Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 119, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

POLYHARMONIC SYSTEMS INVOLVING CRITICAL NONLINEARITIES WITH SIGN-CHANGING WEIGHT FUNCTIONS

ANU RANI, SARIKA GOYAL

Abstract. This article concerns the existence of multiple solutions of the polyharmonic system involving critical nonlinearities with sign-changing weight functions

$$
\begin{gathered}
(-\Delta)^m u = \lambda f(x)|u|^{r-2}u + \frac{\beta}{\beta+\gamma} h(x)|u|^{\beta-2}u|v|^{\gamma} \quad \text{in } \Omega, \\
(-\Delta)^m v = \mu g(x)|v|^{r-2}v + \frac{\gamma}{\beta+\gamma} h(x)|u|^{\beta}|v|^{\gamma-2}v \quad \text{in } \Omega, \\
D^k u = D^k v = 0 \quad \text{for all } |k| \le m-1 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where $(-\Delta)^m$ denotes the polyharmonic operators, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $m \in \mathbb{N}$, $N \ge 2m+1$, $1 < r < 2$ and $\beta > 1$, $\gamma > 1$ satisfying $2 < \beta + \gamma \le 2_m^*$ with $2_m^* = \frac{2N}{N-2m}$ as a critical Sobolev exponent and λ, $\mu > 0$. The functions f, g and $h : \overline{\Omega} \to \mathbb{R}$ are sign-changing weight functions satisfying f, $g \in L^{\alpha}(\Omega)$ and $h \in L^{\infty}(\Omega)$ respectively. Using the variational methods and Nehari manifold, we prove that the system admits at least two nontrivial solutions with respect to parameter $(\lambda,\mu) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $m \in \mathbb{N}$, $N \ge 2m+1$. We consider the following polyharmonic system involving concave-convex nonlinearities with critical exponent and sign-changing weight functions

$$
\begin{gathered}
(-\Delta)^m u = \lambda f(x)|u|^{r-2}u + \frac{\beta}{\beta+\gamma} h(x)|u|^{\beta-2}u|v|^{\gamma} \quad \text{in } \Omega, \\
(-\Delta)^m v = \mu g(x)|v|^{r-2}v + \frac{\gamma}{\beta+\gamma} h(x)|u|^{\beta}|v|^{\gamma-2}v \quad \text{in } \Omega, \\
D^k u = D^k v = 0 \quad \text{for all } |k| \le m-1 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}
$$

where $1 < r < 2$, $\beta > 1$, $\gamma > 1$ satisfying $2 < \beta + \gamma \le 2_m^*$ with $2_m^* = \frac{2N}{N-2m}$ as a critical Sobolev exponent and λ, μ are the parameter such that $(\lambda,\mu) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$.

2010 *Mathematics Subject Classification.* 35A15, 35B33, 35J91.
Key words and phrases. Polyharmonic operator system; sign-changing weight functions; critical exponent; Nehari manifold; concave-convex nonlinearities.

Submitted August 31, 2020. Published December 10, 2020.

Here Δ^m denotes the polyharmonic operators which is defined as

$$\Delta^m u = \begin{cases} \Delta^j(\Delta^j u) & \text{if } m = 2j,\ j = 1, 2, \dots \\ \nabla\cdot(\Delta^{j-1}\nabla\Delta^{j-1}u) & \text{if } m = 2j-1,\ j = 1, 2, \dots. \end{cases}$$

To construct our problem more precise, we give the following assumptions on the weight functions f, g and h:

- (A1) $f,\ g \in L^\alpha(\Omega)$ with $\alpha = \frac{\beta+\gamma}{\beta+\gamma-r}$, $f^\pm = \max\{\pm f, 0\} \not\equiv 0$ in $\overline{\Omega}$ and $g^\pm = \max\{\pm g, 0\} \not\equiv 0$ in $\overline{\Omega}$ i.e. (f and g are possibly sign-changing on $\overline{\Omega}$);
- (A2) $h \in L^\infty(\Omega)$ and $h^+ = \max\{h, 0\} \not\equiv 0$ in Ω.

When $\beta = \gamma$, $\beta + \gamma = 2^*_m$, $\lambda = \mu$, $u = v$ and $f \equiv g$, problem (1.1) reduces to the polyharmonic equation

$$\begin{gathered} (-\Delta)^m u = \lambda f(x)|u|^{r-2}u + h(x)|u|^{2^*_m-2}u \quad \text{in } \Omega, \\ D^k u = 0 \quad \text{for all } |k| \le m-1 \quad \text{on } \partial\Omega, \end{gathered}$$

which was investigated in [30] when f and h are continuous functions. Recently, a lot of attention has been directed to the study of biharmonic and polyharmonic equations, both from concrete applications and for pure mathematical point of view. Such models naturally arise in many applications, such as micro electro-mechanical system, phase field models of multi-phase systems, in thin film theory, nonlinear surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, and the deformation of a nonlinear elastic beam (see [17, 27]).

Starting with the pioneering work of Ambrosetti et al. [3] on Laplacian involving convex concave type nonlinearities, an enormous amount of work has been examined by authors such as Bartsch-Willem [4], Figueiredo et al [11], Brown and Zhang [10], Hamidi [21] and Hsu [23] in this direction. Brézis and Nirenberg [8] studied the problem with critical nonlinearity

$$\begin{gathered} -\Delta u = u^{\frac{N+2}{N-2}} + \lambda u, \quad u > 0 \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $N \ge 3$. They showed that for $N \ge 4$, (1.2) has positive solution if and only if $\lambda \in (0, \lambda_1)$. For $N = 3$ and $\Omega = B_1$ is unit ball in $\mathbb{R}^N$, problem (1.2) has a positive solution if and only if $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$, where $\lambda_1 > 0$ is first eigenvalue of $-\Delta$ in Ω. If Ω is star shaped, then (1.2) has no solution for $\lambda \le 0$. Moreover, a great amount of mathematical effort has been demonstrated by many authors involving biharmonic equation with critical nonlinearity (see [5, 12, 13, 15, 26, 29]).

