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STABILITY FOR CONFORMABLE IMPULSIVE DIFFERENTIAL EQUATIONS

YUANLIN DING, MICHAL FEČKAN, JINRONG WANG

Abstract. In this article, we study impulsive differential equations with conformable derivatives. Firstly, we derive suitable formulas for solving linear impulsive conformable Cauchy problems. Then, we show that the linear problem has asymptotic stability, and the nonlinear problem has generalized Ulam-Hyers-Rassias stability. Also we illustrate our results with examples.

1. Introduction

Among the new mathematical tools, we have the conformable derivative which was introduced in [1, 14]. It has been used in Newton mechanics [9], cobweb models [7], logistic models [2], and other branches of physics [20] and mathematics [4, 18, 22, 24, 25, 26].

Impulsive differential equations have been applied to many problems; see [5, 6, 12, 29, 30]. In particular, [3, 8, 19] consider impulsive differential equations with a conformable derivative of the form

$$\begin{gathered}\mathfrak{D}^a_\beta y(t) = g(t, y(t)), \quad t \in I := [a,b]\backslash\{t_1,\dots,t_m\},\ 0<\beta<1,\\ \Delta y(t_k) = I_k(y(t_k^-)), \quad k=1,2,\dots,m,\end{gathered}$$

where $\mathfrak{D}^a_\beta$ is called the conformable derivative with low index a, the function $g : [a,b]\times\mathbb{R}\to\mathbb{R}$ is continuous, $I_k:\mathbb{R}\to\mathbb{R}$ is an (instantaneous) impulsive function, $a = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, $b>0$, $y(t_k^-) = \lim_{\epsilon\to 0^-} y(t_k+\epsilon)$ and $y(t_k^+) = \lim_{\epsilon\to 0^+} y(t_k+\epsilon)$.

Motivated by the works [13, 16, 17, 23, 27, 28, 32], we consider the conformable linear non-instantaneous impulsive differential equation

$$\begin{gathered}\mathfrak{D}^a_\beta y(t) = \mu y(t), \quad t\in(s_k,t_{k+1}],\ k=0,1,2,\dots,m,\\ y(t_k^+) = \xi y(t_k^-), \quad k=1,2,\dots,m,\\ y(t) = \xi y(t_k^-), \quad t\in(t_k,s_k],\ k=1,2,\dots,m,\\ y(s_k^+) = y(s_k^-), \quad k=1,2,\dots,m.\end{gathered} \tag{1.1}$$

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Note that $y(t_k^+) = \xi y(t_k^-)$ is the classical impulsive condition that affects y at the point t_k; meanwhile $y(t) = \xi y(t_k^-)$ for $t \in (t_k, s_k]$ affects y on the interval $(t_k, s_k]$ and is called non-instantaneous impulsive equation.

Next, we consider the conformable non-linear non-instantaneous impulsive differential equation

$$\begin{gathered}
\mathfrak{D}_\beta^a y(t) = g(t, y(t)), \quad t \in (s_k, t_{k+1}],\ k = 0, 1, \dots, m, \\
y(t_k^+) = h_k(t_k, y(t_k^-)), \quad k = 1, \dots, m, \\
y(t) = h_k(t, y(t)), \quad t \in (t_k, s_k],\ k = 1, \dots, m, \\
y(s_k^+) = y(s_k^-), \quad k = 1, 2, \dots, m,
\end{gathered} \tag{1.2}$$

where μ and ξ are constants, $0 < \beta < 1$. For $k = 1, 2, \dots, m$: the s_k are called junction points while the t_k are called impulse points, $t_0 = s_0 = a < t_1 < s_1 < t_2 \cdots < s_m < t_{m+1} = b$, $b > 0$, $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous, $h_k : [t_k, s_k] \times \mathbb{R} \to \mathbb{R}$ is continuous and is called a non-instantaneous impulsive function. For details on the non-instantaneous impulsive equations, see [27, eq. (1.6)]. Equations (1.1), (1.2) are used in the dynamics of evolution processes in pharmacotherapy: the first equation denotes the health status of a patient; the second equation denotes the doctor takes some actions to test medicine for the patient practicably; the third equation denotes the testing medicine is valid for this patient and then begin to deal with the effect of patient for some time. The final equation shows the effect of testing medicine disappeared in the health of the patient.

The article is organized as follows. In Section 2, we present some basic definitions, and derive the solutions for two kinds of non-instantaneous impulsive fractional Cauchy problems. In Section 3, we define asymptotic stability and give some conditions for (1.1) to be asymptotically stable. In Section 4, we define generalized Ulam-Hyers-Rassias stability for (1.2), and use a fixed point theorem to study this stability. In Section 5, we illustrates our main results by examples.

2. Preliminaries

Let $PC(I, \mathbb{R}) = \{y : I \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, 1, \dots,\ y(t_k^-) = y(t_k)\}$, where $C((t_k, t_{k+1}], \mathbb{R})$. This is the space of piecewise continuous functions endowed with the norm $\|y\| = \sup_{t \in I} |y(t)|$.

Definition 2.1 ([15, Definition 2.1]). The conformable derivative with lower index a of a function $y : [a, b] \to \mathbb{R}$ is defined as

$$\begin{gathered}
\mathfrak{D}_\beta^a y(t) = \lim_{\varepsilon \to 0} \frac{y(t + \varepsilon (t-a)^{1-\beta}) - y(t)}{\varepsilon}, \quad a < t,\ 0 < \beta < 1, \\
\mathfrak{D}_\beta^a y(a) = \lim_{t \to a^+} \mathfrak{D}_\beta^a y(t).
\end{gathered}$$

A function y is called β-differentiable at t_0 if $\mathfrak{D}_\beta^a y(t_0)$ exists and is finite.

If $y \in C^1([a, b], \mathbb{R})$, then $\mathfrak{D}_\beta^a y(t) = (t-a)^{1-\beta} y'(t)$. For $t > a$ the conformable derivative $\mathfrak{D}_\beta^a y(t)$ exists if and only if y is differentiable at t and $\mathfrak{D}_\beta^a y(t) = (t-a)^{1-\beta} y'(t)$; see [1]

Definition 2.2 (see [15, Definition 2.3]). The conformable integral with lower index a of a function $y : [a, b] \to \mathbb{R}$ is defined as

$$\mathfrak{I}_\beta^a y(t) = \int_a^t y(s) d_\beta(s, a) = \int_a^t (s-a)^{\beta-1} y(s) ds, \quad a \le t;\ 0 < \beta < 1.$$

When $a = 0$, we write $d_\beta(s) = d_\beta(s, 0)$.

Lemma 2.3 (see [15, Definition 3.3])**.** *Let $y : I \to \mathbb{R}$ be a continuous function. A solution $y \in C(I, \mathbb{R})$ of the linear problem*

$$\begin{gathered}\mathfrak{D}^a_\beta y(t) = \mu y(t) + g(t), \quad t \in I,\ 0 < \beta < 1,\\ y(a) = y_a\end{gathered}$$

has the form

$$y(t) = y_a e^{\mu(t-a)^\beta/\beta} + \int_a^t e^{\mu(t-a)^\beta/\beta} e^{-\mu(s-a)^\beta/\beta} g(s)(s-a)^{\beta-1} ds.$$

The result in Lemma 2.3 is also valid when continuous function is replaced by integrable functions with finitely many points of discontinuity.

