differentiable on* $W^{1,p}(\mathrm{curl},\Omega)$*.*

Proof. Let $\mathbf{u},\mathbf{v},\mathbf{w} \in W^{1,p}(\mathrm{curl},\Omega)$. Then we have from our previous calculations that

$$\begin{aligned}
&|\langle \psi'(\mathbf{u}) - \psi'(\mathbf{v}), \mathbf{w}\rangle| \\
&= \big|\langle |\mathbf{u}|^{p-2}\mathbf{u} - |\mathbf{v}|^{p-2}\mathbf{v}, \mathbf{w}\rangle + \langle \mathrm{curl}_p(\mathbf{u}) - \mathrm{curl}_p(\mathbf{v}), \mathbf{w}\rangle\big| \\
&= \Big|\int_\Omega \big(|\mathbf{u}|^{p-2}\mathbf{u} - |\mathbf{v}|^{p-2}\mathbf{v}\big)\cdot\mathbf{w}\,dx \\
&\quad + \int_\Omega \big(|\nabla\times\mathbf{u}|^{p-2}\nabla\times\mathbf{u} - |\nabla\times\mathbf{v}|^{p-2}\nabla\times\mathbf{v}\big)\cdot\nabla\times\mathbf{w}\,dx\Big| \\
&\le \|\,|\mathbf{u}|^{p-2}\mathbf{u} - |\mathbf{v}|^{p-2}\mathbf{v}\|_{L^{p'}}\|\mathbf{w}\|_{L^p} + \|\,|\nabla\times\mathbf{u}|^{p-2}\nabla\times\mathbf{u} \\
&\quad - |\nabla\times\mathbf{v}|^{p-2}\nabla\times\mathbf{v}\|_{L^{p'}}\|\nabla\times\mathbf{w}\|_{L^p},
\end{aligned} \tag{3.6}$$

where we have used the Hölder's inequality. We start by estimating the $L^{p'}$-norm of the first term on the right hand side of (3.6). Using Hölder's inequality coupled with Lemma 3.8, we obtain

$$\begin{aligned}
\|\,|\mathbf{u}|^{p-2}\mathbf{u} - |\mathbf{v}|^{p-2}\mathbf{v}\|_{L^{p'}}^{p'} &= \int_\Omega \big||\mathbf{u}|^{p-2}\mathbf{u} - |\mathbf{v}|^{p-2}\mathbf{v}\big|^{p'}\,dx \\
&\le C\int_\Omega |\mathbf{u}-\mathbf{v}|^{p'}\,(|\mathbf{u}|+|\mathbf{v}|)^{(p-2)p'}\,dx
\end{aligned}$$

$$\begin{aligned}&\le C\| \, |\mathbf{u}-\mathbf{v}|^{p'} \|_{L^{p-1}} \| \, (|\mathbf{u}|+|\mathbf{v}|)^{(p-2)p'} \, \|_{L^{\frac{p-1}{p-2}}} \\ &= C\|\mathbf{u}-\mathbf{v}\|_{L^p}^{\frac{p}{p-1}} \| \, |\mathbf{u}|+|\mathbf{v}| \|_{L^p}^{(p-2)p'},\end{aligned}$$

which implies

$$\| \, |\mathbf{u}|^{p-2}\mathbf{u}-|\mathbf{v}|^{p-2}\mathbf{v}\|_{L^{p'}} \le C\|\mathbf{u}-\mathbf{v}\|_{L^p} \| \, |\mathbf{u}|+|\mathbf{v}| \, \|_{L^p}^{(p-2)}. \tag{3.7}$$

In a similar fashion, we can find a constant $C' > 0$ such that

$$\begin{aligned}&\| \, |\nabla\times\mathbf{u}|^{p-2}\nabla\times\mathbf{u}-|\nabla\times\mathbf{v}|^{p-2}\nabla\times\mathbf{v}\|_{L^{p'}} \\ &\le C'\|\nabla\times(\mathbf{u}-\mathbf{v})\|_{L^p} \| \, |\nabla\times\mathbf{u}|+|\nabla\times\mathbf{v}| \, \|_{L^p}^{p-2}.\end{aligned} \tag{3.8}$$

Combining estimates (3.7) and (3.8) allows us to conclude that there exists some constant C (after renaming) such that

$$|\langle \psi'(\mathbf{u})-\psi'(\mathbf{v}),\mathbf{w}\rangle| \le C\|\mathbf{u}-\mathbf{v}\|_{W^{1,p}(\mathrm{curl},\Omega)}\|\mathbf{w}\|_{W^{1,p}(\mathrm{curl},\Omega)}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in W^{1,p}(\mathrm{curl},\Omega)$, which establishes the desired result since a functional is continuously Fréchet differentiable if and only if it is continuously Gâteaux differentiable. $\square$

4. Duality mappings on W_N: part I

We would like to compute J_ϕ for $W_0^{1,p}(\mathrm{curl},\Omega)$ with $\phi(t)=t^{p-1}$. In the case of the standard Sobolev spaces $W_0^{1,p}(\Omega)$, a key result that was used in [11, 13] to compute the duality mapping was the Poincaré inequality

$$\|u\|_{L^p} \le C(\Omega,n)\|\nabla u\|_{L^p} \quad \forall\, u\in W_0^{1,p}(\Omega).$$

It was shown in [3] that for $\mathbf{u}\in W_0^{1,p}(\mathrm{curl},\Omega)$ on a $C^{1,1}$ domain,

$$\|\mathbf{u}\|_{L^p} \le C\left(\|\nabla\times\mathbf{u}\|_{L^p}+\|\nabla\cdot\mathbf{u}\|_{L^p}+|\langle \mathbf{u}\cdot\boldsymbol{\nu},1\rangle_{\partial\Omega}|\right).$$

Thus, we can generally estimate the L^p-norm of a function u not only in terms of its curl, but also its divergence and a certain boundary trace. So, an equivalent norm on $W_0^{1,p}(\mathrm{curl},\Omega)$ is given by

$$\mathbf{w}\mapsto \|\nabla\times\mathbf{w}\|_{L^p}+\|\nabla\cdot\mathbf{w}\|_{L^p}+|\langle \mathbf{w}\cdot\boldsymbol{\nu},1\rangle_{\partial\Omega}|.$$

In any case, one must have a well-defined divergence, and hence if we are interested in computing a duality mapping on a "trace-zero" space, it must be W_N.