Pucci-Serrin [28] considered the polyharmonic equation with critical nonlinearity

$$\begin{gathered} (-\Delta)^m u = |u|^{2^*_m-2}u + \lambda u \quad \text{in } \Omega, \\ D^k u = 0 \quad \text{for all } |k| \le m-1 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.3}$$

They found that if $N \ge 4m$ and $\Omega = B_1$, then (1.3) has positive solution, for all $\lambda \in (0, \lambda_1^{(m)})$, where $\lambda_1^{(m)}$ is the first eigenvalue of polyharmonic operator $(-\Delta)^m$. If $N = 2m + 1$ and $\Omega = B_1$, then (1.3) admits the existence of a nonnegative, nontrivial solution if ($\lambda \in (2m - \frac{1}{2})\lambda_1^{(m-1)}, \lambda_1^{(m)})$. If $\lambda < 0$ and Ω is star shaped, then (1.3) has the trivial solution. Later Edmunds et al [15] extended the results of problem (1.3) for biharmonic operator ($m = 2$) and showed that (1.3) has a nontrivial solution if $\lambda \in (0, \lambda_1)$ and $N \ge 8$. When $N = 5$, 6 or 7, problem (1.3) has a nontrivial solution, for all $\lambda \in (\bar{\lambda}, \lambda_1)$, where $\bar{\lambda} = \lambda_1 - S|\Omega|^{-\frac{4}{N}}$ and S is the best constant for Sobolev embedding of $H_0^2(\Omega)$ in $L^{\frac{2N}{N-4}}(\Omega)$. Also, Grunau [19] studied (1.3) in case of ball and proved that, if $2m+1 \leq N \leq 4m-1$, (1.3) has a positive solution for $\lambda \in (\bar{\lambda}, \lambda_1^{(m)})$ for some $\bar{\lambda} = \bar{\lambda}(N,m) \in (0, \lambda_1^{(m)})$. Thereafter, Gazzola [18] contributed for polyhamonic operators with critical growth.

During the previous decades many authors have paid attention to semilinear and quasilinear elliptic equations involving sign-changing weight functions with subcritical and critical nonlinearity using Nehari manifold. Reader is referred to [1, 2, 6, 9, 20, 31, 34, 35] and references therein. Further, Hsu [22, 24] proved the multiplicity results for elliptic system and quasilinear elliptic system involving convex-concave nonlinearities with sign-changing weight function respectively. Ji and Wang [25] studied the p-biharmonic equation involving subcritical nonlinearity with sign-changing weight function and showed the existence of two nontrivial solution by Nehari manifold and fibering map analysis. Recently, in 2014, Shang and Li [30] investigated the multiplicity of nontrivial solutions of polyharmonic equation with critical exponents and sign-changing weight functions. To the best of our knowledge, there is no result so far concerning polyharmonic system involving critical nonlinearities with sign-changing weight functions. Apart from this, the results obtained here are new for linear case ($m = 2$).

In this article, using the Nehari manifold and fibering map analysis, we establish the existence of at least two nontrivial solutions for a polyharmonic system involving critical nonlinearities with sign-changing weight functions with respect to the pair of parameters λ, μ belongs to a suitable subset of $\mathbb{R}^2$. Since the embedding $H_0^m(\Omega) \hookrightarrow L^{2_m^*}(\Omega)$ is not compact, so the corresponding energy functional does not satisfy the Palais-Smale condition in general. Therefore, it is difficult to obtain the critical points of energy functional by simple arguments, which are based on the compactness of the Sobolev embedding. To overcome this difficulty, we extract a Palais-Smale sequence in the Nehari manifold and show that the weak limit of this sequence is the required solution of problem (1.1).

To state our main results, we introduce

$$\Lambda_1 := \Big(\frac{2-r}{(\beta+\gamma-r)|h|_\infty}\Big)^{\frac{2}{\beta+\gamma-2}} \Big(\frac{\beta+\gamma-r}{\beta+\gamma-2}\Big)^{-\frac{2}{2-r}} S^{\frac{2(\beta+\gamma-r)}{(2-r)(\beta+\gamma-2)}} > 0, \tag{1.4}$$

where S is the best constant that will be introduced in next section. Then we obtain the following existence results.

Theorem 1.1. *Assume that* (A1), (A2) *hold. If* $1 \leq r < 2 < \frac{N}{m}$, $2 < \beta+\gamma \leq 2_m^*$, *and* λ, $\mu > 0$ *satisfy* $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, *then system* (1.1) *has at least one nontrivial solution in* $H_0^m(\Omega) \times H_0^m(\Omega)$.

Theorem 1.2 (Second nontrivial solution in subcritical case)**.** *Assume that* (A1), (A2) *hold. If* $1 \leq r < 2 < \frac{N}{m}$, $2 < \beta+\gamma < 2_m^*$, *and* λ, $\mu > 0$ *satisfy* $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, *then system* (1.1) *has at least two nontrivial solution in* $H_0^m(\Omega) \times H_0^m(\Omega)$.

To obtain the second nontrivial in critical case $\beta+\gamma = 2_m^*$, we need the following extra assumptions on f, g and h:

(A3) There exist a_0, b_0 and $r_0 > 0$ such that $B(x_0, 2r_0) \subset \Omega$ and $f(x) \geq a_0$, $g(x) \geq b_0$ for all $x \in B(0, 2r_0)$;

(A4) there exists $\delta_0 > 0$ such that $|h|_\infty = h(0) = \max_{x\in\overline{\Omega}} h(x)$, $h(x) > 0$ for all $x \in B(0, 2r_0)$ and

$$h(x) = h(0) + o(|x|^{\delta_0}) \quad \text{as } x \to 0.$$

Theorem 1.3 (Second nontrivial solution in critical case)**.** *Assume that* (A1)–(A4) *hold. If* $1 \le r < 2 < N/m$, *and* λ, $\mu > 0$ *satisfy* $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, *then system* (1.1) *has at least two nontrivial solution in* $H_0^m(\Omega)\times H_0^m(\Omega)$.

The article is organized as follows: In section 2, variational setting of problem (1.1) and some preliminary results are introduced. In section 3, we show that the Palais-Smale condition holds for the energy functional associated with (1.1) at energy level in a suitable range related to the best Sobolev constant. Some results about the Nehari manifold and fibering map analysis are discussed in section 4. In section 5, we prove the existence of Palais-Smale sequences and proof of Theorems 1.1 and 1.2. We give the detail of proof of Theorem 1.3 in section 6.

Notation.

