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EXISTENCE AND NONEXISTENCE OF RADIAL SOLUTIONS FOR SEMILINEAR EQUATIONS WITH BOUNDED NONLINEARITIES ON EXTERIOR DOMAINS

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Abstract. In this article we study radial solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ where f is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, δ), $f \equiv 0$ for $u > \delta$, and where the function $K(r)$ is assumed to be positive and $K(r) \to 0$ as $r \to \infty$. The primitive $F(u) = \int_0^u f(t)\,dt$ has a "hilltop" at $u = \delta$. With mild assumptions on f we prove that if $K(r) \sim r^{-\alpha}$ with $2 < \alpha < 2(N-1)$ then there are n solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius R such that $u \to 0$ as $r \to \infty$ if $R > 0$ is sufficiently small. We also show there are no solutions if $R > 0$ is sufficiently large.

1. Introduction

In this article we study radial solutions of

$$\Delta u + K(r)f(u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

$$u \to 0 \quad \text{as } |x| \to \infty \tag{1.3}$$

where $x \in \Omega = \mathbb{R}^N \backslash B_R(0)$ is the complement of the ball of radius $R > 0$ centered at the origin. We assume $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and there exist β, δ with $0 < \beta < \delta$ such that $f(0) = f(\beta) = f(\delta) = 0$ where:

- (H1) f is odd, $f'(0) < 0$, $f < 0$ on $(0, \beta)$, $f > 0$ on (β, δ), $f'(\delta^-) < 0$, $f \equiv 0$ on (δ, ∞).

It follows that $F(u) = \int_0^u f(s)\,ds$ is even. We also assume that F has a unique positive zero, γ, with $\beta < \gamma < \delta$ such that

- (H2) $F < 0$ on $(0, \gamma)$, $F > 0$ on (γ, ∞).

Note from (H1) and (H2) it follows that F is bounded.

In an earlier paper [6] we studied (1.1), (1.3) when $\Omega = \mathbb{R}^N$ and $K(r) \equiv 1$. Interest in the topic for this paper comes from recent papers [5, 12, 14] about solutions of differential equations on exterior domains. In [7] we studied (1.1)-(1.3) with $K(r) \equiv 1$ and $\Omega = \mathbb{R}^N \backslash B_R(0)$, in [8] we studied the case when $K(r) \sim r^{-\alpha}$ with $0 < \alpha < 2$ and in [9] with $\alpha > 2(N-1)$. In [7, 8, 9] we proved existence of an

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infinite number of solutions - one with exactly n zeros for each nonnegative integer n such that $u \to 0$ as $|x| \to \infty$.

When f grows superlinearly at infinity - i.e. $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, and $\Omega = \mathbb{R}^N$. problem (1.1), (1.3) has been extensively studied in [1, 2, 3, 11, 13, 15]. The type of nonlinearity addressed here has not been studied as extensively [6, 7, 8].

When f grows sublinearly at infinity - i.e. $\lim_{u\to\infty} \frac{f(u)}{u} = 0$, but $\lim_{u\to\infty} f(u) = \infty$ and $\Omega = \mathbb{R}^N$, problem (1.1), (1.3) has also been studied in [9, 10].

Since we are interested in radial solutions of (1.1)-(1.3) we assume that $u(x) = u(|x|) = u(r)$ where $x \in \mathbb{R}^N$ and $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$ so that u solves

$$u''(r) + \frac{N-1}{r}u'(r) + K(r)f(u(r)) = 0 \quad \text{on } (R,\infty) \text{ where } R > 0, \tag{1.4}$$

$$u(R) = 0, u'(R) = a > 0. \tag{1.5}$$

We will assume that there exist constants $k_1 > 0$, $k_2 > 0$, and $\alpha > 0$ such that

(H3) $k_1 r^{-\alpha} \le K(r) \le k_2 r^{-\alpha}$ for $2 < \alpha < 2(N-1)$ on $[R,\infty)$.

In addition, we assume that

(H4) K is differentiable, $\lim_{r\to\infty} \frac{rK'}{K} = -\alpha$ and $\frac{rK'}{K} + 2(N-1) > 0$ on $[R,\infty)$.

Note that (H4) implies $r^{2(N-1)}K(r)$ is increasing. Also since $f'(0) < 0$ and $f'(\delta^-) < 0$ then it follows from (H1) that there exist positive constants $f_0, \bar{f}_0, f_1, \bar{f}_1$ such that

$$f_0 = \inf_{(0,\beta/2]}\Big(-\frac{f(u)}{u}\Big), \quad \bar{f}_0 = \sup_{u\neq 0}\Big(-\frac{f(u)}{u}\Big), \tag{1.6}$$

$$f_1 = \inf_{[\gamma,\delta)}\Big(\frac{f(u)}{\delta - u}\Big), \quad \bar{f}_1 = \sup_{[\beta',\delta)}\Big(\frac{f(u)}{\delta - u}\Big) \tag{1.7}$$

where $\beta < \beta' < \gamma$ and $F(\frac{\beta}{2}) = F(\beta')$.

Theorem 1.1. *Let $N > 2$, $R > 0$, $2 < \alpha < 2(N-1)$ and (H1)–(H4) hold.*

(a) *There are n solutions of (1.1)-(1.3) on $[R,\infty)$ - one with exactly n zeros for each nonnegative integer n if*

$$\gamma\Big(1 + \Big(\frac{h_2\bar{f}_0}{h_1 f_1}\Big)^{1/2}\Big) < \delta$$

and if $R > 0$ is sufficiently small.

(b) *There are no solutions for any value of $R > 0$ of (1.1)-(1.3) if*

$$\beta' + \frac{\beta}{2}\frac{h_1}{h_2}\Big(\frac{f_0}{\bar{f}_1}\Big)^{1/2} > \delta.$$

(c) *There are no solutions of (1.1)-(1.3) on $[R,\infty)$ if $R > 0$ is sufficiently large.*

We note that in Sankar, Sasi, and Shivaji [14] established existence of a *positive* solution to a semipositone version of this problem using sub and super solutions. We use different techniques here and are able to establish existence of multiple solutions.

2. Preliminaries

We first suppose that $U(r)$ solves (1.4) and then make the change of variables:
$$U(r) = u(r^{2-N}).$$
Then for $0 < t < \infty$ we see u satisfies
$$u'' + h(t)f(u) = 0, \tag{2.1}$$
where
$$h(t) = \frac{t^{\frac{2(N-1)}{2-N}} K(t^{\frac{1}{2-N}})}{(N-2)^2}.$$
It follows from (H3) and (H4) that
$$\begin{gathered} h(t) > 0, \quad h'(t) < 0, \quad \lim_{t\to 0^+} \frac{th'}{h} = -q, \quad h_1 t^{-q} < h(t) < h_2 t^{-q} \\ \text{for } t > 0, \quad q = \frac{2(N-1) - \alpha}{N-2}, \quad h_i = \frac{k_i}{(N-2)^2}. \end{gathered} \tag{2.2}$$
In addition, it follows from (H3), (H4) and (2.2) that
$$0 < q < 2. \tag{2.3}$$
We also assume that
$$u(0) = 0, u'(0) = b > 0. \tag{2.4}$$
We want to find $b > 0$ such that $u(R^{2-N}) = 0$ then $U(r) = u(r^{2-N})$ will satisfy (1.1)-(1.3). Therefore for the rest of this paper we will study (2.1), (2.4) with (H1)–(H4) and attempt to find solutions u such that $u(R^{2-N}) = 0$.