If we assume that Ω has a $C^{1,1}$ boundary, then W_N in the case when $p=2$ can be identified with the Sobolev space $(H^1(\Omega))^3$ with equivalent norms; see [12, Theorem 3, p. 209]. This relies on the vector having zero tangential trace (a similar analysis works if the normal trace vanishes).

We want to extend this result for more general p. For this we use Peetre's lemma from [27].

Lemma 4.1. *Let E_0, E_1, E_2 be three Banach spaces, and let $A_1 : E_0 \to E_1, A_2 : E_0 \to E_2$ be continuous linear maps, such that*

1. *A_2 is compact and*
2. *there exists $C > 0$ such that*

$$\|v\|_{E_0} \le C\left(\|A_1 v\|_{E_1}+\|A_2 v\|_{E_2}\right) \quad \forall v\in E_0. \tag{4.1}$$

Then $\ker(A_2)$ *has finite dimension and* $\operatorname{Im}(A_1)$ *is closed, and there exists* $C_0 > 0$ *such that*

$$\inf_{w\in\ker(A_1)} \|v + w\|_{E_0} \le C_0\|A_1 v\|_{E_1}. \tag{4.2}$$

Now we show the equivalence of W_N with the space

$$W_{N,0} := \{\mathbf{u} \in W^{1,p}(\Omega) : \boldsymbol{\gamma}_t(\mathbf{u}) = \mathbf{0}\},$$

where the norm here is $\|\mathbf{u}\|_{W_{N,0}} := \|\mathbf{u}\|_{W^{1,p}(\Omega)}$.

Theorem 4.2. *Let* Ω *be a* $C^{1,1}$ *domain. Then the spaces* W_N *and* $W_{N,0}$ *can be identified and have equivalent norms.*

Proof. We will apply Lemma 4.1 with $E_0 = W_{N,0}$, $E_1 = (L^p(\Omega))^3 \times L^p(\Omega) \times (L^p(\Omega))^3$, and $E_2 = B^{-1/p'}_{p',p'}(\partial\Omega)$. The operators we take are given by

$$A_1(\mathbf{v}) = (\nabla \times \mathbf{v}, \nabla \cdot \mathbf{v}, \mathbf{v}), \quad A_2(\mathbf{v}) \equiv 0.$$

We need to prove an estimate of the form (4.1), which boils down to an L^p-estimate for ∇v. It is precisely here that we require the domain to be $C^{1,1}$. Given $\mathbf{v} \in E_0$, by [3, Theorem 3.1], we can find $C > 0$ such that

$$\|\nabla\mathbf{v}\|^p_{L^p} \le C\left(\|\mathbf{v}\|^p_{L^p} + \|\nabla \times \mathbf{v}\|^p_{L^p} + \|\nabla \cdot \mathbf{v}\|^p_{L^p}\right), \tag{4.3}$$

By using (4.3), we have

$$\|\mathbf{v}\|^p_{W_{N,0}} = \|\mathbf{v}\|^p_{L^p} + \|\nabla\mathbf{v}\|^p_{L^p} \le C\left(\|\mathbf{v}\|^p_{L^p} + \|\nabla \times \mathbf{v}\|^p_{L^p} + \|\nabla \cdot \mathbf{v}\|^p_{L^p}\right) = C\|\mathbf{v}\|_{W_N}.$$

Thus, since $\ker(A_1) = \{0\}$, (4.2) implies the equivalence of the norms $\|\cdot\|_{W_{N,0}}$ and $\|\cdot\|_{W_N}$. □

Remark 4.3. Let us briefly discuss the assumptions needed in Theorem 4.2. We have used the L^p-estimate from [3], which actually gives

$$\|\nabla\mathbf{v}\|_{L^p} \le C\left(\|\nabla \cdot \mathbf{v}\|_{L^p} + \|\nabla \times \mathbf{v}\|_{L^p} + |\langle \mathbf{v} \cdot \boldsymbol{\nu}, 1\rangle_{\partial\Omega}|\right). \tag{4.4}$$

This coupled with the following estimate

$$\Big|\int_{\partial\Omega} (\operatorname{Tr}(\mathcal{B}))\,(\mathbf{v}\cdot\boldsymbol{\nu})^2 d\sigma\Big| \le C\int_{\partial\Omega} |\mathbf{v}|^2 d\sigma \le \frac{1}{2}\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + C\|\mathbf{v}\|^2_{L^2(\Omega)}$$

yields (4.3). Above, $\mathcal{B}$ denotes the curvature tensor of $\partial\Omega$ and Tr denotes the trace. It is unclear if these estimates would hold if the domain were merely Lipschitz. Our thought is likely it is not possible, since to get solvability in $W^{1,p}$, the domain should be at least $C^{1,1}$. In [18], particularly Lemma 3.1.1.2, the domain is Lipschitz with the additional assumption that it is piecewise C^2. Thus, the Banach space E_2 we have taken is somewhat arbitrary due to $A_2 \equiv 0$. However, if we were to use the estimate (4.4), then the boundary term would have to be incorporated into A_2 on E_2, and in order for this to be compact, one would need compact embedding of certain Besov spaces.

Finally, instead of $C^{1,1}$, one could take the domain to be convex. The idea here is that one can approximate a convex domain by an increasing sequence of convex, $C^{1,1}$ open sets [18, Lemma 3.2.1.1].

Now that we have the equivalence of norms, we can appeal to the known result [11] for the duality mapping on $W^{1,p}(\Omega)$. First, we need the following theorem.

Theorem 4.4. *Let* Ω *be a* $C^{1,1}$ *domain. Then the space* $(W_N, \|\cdot\|_{W^{1,p}})$ *is uniformly convex, reflexive, and separable.*

For completeness, we prove the Theorem for $1 < p < \infty$.

Proof. It is well-known that if X is uniformly convex, then it is reflexive. To show uniform convexity, first let $p \geq 2$.