Remark 2.4. Consider the multi-dimensional case

$$\begin{gathered}\mathfrak{D}^a_\beta y(t) = f(y(t), t), \quad t \ge a\\ y(a) = y_a,\end{gathered} \tag{2.1}$$

where $f \in C(\mathbb{R}^n \times [a, \infty), \mathbb{R}^n)$. Then we consider the associate ODE

$$\begin{gathered}Y'(z) = f\big(Y(z), \sqrt[\beta]{\beta z} + a\big), \quad z \ge 0\\ Y(0) = y_a.\end{gathered} \tag{2.2}$$

For a solution $Y(z)$ of (2.2), by defining

$$y(t) = Y\Big(\frac{(t-a)^\beta}{\beta}\Big), \tag{2.3}$$

for $t > a$, we obtain

$$\begin{aligned}\mathfrak{D}^a_\beta y(t) &= (t-a)^{1-\beta} y'(t)\\ &= (t-a)^{1-\beta} Y'\Big(\frac{(t-a)^\beta}{\beta}\Big)(t-a)^{\beta-1}\\ &= f\Big(Y\Big(\frac{(t-a)^\beta}{\beta}\Big), \sqrt[\beta]{\beta\frac{(t-a)^\beta}{\beta}} + a\Big),\\ &= f(y(t), t),\\ y(a) &= Y(0) = y_a.\end{aligned}$$

Note that

$$\mathfrak{D}^a_\beta y(a) = \lim_{t \to a^+} \mathfrak{D}^a_\beta y(t) = f(y(a), a) = f(y_a, a).$$

So all solutions of (2.1) are determined by (2.2) and viceversa. For instance, when

$$f(y, t) = Ay + g(t),$$

for a matrix A. Then (2.1) becomes

$$\begin{gathered}y'(z) = Ay(t) + g(t), \quad t \ge a\\ y(t) = y_a,\end{gathered} \tag{2.4}$$

and (2.2) becomes

$$\begin{gathered}Y'(z) = AY(z) + g(\sqrt[\beta]{\beta z} + a), \quad z \ge 0\\ Y(0) = y_a,\end{gathered}$$

with solution

$$Y(z) = e^{Az} y_a + \int_0^z e^{A(z-u)} g(\sqrt[\beta]{\beta u} + a) du.$$

Thus by (2.3), a solution of (2.4) is

$$\begin{aligned} y(t) &= e^{A\frac{(t-a)^\beta}{\beta}} y_a + \int_0^{\frac{(t-a)^\beta}{\beta}} e^{A(\frac{(t-a)^\beta}{\beta}-u)} g(\sqrt[\beta]{\beta u} + a) du \quad \big(u = \frac{(s-a)^\beta}{\beta}\big) \\ &= e^{A\frac{(t-a)^\beta}{\beta}} y_a + \int_a^t e^{A(\frac{(t-a)^\beta}{\beta} - \frac{(s-a)^\beta}{\beta})} g\Big(\sqrt[\beta]{\beta\frac{(s-a)^\beta}{\beta}} + a\Big)(s-a)^{\beta-1} ds, \\ &= e^{\frac{1}{\beta}A(t-a)^\beta} y_a + \int_a^t e^{\frac{1}{\beta}A((t-a)^\beta - (s-a)^\beta)} g(s)(s-a)^{\beta-1} ds. \end{aligned}$$

This is a generalization of Lemma 2.3 to higher dimensions.

Next, we establish two standard frameworks and derive appropriate formulas for solving the impulsive Cauchy problem (1.1), and the problem

$$\begin{gathered} \mathfrak{D}_\beta^a y(t) = g(t), \quad t \in (s_k, t_{k+1}],\ k = 0, 1, \dots, m,\ 0 < \beta < 1, \\ y(t) = h_k(t), \quad t \in (t_k, s_k],\ k = 1, \dots, m, \\ y(a) = y_a. \end{gathered} \tag{2.5}$$

Lemma 2.5. *Let $y(t, s, y_s)$ be the solution of (1.1) with initial value $y(s) = y_s$. Then*

$$y(t) := y(t, s, y_s) = W(t, s) y_s, \quad 0 \le s \le t,$$

where

$$\begin{aligned} W(t,s) = \xi^{n(a,t)-n(a,s)} \exp\Big(\frac{\mu}{\beta}\big[\big(((t-a)^\beta - (s_{n(a,t)} - a)^\beta)^+ - \big((s-a)^\beta \\ - (s_{n(a,s)} - a)^\beta\big)^+\big) + \sum_{k=n(a,s)}^{n(a,t)-1} ((t_{k+1} - a)^\beta - (s_k - a)^\beta)\big]\Big), \end{aligned}$$

where $n(a,t)$ denotes the number of the impulse points that belong to (a,t) and $z^+ := \max\{0, z\}$, $z \in \mathbb{R}$. Note that when $n(a,t) = n(a,s)$, we have $\sum_{k=n(a,s)}^{n(a,t)-1} = 0$. In particular,

$$y(t) = \xi^{n(a,t)} e^{\frac{\mu}{\beta}\left[((t-a)^\beta - (s_{n(a,t)} - a)^\beta)^+ + \sum_{k=n(a,s)}^{n(a,t)-1} ((t_{k+1}-a)^\beta - (s_k - a)^\beta)\right]} y_a.$$

Proof. Depending on the number of pulse and junction points between times t and s, we have the following 8 cases.

Case 1: There are no pulse or junction points between t and s, i.e. $n(a,t) = n(a,s)$.

(i) Let $t, s \in (s_k, t_{k+1}]$ for $k = 0, 1, 2, \dots, n(a,t)$. When $t \in (a, t_1]$, we have

$$y(t) = y_a e^{\mu(t-a)^\beta/\beta}.$$

When $t \in (t_1, s_1]$, we have

$$y(t) = \xi y(t_1^-) = \xi e^{\mu(t_1-a)^\beta/\beta} y_a.$$

When $t \in (s_{1,2}]$, according to

$$y(s_1) = \xi y(t_1^-) = \xi e^{\mu(t_1-a)^\beta/\beta} y_a = e^{\mu(s_1-a)^\beta/\beta} y_{a_1},$$

we have

$$y(t)=e^{\mu(t-a)^\beta/\beta}y_{a_1}=e^{\mu(t-a)^\beta/\beta}\frac{\xi e^{\mu(t_1-a)^\beta/\beta}y_a}{e^{\mu(s_1-a)^\beta/\beta}},$$

$$y(s)=e^{\frac{\mu}{\beta}(s-a)^\beta}y_{a_1}=e^{\mu(t-a)^\beta/\beta}\frac{\xi e^{\mu(t_1-a)^\beta/\beta}y_a}{e^{\mu(s_1-a)^\beta/\beta}},$$

so

$$W(t,s)=e^{\mu\big((t-a)^\beta-(s-a)^\beta\big)/\beta}.$$

(ii) Let us set $t,s\in(t_k,s_k]$, $k=1,2,\dots,n(a,t)$. From

$$y(t)=\xi y(t_k^-),\ k=1,2,\dots,n(a,t),$$

we obtain $y(t)=y(s)$, so $W(t,s)=1$, a constant.