We first prove existence of a solution of (2.1), (2.4) assuming (H1)–(H4) on $[0, \epsilon]$ for some $\epsilon > 0$. Integrating (2.1) twice on $(0, t)$ and using (2.4) gives
$$u(t) = bt - \int_0^t \int_0^s h(x) f(u(x))\, dx\, ds. \tag{2.5}$$
Letting $y(t) = \frac{u(t)}{t}$ and $y(0) = b > 0$ gives
$$y(t) = b - \frac{1}{t} \int_0^t \int_0^s h(x) f(xy(x))\, dx\, ds. \tag{2.6}$$
Now let $S = \{y \in C[0, \epsilon] : y(0) = b > 0\}$ with the supremum norm, $\|\cdot\|$, and define $T : S \to C[0, \epsilon]$ by
$$T(y) = b - \frac{1}{t} \int_0^t \int_0^s h(x) f(xy(x))\, dx\, ds. \tag{2.7}$$
We first observe that $T : S \to S$. Next let K be the Lipschitz constant for $f(u)$ in a neighborhood of $u = 0$ and suppose $0 \le t \le \epsilon$. Then
$$\begin{aligned} |Ty_1 - Ty_2| &\le \frac{1}{t} \int_0^t \int_0^s h_2 K |xy_1 - xy_2| x^{-q}\, dx\, ds \\ &\le \int_0^t h_2 K x^{1-q} |y_1 - y_2|\, dx \\ &\le \frac{h_2 K}{2-q} \epsilon^{2-q} \|y_1 - y_2\|. \end{aligned}$$

It follows from this and (2.3) that T is a contraction if $\epsilon > 0$ is sufficiently small. Thus by the contraction mapping principle [4] it follows that (2.7) has a fixed point y in S and therefore $u = ty$ is a solution of (2.5) on $[0, \epsilon]$ for some $\epsilon > 0$.

Next let

$$E_0(t) = \frac{1}{2}u'^2 + h(t)F(u). \tag{2.8}$$

By (2.1) we have $E_0' = h'(t)F(u)$ and thus on $(\frac{\epsilon}{2}, t)$ we obtain

$$\frac{1}{2}u'^2 + h(t)F(u) = \frac{1}{2}u'^2(\epsilon/2) + h(\epsilon/2)F(u(\epsilon/2)) + \int_{\frac{\epsilon}{2}}^{t} h'(s)F(u(s))\,ds.$$

Since F is bounded and since h, h' are bounded on $[\epsilon/2, \infty)$ it follows that u' is bounded on $[\epsilon/2, \infty)$. It then follows that the solution of (2.1), (2.4) exists on $[0, Q)$ for all $Q > 0$ and thus we obtain a solution of (2.1), (2.4) on $[0, \infty)$.

Next let

$$E(t) = \frac{1}{2}\frac{u'^2}{h(t)} + F(u). \tag{2.9}$$

Using (2.1)-(2.2) and (2.4) we see that $\lim_{t\to 0^+} E(t) = 0$ and

$$E' = -\frac{u'^2 h'(t)}{h^2(t)} \geq 0 \quad \text{for } t > 0. \tag{2.10}$$

Thus E is nondecreasing and $E(t) > 0$ for $t > 0$.

Lemma 2.1. *Assume* (H1)–(H4) *and let u solve* (2.1), (2.4)*. Then there exists $t_{\gamma,b} > 0$ such that $u(t_{\gamma,b}) = \gamma$, $u'(t_{\gamma,b}) > 0$, and $0 < u < \gamma$ on $(0, t_{\gamma,b})$. In addition, there exists $t_{2,b}$ with $0 < t_{2,b} < t_{\gamma,b}$ such that $u(t_{2,b}) = \beta/2$.*

Proof. We first observe from (2.4) that u is initially positive and increasing for $t > 0$ small. If u has a local maximum M then $F(u(M)) = E(M) > 0$ thus $u(M) > \gamma$ by (H2) and so the existence of $t_{\gamma,b}$ follows. So now let us assume u is positive, increasing, and $0 < u < \gamma$ for all $t > 0$. From (2.10) we have $\frac{1}{2}\frac{u'^2}{h(t)} + F(u) = E(t) \geq E(\epsilon) > 0$ for $t \geq \epsilon > 0$. Since $0 < u < \gamma$ then $F(u) \leq 0$ so $\frac{1}{2}\frac{u'^2}{h(t)} \geq E(\epsilon)$ for $t \geq \epsilon$. Thus

$$|u'| \geq \sqrt{2E(\epsilon)h(t)} \geq \sqrt{2E(\epsilon)h_1}\,t^{-q/2} > 0 \quad \text{for } t \geq \epsilon. \tag{2.11}$$

Therefore $u' > 0$ for $t \geq \epsilon$. Integrating (2.11) on (ϵ, t) gives

$$\gamma \geq u(t) - u(\epsilon) \geq \frac{\sqrt{2E(\epsilon)h_1}}{1 - \frac{q}{2}}(t^{1-\frac{q}{2}} - \epsilon^{1-\frac{q}{2}}) \quad \text{for } t \geq \epsilon. \tag{2.12}$$

Recall $0 < q < 2$ by (2.3) and so the left-hand side of (2.12) is bounded but the right-hand side goes to infinity as $t \to \infty$. Therefore we obtain a contradiction and so there exists $t_{\gamma,b} > 0$ such that $u(t_{\gamma,b}) = \gamma$ and $0 < u < \gamma$ for $0 < t < t_{\gamma,b}$. In addition, $\frac{1}{2}\frac{u'^2(t_{\gamma,b})}{h(t_{\gamma,b})} = E(t_{\gamma,b}) > 0$ hence $u'(t_{\gamma,b}) > 0$. Since $u(0) = 0$ it then follows by the intermediate value theorem that there exists $t_{2,b}$ with $0 < t_{2,b} < t_{\gamma,b}$ such that $u(t_{2,b}) = \frac{\beta}{2}$. This completes the proof. □

Lemma 2.2. *Assume* (H1)–(H4) *and let u solve* (2.1), (2.4)*. If $\lim_{t\to\infty} u(t) = L \in \mathbb{R}$ then $f(L) = 0$.*

Proof. Since $\lim_{t\to\infty} u(t) = L$ and $u(0) = 0$ then it follows that u is bounded for all $t \geq 0$. Also $E' \geq 0$ implies $\frac{1}{2}\frac{u'^2}{h(t)} + F(u) \to A \leq \infty$ as $t \to \infty$ and thus $\frac{1}{2}\frac{u'^2}{h(t)} \to A - F(L)$. If $A - F(L) > 0$ then we obtain $|u'| \geq A_1 t^{-q/2}$ for some $A_1 > 0$ and for large t. Thus $|u'| > 0$ and so without loss of generality suppose that $u' > 0$. Integrating $u' \geq A_1 t^{-q/2}$ on (t_0, t) gives $u(t) - u(t_0) \geq \frac{A_1}{1-\frac{q}{2}}(t^{1-\frac{q}{2}} - t_0^{1-\frac{q}{2}}) \to \infty$ as $t \to \infty$ but the left-hand side is bounded since $\lim_{t\to\infty} u(t) = L$. Thus we obtain a contradiction and so we see that $A - F(L) = 0$. Therefore $\frac{1}{2}\frac{u'^2}{h(t)} + F(u) \to F(L)$ and since $F(u) \to F(L)$ it then follows that $\lim_{t\to\infty} \frac{u'^2}{h(t)} = 0$. Therefore by (2.2) we have

$$\lim_{t\to\infty} t^{q/2} u' = 0. \tag{2.13}$$

Next note that $(\frac{u'}{h})' = \frac{u''}{h} - \frac{u'h'}{h^2}$. Rewriting (2.1) we see $\lim_{t\to\infty} \frac{u''}{h} = -f(L)$. Also by (2.2) and (2.13) for large t we have $|\frac{u'h'}{h^2}| \leq \frac{2q}{h_1} t^{q-1}|u'| = \frac{2q}{h_1}(t^{q/2}u')\frac{1}{t^{1-\frac{q}{2}}} \to 0$ as $t \to \infty$ since $0 < q < 2$. Therefore $\lim_{t\to\infty}(\frac{u'}{h})' = -f(L)$. Then by L'Hôpital's rule

$$\lim_{t\to\infty} \frac{u'}{th} = \lim_{t\to\infty} \frac{(\frac{u'}{h})}{t} = \lim_{t\to\infty} \frac{(\frac{u'}{h})'}{(t)'} = -f(L). \tag{2.14}$$