Take $\mathbf{u}, \mathbf{v} \in W_N$ with $\|\mathbf{u}\|_{W^{1,p}} = \|\mathbf{v}\|_{W^{1,p}} = 1$, and $\|\mathbf{u}-\mathbf{v}\|_{W^{1,p}} \geq \epsilon \in (0,2]$. For $z, w \in \mathbb{R}^n$ we know that (see [1])

$$\big|\frac{z+w}{2}\big|^p + \big|\frac{z-w}{2}\big|^p \leq \frac{1}{2}\left(|z|^p + |w|^p\right).$$

Then we have

$$\begin{aligned}
&\|\frac{\mathbf{u}+\mathbf{v}}{2}\|^p_{W^{1,p}} + \|\frac{\mathbf{u}-\mathbf{v}}{2}\|^p_{W^{1,p}}\\
&= \int_\Omega \Big(\big|\frac{\mathbf{u}+\mathbf{v}}{2}\big|^p + \big|\frac{\mathbf{u}-\mathbf{v}}{2}\big|^p\Big)dx + \int_\Omega \Big(\big|\frac{\nabla\mathbf{u}+\nabla\mathbf{v}}{2}\big|^p + \big|\frac{\nabla\mathbf{u}-\nabla\mathbf{v}}{2}\big|^p\Big)dx\\
&\leq \frac{1}{2}\int_\Omega \left(|\mathbf{u}|^p + |\mathbf{v}|^p\right)dx + \frac{1}{2}\int_\Omega \left(|\nabla\mathbf{u}|^p + |\nabla\mathbf{v}|^p\right)dx\\
&= \frac{1}{2}\left(\|\mathbf{u}\|_{W^{1,p}} + \|\mathbf{v}\|_{W^{1,p}}\right) = 1,
\end{aligned}$$

and hence

$$\|\frac{\mathbf{u}+\mathbf{v}}{2}\|^p_{W^{1,p}} \leq 1 - \big(\frac{\epsilon}{2}\big)^p. \tag{4.5}$$

When $1 < p < 2$, then it is also known [1] that for $z, w \in \mathbb{R}^n$,

$$\big|\frac{z+w}{2}\big|^{p'} + \big|\frac{z-w}{w}\big|^{p'} \leq \big[\frac{1}{2}\left(|z|^p + |w|^p\right)\big]^{\frac{1}{p-1}}.$$

Take $\mathbf{u}, \mathbf{v}$ as above. First, notice that $\|\cdot\|^p_{L^p} = \||\cdot|^{p'}\|^{p-1}_{L^{p-1}}$. We then have that

$$\begin{aligned}
&\|\frac{\mathbf{u}+\mathbf{v}}{2}\|^p_{W^{1,p}} + \|\frac{\mathbf{u}-\mathbf{v}}{2}\|^p_{W^{1,p}}\\
&= \|\frac{\mathbf{u}+\mathbf{v}}{2}\|^p_{L^p} + \|\frac{\mathbf{u}-\mathbf{v}}{2}\|^p_{L^p} + \|\frac{\nabla\mathbf{u}+\nabla\mathbf{v}}{2}\|^p_{L^p} + \|\frac{\nabla\mathbf{u}-\nabla\mathbf{v}}{2}\|^p_{L^p}\\
&= \|\,\big|\frac{\mathbf{u}+\mathbf{v}}{2}\big|^{p'}\|^{p-1}_{L^{p-1}} + \|\,\big|\frac{\mathbf{u}-\mathbf{v}}{2}\big|^{p'}\|^{p-1}_{L^{p-1}}\\
&\quad + \|\,\big|\frac{\nabla\mathbf{u}+\nabla\mathbf{v}}{2}\big|^{p'}\|^{p-1}_{L^{p-1}} + \|\,\big|\frac{\nabla\mathbf{u}+\nabla\mathbf{v}}{2}\big|^{p'}\|^{p-1}_{L^{p-1}}\\
&\leq \|\,\big|\frac{\mathbf{u}+\mathbf{v}}{2}\big|^{p'} + \big|\frac{\mathbf{u}-\mathbf{v}}{2}\big|^{p'}\|^{p-1}_{L^{p-1}} + \|\big|\frac{\nabla\mathbf{u}+\nabla\mathbf{v}}{2}\big|^{p'} + \big|\frac{\nabla\mathbf{u}-\nabla\mathbf{v}}{2}\big|^{p'}\|^{p-1}_{L^{p-1}}\\
&\leq \frac{1}{2}\left(\|\mathbf{u}\|^p_{L^p} + \|\nabla\mathbf{u}\|^p_{L^p} + \|\mathbf{v}\|^p_{L^p} + \|\nabla\mathbf{v}\|^p_{L^p}\right)\\
&= \frac{1}{2}(1+1) = 1,
\end{aligned}$$

where in the first inequality we have used that $0 < p-1 < 1$. Therefore, since $\|\mathbf{u}-\mathbf{v}\|_{W^{1,p}} \geq \epsilon$, again we obtain that

$$\|\frac{\mathbf{u}+\mathbf{v}}{2}\|^p_{W^{1,p}} \leq 1 - \big(\frac{\epsilon}{2}\big)^p \tag{4.6}$$

and uniform convexity is proved. Finally, for separability, we require $p < \infty$, since then L^p is separable. Then it's easy to construct an isometry from $W^{1,p}$ onto a subspace of L^p, so separability follows immediately. $\square$

Theorem 4.5. *Let Ω be a $C^{1,1}$ domain and $p \geq 2$. Then the duality mapping $J_\phi : (W_N, \|\cdot\|_{W^{1,p}}) \to (W_N, \|\cdot\|_{W^{1,p}})^*$ corresponding to the normalization function $\phi(t) = t^{p-1}$ is given by*

$$J_\phi \mathbf{u} = -\Delta_p \mathbf{u} + |\mathbf{u}|^{p-2}\mathbf{u}$$

for each $\mathbf{u} \in W_N$. Above, Δ_p denotes the p-Laplace operator on $(W_N, \|\cdot\|_{W^{1,p}})$.