- $L^p(\Omega)$, $1 \le p < \infty$, denote Lebesgue spaces; the norm L^p is denoted by $|\cdot|_p$;
- $Q_{\lambda,\mu}(u, v) = \int_\Omega (\lambda f(x)|u|^r + \mu g(x)|v|^r)dx$;
- $B(x_0, r) = B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$ is the ball in $\mathbb{R}^N$;
- $O(\epsilon^t)$ denotes $|O(\epsilon^t)/\epsilon^t| \le C$ as $\epsilon \to 0$ for $t \ge 0$;
- $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$;
- $O_1(\epsilon^t)$ denotes that there exist the constants $C_1, C_2 > 0$ such that $C_1\epsilon^t \le O_1(\epsilon^t) \le C_2\epsilon^t$ as ϵ small enough. C, C_i's are positive constants.

2. Preliminaries

In this section, we firstly define the function space corresponding to problem (1.1), posed in framework of Sobolev space $\mathcal{H} := H_0^m(\Omega) \times H_0^m(\Omega)$ with standard norm

$$\|(u, v)\| = (\|D^m u\|^2 + \|D^m v\|^2)^{1/2},$$

where

$$\|D^m u\|^2 = \begin{cases} \|(-\Delta)^{\frac{m}{2}} u\|^2 & \text{if } m = 2j, j = 1, 2, \dots, \\ \|\nabla(-\Delta)^{\frac{m-1}{2}} u\|^2 & \text{if } m = 2j-1, j = 1, 2, \dots. \end{cases}$$

Then $\mathcal{H}$ is a Hilbert space.

Definition 2.1. A pair of functions $(u, v) \in \mathcal{H}$ is said to be a weak solution of (1.1) if for all $(\phi_1, \phi_2) \in \mathcal{H}$,

(i) when m is even,

$$\begin{aligned} &\int_\Omega (-\Delta)^{\frac{m}{2}} u(-\Delta)^{\frac{m}{2}} \phi_1 + \int_\Omega (-\Delta)^{\frac{m}{2}} v(-\Delta)^{\frac{m}{2}} \phi_2 - \lambda \int_\Omega f(x)|u|^{r-2}u\phi_1 \\ &- \mu \int_\Omega g(x)|v|^{r-2}v\phi_2 - \frac{\beta}{\beta+\gamma} \int_\Omega h(x)|u|^{\beta-2}u|v|^\gamma \phi_1 \\ &- \frac{\gamma}{\beta+\gamma} \int_\Omega h(x)|u|^\beta|v|^{\gamma-2}v\phi_2 = 0\,; \end{aligned}$$

(ii) when m is odd,

$$\begin{aligned}&\int_\Omega \nabla(-\Delta)^{\frac{m-1}{2}}u\cdot\nabla(-\Delta)^{\frac{m-1}{2}}\phi_1+\int_\Omega \nabla(-\Delta)^{\frac{m-1}{2}}u\cdot\nabla(-\Delta)^{\frac{m-1}{2}}\phi_2\\&-\lambda\int_\Omega f(x)|u|^{r-2}u\phi_1-\mu\int_\Omega g(x)|v|^{r-2}v\phi_2-\frac{\beta}{\beta+\gamma}\int_\Omega h(x)|u|^{\beta-2}u|v|^\gamma\phi_1\\&-\frac{\gamma}{\beta+\gamma}\int_\Omega h(x)|u|^\beta|v|^{\gamma-2}v\phi_2=0.\end{aligned}$$

Now, we define the energy functional $I_{\lambda,\mu}:\mathcal{H}\to\mathbb{R}$ associated with problem (1.1) as

$$I_{\lambda,\mu}(u,v)=\frac{1}{2}\|(u,v)\|^2-\frac{1}{r}\int_\Omega(\lambda f(x)|u|^r+\mu g(x)|v|^r)dx-\frac{1}{\beta+\gamma}\int_\Omega h(x)|u|^\beta|v|^\gamma dx.$$

Then $I_{\lambda,\mu}$ is well defined in $\mathcal{H}$ and $I_{\lambda,\mu}\in C^1(\mathcal{H},\mathbb{R})$. Moreover, the critical points of the functional $I_{\lambda,\mu}$ are the weak solutions of (1.1).

Further, we will prove a lemma which will be used to prove the second solution in critical case. For this, let S be the best Sobolev constant defined as

$$S:=\inf_{u\in H_0^m(\Omega)\setminus\{0\}}\frac{\|D^m u\|^2}{(\int_\Omega|u|^{\beta+\gamma})^{\frac{2}{\beta+\gamma}}}\,, \tag{2.1}$$

where $\beta+\gamma=2_m^*$. Then it is well known that S is achieved if and only if $\Omega=\mathbb{R}^N$, by the function

$$U(x)=\frac{C_{N,m}^{\frac{N-2m}{4m}}}{(1+|x|^2)^{\frac{N-2m}{2}}}$$

(see[33]). Moreover, all the minimizers of S are obtained by

$$U_\epsilon(x)=\epsilon^{\frac{2m-N}{2}}U(\frac{x}{\epsilon})=\frac{C_{N,m}^{\frac{N-2m}{4m}}\epsilon^{\frac{N-2m}{2}}}{(\epsilon^2+|x|^2)^{\frac{N-2m}{2}}},\quad\text{where }\epsilon>0. \tag{2.2}$$

The normalizing constant $C_{N,m}:=C(N,m)=\prod_{j=1-m}^m(N-2j)$ and is chosen in such a way that $U_\epsilon(x)$ solves the equation

$$(-\Delta)^m u=|u|^{2_m^*-2}u\quad\text{in }\mathbb{R}^N,$$

and satisfies

$$\|U_\epsilon(x)\|^2=|U_\epsilon(x)|_{2_m^*}^{2_m^*}=S^{\frac{N}{2m}}.$$

Now, consider the minimization problem

$$S_{\beta,\gamma}=\inf_{(u,v)\in\mathcal{H}\setminus\{(0,0)\}}\frac{\|D^m u\|^2+\|D^m v\|^2}{(\int_\Omega|u|^\beta|v|^\gamma dx)^{\frac{2}{\beta+\gamma}}}. \tag{2.3}$$

Then we establish the following relationship between $S_{\beta,\gamma}$ and S, using an idea from [2].

Lemma 2.2. *For the constants $S_{\beta,\gamma}$ and S given in* (2.1) *and* (2.3), *it holds*

$$S_{\beta,\gamma}=\Big[(\frac{\beta}{\gamma})^{\frac{\gamma}{\beta+\gamma}}+(\frac{\gamma}{\beta})^{\frac{\beta}{\beta+\gamma}}\Big]S. \tag{2.4}$$

In particular, the constant $S_{\beta,\gamma}$ is achieved for $\Omega=\mathbb{R}^N$.