Now suppose without loss of generality that $f(L) > 0$. Then from (2.2) and (2.14) it follows $-u' \geq \frac{|f(L)|h_1}{2} t^{1-q}$ for large t and so integrating on (t_0, t) gives $u(t_0) - u(t) \geq \frac{|f(L)|h_1}{2(2-q)}(t^{2-q} - t_0^{2-q}) \to \infty$ as $t \to \infty$ so $u(t) \to -\infty$ which contradicts that u is bounded. Thus $f(L) \leq 0$. A similar argument shows $f(L) \geq 0$ hence $f(L) = 0$. This completes the proof. □

Lemma 2.3. *Assume* (H1)–(H4) *and let* u *solve* (2.1), (2.4). *Then* $\lim_{b\to 0^+} t_{2,b} = \lim_{b\to 0^+} t_{\gamma,b} = \infty$ *and*

$$\liminf_{b\to 0^+} t_{2,b}^{q/2} u'(t_{2,b}) \geq \frac{\beta}{2}\sqrt{h_1 f_0}, \tag{2.15}$$

$$\limsup_{b\to 0^+} t_{\gamma,b}^{q/2} u'(t_{\gamma,b}) \leq \gamma\sqrt{h_2 \bar{f}_0}. \tag{2.16}$$

Proof. We rewrite (2.1) as

$$u'' = h(t)\Big(-\frac{f(u)}{u}\Big)u. \tag{2.17}$$

Thus by (1.6), (2.2), and (2.17) we see that

$$u'' \leq \frac{h_2 \bar{f}_0 u}{t^q} \text{ when } u > 0.$$

Now let v_2 solve

$$v_2'' = \frac{h_2 \bar{f}_0}{t^q} v_2, \tag{2.18}$$

$$v_2(0) = 0, \quad v_2'(0) = b > 0. \tag{2.19}$$

Then v_2 is positive and increasing for $t > 0$. Also by (1.6) and (2.2) we see that

$$(u'v_2 - uv_2')' = \Big(h(t)\big(-\frac{f(u)}{u}\big) - \frac{h_2\bar{f}_0}{t^q}\Big)uv_2 \leq 0 \quad \text{while } u > 0.$$

Since $u(0) = v_2(0) = 0$ we see then that $u'v_2 - uv_2' \le 0$ while $u > 0$ and thus $(u/v_2)' \le 0$. Since $u'(0) = v_2'(0) = b$ we see then that

$$0 < u \le v_2. \tag{2.20}$$

Also $u'v_2 - uv_2' \le 0$ and $0 < u \le v_2$ imply that

$$\frac{u'}{u} \le \frac{v_2'}{v_2} \quad \text{for } u > 0. \tag{2.21}$$

Next (2.18)-(2.19) can be solved explicitly and we obtain

$$v_2 = bC\sqrt{t} I_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_2\bar{f}_0}}{2-q} t^{\frac{2-q}{2}}\Big) \tag{2.22}$$

where $I_{\frac{1}{2-q}}$ is the modified Bessel function of order $\frac{1}{2-q}$ with $\lim_{t\to 0^+} I_{\frac{1}{2-q}}(t) = 0$. A well-known fact is that $\lim_{t\to 0^+} \frac{I_\nu(t)}{t^\nu} = \frac{1}{2^\nu\Gamma(\nu+1)}$ where I_ν is the modified Bessel function of order ν with $\lim_{t\to 0^+} I_\nu(t) = 0$ and thus from this and (2.22) we see $C = \Gamma(\frac{3-q}{2-q})(\frac{\sqrt{h_2\bar{f}_0}}{2-q})^{-\frac{1}{2-q}} > 0$. (Here $\Gamma(x)$ is the Gamma function). It is also known that $I_\nu > 0$, $I_\nu' > 0$, and $\lim_{t\to\infty} \frac{I_\nu'(t)}{I_\nu(t)} = 1$. (Some other general facts about the modified Bessel functions are included in the appendix).

Now using (2.20) we see that

$$\frac{\beta}{2} = u(t_{2,b}) \le v_2(t_{2,b}) = bC\sqrt{t_{2,b}} I_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_2\bar{f}_0}}{2-q} t_{2,b}^{\frac{2-q}{2}}\Big). \tag{2.23}$$

If the $t_{2,b}$ are bounded as $b \to 0^+$ then the right-hand side of (2.23) goes to zero which contradicts that $\beta > 0$. Thus it must be that $\lim_{b\to 0^+} t_{2,b} = \infty$. Since $t_{\gamma,b} > t_{2,b}$ then also $\lim_{b\to 0^+} t_{\gamma,b} = \infty$. This completes the first part of the lemma.

Denoting

$$s = \frac{2\sqrt{h_2\bar{f}_0}}{2-q} t^{1-\frac{q}{2}} \quad \text{and} \quad s_{\gamma,b} = \frac{2\sqrt{h_2\bar{f}_0}}{2-q} t_{\gamma,b}^{1-\frac{q}{2}} \tag{2.24}$$

It follows from (2.22) that

$$v_2'(t) = \frac{v_2(t)}{2t} + \sqrt{h_2\bar{f}_0}\, t^{-q/2} v_2(t) \frac{I'_{\frac{1}{2-q}}(s)}{I_{\frac{1}{2-q}}(s)}.$$

Therefore

$$\frac{t^{q/2} v_2'(t)}{v_2(t)} = \frac{1}{2t^{1-\frac{q}{2}}} + \sqrt{h_2\bar{f}_0} \frac{I'_{\frac{1}{2-q}}(s)}{I_{\frac{1}{2-q}}(s)}. \tag{2.25}$$

Evaluating at $t_{\gamma,b}$ it follows from (2.21) and (2.25) that

$$\frac{t_{\gamma,b}^{q/2} u'(t_{\gamma,b})}{u(t_{\gamma,b})} \le \frac{1}{2t_{\gamma,b}^{1-\frac{q}{2}}} + \sqrt{h_2\bar{f}_0} \quad \frac{I'_{\frac{1}{2-q}}(s_{\gamma,b})}{I_{\frac{1}{2-q}}(s_{\gamma,b})}. \tag{2.26}$$

As mentioned earlier it is well-known that $\lim_{s\to\infty} \frac{I_\nu'(s)}{I_\nu(s)} = 1$. Recalling that $0 < q < 2$ and that $t_{\gamma,b} \to \infty$ as $b \to 0^+$ then we see from (2.26) that

$$\limsup_{b\to 0^+} t_{\gamma,b}^{q/2} u'(t_{\gamma,b}) \le \gamma\sqrt{h_2\bar{f}_0}.$$

In a similar way let v_1 solve

$$v_1'' = \frac{h_1 f_0}{t^q} v_1, \tag{2.27}$$

$$v_1(0) = 0, \quad v_1'(0) = b > 0. \tag{2.28}$$

We note that $v_1 > 0$ and $v_1' > 0$ for $t > 0$. Then we can similarly show that

$$\frac{v_1'}{v_1} \le \frac{u'}{u} \quad \text{for } 0 < u < \frac{\beta}{2}. \tag{2.29}$$