Case 2: There is only one junction point between t and s, i.e. $n(a,t)=n(a,s)$. For every $s\in(t_{n(a,s)},s_{n(a,s)}]$ and $t\in(s_{n(a,t)},t_{n(a,t)+1})$, we have

$$\begin{aligned}y(t)&=e^{\mu\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)/\beta}y(s_{n(a,t)}^+)\\&=e^{\mu\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)/\beta}y(s_{n(a,t)}^-)\\&=e^{\mu\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)/\beta}y(s),\end{aligned}$$

so

$$W(t,s)=e^{\frac{\mu}{\beta}\big((t-a)^\beta-((s_{n(a,t)}-a)^\beta\big)}.$$

Case 3: There is only one pulse point between time t and s, i.e. $n(a,t)=n(a,s)+1$. Let us select every $s\in(s_{n(a,s)},t_{n(a,s)+1}]$ and $t\in(t_{n(a,t)},s_{n(a,t)}]$. When $y(t)=\xi y(t_{n(a,t)}^-)$, we have

$$y(t)=\xi y(t_{n(a,t)}^-)=\xi e^{\mu\big((t_{n(a,t)}-a)^\beta-(s-a)^\beta\big)/\beta}y(s),$$

so

$$W(t,s)=\xi e^{\mu\big((t_{n(a,t)}-a)^\beta-(s-a)^\beta\big)/\beta}.$$

Case 4: There are one pulse and one junction points between t and s, i.e. $n(a,t)=n(a,s)+1$.

(i) By selecting every $s\in(s_{n(a,s)},t_{n(a,s)+1}]$ and $t\in(s_{n(a,t)},t_{n(a,t)+1}]$, we have

$$\begin{aligned}y(t)&=e^{\mu\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)/\beta}y(s_{n(a,t)}^+)\\&=e^{\mu\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)/\beta}y(s_{n(a,t)}^-)\\&=e^{\mu\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)/\beta}\xi y(t_{n(a,t)}^-)\\&=e^{\mu\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)/\beta}\xi e^{\mu\big((t_{n(a,t)}-a)^\beta-(s-a)^\beta\big)/\beta}y(s),\end{aligned}$$

so

$$W(t,s)=e^{\mu\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)/\beta}\xi e^{\mu\big((t_{n(a,t)}-a)^\beta-(s-a)^\beta\big)/\beta}.$$

(ii) For every $s\in(t_{n(a,s)},s_{n(a,s)}]$ and $t\in(t_{n(a,t)},s_{n(a,t)}]$, we have

$$y(t)=\xi y(t_{n(a,t)}^-)$$

$$
\begin{aligned}
&= \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)}-a)^\beta\right)/\beta} y(s^+_{n(a,s)})\\
&= \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)}-a)^\beta\right)/\beta} y(s^-_{n(a,s)})\\
&= \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)}-a)^\beta\right)/\beta} y(s),
\end{aligned}
$$

so

$$
W(t,s) = \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)}-a)^\beta\right)/\beta}.
$$

Case 5: There are two pulse and one junction points between t and s, i.e. $n(a,t) = n(a,s)+2$. For every $t \in (t_{n(a,t)}, s_{n(a,t)}]$ and $s \in (s_{n(a,s)}, t_{n(a,s)+1}]$, we have

$$
\begin{aligned}
y(t) &= \xi y(t^-_{n(a,t)})\\
&= \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\right)/\beta} y(s^+_{n(a,s)+1})\\
&= \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\right)/\beta} y(s^-_{n(a,s)+1})\\
&= \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\right)/\beta} \xi y(t^-_{n(a,s)+1})\\
&= \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\right)/\beta} \xi e^{\frac{\mu}{\beta}\left((t_{n(a,s)+1}-a)^\beta-(s-a)^\beta\right)} y(s)\\
&= \xi^2 \exp\Big(\frac{\mu}{\beta}\beta\big((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\big)\\
&\quad + \big((t_{n(a,s)+1}-a)^\beta-(s_{n(a,s)}-a)^\beta\big) - \big((s-a)^\beta-(s_{n(a,s)}-a)^\beta\big)\Big) y(s),
\end{aligned}
$$

so

$$
\begin{aligned}
W(t,s) &= \xi^2 \exp\Big(\frac{\mu}{\beta}\big((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\big)\\
&\quad + \big((t_{n(a,s)+1}-a)^\beta-(s_{n(a,s)}-a)^\beta\big) - \big((s-a)^\beta-(s_{n(a,s)}-a)^\beta\big)\Big).
\end{aligned}
$$

Case 6: There are one pulse and two junction points between t and s, i.e. $n(a,t) = n(a,s)+1$. For every $s \in (t_{n(a,s)}, s_{n(a,s)}]$ and $t \in (s_{n(a,t)}, t_{n(a,t)+1})$, we have

$$
\begin{aligned}
y(t) &= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} y(s^+_{n(a,t)})\\
&= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} y(s^-_{n(a,t)})\\
&= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} \xi y(t^-_{n(a,t)})\\
&= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)}-a)^\beta\right)/\beta} y(s^+_{(a,t)})\\
&= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)}-a)^\beta\right)/\beta} y(s),
\end{aligned}
$$

so

$$
W(t,s) = e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\right)/\beta}.
$$

Case 7: There are two pulse and two junction points between t and s, i.e. $n(a,t) = n(a,s)+2$.

(i) For every $s \in (s_{n(a,s)}, t_{n(a,s)+1}]$ and $t \in (s_{n(a,t)}, t_{n(a,t)+1})$, we have

$$
\begin{aligned}
y(t) &= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} y(s^+_{n(a,t)})\\
&= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} y(s^-_{n(a,t)})\\
&= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} \xi y(t^-_{n(a,t)})\\
&= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\right)/\beta} y(s^+_{n(a,t)-1})\\
&= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\right)/\beta} y(s^-_{n(a,s)+1})\\
&= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\right)/\beta} \xi y(t^-_{n(a,s)+1})\\
&= e^{\mu\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)/\beta} \xi e^{\mu\left((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\right)/\beta}\\
&\quad \times \xi e^{\frac{\mu}{\beta}\left((t_{n(a,t)-1}-a)^\beta-(s-a)^\beta\right)} y(s)\\
&= \xi^2 \exp\Big(\frac{\mu}{\beta}\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)+\big((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\big)\\
&\quad + \big((t_{n(a,t)-1}-a)^\beta-(s-a)^\beta\big)\Big) y(s),
\end{aligned}
$$

so

$$
\begin{aligned}
W(t,s) &= \xi^2 \exp\Big(\frac{\mu}{\beta}\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)\\
&\quad + \big((t_{n(a,t)}-a)^\beta-(s_{n(a,t)-1}-a)^\beta\big)+\big((t_{n(a,t)-1}-a)^\beta-(s-a)^\beta\big)\Big).
\end{aligned}
$$

(ii) For every $s \in (t_{n(a,s)}, s_{n(a,s)}]$ and $t \in (t_{n(a,t)}, s_{n(a,t)}]$, we have

$$
\begin{aligned}
y(t) &= \xi y(t^-_{n(a,t)})\\
&= \xi e^{\frac{\mu}{\beta}\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)+1}-a)^\beta\right)} y(s^+_{n(a,s)+1})\\
&= \xi e^{\frac{\mu}{\beta}\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)+1}-a)^\beta\right)} y(s^-_{n(a,s)+1})\\
&= \xi e^{\frac{\mu}{\beta}\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)+1}-a)^\beta\right)} \xi y(t^-_{n(a,s)+1})\\
&= \xi e^{\frac{\mu}{\beta}\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)+1}-a)^\beta\right)} \xi e^{\frac{\mu}{\beta}\left((t_{n(a,s)+1}-a)^\beta-(s_{n(a,s)}-a)^\beta\right)} y(s)\\
&= \xi^2 e^{\frac{\mu}{\beta}\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)+1}-a)^\beta\right)+\left((t_{n(a,s)+1}-a)^\beta-(s_{n(a,s)}-a)^\beta\right)} y(s),
\end{aligned}
$$

so

$$
W(t,s) = \xi^2 e^{\frac{\mu}{\beta}\left((t_{n(a,t)}-a)^\beta-(s_{n(a,s)+1}-a)^\beta\right)+\left((t_{n(a,s)+1}-a)^\beta-(s_{n(a,s)}-a)^\beta\right)}.
$$

Case 8: There are several pulse and several junction points between t and s.