Proof. Let $\{w_n\} \subset H_0^m(\Omega)$ be a minimizing sequence for S. Then take the sequences $u_n = sw_n$ and $v_n = tw_n$ in $H_0^m(\Omega)$, where $s,\, t > 0$. By definition of $S_{\beta,\gamma}$, we have

$$S_{\beta,\gamma} \le \frac{\|(u_n,v_n)\|^2}{(\int_\Omega |u_n|^\beta |v_n|^\gamma dx)^{\frac{2}{\beta+\gamma}}}.$$

Therefore

$$S_{\beta,\gamma} \le \frac{(s^2+t^2)S}{s^{\frac{2\beta}{\beta+\gamma}} t^{\frac{2\gamma}{\beta+\gamma}}} = \Big[(\frac{s}{t})^{\frac{2\gamma}{\beta+\gamma}} + (\frac{t}{s})^{\frac{2\beta}{\beta+\gamma}}\Big] S.$$

Now, define a function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Upsilon(x) = x^{\frac{2\gamma}{\beta+\gamma}} + x^{\frac{-2\beta}{\beta+\gamma}}$. Then $\Upsilon(\frac{s}{t}) = (\frac{s}{t})^{\frac{2\gamma}{\beta+\gamma}} + (\frac{t}{s})^{\frac{2\beta}{\beta+\gamma}}$ and Υ attains its minimum at $x_0 = \big(\frac{\beta}{\gamma}\big)^{\frac{1}{2}}$. So, we have

$$\min_{x\in\mathbb{R}^+} \Upsilon(x) = \Upsilon(x_0) = \big(\frac{\beta}{\gamma}\big)^{\frac{\gamma}{\beta+\gamma}} + \big(\frac{\gamma}{\beta}\big)^{\frac{\beta}{\beta+\gamma}}.$$

Choosing s, t such that $\frac{s}{t} = \big(\frac{\beta}{\gamma}\big)^{\frac{1}{2}}$ and letting $n \to \infty$ yields

$$S_{\beta,\gamma} \le \Big[\big(\frac{\beta}{\gamma}\big)^{\frac{\gamma}{\beta+\gamma}} + \big(\frac{\gamma}{\beta}\big)^{\frac{\beta}{\beta+\gamma}}\Big] S. \tag{2.5}$$

On the other hand, let $\{(u_n, v_n)\}$ be a minimizing sequence for $S_{\beta,\gamma}$. Define $a_n = s_n v_n$ for some $s_n > 0$ such that $\int_\Omega |u_n|^{\beta+\gamma} dx = \int_\Omega |a_n|^{\beta+\gamma} dx$. Then Young's inequality implies that

$$\begin{aligned}\int_\Omega |u_n|^\beta |a_n|^\gamma dx &\le \frac{\beta}{\beta+\gamma}\int_\Omega |u_n|^{\beta+\gamma} dx + \frac{\gamma}{\beta+\gamma}\int_\Omega |a_n|^{\beta+\gamma} dx \\ &= \int_\Omega |a_n|^{\beta+\gamma} dx = \int_\Omega |u_n|^{\beta+\gamma} dx.\end{aligned}$$

Thus, using this we obtain

$$\begin{aligned}\frac{\|(u_n,v_n)\|^2}{(\int_\Omega |u_n|^\beta |v_n|^\gamma dx)^{\frac{2}{\beta+\gamma}}} &= s_n^{\frac{2\gamma}{\beta+\gamma}}\Big[\frac{\|D^m u_n\|^2}{(\int_\Omega |u_n|^\beta |a_n|^\gamma dx)^{\frac{2}{\beta+\gamma}}} + \frac{\|D^m v_n\|^2}{(\int_\Omega |u_n|^\beta |a_n|^\gamma dx)^{\frac{2}{\beta+\gamma}}}\Big] \\ &\ge s_n^{\frac{2\gamma}{\beta+\gamma}} \frac{\|D^m u_n\|^2}{(\int_\Omega |u_n|^{\beta+\gamma} dx)^{\frac{2}{\beta+\gamma}}} + s_n^{\frac{2\gamma}{\beta+\gamma}-2} \frac{\|D^m a_n\|^2}{(\int_\Omega |a_n|^{\beta+\gamma} dx)^{\frac{2}{\beta+\gamma}}} \\ &\ge \Big(s_n^{\frac{2\gamma}{\beta+\gamma}} + s_n^{\frac{2\gamma}{\beta+\gamma}-2}\Big) S \ge \Upsilon(x_0) S.\end{aligned}$$

On passing to the limit as $n \to \infty$, we obtain

$$S_{\beta,\gamma} \ge \Big[\big(\frac{\beta}{\gamma}\big)^{\frac{\gamma}{\beta+\gamma}} + \big(\frac{\gamma}{\beta}\big)^{\frac{\beta}{\beta+\gamma}}\Big] S. \tag{2.6}$$

Hence, from (2.5) and (2.6), we obtain the required result. □

Definition 2.3. Let $J : X \to \mathbb{R}$ be a C^1 functional on a Banach space X.

- For $c \in \mathbb{R}$, a sequence $\{u_k\} \subset X$ is a Palais-Smale sequence at level c ($(PS)_c$) in X for J if $J(u_k) = c + o_k(1)$ and $J'(u_k) \to 0$ in X^{-1} as $k \to \infty$.
- We say J satisfies $(PS)_c$ condition if for any Palais-Smale sequence $\{u_k\}$ in X for J has a convergent subsequence.