Solving for v_1 explicitly we have

$$v_1 = bC_1\sqrt{t} I_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_1 f_0}}{2-q} t^{\frac{2-q}{2}}\Big) \quad \text{where } C_1 = \Gamma(\frac{3-q}{2-q})(\frac{\sqrt{h_1 f_0}}{2-q})^{-\frac{1}{2-q}} > 0. \tag{2.30}$$

It follows from (2.29) and (2.30) that

$$\frac{t_{2,b}^{q/2} u'(t_{2,b})}{u(t_{2,b})} \ge \frac{t_{2,b}^{q/2} v_1'(t_{2,b})}{v_1(t_{2,b})} = \frac{1}{2t_{2,b}^{1-\frac{q}{2}}} + \sqrt{h_1 f_0}\frac{I'_{\frac{1}{2-q}}(p_{2,b})}{I_{\frac{1}{2-q}}(p_{2,b})} \tag{2.31}$$

where $p_{2,b} = \frac{2\sqrt{h_1 f_0}}{2-q} t_{2,b}^{1-\frac{q}{2}}$.

It is shown in the appendix that

$$\frac{I_\nu'}{I_\nu} + \frac{\nu}{t} > 1 \quad \text{for } t > 0 \text{ and } \nu > 1/2$$

from which it follows using (2.31) that

$$\liminf_{b\to 0^+} t_{2,b}^{q/2} u'(t_{2,b}) \ge \frac{\beta}{2}\sqrt{h_1 f_0}.$$

This completes the proof. □

Next we rewrite (2.1) as

$$u'' + h(t)\left(\frac{f(u)}{\delta - u}\right)(\delta - u) = 0. \tag{2.32}$$

From (1.7) and (2.2) we have

$$h(t)\Big(\frac{f(u)}{\delta - u}\Big) \ge \frac{h_1 f_1}{t^q} \quad \text{on } [\gamma, \delta), \tag{2.33}$$

$$\frac{h_2 \bar{f}_1}{t^q} \ge h(t)\Big(\frac{f(u)}{\delta - u}\Big) \quad \text{for } u \in [\beta', \delta). \tag{2.34}$$

So now we compare (2.32) to

$$w_2'' + \frac{h_1 f_1}{t^q}(\delta - w_2) = 0 \tag{2.35}$$

$$w_2(t_{\gamma,b}) = u(t_{\gamma,b}) = \gamma, w_2'(t_\gamma, b) = u'(t_{\gamma,b}). \tag{2.36}$$

and

$$w_1'' + \frac{h_2 \bar{f}_1}{t^q}(\delta - w_1) = 0 \tag{2.37}$$

$$w_1(t_{b'}) = u(t_{b'}) = \beta', w_1'(t_{b'}) = u'(t_{b'}). \tag{2.38}$$

Lemma 2.4. *Assume* (H1)–(H4) *and let u solve* (2.1), (2.4). *Then $w_1 \le u$ when $u, w_1 \in [\beta', \delta)$ where w_1 is the solution of* (2.37), (2.38). *Also $u \le w_2$ when $u, w_2 \in [\gamma, \delta)$ where w_2 is the solution of* (2.35)-(2.36).

Proof. It follows from (2.32) and (2.35) that

$$\big((\delta - w_2)u' - (\delta - u)w_2'\big)' + \Big(h(t)\big(\frac{f(u)}{\delta - u}\big) - \frac{h_1 f_1}{t^q}\Big)(\delta - u)(\delta - w_2) = 0. \tag{2.39}$$

By (2.33) it follows that the second term in (2.39) is ≥ 0 when $u, w_2 \in [\gamma, \delta)$. Therefore integrating (2.39) on $(t_{\gamma,b}, t)$ gives

$$(\delta - w_2)u' - (\delta - u)w_2' \leq 0. \tag{2.40}$$

Thus

$$\Big(\frac{\delta - w_2}{\delta - u}\Big)' \leq 0.$$

Integrating on $(t_{\gamma,b}, t)$ gives

$$\frac{\delta - w_2}{\delta - u} - 1 \leq 0$$

which implies $u \leq w_2$ when $u, w_2 \in [\gamma, \delta)$.

A nearly identical argument proves that

$$w_1 \leq u \text{ when } u, w_1 \in [\beta', \delta)$$

and

$$(\delta - w_1)u' - (\delta - u)w_1' \geq 0. \tag{2.41}$$

This completes the proof. □

Now (2.35) can be solved explicitly and we obtain

$$w_2 = \delta + \sqrt{t}\Big(c_1 I_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_1 f_1}}{2-q} t^{\frac{2-q}{2}}\Big) + c_2 K_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_1 f_1}}{2-q} t^{\frac{2-q}{2}}\Big)\Big) \tag{2.42}$$

where $I_{\frac{1}{2-q}}$ and $K_{\frac{1}{2-q}}$ are the modified Bessel functions of order $\frac{1}{2-q}$ and c_1, c_2 are constants. It is well-known for $t > 0$ that: $I_\nu > 0, I_\nu' > 0$, $K_\nu > 0$ and $K_\nu' < 0$.

We rewrite (2.42) as

$$w_2 - \delta = c_1 y_1 + c_2 y_2$$

where

$$y_1(t) = \sqrt{t} I_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_1 f_1}}{2-q} t^{\frac{2-q}{2}}\Big), \quad y_2(t) = \sqrt{t} K_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_1 f_1}}{2-q} t^{\frac{2-q}{2}}\Big). \tag{2.43}$$

A straightforward computation shows

$$c_1 = \frac{y_2'(t_{\gamma,b})(w_2(t_{\gamma,b}) - \delta) - y_2(t_{\gamma,b})w_2'(t_{\gamma,b})}{y_1(t_{\gamma,b})y_2'(t_{\gamma,b}) - y_1'(t_{\gamma,b})y_2(t_{\gamma,b})}, \tag{2.44}$$

$$c_2 = \frac{-y_1'(t_{\gamma,b})(w_2(t_{\gamma,b}) - \delta) + y_1(t_{\gamma,b})w_2'(t_{\gamma,b})}{y_1(t_{\gamma,b})y_2'(t_{\gamma,b}) - y_1'(t_{\gamma,b})y_2(t_{\gamma,b})}. \tag{2.45}$$

Another well-known fact about the modified Bessel functions I_ν and K_ν is that

$$I_\nu(t)K_\nu'(t) - I_\nu'(t)K_\nu(t) = -\frac{1}{t} \quad \text{for } t > 0. \tag{2.46}$$

Next a straightforward computation using (2.43) and (2.46) shows

$$y_1(t)y_2'(t) - y_1'(t)y_2(t) = -(1 - \frac{q}{2}).$$

And so we see from (2.36), (2.44)-(2.45) that

$$c_1 = \frac{y_2'(t_{\gamma,b})(\delta - \gamma) + y_2(t_{\gamma,b})u'(t_{\gamma,b})}{1 - \frac{q}{2}}, \tag{2.47}$$

$$c_2 = \frac{-y_1'(t_{\gamma,b})(\delta-\gamma) - y_1(t_{\gamma,b})u'(t_{\gamma,b})}{1-\frac{q}{2}}. \tag{2.48}$$

Note that $y_1(t) > 0$ and $y_1'(t) > 0$. In addition, $u'(t_{\gamma,b}) > 0$ and $\delta - \gamma > 0$ so it follows from (2.48) that

$$c_2 < 0. \tag{2.49}$$

Lemma 2.5. *Assume* (H1)–(H4) *and let u solve* (2.1), (2.4). *If $b > 0$ is sufficiently small and if*

$$\gamma\Big(1 + \Big(\frac{h_2\bar{f}_0}{h_1 f_1}\Big)^{1/2}\Big) < \delta \tag{2.50}$$

then $c_1 < 0$.