(i) For every $s \in (t_{n(a,s)}, s_{n(a,s)}]$ and $t \in (s_{n(a,t)}, t_{n(a,t)+1})$ we have

$$
W(t,s) = \xi^{n(a,t)-n(a,s)} e^{\frac{\mu}{\beta}\left[\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)+\sum_{k=n(a,s)}^{n(a,t)-1}\left((t_{k+1}-a)^\beta-(s_k-a)^\beta\right)\right]}.
$$

(ii) For every $s \in (s_{n(a,s)}, t_{n(a,s)+1}]$ and $t \in (t_{n(a,t)}, s_{n(a,t)}]$, we have

$$
W(t,s) = \xi^{n(a,t)-n(a,s)} e^{\frac{\mu}{\beta}\left[\sum_{k=n(a,s)}^{n(a,t)-1}\left((t_{k+1}-a)^\beta-(s_k-a)^\beta\right)-\left((s-a)^\beta-(s_{n(a,s)}-a)^\beta\right)\right]}.
$$

(iii) For every $s \in (s_{n(a,s)}, t_{n(a,s)+1}]$ and $t \in (s_{n(a,t)}, t_{n(a,t)+1}]$, we have

$$W(t,s) = \xi^{n(a,t)-n(a,s)} \exp\Big(\frac{\mu}{\beta}\big[\big(((t-a)^\beta - (s_{n(a,t)}-a)^\beta) - ((s-a)^\beta - (s_{n(a,s)}-a)^\beta)\big) + \sum_{k=n(a,s)}^{n(a,t)-1} ((t_{k+1}-a)^\beta - (s_k-a)^\beta)\big]\Big).$$

(iv) For every $s \in (t_{n(a,s)}, s_{n(a,s)}]$ and $t \in (t_{n(a,t)}, s_{n(a,t)}]$, we have

$$W(t,s) = \xi^{n(a,t)-n(a,s)} e^{\frac{\mu}{\beta}\sum_{k=n(a,s)}^{n(a,t)-1}\left((t_{k+1}-a)^\beta - (s_k-a)^\beta\right)}.$$

Summarizing the 8 cases above, we can write

$$\begin{aligned} W(t,s) = \xi^{n(a,t)-n(a,s)} \exp\Big(\frac{\mu}{\beta}\big[\big(((t-a)^\beta - (s_{n(a,t)}-a)^\beta)^+ \\ - ((s-a)^\beta - (s_{n(a,s)}-a)^\beta)^+\big) \\ + \sum_{k=n(a,s)}^{n(a,t)-1} ((t_{k+1}-a)^\beta - (s_k-a)^\beta)\big]\Big). \end{aligned} \tag{2.6}$$

In particular when $s = a$,

$$W(t,a) = \xi^{n(a,t)} e^{\frac{\mu}{\beta}\left[\left((t-a)^\beta - (s_{n(a,t)}-a)^\beta\right)^+ + \sum_{k=n(a,s)}^{n(a,t)-1}((t_{k+1}-a)^\beta - (s_k-a)^\beta)\right]}.$$

The proof is complete. □

Lemma 2.6. *A function $y \in PC(I,\mathbb{R})$, is a solution of the fractional integral equations*

$$y(t) = \int_a^t (s-a)^{\beta-1} g(s)ds + y_a, \quad t \in (a, t_1];$$

$$y(t) = \int_{s_k}^t (s-a)^{\beta-1} g(s)ds + h_k(s_k), \quad t \in (s_k, t_{k+1}],\ k = 1,\dots,m;$$

$$y(t) = h_k(t), \quad t \in (t_k, s_k],\ k = 1,\dots,m,$$

if and only if y is a solution of (2.5).

Proof. Assume y is the solution of (2.5). When $t \in [a, t_1]$, we have

$$\mathfrak{D}^a_\beta y(t) = g(t),\ t \in (a, t_1] \quad \text{with } y(a) = y_a. \tag{2.7}$$

By Definition 2.2 and integrating (2.7), we obtain

$$y(t) = \int_a^t (s-a)^{\beta-1} g(s)ds + c.$$

Obviously, $y(a) = y_a$ so $c = y_a$. Therefore

$$y(t) = \int_a^t (s-a)^{\beta-1} g(s)ds + y_a, \quad t \in [a, t_1].$$

Note that when $t \in (t_1, s_1]$, we have $y(t) = h_1(t)$. Also when $t \in (s_{1,2}]$, we have

$$\mathfrak{D}^a_\beta y(t) = g(t),\ t \in (s_{1,2}] \quad \text{with } y(s_1) = h_1(s_1).$$

Similarly, we have

$$y(t) = \int_{s_1}^t (s-a)^{\beta-1} g(s)ds + h_1(s_1), \quad \text{for } t \in (s_{1,2}].$$

When $t \in (_2, s_2]$, we have $y(t) = h_2(t)$. Also when $t \in (s_2, t_3]$, we have

$$\mathfrak{D}^a_\beta y(t) = g(t),\ t \in (s_2, t_3] \quad \text{with } y(s_2) = h_2(s_2).$$

So, we obtain

$$y(t) = \int_{s_2}^{t} (s-a)^{\beta-1} g(s)ds + h_2(s_2), \quad t \in (s_2, t_3].$$

Summarizing,

$$\mathfrak{D}^a_\beta y(t) = g(t),\ t \in (s_k, t_{k+1}] \quad \text{with } y(s_k) = h_k(s_k).$$

Then

$$y(t) = \int_{s_k}^{t} (s-a)^{\beta-1} g(s)ds + h_k(s_k), \quad t \in (t_k, s_k].$$

The remaining proofs can be done by continuing the standard steps and then verify the conclusions. □

Lemma 2.7 (see [10])**.** *Suppose that (Y,d) is a complete metric space, and that $W : Y \to Y$ is a strictly contractive operator with constant $L < 1$. If there exists a nonnegative integer k such that $d(W^{k+1}y, W^k y) < \infty$ for some $y \in Y$, then:*

- (i) *The sequence $\{W^n y\}$ converges to a fixed point y^* in W;*
- (ii) *y^* is the unique fixed point of W in $Y^* = \{x \in Y : d(W^k y, x) < \infty\}$;*
- (iii) *If $x \in Y^*$, then $d(x, y*) \le \frac{1}{1-L} d(Wx, x)$.*

3. Asymptotic stability for the linear problem

Definition 3.1. The solution $y(t)$ of (1.1) is locally asymptotically stable if there exists $\delta > 0$ such that for any $x_a \in \mathbb{R}$ with $|y_a - x_a| < \delta$, it holds

$$\lim_{t\to\infty} |y(t,a,y_a) - y(t,a,x_a)| = 0.$$

If δ is arbitrary, then $y(t)$ is globally asymptotically stable.