3. The Palais-Smale condition

Lemma 3.1. *Suppose that $\{(u_n,v_n)\}\subset\mathcal{H}$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$ such that $(u_n,v_n)\rightharpoonup(u,v)$ weakly in $\mathcal{H}$. Then $I'_{\lambda,\mu}(u,v)=0$ and there exists a positive constant P_0 depending on m, N, r and S such that*

$$I_{\lambda,\mu}(u,v)\geq -P_0((\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}).$$

Proof. Let $\{(u_n,v_n)\}$ be a $(PS)_c$-sequence in $\mathcal{H}$, then by using the standard argument, one can easily obtain $I'_{\lambda,\mu}(u,v)=0$, i.e. $\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle=0$. Using this, Hölder's and Young's inequalities, we obtain

$$\begin{aligned}
I_{\lambda,\mu}(u,v)&=\Big(\frac{1}{2}-\frac{1}{\beta+\gamma}\Big)\|(u,v)\|^2-\Big(\frac{1}{r}-\frac{1}{\beta+\gamma}\Big)\int_\Omega(\lambda f(x)|u|^r+\mu g(x)|v|^r)dx\\
&\geq\frac{m}{N}\|(u,v)\|^2-\frac{(\beta+\gamma-r)}{r(\beta+\gamma)}S^{-r/2}\\
&\quad\times\Big[\omega^{\frac{2}{2-r}}\big(\frac{2-r}{2}\big)\Big((\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}\Big)+\frac{\omega^{-2/r}}{2}\|(u,v)\|^2\Big]\\
&=\frac{m}{N}\|(u,v)\|^2-\frac{m}{N}\|(u,v)\|^2-P_0\Big(\big((\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}\big)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}\Big),
\end{aligned}$$

where

$$P_0=\frac{(\beta+\gamma-r)(2-r)}{2r(\beta+\gamma)}S^{-r/2}\omega^{\frac{2}{2-r}},\quad \omega=\Big(\frac{N(\beta+\gamma-r)}{2m(\beta+\gamma)}S^{-r/2}\Big)^{r/2}.$$

This completes the proof. □

Lemma 3.2. *If $\{(u_n,v_n)\}\subset\mathcal{H}$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$, then $\{(u_n,v_n)\}$ is bounded in $\mathcal{H}$.*

Proof. Let $\{(u_n,v_n)\}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ in $\mathcal{H}$, then we assume by contradiction that $\|(u_n,v_n)\|\to\infty$ as $n\to\infty$. Define

$$(\widehat{u}_n,\widehat{v}_n):=\frac{(u_n,v_n)}{\|(u_n,v_n)\|}=\Big(\frac{u_n}{\|(u_n,v_n)\|},\frac{v_n}{\|(u_n,v_n)\|}\Big).$$

Then $\{(\widehat{u}_n,\widehat{v}_n)\}$ is a bounded sequence. So, up to a subsequence $(\widehat{u}_n,\widehat{v}_n)\rightharpoonup(\widehat{u},\widehat{v})$ weakly in $\mathcal{H}$. This implies that $\widehat{u}_n\to\widehat{u}$, $\widehat{v}_n\to\widehat{v}$ strongly in $L^s(\Omega)$ for all $1\leq s<2^*_m$ and

$$Q_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n)=Q_{\lambda,\mu}(\widehat{u},\widehat{v})+o_n(1). \tag{3.1}$$

Since $\{(u_n,v_n)\}$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$ and $\|(u_n,v_n)\|\to\infty$ as $n\to\infty$, we obtain

$$\begin{gathered}
\frac{1}{2}\|(\widehat{u}_n,\widehat{v}_n)\|^2-\frac{\|(u_n,v_n)\|^{r-2}}{r}Q_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n)\\
-\frac{\|(u_n,v_n)\|^{\beta+\gamma-2}}{\beta+\gamma}\int_\Omega h(x)|\widehat{u}_n|^\beta|\widehat{v}_n|^\gamma dx=o_n(1),
\end{gathered}$$

and

$$\begin{gathered}
\|(\widehat{u}_n,\widehat{v}_n)\|^2-\|(u_n,v_n)\|^{r-2}Q_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n)\\
-\|(u_n,v_n)\|^{\beta+\gamma-2}\int_\Omega h(x)|\widehat{u}_n|^\beta|\widehat{v}_n|^\gamma dx=o_n(1).
\end{gathered}\tag{3.2}$$

From (3.1) and (3.2), we can deduce that

$$\|(\widehat{u}_n,\widehat{v}_n)\|^2=\frac{2(\beta+\gamma-r)}{r(\beta+\gamma-2)}\|(u_n,v_n)\|^{r-2}Q_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n)+o_n(1). \tag{3.3}$$

Since $1 \le r < 2$ and $\|(u_n, v_n)\| \to \infty$, then (3.3) implies $\|(\widehat{u}_n, \widehat{v}_n)\|^2 \to 0$ as $n \to \infty$, which is a contradiction to the fact that $\|(\widehat{u}_n, \widehat{v}_n)\| = 1$. □

Lemma 3.3. *$I_{\lambda,\mu}$ satisfies the $(PS)_c$-condition with c satisfying $c \in (0, c_\infty)$, where*

$$c_\infty = \frac{m}{N} S_{\beta,\gamma}^{\frac{N}{2m}} |h|_\infty^{-\frac{N-2m}{2m}} - P_0\big((\lambda |f|_\alpha)^{\frac{2}{2-r}} + (\mu |g|_\alpha)^{\frac{2}{2-r}}\big),$$

and P_0 is given in Lemma 3.1.

Proof. Let $\{(u_n, v_n)\} \subset \mathcal{H}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ with $0 < c < c_\infty$. Then by Lemma 3.2, $\{(u_n, v_n)\}$ is a bounded sequence in $\mathcal{H}$. Hence, up to a subsequence, $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$. So $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $H_0^m(\Omega)$, $u_n \to u$ and $v_n \to v$ strongly in $L^s(\Omega)$ for all $1 \le s < 2_m^*$ and $u_n \to u$, $v_n \to v$ pointwise a.e. in Ω. Thus

$$\int_\Omega (\lambda f(x)|u_n|^r + \mu g(x)|v_n|^r) dx = \int_\Omega (\lambda f(x)|u|^r + \mu g(x)|v|^r) dx + o_n(1). \tag{3.4}$$

Also, $I'_{\lambda,\mu}(u, v) = 0$, follows from Lemma 3.1. Now, define $(\tilde{u}_n, \tilde{v}_n)$, where $\tilde{u}_n = u_n - u$, $\tilde{v}_n = v_n - v$. Then by Brézis-Lieb Lemma [7] and Vitali theorem, we have

$$\|(\tilde{u}_n, \tilde{v}_n)\|^2 = \|(u_n, v_n)\|^2 - \|(u, v)\|^2 + o_n(1), \tag{3.5}$$

$$\int_\Omega h(x)|\tilde{u}_n|^\beta |\tilde{v}_n|^\gamma dx = \int_\Omega h(x)|u_n|^\beta |v_n|^\gamma dx - \int_\Omega h(x)|u|^\beta |v|^\gamma dx + o_n(1). \tag{3.6}$$