Proof. Letting $J_\phi : (W_N, \|\cdot\|_{W^{1,p}}) \to (W_N, \|\cdot\|_{W^{1,p}})^*$, with $\phi(t) = t^{p-1}$, we know that

$$J_\phi(\mathbf{u}) = \partial\Phi(\|\mathbf{u}\|) = \partial\Big(\int_0^{\|\mathbf{u}\|_{W^{1,p}}} t^{p-1}\,dt\Big) = \Phi_1(\mathbf{u}) + \Phi_2(\mathbf{u}),$$

where

$$\Phi_1(\mathbf{u}) = \frac{1}{p}\|\mathbf{u}\|_{L^p}^p \quad \text{and} \quad \Phi_2(\mathbf{u}) = \frac{1}{p}\|\nabla\mathbf{u}\|_{L^p}^p$$

and ∂ denotes the subdifferential. We calculate $\Phi_2'(\mathbf{u})$. Simple computations using (3.3) imply that

$$\langle \Phi_2'(\mathbf{u}), \mathbf{v}\rangle = \sum_{j=1}^3 \langle |\nabla u_j|^{p-2}\nabla u_j, \nabla v_j\rangle.$$

Thus, we conclude that

$$\begin{aligned}
\langle \Phi'(\mathbf{u}), \mathbf{v}\rangle &= \sum_{j=1}^3 \big(\langle |u_j|^{p-2}u_j, v_j\rangle + \langle |\nabla u_j|^{p-2}\nabla u_j, \nabla v_j\rangle\big)\\
&= \sum_{j=1}^3 \Big[\int_\Omega |u_j|^{p-2}u_j v_j\,dx + \int_\Omega |\nabla u_j|^{p-2}\nabla u_j\cdot\nabla v_j\,dx\Big].
\end{aligned}$$

□

We now make precise the way the p-Laplacian Δ_p acts on $(W_N, \|\cdot\|_{W^{1,p}})$. If $\mathbf{u} \in W_N$ and

$$\nabla\cdot\big(|\nabla u_j|^{p-2}\nabla u_j\big) \in (L^p(\Omega))^3, \quad j = 1, 2, 3,$$

then the traces $\gamma_n(\mathbf{u})$ and $\gamma_n(|\nabla u_j|^{p-2}\nabla u_j)$ make sense. Setting

$$\gamma_n(|\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}) := \big(\gamma_n(|\nabla u_1|^{p-2}\nabla u_1), \gamma_n(|\nabla u_2|^{p-2}\nabla u_2), \gamma_n(|\nabla u_3|^{p-2}\nabla u_3)\big),$$

Theorem 2.9 then implies that

$$\begin{aligned}
&\langle \gamma_n(|\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}), \boldsymbol{\phi}\rangle\\
&= \sum_{j=1}^3 \Big[\int_\Omega \big(\nabla\cdot|\nabla u_j|^{p-2}\nabla u_j\big)\,\phi_j\,dx + \int_\Omega |\nabla u_j|^{p-2}\nabla u_j\cdot\nabla\phi_j\,dx\Big]
\end{aligned}$$

for all $\mathbf{u}, \boldsymbol{\phi} \in W_N$. If each $|\nabla u_j|^{p-2}\nabla u_j \in \ker(\gamma_n)$, $j = 1, 2, 3$, then

$$\sum_{j=1}^3 \int_\Omega -\big(\nabla\cdot|\nabla u_j|^{p-2}\nabla u_j\big)\,\phi_j\,dx = \sum_{j=1}^3 \int_\Omega |\nabla u_j|^{p-2}\nabla u_j\cdot\nabla\phi_j\,dx \quad \forall\boldsymbol{\phi} \in W_N.$$

Note that the integral on the right-hand side above exists for all $\mathbf{u}, \boldsymbol{\phi} \in W_N$. Thus, we denote

$$\Delta_p\mathbf{u} := \nabla\cdot\big(|\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}\big), \quad \mathbf{u} \in W_N.$$

This should be understood componentwise, so that

$$(\Delta_p\mathbf{u})_j = \nabla\cdot\big(|\nabla u_j|^{p-2}\nabla u_j\big), \quad j = 1, 2, 3.$$

Remark 4.6. Suppose $\mathbf{u}(x_1, x_2, x_3) = (u(x_1, x_2, x_3)\ 0\ 0)^t$. Then the previous calculations for the derivative of $\Phi(\mathbf{u})$ reduce to

$$\langle \Phi'(u), v\rangle = \int_\Omega |u|^{p-2} uv\,dx + \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx \quad \forall u, v\in W^{1,p}(\Omega).$$

Thus, the p-Laplacian Δ_p in this case becomes the usual p-Laplacian on the space $W^{1,p}(\Omega)$ because the space W_N reduces to the (scalar) space $W^{1,p}(\Omega)$. The duality mapping from Theorem 4.5 then agrees with that in [11, Theorem 3.1].

In the next section, we consider duality mappings on W_N endowed with its graph norm.

5. Duality mappings on W_N: part 2

In this section we can take Ω to be a bounded $C^{1,1}$ domain; the reason for this is to ensure the definition (5.2) below makes sense. We have

$$\|\mathbf{u}\|_{W_N}^p = \|\mathbf{u}\|_{L^p}^p + \|\nabla\times\mathbf{u}\|_{L^p}^p + \|\nabla\cdot\mathbf{u}\|_{L^p}^p.$$

To compute the duality mapping on W_N, define $\psi : W_N \to \mathbb{R}$ by

$$\psi(\mathbf{u}) = \frac{1}{p}\left(\|\mathbf{u}\|_{L^p}^p + \|\nabla\times\mathbf{u}\|_{L^p}^p + \|\nabla\cdot\mathbf{u}\|_{L^p}^p\right) := \psi_1(\mathbf{u}) + \psi_2(\mathbf{u}) + \psi_3(\mathbf{u}).$$

By Theorem 3.1, part (3), we know that $J_\phi = \partial\psi(x)$. We have previously computed $\psi_1'(\mathbf{u})$ and $\psi_2'(\mathbf{u})$. Thus, it remains to compute the Gâteaux derivative of ψ_3. We can write $\psi_3 = FH$, where $H(\mathbf{v}) = |\nabla\cdot\mathbf{v}|$ and $F(\mathbf{v}) = p^{-1}\|\mathbf{v}\|_{L^p}^p$, to obtain

$$H'(\mathbf{u})\cdot\mathbf{v} = \frac{(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})}{|\nabla\cdot\mathbf{u}|}.$$

Thus, we get

$$\begin{aligned}\langle\psi'(\mathbf{u}), \mathbf{v}\rangle &= \langle\psi_1'(\mathbf{u}), \mathbf{v}\rangle + \langle\psi_2'(\mathbf{u}), \mathbf{v}\rangle + \langle\psi_3'(\mathbf{u}), \mathbf{v}\rangle\\ &= \langle|\mathbf{u}|^{p-2}\mathbf{u} + \operatorname{curl}_p(\mathbf{u}) + \operatorname{div}_p(\mathbf{u}), \mathbf{v}\rangle,\end{aligned}\tag{5.1}$$

where we have defined

$$\operatorname{div}_p : W_N^0 \to \left(W_N^0\right)^*, \quad \operatorname{div}_p(\mathbf{u}) := -\nabla(|\nabla\cdot\mathbf{u}|^{p-2}\nabla\cdot\mathbf{u}),\tag{5.2}$$

in the sense that div_p acts, in view of Theorems 2.9 and 2.11, by

$$\langle\operatorname{div}_p(\mathbf{u}), \mathbf{v}\rangle = \int_\Omega |\nabla\cdot\mathbf{u}|^{p-2}\,(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})\,dx \quad \forall\mathbf{u}, \mathbf{v}\in W_N^0.$$

Hence we have shown the following theorem.