Proof. We let

$$r = \frac{2\sqrt{h_1 f_1}}{2-q} t^{1-\frac{q}{2}}, \quad r_{\gamma,b} = \frac{2\sqrt{h_1 f_1}}{2-q} t_{\gamma,b}^{1-\frac{q}{2}}. \tag{2.51}$$

It follows from (2.43) and (2.24) that

$$\begin{aligned} c_1 = \frac{1}{1-\frac{q}{2}} \Big[(\delta-\gamma)\Big(\frac{1}{2\sqrt{t_{\gamma,b}}} K_{\frac{1}{2-q}}(r_{\gamma,b}) + \sqrt{h_1 f_1}\, t_{\gamma,b}^{\frac{1-q}{2}} K'_{\frac{1}{2-q}}(r_{\gamma,b})\Big) \\ + \sqrt{t_{\gamma,b}} K_{\frac{1}{2-q}}(r_{\gamma,b}) u'(t_{\gamma,b})\Big]. \end{aligned}$$

Therefore

$$\begin{aligned} c_1 = \frac{1}{1-\frac{q}{2}} t_{\gamma,b}^{\frac{1-q}{2}} K_{\frac{1}{2-q}}(r_{\gamma,b}) \Big[(\delta-\gamma)\Big(\frac{1}{2t_{\gamma,b}^{1-\frac{q}{2}}} + \sqrt{h_1 f_1}\frac{K'_{\frac{1}{2-q}}(r_{\gamma,b})}{K_{\frac{1}{2-q}}(r_{\gamma,b})}\Big) \\ + t_{\gamma,b}^{q/2} u'(t_{\gamma,b})\Big]. \end{aligned} \tag{2.52}$$

Another well-known fact about the modified Bessel function is that $\lim_{t\to\infty} \frac{K_\nu'(t)}{K_\nu(t)} = -1$. We also know that $t_{\gamma,b} \to \infty$ as $b \to 0^+$ by Lemma 2.3 and thus by (2.51) we see $r_{\gamma,b} \to \infty$ as $b \to 0^+$. Thus from Lemma 2.3, (2.16), (2.50), and taking the limit superior of the bracketed term in (2.52) gives

$$\begin{aligned} &\limsup_{b\to 0^+} \Big[(\delta-\gamma)\Big(\frac{1}{2t_{\gamma,b}^{1-\frac{q}{2}}} + \sqrt{h_1 f_1}\frac{K'_{\frac{1}{2-q}}(r_{\gamma,b})}{K_{\frac{1}{2-q}}(r_{\gamma,b})}\Big) + t_{\gamma,b}^{q/2} u'(t_{\gamma,b})\Big] \\ &\le (\delta-\gamma)(-\sqrt{h_1 f_1}) + \gamma\sqrt{h_2\bar{f}_0} = \sqrt{h_1 f_1}\Big[\gamma\Big(1 + \sqrt{\frac{h_2\bar{f}_0}{h_1 f_1}}\Big) - \delta\Big] < 0. \end{aligned}$$

It follows from this and (2.52) that $c_1 < 0$. This completes the proof. $\square$

Lemma 2.6. *Assume* (H1)–(H4) *and let u solve* (2.1), (2.4). *Let n be a positive integer. If $\gamma\big(1 + \sqrt{\frac{h_2\bar{f}_0}{h_1 f_1}}\big) < \delta$ and $b > 0$ is sufficiently small then u has n zeros on $(0,\infty)$.*

Proof. From Lemma 2.5 it follows that $c_1 < 0$ if $b > 0$ is sufficiently small and (2.50) holds. In addition, $c_2 < 0$ by (2.49). Since $I_\nu \to \infty$ as $t \to \infty$ and $K_\nu > 0$ then we see from (2.42) that $w_2 < \delta$ for all $t > 0$. Since $c_1 < 0$ and $I_\nu \to \infty$ as $t \to \infty$ it follows from (2.42) that $w_2 \to -\infty$ as $t \to \infty$ so w_2 must have a local maximum, M_{w_2}, and that $w_2(M_{w_2}) < \delta$. Since $u \le w_2$ by Lemma2.4 it follows that $u(t) \le w_2(t) \le w_2(M_{w_2}) < \delta$. This implies that u also has a local maximum for otherwise u would be increasing and have a limit, L, with $\gamma < L < \delta$ which is impossible by Lemma 2.2. Thus u has a local max, M_b, and since $F(u(M_b)) = E(M_b) > 0$ we have $\beta < \gamma < u(M_b) \le w_2(M_b) \le w_2(M_{w_2}) < \delta$. Then from (2.1) we see u is concave down while $\beta < u < \delta$ and so there exists $x_b > M_b$ such that $u(x_b) = \beta$ and $u'(x_b) < 0$. Next recall from (2.10) that $E(t) \ge E(M_b)$ for $t > M_b$ and so

$$\frac{1}{2}\frac{u'^2}{h(t)} + F(u) \ge F(u(M_b)) \quad \text{for } t > M_b. \tag{2.53}$$

Now for $t > x_b$ we have $F(u) \le 0$ and so from (2.53) we have

$$\frac{1}{2}\frac{u'^2}{h(t)} \ge F(u(M_b)) \quad \text{for } t > x_b.$$

Thus by (2.2),

$$-u' \ge \sqrt{2F(u(M_b))h(t)} \ge \sqrt{2h_1 F(u(M_b))}\, t^{-q/2} \quad \text{for } t > x_b.$$

Integrating this on (x_b, t) gives

$$-u(t) + \beta \ge \frac{\sqrt{2h_1 F(u(M_b))}}{1 - \frac{q}{2}}\Big(t^{1-\frac{q}{2}} - x_b^{1-\frac{q}{2}}\Big) \to \infty \text{ as } t \to \infty$$

and so u must be negative. Thus there exists $z_{1,b} > x_b$ such that $u(z_{1,b}) = 0$. In addition, $\frac{1}{2}u'^2(z_{1,b}) = E(z_{1,b}) > 0$ so $u'(z_{1,b}) < 0$.

Further, $u'(z_{1,b}) \to 0$ as $b \to 0^+$. To see this, recall from (2.8) that $E_0' = h'(t)F(u)$ and so integrating this on $(t_{\gamma,b}, z_{1,b})$ gives

$$\begin{aligned} \frac{1}{2}u'^2(z_{1,b}) &= \frac{1}{2}u'^2(t_{\gamma,b}) + \int_{t_{\gamma,b}}^{z_{1,b}} h'(x)F(u(x))\,dx \\ &\le \frac{1}{2}u'^2(t_{\gamma,b}) + F_1[h(t_{\gamma,b}) - h(z_{1,b})] \end{aligned} \tag{2.54}$$

where $|F(u)| \le F_1$ for some constant F_1. (Recall from (H1) and (H2) that F is bounded). Since $t_{\gamma,b}$ and $z_{1,b}$ go to infinity as $b \to 0^+$ by Lemma 2.3 we see by (2.2) that the second term in (2.54) goes to 0 as $b \to 0^+$. Also from (2.16) we see that $u'(t_{\gamma,b}) \to 0$ as $b \to 0^+$. Thus from (2.54) we see $u'(z_{1,b}) \to 0$ as $b \to 0^+$.

Next, let $u_1(t) = -u(t)$. Then since $f(u)$ is odd we see that u_1 also solves (2.1). Further $u_1(z_{1,b}) = 0$, $u_1'(z_{1,b}) = -u'(z_{1,b}) > 0$, and $u_1'(z_{1,b}) \to 0$ as $b \to 0^+$.