Using $I_{\lambda,\mu}(u_n, v_n) = c + o_n(1)$, $I'_{\lambda,\mu}(u_n, v_n) = o_n(1)$, (3.4) and (3.6), we obtain

$$\frac{1}{2}\|(\tilde{u}_n, \tilde{v}_n)\|^2 - \frac{1}{\beta+\gamma} \int_\Omega h(x)|\tilde{u}_n|^\beta |\tilde{v}_n|^\gamma dx = c - I_{\lambda,\mu}(u, v) + o_n(1), \tag{3.7}$$

and

$$\|(\tilde{u}_n, \tilde{v}_n)\|^2 - \int_\Omega h(x)|\tilde{u}_n|^\beta |\tilde{v}_n|^\gamma dx = \langle I'_{\lambda,\mu}(u, v), (u_n - u, v_n - v)\rangle + o_n(1) = o_n(1).$$

Therefore, we assume that

$$\|(\tilde{u}_n, \tilde{v}_n)\|^2 \to l, \quad \int_\Omega h(x)|\tilde{u}_n|^\beta |\tilde{v}_n|^\gamma dx \to l. \tag{3.8}$$

If $l = 0$, then proof is complete. If $l > 0$, then by definition of $S_{\beta,\gamma}$ and (3.8), we obtain

$$\begin{aligned} S_{\beta,\gamma} l^{\frac{2}{\beta+\gamma}} &\le S_{\beta,\gamma} \lim_{n\to\infty} \Big(|h|_\infty \int_\Omega |u_n|^\beta |v_n|^\gamma dx\Big)^{2/2_m^*} \\ &\le |h|_\infty^{\frac{2}{\beta+\gamma}} \lim_{n\to\infty} \|(\tilde{u}_n, \tilde{v}_n)\|^2 = |h|_\infty^{\frac{2}{\beta+\gamma}} l. \end{aligned}$$

As $\beta + \gamma = 2_m^*$, so the above relation gives

$$l \ge S_{\beta,\gamma}^{\frac{N}{2m}} |h|_\infty^{-\frac{(N-2m)}{2m}}.$$

Now, by (3.7), (3.8) and Lemma 3.1, we obtain

$$\begin{aligned} c &= \big(\frac{1}{2} - \frac{1}{\beta+\gamma}\big) l + I_{\lambda,\mu}(u, v) \\ &\ge \frac{m}{N} S_{\beta,\gamma}^{\frac{N}{2m}} |h|_\infty^{-\frac{N-2m}{2m}} - P_0((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}}) = c_\infty, \end{aligned}$$

which is a contradiction to $c < c_\infty$. The proof is complete. □

4. Nehari manifold for (1.1)

Since the energy functional $I_{\lambda,\mu}$ is not bounded below on $\mathcal{H}$, it is appropriate to consider the functional on the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \{(u,v) \in \mathcal{H} \setminus \{(0,0)\} : \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\}.$$

Thus, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = \|(u,v)\|^2 - Q_{\lambda,\mu}(u,v) - \int_\Omega h(x)|u|^\beta |v|^\gamma dx = 0. \tag{4.1}$$

It is easy to see that $\mathcal{N}_{\lambda,\mu}$ contains every nonzero solution of (1.1). In fact, we will show later that local minimizers of $\mathcal{N}_{\lambda,\mu}$ are the critical points of $I_{\lambda,\mu}$.

Lemma 4.1. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Let $(u,v) \in \mathcal{N}_{\lambda,\mu}$, then by (4.1), Hölder inequality and Sobolev embedding theorem, we have

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \frac{\beta+\gamma-2}{2(\beta+\gamma)}\|(u,v)\|^2 - \frac{\beta+\gamma-r}{r(\beta+\gamma)}Q_{\lambda,\mu}(u,v) \\ &\geq \frac{\beta+\gamma-2}{2(\beta+\gamma)}\|(u,v)\|^2 \\ &\quad - \frac{\beta+\gamma-r}{r(\beta+\gamma)}S^{-r/2}\Big((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}}\Big)^{\frac{2-r}{2}}\|(u,v)\|^r. \end{aligned} \tag{4.2}$$

Since $1 < r < 2$. Thus, $I_{\lambda,\mu}$ is coercive.

Now, consider the function $\rho : \mathbb{R} \to \mathbb{R}$ as $\rho(t) = at^2 - bt^r$. Then one can easily see that $\rho'(t) = 0$ if and only if $t = (\frac{br}{2a})^{\frac{1}{2-r}} := t^*$ and $\rho''(t^*) > 0$. So ρ attains its minimum at t^*. Moreover,

$$\rho(t) \geq \rho(t^*) = -(2-r)(\frac{b}{2})^{\frac{2}{2-r}}(\frac{r}{a})^{\frac{r}{2-r}}.$$

Taking

$$a = \frac{\beta+\gamma-2}{2(\beta+\gamma)}, \quad b = \frac{\beta+\gamma-r}{r(\beta+\gamma)}S^{-r/2}((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}})^{\frac{2-r}{2}}, \quad t = \|(u,v)\|$$

in the function ρ, we obtain

$$I_{\lambda,\mu}(u,v) \geq \rho(\|(u,v)\|) \geq \rho(t^*).$$

Hence, $I_{\lambda,\mu}$ is bounded below on $\mathcal{N}_{\lambda,\mu}$. □

The Nehari manifold is closely related to the fibering map introduced by Drábek and Pohozaev in [14]. For each (u,v), we define $\Psi_{(u,v)} : t \to I_{\lambda,\mu}(tu,tv)$ given by

$$\begin{gathered} \Psi_{(u,v)}(t) = I_{\lambda,\mu}(tu,tv) = \frac{t^2}{2}\|(u,v)\|^2 - \frac{t^r}{r}Q_{\lambda,\mu}(u,v) - \frac{t^{\beta+\gamma}}{\beta+\gamma}\int_\Omega h(x)|u|^\beta|v|^\gamma dx, \\ \Psi'_{(u,v)}(t) = t\|(u,v)\|^2 - t^{r-1}Q_{\lambda,\mu}(u,v) - t^{\beta+\gamma-1}\int_\Omega h(x)|u|^\beta|v|^\gamma dx, \\ \Psi''_{(u,v)}(t) = \|(u,v)\|^2 - (r-1)t^{r-2}Q_{\lambda,\mu}(u,v) - (\beta+\gamma-1)t^{\beta+\gamma-2}\int_\Omega h(x)|u|^\beta|v|^\gamma dx. \end{gathered}$$