For the next theorem we assume that s_k and t_{k+1} satisfy

$$\eta_1 \le \frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta} \le \eta_2, \quad k = 0,1,2,\dots,m \tag{3.1}$$

and define

$$\eta = \begin{cases} \eta_1, & \mu < 0, \\ \eta_2, & \mu \ge 0. \end{cases}$$

Theorem 3.2. *Assume that* (3.1) *holds. If*

$$\Theta := \mu + \frac{1}{\eta} \ln \xi < 0, \tag{3.2}$$

then (1.1) *is asymptotically stable.*

Proof. From (2.6) and (3.1), we have

$$\begin{aligned} |W(t,a)| &\le e^{\mu\left[(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1} (\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})\right]} \xi^{n(a,t)} \\ &\le e^{\mu\left[(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + n(a,t)\eta\right]} \xi^{n(a,t)} \\ &\le e^{\mu\eta} \big(e^{\mu\eta}\xi\big)^{n(a,t)}. \end{aligned}$$

By (3.2), we have

$$e^{\mu\eta}\xi \le e^{\frac{\eta\Theta}{2}} < 1,$$

so when $t \to \infty$, we have $n(a,t) \to \infty$, and then

$$|W(t,a)| \le e^{\mu\eta} e^{\frac{\eta\Theta}{2} n(a,t)} \to 0, \quad \text{as } t \to \infty.$$

The proof is complete. □

Theorem 3.3. *Assume that $\lambda = \mu + \rho \ln \xi < 0$ and one of the following two conditions holds: $\xi \ge 1$, and*

$$\limsup_{t\to\infty} \frac{n(a,t)}{(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})} := \rho < \infty, \tag{3.3}$$

or $\xi < 1$ and

$$\liminf_{t\to\infty} \frac{n(a,t)}{(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})} := \rho < \infty, \tag{3.4}$$

then (1.1) is asymptotically stable.

Proof. By (2.6), we have

$$|W(t,a)| \le e^{\mu\left[(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})\right]} \xi^{n(a,t)}.$$

When $\xi \ge 1$, by (3.3), we obtain

$$n(a,t) < \rho\Big[(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})\Big],$$

for any t large enough. Then

$$|W(t,a)| \le e^{(\mu+\rho\ln\xi)\left[(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})\right]}.$$

Because $\mu + \rho\ln\xi < \lambda/2 < 0$ we have

$$|W(t,a)| \le e^{\frac{\lambda}{2}\left[(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})\right]} \to 0,$$

as $t \to \infty$. Similarly, when $\xi < 1$, by (3.4), we obtain

$$n(a,t) > \rho\Big[(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})\Big],$$

for any t large enough. When $\xi < 1$, we have

$$|W(t,a)| \le e^{(\mu+\rho\ln\xi)\left[(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})\right]},$$

and satisfy $\mu + \rho\ln\xi < \lambda/2 < 0$, so

$$|W(t,a)| \le e^{\frac{\lambda}{2}\left[(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})\right]} \to 0,$$

as $t \to \infty$. The proof is complete. □

Note that

$$\begin{aligned} &W(t,a)\\ &= e^{\frac{\mu}{\beta}\left[\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)^+ + \sum_{k=0}^{n(a,t)-1}((t_{k+1}-a)^\beta-(s_k-a)^\beta)\right]}\xi^{n(a,t)}\\ &= e^{\frac{\lambda}{\beta}\left[\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)^+ + \sum_{k=0}^{n(a,t)-1}((t_{k+1}-a)^\beta-(s_k-a)^\beta)\right]}\\ &\quad\times e^{\ln\xi\left(n(a,t)-\frac{\rho}{\beta}\left[\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)^+ + \sum_{k=0}^{n(a,t)-1}((t_{k+1}-a)^\beta-(s_k-a)^\beta)\right]\right)}. \end{aligned} \tag{3.5}$$

Next we discuss the condition on $\lambda=\mu+\rho\ln\xi$ directly.

Theorem 3.4. *Assume that*

$$\lim_{t\to\infty}\frac{n(a,t)}{\left(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta}\right)^+ + \sum_{k=0}^{n(a,t)-1}\left(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta}\right)} := \rho<\infty. \tag{3.6}$$

Then:

(i) *If $\lambda<0$, then (1.1) is asymptotically stable.*
(ii) *If $\lambda>0$, then (1.1) is unstable.*

Proof. (i) Since $\lambda<0$, there exists ζ_1 such that

$$|e^{\lambda t}|\le e^{-\zeta_1 t},\quad t\ge 0, \tag{3.7}$$

in which $\zeta_1=-\lambda/2$.

By (3.6), there exist $\omega_1>0$ such that for any $t\ge\omega_1$, we have

$$\Big|\frac{n(a,t)}{\left(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta}\right)^+ + \sum_{k=0}^{n(a,t)-1}\left(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta}\right)}-\rho\Big|\le\frac{\zeta_1}{2|\ln\xi|}.$$

Then

$$\begin{aligned} &\Big|e^{\ln\xi\left(n(a,t)-\rho\left[\left(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta}\right)^+ + \sum_{k=0}^{n(a,t)-1}\left(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta}\right)\right]\right)}\Big|\\ &\le e^{|\ln\xi|\left|n(a,t)-\rho\left[\left(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta}\right)^+ + \sum_{k=0}^{n(a,t)-1}\left(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta}\right)\right]\right|}\\ &\le e^{\frac{\zeta_1}{2}\left[\left(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta}\right)^+ + \sum_{k=0}^{n(a,t)-1}\left(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta}\right)\right]}. \end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.5), we obtain

$$|W(t,a)|\le e^{-\frac{\zeta_1}{2}\left[\left(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta}\right)^+ + \sum_{k=0}^{n(a,t)-1}\left(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta}\right)\right]}\to 0,$$

as $t\to\infty$. Thus (i) is proved.

(ii) We rewrite (3.5) as

$$\begin{aligned} &W(t,a)e^{-\ln\xi\left(n(a,t)-\frac{\rho}{\beta}\left[\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)^+ + \sum_{k=0}^{n(a,t)-1}((t_{k+1}-a)^\beta-(s_k-a)^\beta)\right]\right)}\\ &= e^{\frac{\lambda}{\beta}\left[\left((t-a)^\beta-(s_{n(a,t)}-a)^\beta\right)^+ + \sum_{k=0}^{n(a,t)-1}((t_{k+1}-a)^\beta-(s_k-a)^\beta)\right]}. \end{aligned} \tag{3.9}$$

Since $\lambda>0$, there exists ζ_2 and $y_0\in\mathbb{R}^n$ such that

$$|e^{\lambda t}y_a|\ge e^{\zeta_2 t},\ t\ge 0, \tag{3.10}$$

in which $\zeta_2=\lambda/2$.