Now we can define $\bar{v}_2$ with $\bar{v}_2$ solving (2.18) with $\bar{v}_2(z_{1,b}) = 0, \bar{v}_2'(z_{1,b}) = u_1'(z_{1,b}) > 0$ and as in Lemma 2.1 there exists $\bar{t}_{\gamma,b} > z_{1,b}$ such that $\bar{v}_2(\bar{t}_{\gamma,b}) = \gamma$. As in Lemma 2.3 we can show that

$$\frac{u_1'}{u_1} \le \frac{\bar{v}_2'}{\bar{v}_2}. \tag{2.55}$$

We again can solve for $\bar{v}_2$ explicitly and see that

$$\bar{v}_2 = \bar{c}_1\bar{y}_1 + \bar{c}_2\bar{y}_2 \tag{2.56}$$

where $\bar{y}_1 = \sqrt{t} I_{\frac{1}{2-q}}(s)$ and $\bar{y}_2 = \sqrt{t} K_{\frac{1}{2-q}}(s)$ and:

$$s = \frac{2\sqrt{h_2\bar{f}_0}}{2-q} t^{\frac{2-q}{2}} \text{ with } s_{\gamma,b} = \frac{2\sqrt{h_2\bar{f}_0}}{2-q} t_{\gamma,b}^{\frac{2-q}{2}}.$$

Then

$$t^{q/2}\bar{v}_2' = \bar{c}_1 t^{q/2}\bar{y}_1' + \bar{c}_2 t^{q/2}\bar{y}_2'.$$

As in Lemma 2.3 and with the facts that $\frac{I_\nu'}{I_\nu} \to 1$ and $\frac{K_\nu'}{K_\nu} \to -1$ as $t \to \infty$ then

$$\lim_{b\to 0^+} \frac{\bar t_{\gamma,b}^{q/2} \bar y_1'(\bar t_{\gamma,b})}{\bar y_1(\bar t_{\gamma,b})} = \sqrt{h_2 \bar f_0}, \tag{2.57}$$

$$\lim_{b\to 0^+} \frac{\bar t_{\gamma,b}^{q/2} \bar y_2'(\bar t_{\gamma,b})}{\bar y_2(\bar t_{\gamma,b})} = -\sqrt{h_2 \bar f_0}. \tag{2.58}$$

Thus from (2.56),

$$\begin{aligned} \frac{\bar t_{\gamma,b}^{q/2} \bar v_2'(\bar t_{\gamma,b})}{\bar v_2(\bar t_{\gamma,b})} &= \frac{\bar c_1 \bar t_{\gamma,b}^{q/2} \bar y_1'(\bar t_{\gamma,b}) + \bar c_2 \bar t_{\gamma,b}^{q/2} \bar y_2'(\bar t_{\gamma,b})}{\bar c_1 \bar y_1(\bar t_{\gamma,b}) + \bar c_2 \bar y_2(\bar t_{\gamma,b})} \\ &= \frac{\bar c_1 \frac{\bar t_{\gamma,b}^{q/2} \bar y_1'(\bar t_{\gamma,b})}{\bar y_1(\bar t_{\gamma,b})} + \bar c_2 \frac{\bar t_{\gamma,b}^{q/2} \bar y_2'(\bar t_{\gamma,b})}{\bar y_1(\bar t_{\gamma,b})}}{\bar c_1 + \bar c_2 \frac{\bar y_2(\bar t_{\gamma,b})}{\bar y_1(\bar t_{\gamma,b})}}. \end{aligned} \tag{2.59}$$

We note that $\bar c_1 \neq 0$ for sufficiently small $b > 0$ for if so then

$$\frac{\bar t_{\gamma,b}^{q/2} \bar v_2'(\bar t_{\gamma,b})}{\bar v_2(\bar t_{\gamma,b})} = \frac{\bar t_{\gamma,b}^{q/2} \bar y_2'(\bar t_{\gamma,b})}{\bar y_2(\bar t_{\gamma,b})}$$

for sufficiently small $b > 0$ but the right-hand side goes to $-\sqrt{h_2 \bar f_0} < 0$ while the left-hand side is positive.

Since $\bar y_2 \to 0$, $\bar y_2' \to 0$ and $\bar y_1 \to \infty$ as $t \to \infty$ it follows from (2.57)-(2.59) that $\frac{\bar t_{\gamma,b}^{q/2} \bar v_2'(\bar t_{\gamma,b})}{\bar v_2(\bar t_{\gamma,b})}$ goes to $\sqrt{h_2 \bar f_0}$ as $b \to 0^+$ and so by (2.55) we see that

$$\limsup_{b\to 0} \bar t_{\gamma,b}^{q/2} u_1'(\bar t_{\gamma,b}) \le \gamma \sqrt{h_2 \bar f_0}.$$

As in Lemmas 2.4 and 2.6 it is then possible to show if b is sufficiently small and $\gamma\big(1+\sqrt{\frac{h_2 \bar f_0}{h_1 f_1}}\big) < \delta$ then u_1 will have a zero and hence u will have a second zero, $z_{2,b}$. Continuing in this way we see that if $b > 0$ is sufficiently small and $\gamma\big(1+\sqrt{\frac{h_2 \bar f_0}{h_1 f_1}}\big) < \delta$ then u will have n zeros for any given integer n. This completes the proof. □

Lemma 2.7. *Assume* (H1)–(H4) *and let u solve* (2.1), (2.4). *If*

$$\beta' + \frac{\beta}{2} \frac{h_1}{h_2} \Big(\frac{f_0}{\bar f_1}\Big)^{1/2} > \delta \tag{2.60}$$

then $u(t) > 0$ for $t > 0$.

Proof. Since E is nondecreasing,

$$\frac{1}{2} \frac{u'^2(t_{b'})}{h(t_{b'})} + F(\beta/2) = E(t_{b'}) \ge E(t_{2,b}) = \frac{1}{2} \frac{u'^2(t_{2,b})}{h(t_{2,b})} + F(\beta/2)$$

thus by (2.2) and (2.15),

$$\liminf_{b\to 0^+} t_{b'}^{q/2} u'(t_{b'}) \ge \liminf_{b\to 0^+} \sqrt{\frac{h_1}{h_2}} t_{2,b}^{q/2} u'(t_{2,b}) \ge \sqrt{\frac{h_1}{h_2}} \sqrt{h_1 f_0} \frac{\beta}{2} = h_1 \frac{\beta}{2} \sqrt{\frac{f_0}{h_2}}. \tag{2.61}$$

Now (2.37) can be solved explicitly and we obtain

$$w_1 = \delta + \sqrt{t}\Big(\hat c_1 I_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_2 \bar f_1}}{2-q} t^{\frac{2-q}{2}}\Big) + \hat c_2 K_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_2 \bar f_1}}{2-q} t^{\frac{2-q}{2}}\Big)\Big) \tag{2.62}$$

where $I_{\frac{1}{2-q}}$ and $K_{\frac{1}{2-q}}$ are the modified Bessel functions of order $\frac{1}{2-q}$ and $\hat{c}_1, \hat{c}_2$ are constants. We rewrite this as

$$w_1 - \delta = \hat{c}_1 \hat{y}_1 + \hat{c}_2 \hat{y}_2 \tag{2.63}$$

where

$$\hat{y}_1(t) = \sqrt{t} I_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_2 \bar{f}_1}}{2-q} t^{\frac{2-q}{2}}\Big), \quad \hat{y}_2(t) = \sqrt{t} K_{\frac{1}{2-q}}\Big(\frac{2\sqrt{h_2 \bar{f}_1}}{2-q} t^{\frac{2-q}{2}}\Big). \tag{2.64}$$