It is observed that $\Psi'_{(u,v)}(t) = 0$ if and only if $(tu, tv) \in \mathcal{N}_{\lambda,\mu}$. Thus $(u, v) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\Psi'_{(u,v)}(1) = 0$. Therefore it is natural to split $\mathcal{N}_{\lambda,\mu}$ into three parts corresponding to local minima, local maxima and points of inflexion respectively as

$$\mathcal{N}^{\pm}_{\lambda,\mu} := \{(u, v) \in \mathcal{N}_{\lambda,\mu} : \Psi''_{(u,v)}(1) \gtrless 0\},$$
$$\mathcal{N}^{0}_{\lambda,\mu} := \{(u, v) \in \mathcal{N}_{\lambda,\mu} : \Psi''_{(u,v)}(1) = 0\}.$$

For each $(u, v) \in \mathcal{N}_{\lambda,\mu}$, we have one of the following 3 equalities

$$\Psi''_{(u,v)}(1) = \begin{cases} 2\|(u, v)\|^2 - rQ_{\lambda,\mu}(u, v) - (\beta + \gamma) \int_\Omega h(x)|u|^\beta |v|^\gamma dx\,, \\ (2 - r)\|(u, v)\|^2 - (\beta + \gamma - r) \int_\Omega h(x)|u|^\beta |v|^\gamma dx\,, \\ (\beta + \gamma - r)Q_{\lambda,\mu}(u, v) - (\beta + \gamma - 2)\|(u, v)\|^2. \end{cases} \tag{4.3}$$

Lemma 4.2. *If (u_0, v_0) is the local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ and $(u_0, v_0) \notin \mathcal{N}^0_{\lambda,\mu}$. Then $I'_{\lambda,\mu}((u_0, v_0)) = 0$ in $\mathcal{H}^{-1}$, where $\mathcal{H}^{-1}$ denotes the dual space of $\mathcal{H}$.*

Proof. If (u_0, v_0) is a local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, then (u_0, v_0) is a solution of the problem: minimize $I_{\lambda,\mu}(u, v)$ subject to $\Phi_{\lambda,\mu}(u, v) : \langle I'_{\lambda,\mu}(u, v), (u, v)\rangle = 0$. Hence, by Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that $I'_{\lambda,\mu}((u_0, v_0)) = \theta\Phi'_{\lambda,\mu}((u_0, v_0))$. Thus, $\langle I'_{\lambda,\mu}(u_0, v_0), (u_0, v_0)\rangle = \theta\langle\Phi'_{\lambda,\mu}(u_0, v_0), (u_0, v_0)\rangle$.

Since $(u_0, v_0) \in \mathcal{N}_{\lambda,\mu}$, it follows that

$$\langle\Phi'_{\lambda,\mu}(u_0, v_0), (u_0, v_0)\rangle = (2 - r)\|(u_0, v_0)\|^2 - (\beta + \gamma - r) \int_\Omega h(x)|u_0|^\beta |v_0|^\gamma dx \neq 0,$$

as $(u_0, v_0) \notin \mathcal{N}^0_{\lambda,\mu}$. Hence, we have $\theta = 0$. □

Lemma 4.3. *We have the following*

(i) *If $(u, v) \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, then $Q_{\lambda,\mu}(u, v) > 0$.*
(ii) *If $(u, v) \in \mathcal{N}^-_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, then $\int_\Omega h(x)|u|^\beta |v|^\gamma dx > 0$.*

The proof of the above lemma follows directly from (4.3). Now, we show that $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$ are nonempty. For this we define some notations. For each $(u, v) \in \mathcal{H}$ with $\int_\Omega h(x)|u|^\beta |v|^\gamma dx > 0$

$$t_{\max} = \Big(\frac{(2 - r)\|(u, v)\|^2}{(\beta + \gamma - r) \int_\Omega h(x)|u|^\beta |v|^\gamma dx}\Big)^{\frac{1}{\beta+\gamma-2}} > 0,$$

and for $Q_{\lambda,\mu}(u, v) > 0$,

$$\bar{t}_{\max} = \Big(\frac{(\beta + \gamma - r)Q_{\lambda,\mu}(u, v)}{(\beta + \gamma - 2)\|(u, v)\|^2}\Big)^{\frac{1}{2-r}} > 0.$$

Lemma 4.4. *Suppose that $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$ and $(u, v) \in \mathcal{H}$. Then we have the following:*

(i) *If $\int_\Omega h(x)|u|^\beta |v|^\gamma dx > 0$ and $Q_{\lambda,\mu}(u, v) \leq 0$, then there exists a unique $t^- > t_{\max}$ such that $(t^- u, t^- v) \in \mathcal{N}^-_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^- u, t^- v) = \sup_{t\geq t_{\max}} I_{\lambda,\mu}(tu, tv)$.*
(ii) *If $\int_\Omega h(x)|u|^\beta |v|^\gamma dx > 0$ and $Q_{\lambda,\mu}(u, v) > 0$, then there exists a unique $0 < t^+ < t_{\max} < t^-$ such that $(t^+ u, t^+ v) \in \mathcal{N}^+_{\lambda,\mu}$, $(t^- u, t^- v) \in \mathcal{N}^-_{\lambda,\mu}$. Moreover,*

$$I_{\lambda,\mu}(t^+ u, t^+ v) = \inf_{0\leq t\leq t_{max}} I_{\lambda,\mu}(tu, tv); \quad I_{\lambda,\mu}(t^- u, t^- v) = \sup_{t\geq t_{\max}} I_{\lambda,\mu}(tu, tv).$$

(iii) *If* $Q_{\lambda,\mu}(u,v) > 0$ *and* $\int_\Omega h(x)|u|^\beta |v|^\gamma \le 0$, *then there exists a unique* $0 < t^+ < \bar{t}_{\max}$ *such that* $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$ *and* $I_{\lambda,\mu}(t^+u,t^+v) = \inf_{t\ge 0} I_{\lambda,\mu}(tu,tv)$.