Theorem 5.1. *Let Ω be a bounded $C^{1,1}$ domain. Then, the duality mapping $J_\phi : W_N^0 \to \left(W_N^0\right)^*$ corresponding to the normalization function $\phi(t) = t^{p-1}$ is given by*

$$J_\phi\mathbf{u} = |\mathbf{u}|^{p-2}\mathbf{u} + \operatorname{curl}_p(\mathbf{u}) + \operatorname{div}_p(\mathbf{u})$$

for each $\mathbf{u}\in W_N^0$. In particular, it is coercive and a strictly monotone homeomorphism.

As a result of the surjectivity of the duality mapping, we obtain the following result.

Corollary 5.2. *For each $\mathbf{f}\in\left(W_N^0\right)^*$, the equation $|\mathbf{u}|^{p-2}\mathbf{u} + \operatorname{curl}_p(\mathbf{u}) + \operatorname{div}_p(\mathbf{u}) = \mathbf{f}$ has a unique solution in W_N^0.*

6. Generalizations

This method appears to be generalizable as follows. Let now X be an arbitrary Banach space with norm $\|\cdot\|_X$, such that there exists $a \in [1,\infty)$ and a C^1 functional F so that

$$\|u\|_X^a = \int_\Omega F(u(x))dx. \tag{6.1}$$

We require that $(X, \|\cdot\|_X)$ be uniformly convex, which is equivalent to the following which we further assume that

$$\text{the norm on } X^* \text{ is uniformly Fréchet differentiable on } \{x \in X : \|x\|_X = 1\}. \tag{6.2}$$

Recall that the norm on a Banach space is said to be uniformly Fréchet differentiable on the unit sphere if

$$\lim_{h\to 0} \Big| \frac{\|x+hy\| - \|x\|}{h} - f_x(y) \Big|$$

exists uniformly in x and y on the unit sphere in X. Above $f_x(y)$ denotes a support functional; see [23] for more details. As before, we let $\psi(u) = a^{-1}\|u\|_X^a$. Then our previous calculations show that

$$\langle J_\phi(u), v\rangle = \langle \psi'(u), v\rangle = \int_\Omega F'(u(x))v(x)\,dx. \tag{6.3}$$

This formula agrees with the well-known derivative of the L^p norm by taking $a = p$ and $F(u) = |u|^p$ in $X = L^p(\Omega)$, as well as the result from [13] by taking $a = p$, $F(u) = |\nabla u|^p$ and $X = W_0^{1,p}(\Omega)$.

Theorem 6.1. *Let $(X, \|\cdot\|_X)$ be a uniformly convex Banach space such that* (6.1) *and* (6.2) *hold. Then the duality mapping corresponding to the normalization function $\phi(t) = t^{a-1}$ is the single-valued function*

$$J_\phi : (X, \|\cdot\|_X) \to (X, \|\cdot\|_X)^*$$

satisfying (6.3).

7. On the problem $\mathscr{L}_p(\mathbf{u}) = \mathbf{f}(x,\mathbf{u})$

In this section we assume that Ω has a $C^{1,1}$ boundary or is convex, so that $W_N \hookrightarrow L^p$ compactly. We first define a vector valued variant of Carathéodory functions.

Definition 7.1. A vector-valued function $\mathbf{f} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is called Carathéodory provided

(i) for each $s \in \mathbb{R}^3$, the function $x \mapsto \mathbf{f}(x,s)$ is measurable in Ω; and
(ii) for a.e. $x \in \Omega$, the function $s \mapsto \mathbf{f}(x,s)$ is continuous in $\mathbb{R}^3$.

For a vector-valued Carathéodory function $\mathbf{f}$, for each measurable vector-valued function $u = (u_1, u_2, u_3)$, the function

$$(N_{\mathbf{f}}\mathbf{u})(x) = \mathbf{f}(x, \mathbf{u}(x))$$

is measurable. The operator $N_{\mathbf{f}}$ from the set of measurable functions to itself is called the *Nemytskii operator*. We will consider what conditions on $\mathbf{f}$ are required in order to obtain existence of a $\mathbf{u} \in W_N^0$ of the nonlinear boundary value problem

$$\begin{gathered} |\mathbf{u}|^{p-2}\mathbf{u} + \operatorname{curl}_p(\mathbf{u}) + \operatorname{div}_p(\mathbf{u}) = \mathbf{f}(x,\mathbf{u}) \quad \text{in } \Omega, \\ \boldsymbol{\gamma}_t(\mathbf{u}) = \mathbf{0} \quad \text{on } \partial\Omega. \end{gathered} \tag{7.1}$$

Note that since we are seeking $\mathbf{u} \in W_N^0$ we also have $\gamma_n(\mathbf{u}) = 0$ on $\partial\Omega$.

Remark 7.2. Consider the diagram

$$W_0^{1,p}(\mathrm{curl}, \Omega) \overset{Id}{\hookrightarrow} (L^q(\Omega))^3 \overset{N_{\mathbf{f}}}{\hookrightarrow} \left(L^{q'}(\Omega)\right)^3 \overset{Id^*}{\hookrightarrow} \left(W_0^{1,p}(\mathrm{curl}, \Omega)\right)^*.$$

If $W_0^{1,p}(\mathrm{curl}, \Omega) \hookrightarrow (L^q(\Omega))^3$ compactly, then we would be able to conclude that $N_{\mathbf{f}}$ is a compact operator. Given the results in [22, 6, 7], it is unlikely to expect any compactness without imposing a divergence condition as well. For this reason we seek solutions of (7.1) in W_N.