Again we see as in (2.44)-(2.45),

$$\hat{c}_1 = \frac{\hat{y}_2'(t_{b'})(w_1(t_{b'}) - \delta) - \hat{y}_2(t_{b'}) w_1'(t_{b'})}{\hat{y}_1(t_{b'})\hat{y}_2'(t_{b'}) - \hat{y}_1'(t_{b'})\hat{y}_2(t_{b'})}, \tag{2.65}$$

$$\hat{c}_2 = \frac{-\hat{y}_1'(t_{b'})(w_1(t_{b'}) - \delta) + \hat{y}_1(t_{b'}) w_1'(t_{b'})}{\hat{y}_1(t_{b'})\hat{y}_2'(t_{b'}) - \hat{y}_1'(t_{b'})\hat{y}_2(t_{b'})}. \tag{2.66}$$

So we see from (2.46) and (2.64) that

$$\hat{y}_1(t)\hat{y}_2'(t) - \hat{y}_1'(t)\hat{y}_2(t) = -(1 - \frac{q}{2}).$$

Then we see from (2.65)-(2.66) that

$$\hat{c}_1 = \frac{\hat{y}_2'(t_{b'})(\delta - \beta') + \hat{y}_2(t_{b'}) u'(t_{b'})}{1 - \frac{q}{2}}, \tag{2.67}$$

$$\hat{c}_2 = \frac{-\hat{y}_1'(t_{b'})(\delta - \beta') - \hat{y}_1(t_{b'}) u'(t_{b'})}{1 - \frac{q}{2}}. \tag{2.68}$$

Note that $\hat{y}_1(t) > 0$ and $\hat{y}_1'(t) > 0$. In addition, $u'(t_{b'}) > 0$ and $\delta - \beta' > 0$ so it follows that

$$\hat{c}_2 < 0. \tag{2.69}$$

Also

$$\hat{c}_1 = \frac{1}{1 - \frac{q}{2}} t_{b'}^{\frac{1-q}{2}} K_{\frac{1}{2-q}}(r_{b'}) \Big[(\delta - \beta')\Big(\frac{1}{2t_{b'}^{1-\frac{q}{2}}} + \sqrt{h_2 f_1} \frac{K'_{\frac{1}{2-q}}(r_{b'})}{K_{\frac{1}{2-q}}(r_{b'})}\Big) + t_{b'}^{q/2} u'(t_{b'})\Big], \tag{2.70}$$

$$\text{with } r_{b'} = \frac{2}{2-q}\sqrt{h_2 \bar{f}_1}\; {t_{b'}}^{1-\frac{q}{2}}. \tag{2.71}$$

We show in the appendix that

$$\Big(\frac{K_\nu'}{K_\nu} + \frac{\nu}{t}\Big) > -1 \quad \text{for } t > 0 \text{ and } \nu > \frac{1}{2}. \tag{2.72}$$

Now here we have $\nu = \frac{1}{2-q} > \frac{1}{2}$ since $q > 0$ thus using (2.60) and (2.61) we obtain in the bracketed term in (2.70),

$$\begin{aligned} &(\delta - \beta')\Big(\frac{1}{2t_{b'}^{1-\frac{q}{2}}} + \sqrt{h_1 f_1} \frac{K'_{\frac{1}{2-q}}(r_{b'})}{K_{\frac{1}{2-q}}(r_{b'})}\Big) + t_{b'}^{q/2} u'(t_{b'}) \\ &\geq (\delta - \beta')(-\sqrt{h_2 \bar{f}_1}) + h_1 \frac{\beta}{2}\Big(\frac{f_0}{h_2}\Big)^{1/2} \\ &= \sqrt{h_2 \bar{f}_1}\Big[-(\delta - \beta') + \frac{\beta}{2}\frac{h_1}{h_2}\Big(\frac{f_0}{\bar{f}_1}\Big)^{1/2}\Big] > 0. \end{aligned} \tag{2.73}$$

It follows from this that $\hat{c}_1 > 0$.

Now recall from (2.63) that $w_1 = \delta + \hat{c}_1\hat{y}_1 + \hat{c}_2\hat{y}_2$ and $w_1(t_{b'}) = \beta' < \delta$, $w_1'(t_{b'}) > 0$. It follows from (2.37) that w_1 is concave up when $w_1 > \delta_1$ and w_1 is concave down when $w_1 < \delta_1$. Since $\hat{c}_1 > 0$, $\hat{c}_2 < 0$, $\hat{y}_1 \to \infty$ as $t \to \infty$, and $\hat{y}_2 \to 0$ as $t \to \infty$ it follows therefore that it must be the case that $w_1 \to \infty$ as $t \to \infty$ and thus there exists $t_d > t_{b'}$ with $w_1(t_d) = \delta$ and $w_1 \geq \delta$ for $t \geq t_d$. By Lemma 2.4 it follows that there exists $t_\delta < t_d$ such that $u(t_\delta) = \delta$ and $u \geq \delta$ for $t > t_\delta$. It also follows from Lemma 2.4 that $u \geq w_1 > 0$ for $t_{b'} \leq t \leq t_\delta$. From Lemma 2.1 we know $u > 0$ on $(0, t_{\gamma,b})$ and since $t_{b'} < t_{\gamma,b}$ it follows that $u(t) > 0$ for $t > 0$. This completes the proof. □

3. Proof of Theorem 1.1

Proof. For the proof of part (a), from Lemma 2.6 we see that if $R > 0$ is sufficiently small then R^{2-N} is very large and so $z_{1,b} < R^{2-N}$. We also know that $t_{\gamma,b} \to \infty$ as $b \to 0^+$ and since $z_{1,b} > t_{\gamma,b}$ it follows that $u(t) > 0$ on $(0, R^{2-N})$ if $b > 0$ is sufficiently small. Thus by continuity with respect to initial conditions it follows that there is $b_0 > 0$ such that $u(R^{2-N}) = 0$. Thus we obtain a positive solution, u_0, of (2.1), (2.4) if $R > 0$ is sufficiently small and if $\gamma\big(1 + \sqrt{\frac{h_2 \bar{f}_0}{h_1 f_1}}\big) < \delta$. Similarly if $R > 0$ is sufficiently small then $z_{2,b} < R^{2-N}$ and if $b > 0$ is sufficiently small then $z_{2,b} > R^{2-N}$. Then by continuity there exists a b_1 such that $u_1(R^{2-N}) = 0$. Thus u_1 is a solution with exactly one zero on $(0, R^{2-N})$. Continuing in this way we see that if R is sufficiently small then there exists $u_0, u_1, \dots, u_n$ such that u_k has k zeros on $(0, R^{2-N})$ and $u_k(R^{2-N}) = 0$. This completes the proof part (a).

The proof of part (b) follows immediately from Lemma 2.7.