By (3.6), there exist $\omega_2 > 0$ such that for any $t > \omega_2$, we obtain

$$\begin{aligned}&\Big|e^{-\ln\xi\big(n(a,t)-\frac{\rho}{\beta}\big[\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)^+ +\sum_{k=0}^{n(a,t)-1}((t_{k+1}-a)^\beta-(s_k-a)^\beta)\big]\big)}\Big|\\ &\le e^{\frac{\zeta_2}{2}\big[\big(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta}\big)^+ +\sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta})\big]}.\end{aligned} \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we obtain

$$\begin{aligned}&e^{\zeta_2\big[(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ +\sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta})\big]}\\ &\le \Big|e^{\lambda\big[(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ +\sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta})\big]}y_0\Big|\\ &\le |W(t,a)y_a|\Big|\exp\Big(-\ln\xi\big(n(a,t)-\frac{\rho}{\beta}\big[\big((t-a)^\beta-(s_{n(a,t)}-a)^\beta\big)^+\\ &\quad+\sum_{k=0}^{n(a,t)-1}((t_{k+1}-a)^\beta-(s_k-a)^\beta)\big]\big)\Big)\Big|\\ &\le |W(t,a)y_a|e^{\frac{\zeta_2}{2}\big[\big(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta}\big)^+ +\sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta})\big]},\end{aligned}$$

so

$$|W(t,a)y_a| \ge e^{\frac{\zeta_2}{2}\big[(\frac{(t-a)^\beta}{\beta}-\frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ +\sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta}-\frac{(s_k-a)^\beta}{\beta})\big]} \to \infty,$$

as $t \to \infty$. The proof is complete. □

4. Generalized Ulam-Hyers-Rassias stability for the nonlinear problem

We introduce the concept of generalized Ulam-Hyers-Rassias stability through the concept of stability in [21, 31].

Let $\epsilon > 0$, $\psi \ge 0$ and $\phi \in PC(I, \mathbb{R}_+)$ be nondecreasing, in the conditions

$$\begin{gathered}|\mathfrak{D}^a_\beta x(t) - g(t, x(t))| \le \phi(t), \quad t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, m,\ 0 < \beta < 1,\\ |x(t) - h_k(t, x(t))| \le \varphi, \quad t \in (t_k, s_k],\ k = 1, 2, \dots, m.\end{gathered} \tag{4.1}$$

Definition 4.1. Equation (1.2) has generalized Ulam-Hyers-Rassias stability if there exists $c_{g,\beta,h_k,\phi} > 0$ such that for each solution $x \in PC(I, \mathbb{R})$ of inequality (4.1), there exists a solution $y \in PC(I, \mathbb{R})$ of (1.2) with

$$|x(t) - y(t)| \le c_{g,\beta,h_k,\phi}(\phi(t) + \varphi), \quad t \in I.$$

When $\varepsilon = 1$, the generalized Ulam-Hyers-Rassias stability reduces to the classical Ulam-Hyers-Rassias stability, see [27, Remark 3.5].

Remark 4.2. A function $x \in PC(I, \mathbb{R})$ is a solution of (4.1) if and only if there exists $H \in PC(I, \mathbb{R})$ and a sequence H_k, $k = 1, 2, \dots, m$ which depends on x such that

(i) $|H(t)| \le \phi(t)$ for $t \in I$, and $|H_k| \le \varphi$ for $k = 1, 2, \dots, m$;
(ii) $\mathfrak{D}^a_\beta x(t) = g(t, x(t)) + H(t)$ for $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \dots, m$;
(iii) $x(t) = h_k(t, x(t)) + H_k$, $t \in (s_{k-1}, t_k]$, $k = 1, 2, \dots, m$.

Remark 4.3. If $x \in PC(I, \mathbb{R})$ is the solution of (4.1) then x satisfies the following integral inequalities:

$$
\begin{gathered}
|x(t) - h_k(t, x(t))| \le \varphi, \quad t \in (t_k, s_k],\ k = 1, 2, \dots, m, \\
|x(t) - x(a) - \int_a^t (s-a)^{\beta-1} g(s, x(s)ds| \le \int_a^t (s-a)^{\beta-1}\phi(s)ds, \quad t \in (a, t_1], \\
|x(t) - \int_{s_k}^t (s-a)^{\beta-1} g(s, x(s))ds - h_k(s_k, x(s_k))| \le \int^t (s-a)^{\beta-1}\phi(s)ds + \varphi, \\
t \in (s_k, t_{k+1}],\ k = 1, 2, \dots, m.
\end{gathered}
\tag{4.2}
$$

By Remark 4.2 (i), we have

$$
\begin{gathered}
\mathfrak{D}^a_\beta x(t) = g(t, x(t)) + H(t), \quad t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, m, \\
x(t) = h_k(t, x(t)) + H_k, \quad t \in (t_k, s_k],\ k = 1, 2, \dots, m.
\end{gathered}
\tag{4.3}
$$

Obviously,

$$
\begin{gathered}
x(t) = h_k(t, x(t)) + H_k, \quad t \in (t_k, s_k],\ k = 1, 2, \dots, m, \\
x(t) = \int_a^t (s-a)^{\beta-1}\big(g(s, x(s)) + H(s)\big)ds + y_a,\ t \in (a, t_1], \\
x(t) = \int_{s_k}^t (s-a)^{\beta-1}\big(g(s, x(s)) + H(s)\big)ds + h_k(s_k, x(s_k)) + H_k, \\
t \in (t_k, s_k],\ k = 1, 2, \dots, m
\end{gathered}
$$

is the solution of (4.3). For $t \in (s_k, t_{k+1}],\ k = 1, 2, \dots, m$, we have

$$
\begin{aligned}
&\Big|x(t) - \int_{s_k}^t (s-a)^{\beta-1} g(s, x(s))ds - h_k(s_k, x(s_k))\Big| \\
&\le \Big|\int_{s_k}^t (s-a)^{\beta-1} H(s)ds\Big| + |H_k| \\
&\le \int_{s_k}^t (s-a)^{\beta-1}\phi(s)ds + \varphi.
\end{aligned}
$$

As mentioned above, we can obtain

$$
|x(t) - h_k(t, x(t))| \le |H_k| \le \varphi, \quad t \in (t_k, s_k],\ k = 1, 2, \dots, m,
$$

and

$$
\begin{aligned}
\Big|x(t) - x(a) - \int_a^t (s-a)^{\beta-1} g(s, x(s))ds\Big| &\le \Big|\int_a^t (s-a)^{\beta-1} H(s)ds\Big| \\
&\le \int_a^t (s-a)^{\beta-1}\phi(s)ds,\ t \in (a, t_1].
\end{aligned}
$$

For using a fixed point theorem of the alternative and for deriving our main result, which is about contractions on a complete metric space, we consider the following assumptions:

- (H1) $g \in C(I \times \mathbb{R}, \mathbb{R})$.
- (H2) There exists a positive constant L_g such that
$$
|g(t, v_1) - g(t, v_2)| \le L_g |v_1 - v_2|,
$$
for each $t \in I$ and all $v_1, v_2 \in \mathbb{R}$.

(H3) $h_k \in C([t_k, s_k] \times \mathbb{R}, \mathbb{R})$ and there are positive constants L_{h_k}, $k = 1, 2, \dots, m$ such that
$$|g_k(t, v_1) - g_k(t, v_2)| \le L_{h_k}|v_1 - v_2|,$$
for each $t \in [t_k, s_k]$ and all $v_1, v_2 \in \mathbb{R}$.

(H4) $\phi \in C(I, \mathbb{R}_+)$ is a nondecreasing function, and there exists $c_\phi > 0$ such that
$$\Big(\int_a^t \big(\phi(s)\big)^{1/p} ds\Big)^p \le c_\phi \phi(t), \quad p \in (0,1), \text{ for each } t \in I.$$

We use the concept of generalized Ulam-Hyers-Rassias to show stability of (1.2) in the following section.