Note that (7.1) is understood in the sense of $\left(W_N^0\right)^*$:

$$\langle |\mathbf{u}|^{p-2}\mathbf{u} + \mathrm{curl}_p(\mathbf{u}) + \mathrm{div}_p(\mathbf{u}), \mathbf{v}\rangle = \langle N_{\mathbf{f}}(\mathbf{u}), \mathbf{v}\rangle \quad \forall \mathbf{v} \in W_N^0. \tag{7.2}$$

The following result will be useful to establish the compactness of $N_{\mathbf{f}}$, see [30, Theorem 19.1].

Proposition 7.3. *Let $\mathbf{f} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ be Carathéodory such that*

$$|f_i(x, u_1, u_2, u_3)| \le C \sum_{k=1}^{3} |u_k|^r + b_i(x), \quad x \in \Omega, \quad i = 1, 2, 3,$$

where $r > 0$, $\mathbf{f}(x, \mathbf{u}(x)) = (f_1(x, \mathbf{u}(x)), f_2(x, \mathbf{u}(x)), f_3(x, \mathbf{u}(x)))$ with $\mathbf{u} = (u_1, u_2, u_3)$, and each $b_i \in L^q(\Omega)$, $1 \le q < \infty$. Then $N_{\mathbf{f}}\big((L^{qr}(\Omega))^3 \big) \hookrightarrow (L^q(\Omega))^3$ continuously and maps bounded sets into bounded sets.

Note that if $r = q - 1$ and $\mathbf{b} \in \left(L^{q'}(\Omega)\right)^3$, then from Proposition 7.3 we have $N_{\mathbf{f}}\big((L^q(\Omega))^3 \big) \hookrightarrow \left(L^{q'}(\Omega)\right)^3$ and $N_{\mathbf{f}}\big((L^q(\Omega))^3 \big) \hookrightarrow \left(L^1(\Omega)\right)^3$ continuously. Thus, we will assume that the right hand side $\mathbf{f}$ in (7.1) is Carathéodory as well as satisfies the growth condition

$$|f_i(x, u_1, u_2, u_3)| \le C \sum_{k=1}^{3} |u_k|^{p-1} + b_i(x), \quad x \in \Omega,\ i = 1, 2, 3 \tag{7.3}$$

for some $C \ge 0$, with $b_i \in L^{p'}(\Omega)$. Thus, by considering

$$W_N^0 \overset{Id}{\hookrightarrow} (L^p(\Omega))^3 \overset{N_{\mathbf{f}}}{\hookrightarrow} \left(L^{p'}(\Omega)\right)^3 \overset{Id^*}{\hookrightarrow} \left(W_N^0\right)^*,$$

under the previous assumptions we have that $N_{\mathbf{f}}$ is compact.

For $\mathbf{u} \in W_N^0$, let $\psi(\mathbf{u}) = \frac{1}{p}\left(\|\mathbf{u}\|_{L^p}^p + \|\nabla \times \mathbf{u}\|_{L^p}^p + \|\nabla \cdot \mathbf{u}\|_{L^p}^p\right)$ as in Section 5. Then, using the method of proof from Theorem 3.9, it can be shown that ψ is continuously Fréchet differentiable on W_N^0.

Next we are interested in seeing when the Nemytskii operator $N_{\mathbf{f}}$ can be written as the gradient of some functional. From [30, Theorem 21.1], we know that if for some real-valued $F(x, u_1, u_2, u_3)$,

$$f_i(x, u_1, u_2, u_3) = \frac{\partial}{\partial u_i} F(x, u_1, u_2, u_3), \quad F(x, 0, 0, 0) = 0, \quad i = 1, 2, 3 \tag{7.4}$$

with each f_i satisfying (7.3), then the functional $\Phi : W_N^0 \to \mathbb{R}$ given by

$$\Phi(\mathbf{u}) = \int_\Omega F(x, \mathbf{u}(x))\, dx \tag{7.5}$$

satisfies $\nabla\Phi = N_{\mathbf{f}}$.

Thus, the functional $\mathscr{F} : W_N^0 \to \mathbb{R}$ given by

$$\mathscr{F}(\mathbf{u}) = \psi(\mathbf{u}) - \Phi(\mathbf{u}) = \frac{1}{p}\left(\|\mathbf{u}\|_{L^p}^p + \|\nabla \times \mathbf{u}\|_{L^p}^p + \|\nabla \cdot \mathbf{u}\|_{L^p}^p\right) - \int_\Omega F(x, \mathbf{u})\, dx$$

is continuously differentiable on W_N^0 and

$$\mathscr{F}'(\mathbf{u}) = \mathscr{L}_p(\mathbf{u}) - N_{\mathbf{f}}(\mathbf{u}),$$

where we have defined

$$\mathscr{L}_p(\mathbf{u}) = |\mathbf{u}|^{p-2}\mathbf{u} + \operatorname{curl}_p(\mathbf{u}) + \operatorname{div}_p(\mathbf{u}). \tag{7.6}$$

Thus, problem (7.1) is reduced to finding critical points of $\mathscr{F}$ on W_N^0. Note that since $W_N \hookrightarrow (L^p(\Omega))^3$, it holds that $\mathscr{F}$ is weakly lower semicontinuous, i.e. whenever $\mathbf{u}_j \to \mathbf{u}$ strongly in W_N, it holds that $\liminf_{j\to\infty} \mathscr{F}(\mathbf{u}_j) \geq \mathscr{F}(\mathbf{u})$.

It is enough now to prove that $\mathscr{F}$ is coercive. One way to show this is via the method of Anane and Gossez for the p-Laplace operator [4]. To this end, we proceed to understand the first eigenvalue of $\mathscr{L}_p$. We see that the equation $\mathscr{L}_p(\mathbf{u}) = \mathbf{0}$ arises as the Euler-Lagrange equation for the integral

$$I(\mathbf{u}) = \int_\Omega \left(|\mathbf{u}|^p + |\nabla \times \mathbf{u}|^p + |\nabla \cdot \mathbf{u}|^p\right) dx, \quad 1 < p < \infty. \tag{7.7}$$

Consider now the Rayleigh quotient

$$R(\mathbf{u}) = \frac{\int_\Omega \left(|\mathbf{u}|^p + |\nabla \times \mathbf{u}|^p + |\nabla \cdot \mathbf{u}|^p\right) dx}{\int_\Omega |\mathbf{u}|^p\, dx}. \tag{7.8}$$

The minimization of this quotient in W_N^0 leads to a nonlinear eigenvalue problem. The corresponding Euler-Lagrange equation is

$$\mathscr{L}_p(\mathbf{u}) - \lambda |\mathbf{u}|^{p-2}\mathbf{u} = \mathbf{0}. \tag{7.9}$$

This is obtained by minimizing the functional $I(\mathbf{u})$ subject to the constraint $G(\mathbf{u}) = \int_\Omega |u|^p\, dx = 1$.