A proof of part(c) c can be found in [10] but we include it here for completeness. Suppose there is a solution of (1.4)-(1.5) such that $\lim_{r\to\infty} u = 0$. Then a straightforward computation shows if $E_2(r) = \frac{1}{2}\frac{u'^2}{K} + F(u)$ then $E_2' = -\frac{u'^2}{2K}\big(2(N-1) + \frac{rK'}{K}\big) \leq 0$ for $r \geq R$. Now if $\lim_{r\to\infty} u = 0$ it follows that $E_2(r) > 0$ for $r \geq R$. Now u cannot have an infinite number of extrema, M_k, with $M_k \to \infty$ because if so $F(u(M_k)) = E_2(M_k) > 0$ so $|u(M_k)| > \gamma$ contradicting that $u(r) \to 0$ as $r \to \infty$. Also there could not be an infinite number of extrema with $M_k \leq L < \infty$ for if so then for some subsequence $M_k \to M$ and there would exist $s_k \to M$ such that $|u'(s_k)| \to \infty$ contradicting that $\frac{1}{2}\frac{u'^2}{K} - F_0 \leq E(r) \leq E(R) = \frac{1}{2}\frac{a^2}{K(R)}$ which implies u' is bounded on $[R, M]$. Thus we see that u must have a largest extremum, M, and without loss of generality let us suppose that $M > R$ is a local maximum and $u' < 0$ for $r > M$. Then

$$\frac{1}{2}\frac{u'^2}{K(r)} + F(u) \leq F(u(M)) \quad \text{for } r > M.$$

Rewriting and integrating on (M, ∞) using that $\alpha > 2$ (from (H3)) gives

$$\begin{aligned} \int_0^{u(M)} \frac{dt}{\sqrt{2}\sqrt{F(u(M)) - F(t)}} &= \int_M^\infty \frac{-u'(r)\,dr}{\sqrt{2}\sqrt{F(u(M)) - F(u(r))}} \\ &\leq \int_M^\infty \sqrt{K}\,dr \\ &\leq \frac{\sqrt{k_2}M^{1-\frac{\alpha}{2}}}{\frac{\alpha}{2} - 1} \leq \frac{\sqrt{k_2}R^{1-\frac{\alpha}{2}}}{\frac{\alpha}{2} - 1}. \end{aligned} \tag{3.1}$$

From (H2) we see that F is bounded below so there exists $F_0 > 0$ such that $F(u) \geq -F_0$ for all u. Also, $u(M) > \gamma$ and $F(u(M)) < F(\delta)$ therefore we see that

$$\int_0^{u(M)} \frac{dt}{\sqrt{2}\sqrt{F(u(M)) - F(t)}} \geq \frac{\gamma}{\sqrt{2}\sqrt{F(\delta) + F_0}}. \tag{3.2}$$

Combining (3.1) and (3.2) gives

$$\frac{\gamma}{\sqrt{2}\sqrt{F(\delta) + F_0}} \leq \frac{\sqrt{k_2} R^{1-\frac{\alpha}{2}}}{\frac{\alpha}{2} - 1}. \tag{3.3}$$

The right-hand side of (3.3) goes to zero as $R \to \infty$ which contradicts (3.3) if $R > 0$ is too large. Thus there are no solutions of (1.1)-(1.3) if $R > 0$ is sufficiently large. This completes the proof of part (c). □

4. Appendix - Facts about modified Bessel functions

In this section we collect some facts about modified Bessel functions. There are numerous texts which contain these results such as [4].

The modified Bessel functions I_ν and K_ν are linearly independent solutions of

$$y'' + \frac{1}{t} y' - \left(1 + \frac{\nu^2}{t^2}\right) y = 0 \quad \text{for } t > 0,\ \nu > 0 \tag{4.1}$$

for which $\lim_{t \to 0^+} I_\nu(t) = 0$ and $\lim_{t \to 0^+} K_\nu(t) = \infty$. They are normalized so that

$$\lim_{t \to 0^+} \frac{I_\nu(t)}{t^\nu} = \frac{1}{2^\nu \Gamma(\nu + 1)}, \quad \lim_{t \to 0^+} \frac{K_\nu(t)}{t^{-\nu}} = 2^{\nu-1}\Gamma(\nu).$$

It can in fact be shown that

$$I_\nu(t) = t^\nu \sum_{n=0}^{\infty} a_n t^n, \quad K_\nu(t) = t^{-\nu} \sum_{n=0}^{\infty} b_n t^n$$

for appropriate constants a_n, b_n.

In addition it is known that $I_\nu(t) > 0$, $K_\nu(t) > 0$, $I_\nu'(t) > 0$ and $K_\nu'(t) < 0$ for $t > 0$ and also $I_\nu(t) \sim \frac{e^t}{\sqrt{t}}$, $K_\nu(t) \sim \frac{e^{-t}}{\sqrt{t}}$ for large t.

It is also known that

$$\lim_{t \to \infty} \frac{I_\nu'}{I_\nu} = 1, \quad \lim_{t \to \infty} \frac{K_\nu'}{K_\nu} = -1.$$

Another well-known fact is that

$$I_\nu(t) K_\nu'(t) - I_\nu'(t) K_\nu(t) = -\frac{1}{t} \quad \text{for } t > 0. \tag{4.2}$$

In addition

$$\Big(\frac{K_\nu'}{K_\nu} + \frac{\nu}{t}\Big) > -1 \quad \text{if } \nu > \frac{1}{2},\ t > 0;$$
$$\Big(\frac{I_\nu'}{I_\nu} + \frac{\nu}{t}\Big) > 1 \quad \text{if } \nu > \frac{1}{2},\ t > 0.$$

We prove these last two facts.

Proof. First $\big(\frac{I_\nu'}{I_\nu} + \frac{\nu}{t}\big) > 0$ and $\lim_{t \to \infty} \big(\frac{I_\nu'}{I_\nu} + \frac{\nu}{t}\big) = 1$. From (4.1) we see that

$$\frac{I_\nu''}{I_\nu} + \frac{1}{t}\Big(\frac{I_\nu'}{I_\nu}\Big) = 1 + \frac{\nu^2}{t^2}.$$

Next,

$$\Big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\Big)' = \frac{I''_\nu}{I_\nu} - \Big(\frac{I'_\nu}{I_\nu}\Big)^2 - \frac{\nu}{t^2}.$$

Combining these gives

$$\Big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\Big)' + \Big(\frac{I'_\nu}{I_\nu}\Big)^2 + \frac{1}{t}\frac{I'_\nu}{I_\nu} = 1 + \frac{\nu^2 - \nu}{t^2}.$$

Therefore,

$$\Big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\Big)' + \Big(\frac{I'_\nu}{I_\nu} + \frac{1}{2t}\Big)^2 = 1 + \frac{(\nu - \frac{1}{2})^2}{t^2}.$$

And

$$\Big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\Big)'' + 2\Big(\frac{I'_\nu}{I_\nu} + \frac{1}{2t}\Big)\Big(\Big(\frac{I'_\nu}{I_\nu}\Big)' - \frac{1}{2t^2}\Big) = \frac{-2(\nu - \frac{1}{2})^2}{t^3}. \tag{4.3}$$

Now suppose $\big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\big)$ has a local minimum for $t > 0$. Then $\big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\big)' = 0$ and $\big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\big)'' \geq 0$. Substituting into (4.3) gives

$$\Big(\frac{\nu}{t^2} + \frac{1}{2t}\Big)\frac{(\nu - \frac{1}{2})}{t^2} \leq \frac{-2(\nu - \frac{1}{2})^2}{t^3}$$

which is impossible since $\nu > \frac{1}{2}$. Thus $\big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\big)$ does not have a local minimum. Since

$$\lim_{t\to 0^+} \Big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\Big) = \infty$$

it follows that $\big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\big)$ is a decreasing function and since $\lim_{t\to\infty} \big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\big) = 1$ it follows that $\big(\frac{I'_\nu}{I_\nu} + \frac{\nu}{t}\big) > 1$ for $t > 0$.

Similarly, $\big(\frac{K'_\nu}{K_\nu} + \frac{\nu}{t}\big)$ does not have a local minimum for $\nu > 1/2$. We also know

$$\lim_{t\to\infty} \Big(\frac{K'_\nu}{K_\nu} + \frac{\nu}{t}\Big) = -1.$$

Thus $\big(\frac{K'_\nu}{K_\nu} + \frac{\nu}{t}\big) > -1$ for $t > 0$ and $\nu > 1/2$. □

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