Theorem 4.4. *Assume that* (H1)–(H4) *are satisfied and a function* $x \in PC(I, \mathbb{R})$ *that satisfies* (4.1). *Then there exists a unique solution* x_0 *of* (1.2) *such that*
$$\begin{gathered} x_0(t) = \int_a^t (s-a)^{\beta-1} g(s, x_0(s))ds + y_a, \quad t \in [a, t_1], \\ x_0(t) = h_k(t, x_0(t)), \quad t \in (t_k, s_k], \ k = 1, 2, \dots, m, \\ x_0(t) = \int_{s_k}^t (s-a)^{\beta-1} g(s, x_0(s))ds + h_k(s_k, x_0(s_k)), \\ t \in (s_k, t_{k+1}], \ k = 1, 2, \dots, m, \end{gathered} \tag{4.4}$$
and
$$|x(t) - x_0(t)| \le \frac{\big(2c_\phi\big(\frac{1-p}{\beta-p}\big)^{1-p} b^{\beta-p} + 1\big)(\phi(t) + \varphi)}{1-M}, \tag{4.5}$$
for all $t \in I$ *provided that* $0 < p < \beta < 1$ *and*
$$M = M_1 < 1, \tag{4.6}$$
where
$$M_1 = \max\{L_g c_\phi \Big(\frac{1-p}{\beta-p}\Big)^{1-p} t_{k+1}^{\beta-p} + L_{h_k} : k = 0, 1, 2, \dots, m\}.$$

Proof. Consider the space of piecewise continuous functions $Y = \{f : I \to \mathbb{R} : f \in PC(I, \mathbb{R})\}$, and the generalized metric
$$d(f, h) = \inf\big\{A_1 + A_2 \in [0, +\infty] : |f(t) - h(t)| \le (A_1 + A_2)(\phi(t) + \varphi) \ \forall t \in I\big\}, \tag{4.7}$$
where
$$\begin{gathered} A_1 \in \{A \in [0, +\infty] \big| |f(t) - h(t)| \le A\phi(t) \text{ for all } t \in (s_k, t_{k+1}], \ k = 0, 1, 2, \dots, m\}, \\ A_2 \in \{A \in [0, +\infty] \big| |f(t) - h(t)| \le A\varphi \text{ for all } t \in (t_k, s_k], \ k = 1, 2, \dots, m\}. \end{gathered}$$
This is a generalized metric in the sense that it can have value $+\infty$. For the necessity of introducing such a generalized metric and applications, we refer to [10]. One can easily show that (Y, d) is a complete generalized metric space.

We define an operator $\Upsilon : Y \to Y$ by
$$(\Upsilon y)(t) = \begin{cases} \int_a^t (s-a)^{\beta-1} g(s, y(s))ds + y_a & \text{if } t \in [a, t_1], \\ h_k(t, y(t)) \quad \text{if } t \in (t_k, s_k], \ k = 1, 2, \dots, m, \\ \int_{s_k}^t (s-a)^{\beta-1} g(s, y(s))ds + h_k(s_k, y(s_k)) \\ \quad \text{if } t \in (s_k, t_{k+1}], \ k = 1, 2, \dots, m, \end{cases} \tag{4.8}$$
for all $y \in Y$ and $t \in [a, b]$. Obviously, Υ is a well defined operator by (H1).

Next, we verify that Υ is strictly contractive. We considering the definition of (Y,d), for any $f,h\in Y$, we find a $A_1,A_2\in[0,\infty]$ such that

$$|f(t)-h(t)|\leq\begin{cases}A_1\phi(t), & t\in(s_k,t_{k+1}],\ k=0,1,2,\dots,m,\\ A_2\varphi, & t\in(t_k,s_k],\ k=1,2,\dots,m.\end{cases}\tag{4.9}$$

By the definition of Υ in (4.8), (H2), (H3) and (4.9), we obtain the following three cases:

Case 1: For $t\in[a,t_1]$ we have

$$\begin{aligned}|(\Upsilon f)(t)-(\Upsilon h)(t)|&=\Big|\int_a^t(s-a)^{\beta-1}g(s,f(s))ds-\int_a^t(s-a)^{\beta-1}g(s,h(s))ds\Big|\\
&\leq\int_a^t(s-a)^{\beta-1}|g(s,f(s))-g(s,h(s))|ds\\
&\leq L_g\int_a^t(s-a)^{\beta-1}|f(s)-h(s)|ds\\
&\leq L_gA_1\int_a^t(s-a)^{\beta-1}|\phi(s)|ds\\
&\leq L_gA_1\Big(\int_a^t(s-a)^{\frac{\beta-1}{1-p}}ds\Big)^{1-p}\Big(\int_a^t\big(\phi(s)\big)^{1/p}ds\Big)^p\\
&\leq L_gA_1c_\phi\phi(t)\Big(\frac{1-p}{\beta-p}\Big)^{1-p}t^{\beta-p}\\
&\leq L_gc_\phi\Big(\frac{1-p}{\beta-p}\Big)^{1-p}t_1^{\beta-p}A_1\phi(t).\end{aligned}$$

Case 2: For $t\in(t_k,s_k]$ we have

$$\begin{aligned}|(\Upsilon f)(t)-(\Upsilon h)(t)|&=|h_k(t,f(t))-h_k(t,h(t))|\\
&\leq L_{h_k}|f(t)-h(t)|\leq L_{h_k}A_2\varphi.\end{aligned}$$

Case 3: For $t\in(s_k,t_{k+1}]$ we have

$$\begin{aligned}&|(\Upsilon f)(t)-(\Upsilon h)(t)|\\
&=\Big|\int_{s_k}^t(s-a)^{\beta-1}g(s,f(s))ds+h_k(s_k,f(s_k))\\
&\quad-\int_{s_k}^t(s-a)^{\beta-1}g(s,h(s))ds-h_k(s_k,h(s_k))\Big|\\
&\leq\Big|\int_{s_k}^t(s-a)^{\beta-1}g(s,f(s))ds-\int_{s_k}^t(s-a)^{\beta-1}g(s,h(s))ds\Big|\\
&\quad+\big|h_k(s_k,f(s_k))-h_k(s_k,h(s_k))\big|\\
&\leq L_gc_\phi\Big(\frac{1-p}{\beta-p}\Big)^{1-p}t_{k+1}^{\beta-p}A_1\phi(t)+L_{h_k}A_2\varphi\\
&\leq\Big(L_gc_\phi\Big(\frac{1-p}{\beta-p}\Big)^{1-p}t_{k+1}^{\beta-p}+L_{h_k}\Big)(A_1+A_2)(\phi(t)+\varphi).\end{aligned}$$

In this case we have

$$|(\Upsilon f)(t) - (\Upsilon h)(t)| \le M(A_1 + A_2)(\phi(t) + \psi), \quad t \in I.$$

Then $d(\Upsilon f, \Upsilon h) \le M(A_1 + A_2)$. Therefore,

$$d(\Upsilon f, \Upsilon h) \le M d(f, h),$$

for any $f, h \in Y$, and because of (4.6), we verify the strictly continuous property.