Definition 7.4. A function $\mathbf{u} \in W_N^0 \cap \left(C(\overline{\Omega})\right)^3$ will be called a *p-eigenfunction* if there exists $\lambda \in \mathbb{R}$ so that

$$\begin{aligned}
&\int_\Omega |\mathbf{u}|^{p-2}\mathbf{u} \cdot \mathbf{v}\, dx + \int_\Omega |\nabla \times \mathbf{u}|^{p-2} \nabla \times \mathbf{u} \cdot \nabla \times \mathbf{v}\, dx \\
&\quad + \int_\Omega |\nabla \cdot \mathbf{u}|^{p-2} \nabla \cdot \mathbf{u} \nabla \cdot \mathbf{v}\, dx \\
&= \lambda \int_\Omega |\mathbf{u}|^{p-2}\mathbf{u} \cdot \mathbf{v}\, dx
\end{aligned} \tag{7.10}$$

for all $\mathbf{v} \in W_N^0$. The associated λ will be called a *p-eigenvalue*.

Now, notice that if $\mathbf{u}$ is a solution to (7.9), then $\lambda = R(\mathbf{u})$ as expected. Thus, we see that $\lambda > 0$. The smallest eigenvalue of (7.9) is

$$\begin{aligned}
\lambda_1 =&\lambda_1(\Omega) \\
:=& \inf\Big\{ \int_\Omega \left(|\mathbf{u}|^p + |\nabla \times \mathbf{u}|^p + |\nabla \cdot \mathbf{u}|^p\right) dx : \mathbf{u} \in W_N^0,\ \|\mathbf{u}\|_{L^p} = 1\Big\}.
\end{aligned} \tag{7.11}$$

We assume that

$$\text{the infimum in (7.11) is attained when } \mathbf{u} \text{ is a multiple of some } \mathbf{u}_1 > \mathbf{0}. \tag{7.12}$$

Assume, further, that there exists a function $\alpha(x) \in L^\infty(\Omega)$ with $\alpha(x) < \lambda_1$ on a set of positive measure, such that

$$\limsup_{|s|\to\pm\infty} \frac{pF(x,s)}{|s|^p} \le \alpha(x) \le \lambda_1 \quad \text{uniformly in } \Omega. \tag{7.13}$$

Under these two assumptions, the following theorem holds.

Theorem 7.5. *Let $\mathbf{f} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ be Carathéodory and satisfy* (7.3) *as well as* (7.4)*. Suppose there exists a function $\alpha(x) \in L^\infty(\Omega)$ with $\alpha(x) < \lambda_1$ on a set of positive measure such that* (7.13) *holds. Finally, assume the infimum condition* (7.12) *holds. Then $\mathscr{F}$ is coercive, and so* (7.1) *has solutions in W_N^0.*

Our next goal is to show that the infimum in (7.11) is attained by $\mathbf{u}_0$ which is a weak solution of the Euler-Lagrange equation (7.9). Since $W_N \hookrightarrow (L^p(\Omega))^3$ is compact and W_N is reflexive, this is enough to guarantee the existence of a minimizer of (7.11). Indeed, let $m \ge 0$ denote the infimum, and suppose $\{\mathbf{u}_n\}$ is a minimizing sequence of λ_1, so that

$$\int_\Omega (|\mathbf{u}_n|^p + |\nabla \times \mathbf{u}_n|^p + |\nabla \cdot \mathbf{u}_n|^p)\, dx \longrightarrow \lambda_1, \quad \|\mathbf{u}_n\|_{L^p} = 1.$$

Then clearly $\{\mathbf{u}_n\}$ is bounded in W_N^0, and since W_N^0 is reflexive, there exists a subsequence $\mathbf{u}_{n_k}$ converging weakly to some $\mathbf{u}_0$ in W_N^0. Since $W_N^0 \hookrightarrow (L^p(\Omega))^3$ is compact, it follows that $\|\mathbf{u}_{n_k} - \mathbf{u}_0\|_{L^p} \to 0$, and so $\|\mathbf{u}_0\|_{L^p} = 1$. Thus, we have by lower semicontinuity of the L^p-norm that

$$\begin{aligned} m &\le \int_\Omega (|\mathbf{u}|^p + |\nabla \times \mathbf{u}|^p + |\nabla \cdot \mathbf{u}|^p)\, dx \\ &\le \liminf_{k\to\infty} \int_\Omega (|\mathbf{u}_{n_k}|^p + |\nabla \times \mathbf{u}_{n_k}|^p + |\nabla \cdot \mathbf{u}_{n_k}|^p)\, dx \\ &= m, \end{aligned}$$

and so $\mathbf{u}_0 \in W_N^0$ is a minimizer. As such it satisfies (7.10) with $\lambda = \lambda_1$.

With this in hand, inspired by the proof of [13, Theorem 13], we turn to the proof of Theorem 7.5.

Proof of Theorem 7.5. Define $N : W_N^0 \to \mathbb{R}$ by

$$N(\mathbf{v}) = \|\mathbf{v}\|_{W_N^0}^p - \int_\Omega \alpha(x)|\mathbf{v}(x)|^p\, dx.$$

From (7.11) and the assumption (7.13), we have that $N(\mathbf{v}) \ge 0$ for all $\mathbf{v} \in W_N^0$.

Suppose now that there exists a sequence $\{\mathbf{v}_n\} \in W_N^0$ such that $\|\mathbf{v}_n\|_{W_N^0} = 1$ and $N(\mathbf{v}_n) \to 0$. Since W_N^0 is reflexive, we can find a subsequence of $\{\mathbf{v}_n\}$ (still denoted $\{\mathbf{v}_n\}$) and some $\mathbf{v}_0 \in W_N^0$ such that $\mathbf{v}_n \rightharpoonup \mathbf{v}_0$ weakly in W_N^0 and $\mathbf{v}_n \to \mathbf{v}_0$ strongly in $L^p(\Omega)$.