Let us take $f_0 \in Y$. By the piecewise continuous property of f_0 and Υf_0, there exists a constant $0 < F_1 < \infty$ such that

$$\begin{aligned} |(\Upsilon f_0)(t) - f_0(t)| &= \Big| \int_a^t (s-a)^{\beta-1} g(s, f_0(s)) ds + y_a - f_0(t) \Big| \\ &\le F_1 \phi(t) \le F_1(\phi(t) + \varphi), \quad t \in [a, t_1]. \end{aligned}$$

Then there exists a constant $0 < F_2 < \infty$ such that

$$|(\Upsilon f_0)(t) - f_0(t)| = \big| h_k(t, f_0(t)) - f_0(t) \big| \le F_2 \varphi \le F_2(\phi(t) + \varphi),$$

for $t \in (t_k, s_k]$ and $k = 1, 2, \dots, m$. Also there exists a constant $0 < F_3 < \infty$ such that

$$\begin{aligned} |(\Upsilon f_0)(t) - f_0(t)| &= \Big| \int_{s_k}^t (s-a)^{\beta-1} g(s, f_0) ds + h_k(s_k, f_0(s_k)) - f_0(t) \Big| \\ &\le F_3(\phi(t) + \varphi), \quad t \in (s_k, t_{k+1}],\ k = 1, 2, \dots, m. \end{aligned}$$

because $g, h_k, f_0 < \infty$ are bounded on I and $\phi(\cdot) + \varphi > 0$. So (4.7) implies that

$$d(\Upsilon f_0, f_0) < \infty.$$

Using the Banach fixed point theorem, we obtain a continuous function $x_0 : I \to \mathbb{R}$ such that $\Upsilon^n(f_0) \to x_0$ in (Y, d) as $n \to \infty$ and $\Upsilon x_0 \to x_0$, and for every $t \in I$, x_0 satisfies (4.4).

Next, we verify that $\{f \in Y | d(f_0, f) < \infty\} = Y$. For any $f \in Y$, because f_0, f are bounded on I and $\min_{t \in I}(\phi(t) + \varphi) > 0$, there is a constant $0 < A_f < \infty$ such that $|f_0(t) - f(t)| \le A_f(\phi(t) + \varphi)$, for any $t \in I$. So we have $d(f_0, f) < \infty$ for any $f \in Y$; that is, $\{f \in Y | d(f_0, f) < \infty\} = Y$. Therefore, we know that x_0 is the unique continuous function and it has the property (4.4). From (4.2) and (H4), we have

$$d(x, \Upsilon x) \le 2c_\phi \Big(\frac{1-p}{\beta-p}\Big)^{1-p} b^{\beta-p} + 1,$$

In summary, we have

$$d(x, x_0) \le \frac{d(\Upsilon y, y)}{1-M} \le \frac{2c_\phi \big(\frac{1-p}{\beta-p}\big)^{1-p} b^{\beta-p} + 1}{1-M},$$

so (4.5) holds for $t \in I$. The proof is complete. □

5. Examples

To illustrate our results we present the following examples.

Example 5.1. Consider the conformable linear non-instantaneous impulsive differential equations

$$\begin{gathered}\mathfrak{D}^a_\beta y(t) = \upsilon y(t), \quad t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, m,\\ y(t_k^+) = \nu y(t_k^-) \quad \text{on } (t_k, s_k],\ k = 1, 2, \dots, m,\\ y(t) = \nu y(t_k^-), \quad t \in (t_k, s_k],\ k = 1, 2, \dots, m,\\ y(s_k^+) = y(s_k^-), \quad k = 1, 2, \dots, m.\end{gathered} \tag{5.1}$$

Let $t_a = s_a = 1$ and $\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta} = 1$ for $k = 0, 1, 2, 3, \dots, m$. Then $\eta = 1$. Note that

$$\begin{aligned}\frac{n(a,t)}{n(a,t)+1} &= \frac{n(a,t)}{\sum_{k=0}^{n(a,t)}\big(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta}\big)}\\ &\le \frac{n(a,t)}{\big(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta}\big)^+ + \sum_{k=0}^{n(a,t)-1}\big(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta}\big)}\\ &\le \frac{n(a,t)}{\sum_{k=0}^{n(a,t)-1}\big(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta}\big)} = 1,\end{aligned}$$

because $n(a,t) > 1$. Then

$$\rho = \lim_{t\to\infty} \frac{n(a,t)}{(\frac{(t-a)^\beta}{\beta} - \frac{(s_{n(a,t)}-a)^\beta}{\beta})^+ + \sum_{k=0}^{n(a,t)-1}(\frac{(t_{k+1}-a)^\beta}{\beta} - \frac{(s_k-a)^\beta}{\beta})} = 1.$$

Next, $\lambda = \upsilon + \ln\nu$. By Theorem 3.4, we know that if $\upsilon < -\ln\nu$, (5.1) is asymptotically stable. Also if $\upsilon > -\ln\nu$, (5.1) is unstable.

Example 5.2. Consider

$$\begin{gathered}\mathfrak{D}^0_\beta y(t) = \frac{|y(t)|}{10 + 4t^2 + 10e^t}, \quad t \in (0, 1] \cup (2, 3],\\ y(t) = \frac{t}{6} e^{-y(t)}, \quad t \in (1, 2],\end{gathered}$$

and

$$\begin{gathered}\big|\mathfrak{D}^0_\beta x(t) - \frac{|x(t)|}{10 + 4t^2 + 10e^t}\big| \le e^t, \quad t \in [0, 1] \cup (2, 3],\\ \big|x(t) - \frac{t}{6}e^{-x(t)}\big| \le 1, \quad t \in (1, 2].\end{gathered}$$

Let $I = [0, 3]$, $\beta = 1/2$, $p = 1/3$ and $0 = t_0 = s_0 < 1 = t_1 < 2 = s_1 <_2 = 3$. Denote $g(t, y(t)) = \frac{|y(t)|}{10+4t^2+10e^t}$ with $L_g = \frac{1}{20}$, for $t \in (0, 1] \cup (2, 3]$ and $h_1(t, y(t)) = \frac{t}{6}e^{-y(t)}$ with $L_{h_1} = \frac{1}{3}$ for $t \in (1, 2]$. Putting $\phi(t) = e^t$, $\varphi = 1$ and $c_\phi = 1$, we have $\big(\int_0^t (e^t)^3 ds\big)^{1/3} \le e^t$. Let $M_1 = \{\frac{1}{20}4^{2/3}3^{1/6} + \frac{1}{3}\} = 0.4846$, so $M = 0.4846 < 1$. By Theorem 4.4, there exists a unique solution $x_0 : [0, 3] \to R$ such that

$$x_0(t) = \begin{cases}\int_0^t s^{-1/2}\frac{|x_0(s)|}{10+4s^2+10e^s} ds + y_0, & t \in [0, 1],\\ \frac{t}{6}e^{-x_0(t)}, & t \in (1, 2],\\ \in_2^t s^{-1/2}\frac{|x_0(s)|}{10+4s^2+10e^s} ds + \frac{2}{6}e^{-x_0(2)}, & t \in (2, 3],\end{cases}$$

and

$$|x(t) - x_0(t)| \le \frac{2 \times 4^{2/3} \times 3^{1/6} + 1}{1 - 0.5}(e^t + 1) \approx 14.1047(e^t + 1),$$

for all $t \in [0, 3]$.

Conclusion. This article gives elementary results for linear and nonlinear non-instantaneous conformable impulsive differential equations keeping the lower limit at a fixed point a. Representation of solutions and asymptotical stability for linear problems are established. The generalized Ulam-Hyers-Rassias stability for nonlinear problems are also derived. In a forthcoming paper, we can extend the current results to higher dimension case based on Remark 2.4. Note there is no nonconstant periodic solution for (2.5). We can consider (2.5) replacing a by s_k, i.e., in each impulse starting at impulsive time. Then when impulses and nonlinear terms are periodic, existence of periodic solutions will be possible by following the idea in [11].

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Yuanlin Ding
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China
Email address: yldingmath@126.com

Michal Fečkan
Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia.
Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia
Email address: Michal.Feckan@fmph.uniba.sk

Jinrong Wang (corresponding author)
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China.
School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China
Email address: wjr9668@126.com