Now, it is clear that the functional $\mathbf{v} \mapsto \int_\Omega \alpha(x)|\mathbf{v}(x)|^p\, dx$ is weakly continuous on W_N^0, which implies that

$$0 \le \|\mathbf{v}_0\|_{W_N^0}^p - \int_\Omega \alpha(x)|\mathbf{v}_0(x)|^p dx \le \liminf_{n\to\infty} N(\mathbf{v}_n) = 0\,,$$

and so $\|\mathbf{v}_0\|^p_{W^0_N} = \int_\Omega \alpha(x)|\mathbf{v}_0(x)|^p dx$. Since $N(\mathbf{v}_n) \to 1 - \int_\Omega \alpha(x)|\mathbf{v}_0(x)|^p\ dx$, we have

$$\|\mathbf{v}_0\|^p_{W^0_N} = \int_\Omega \alpha(x)|\mathbf{v}_0(x)|^p dx = 1$$

so that $\mathbf{v}_0 \neq \mathbf{0}$. Again by (7.11) and (7.13) we see that

$$\lambda_1\|\mathbf{v}_0\|^p_{L^p} \leq \|\mathbf{v}_0\|^p_{W^0_N} \leq \lambda_1\|\mathbf{v}_0\|^p_{L^p} \tag{7.14}$$

since $\|\mathbf{v}_0\|^p_{W^0_N} = \int_\Omega \alpha(x)|\mathbf{v}_0(x)|^p\, dx$. Thus

$$\lambda_1 = \frac{\|\mathbf{v}_0\|^p_{W^0_N}}{\|\mathbf{v}_0\|^p_{L^p}}.$$

By (7.12), we have that $\mathbf{v}_0$ is a multiple of $\mathbf{u}_1 > \mathbf{0}$. Hence $|\mathbf{v}_0(x)| > 0$ a.e. in Ω. Let $\Omega_1 := \{x \in \Omega : \alpha(x) < \lambda_1\}$, which we have assumed to have positive measure. This means that

$$\int_\Omega \alpha(x)|\mathbf{v}_0(x)|^p\, dx < \lambda_1\|\mathbf{v}_0\|^p_{L^p}$$

by splitting Ω into Ω_1 and $\Omega \setminus \Omega_1$. This directly contradicts (7.14).

Thus, we conclude that there must exist some $\epsilon > 0$ such that

$$N(\mathbf{v}) \geq \epsilon \quad \text{for all } \mathbf{v} \in W^0_N \text{ with } \|\mathbf{v}\|^p_{W^0_N} = 1. \tag{7.15}$$

This implies

$$\|\mathbf{v}\|^p_{W^0_N} - \int_\Omega \alpha(x)|\mathbf{v}_0(x)|^p dx \geq \epsilon\|\mathbf{v}\|^p_{W^0_N} \quad \text{for all } \mathbf{v} \in W^0_N. \tag{7.16}$$

Take $\epsilon_1 < \lambda_1\epsilon$. From (7.13) and (7.3), we can find a constant k such that

$$F(x,s) \leq \frac{\alpha + \epsilon_1}{p}|s|^p + k + c(x), \quad x \in \Omega,\ s \in \mathbb{R}^3 \tag{7.17}$$

for some function $c(x) \in L^1(\Omega)$. Then (7.16) together with (7.17) imply that

$$\mathscr{F}(\mathbf{v}) \geq \frac{\lambda_1\epsilon - \epsilon_1}{p}\|\mathbf{v}\|^p_{W^0_N} - k_0 \longrightarrow \infty, \quad k_0 \text{ a constant},$$

as $\|\mathbf{v}\|^p_{W^0_N} \to \infty$. □

Remark 7.6. The condition (7.12) used in Theorem 7.5 could be removed if we had a Harnack inequality for non-negative p-eigenfunctions. We have not yet pursued this direction, but believe it would be of independent interest. Moreover, the conservative condition (7.4) is not required in one dimension, as one can simply take

$$F(x,u) = \int_0^u f(x,s)\, ds.$$

8. The one-dimensional case

In the one-dimensional case, we can find the first eigenvalue explicitly. To this end, let $u(x_1,x_2,x_3) = (u(x_1)\ \ 0\ \ 0)^t$ only depend on x_1. Then the curl of such u vanishes and the divergence of u becomes u' where $' = \frac{d}{dx_1}$. Indeed, the eigenvalue problem in one dimension becomes

$$\big(|u'|^{p-2}u'\big)' = (1-\lambda)|u|^{p-2}u \tag{8.1}$$

subject to a Neumann boundary condition (due to the definition of W_N). For simplicity we take the domain $\Omega = (0,1)$. Such problems have been studied in [15] for the p-Laplacian, but a similar analysis carries over here as well. Indeed, the eigenvalue problem (8.1) is the same as the eigenvalue problem for the p-Laplacian with $\lambda - 1$ as an eigenvalue instead.

Thus, letting $\alpha_1 = \lambda - 1$, in [15, Theorem 3.2] implies that

$$\alpha_1 = \Big(\frac{4\pi}{\sin(\pi/p)}\Big)^p \frac{p-1}{p^p}.$$

Hence the first eigenvalue of (8.1) in one dimension is

$$\lambda_1 = \Big(\frac{4\pi}{\sin(\pi/p)}\Big)^p \frac{p-1}{p^p} + 1, \tag{8.2}$$

and more generally, we have $\lambda_n(p) = n^p \lambda_1(p)$. Thus, since $\lambda_1 > 0$ (for $p > 1$), we also conclude that each $\lambda_n > 0$ as well (again for $p > 1$).

Conclusion. We have proved a number of trace results for Banach spaces related to the p-curl operator. These were used in turn to calculate duality mappings on various Sobolev spaces related to the p-curl system. We have solved a nonlinear p-curl system in a $C^{1,1}$ or convex domain by proving that weak solutions are critical points of a coercive and lower semicontinuous functional. We also have a variational characterization of the first eigenvalue of the p-curl operator, and in one dimension, we have obtained an explicit expression for the first eigenvalue. It was necessary to assume positivity of the first eigenfunction of the p-curl operator in order to prove coercivity of the associated functional. It is expected that this can be removed by proving a Harnack inequality for the p-curl operator, which we leave to future work.

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