(iv) *If* $Q_{\lambda,\mu}(u,v) < 0$ *and* $\int_\Omega h(x)|u|^\beta|v|^\gamma dx < 0$, *then there does not exist any critical point.*

Proof. For $(u,v) \in \mathcal{H}$ with $\int_\Omega h(x)|u|^\beta|v|^\gamma dx > 0$. Define

$$\xi_{(u,v)}(t) = t^{2-r}\|(u,v)\|^2 - t^{\beta+\gamma-r}\int_\Omega h(x)|u|^\beta|v|^\gamma dx, \quad \text{for } t > 0.$$

We have $\xi_{(u,v)}(0) = 0$, $\xi_{(u,v)}(t) \to -\infty$ as $t \to \infty$. Since

$$\xi'_{(u,v)}(t) = (2-r)t^{1-r}\|(u,v)\|^2 - (\beta+\gamma-r)t^{\beta+\gamma-r-1}\int_\Omega h(x)|u|^\beta|v|^\gamma dx,$$

we obtain $\xi'_{(u,v)}(t) = 0$ at $t = t_{\max}$, $\xi'_{(u,v)}(t) > 0$ for $t \in [0, t_{\max})$ and $\xi'_{(u,v)}(t) < 0$ for $t \in (t_{\max}, \infty)$. So $\xi_{(u,v)}(t)$ attains its maximum at $t_{\max}$. $\xi_{(u,v)}(t)$ is increasing function for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Moreover,

$$\begin{aligned}
\xi_{(u,v)}(t_{\max}) &= \Big(\frac{(2-r)\|(u,v)\|^2}{(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx}\Big)^{\frac{2-r}{\beta+\gamma-2}}\|(u,v)\|^2 \\
&\quad - \Big(\frac{(2-r)\|(u,v)\|^2}{(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx}\Big)^{\frac{\beta+\gamma-r}{\beta+\gamma-2}}\int_\Omega h(x)|u|^\beta|v|^\gamma dx \\
&= \|(u,v)\|^r\big(\frac{2-r}{\beta+\gamma-r}\big)^{\frac{2-r}{\beta+\gamma-2}}\Big(\frac{\beta+\gamma-2}{\beta+\gamma-r}\Big)\Big(\frac{\|(u,v)\|^{\beta+\gamma}}{\int_\Omega h(x)|u|^\beta|v|^\gamma dx}\Big)^{\frac{2-r}{\beta+\gamma-2}} \\
&\ge \|(u,v)\|^r\big(\frac{2-r}{\beta+\gamma-r}\big)^{\frac{2-r}{\beta+\gamma-2}}\Big(\frac{\beta+\gamma-2}{\beta+\gamma-r}\Big)\Big(\frac{S^{\frac{\beta+\gamma}{2}}}{|h|_\infty}\Big)^{\frac{2-r}{\beta+\gamma-2}}.
\end{aligned}$$

(i) If $\int_\Omega h(x)|u|^\beta|v|^\gamma dx > 0$ and $Q_{\lambda,\mu}(u,v) \le 0$, there is a unique $t^- > t_{\max} > 0$ such that $\xi_{(u,v)}(t^-) = Q_{\lambda,\mu}(u,v) \le 0$ and $\xi'_{(u,v)}(t^-) < 0$.

$$\begin{aligned}
&\langle I'_{\lambda,\mu}(t^-u,t^-v),(t^-u,t^-v)\rangle \\
&= (t^-)^2\|(u,v)\|^2 - (t^-)^r Q_{\lambda,\mu}(u,v) - (t^-)^{\beta+\gamma}\int_\Omega h(x)|u|^\beta|v|^\gamma dx \\
&= (t^-)^r(\xi_{(u,v)}(t^-) - Q_{\lambda,\mu}(u,v)) = 0.
\end{aligned}$$

Therefore, $(t^-u,t^-v) \in \mathcal{N}_{\lambda,\mu}$.

$$\begin{aligned}
\Psi''_{(u,v)}(t^-) &= (2-r)(t^-)^2\|(u,v)\|^2 - (\beta+\gamma-r)(t^-)^{\beta+\gamma}\int_\Omega h(x)|u|^\beta|v|^\gamma dx \\
&= (t^-)^{1+r}\xi'_{(u,v)}(t^-) < 0.
\end{aligned}$$

Hence, $(t^-u,t^-v) \in \mathcal{N}^-_{\lambda,\mu}$. Since for $t > t_{\max}$, we have

$$\Psi''_{(u,v)}(t) = (2-r)t^2\|(u,v)\|^2 - (\beta+\gamma-r)t^{\beta+\gamma}\int_\Omega h(x)|u|^\beta|v|^\gamma dx = t^{1+r}\xi'_{(u,v)}(t) < 0.$$

$$\frac{d^2}{dt^2}I_{\lambda,\mu}(tu,tv) = (r-1)t^{r-2}[\xi_{(u,v)}(t) - Q_{\lambda,\mu}(u,v)] + t^{r-1}\xi'_{(u,v)}(t) < 0, \quad \text{when } t = t^-,$$

$$\frac{d}{dt}I_{\lambda,\mu}(tu,tv) = t^{r-1}\left[\xi_{(u,v)}(t) - Q_{\lambda,\mu}(u,v)\right] = 0, \text{ when } t = t^-.$$

Thus,

$$I_{\lambda,\mu}(t^-u,t^-v) = \sup_{t \ge t_{\max}} I_{\lambda,\mu}(tu,tv).$$

(ii) If $\int_\Omega h(x)|u|^\beta|v|^\gamma dx > 0$ and $Q_{\lambda,\mu}(u,v) > 0$, then by (4.6),

$$\begin{aligned}
\xi_{(u,v)}(0) &= 0 < Q_{\lambda,\mu}(u,v) \\
&\le S^{-r/2}\big((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}}\big)^{\frac{2-r}{2}}\|(u,v)\|^r \\
&< \Big(\frac{2-r}{\beta+\gamma-r}\Big)^{\frac{2-r}{\beta+\gamma-2}}\Big(\frac{\beta+\gamma-2}{\beta+\gamma-r}\Big)\Big(\frac{S^{\frac{\beta+\gamma}{2}}}{|h|_\infty}\Big)^{\frac{2-r}{\beta+\gamma-2}}\|(u,v)\|^r \\
&\le \xi_{(u,v)}(t_{\max}),
\end{aligned}$$

for $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$. There are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$ with

$$\xi_{(u,v)}(t^+) = Q_{\lambda,\mu}(u,v) = \xi_{(u,v)}(t^-) \quad \xi'_{(u,v)}(t^+) > 0 > \xi'_{(u,v)}(t^-).$$

This implies $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$, $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$ and

$$\begin{gathered}
\frac{d}{dt}I_{\lambda,\mu}(tu,tv) = 0, \quad \text{when } t = t^+ \text{ and } t = t^-, \\
\frac{d^2}{dt^2}I_{\lambda,\mu}(tu,tv) > 0, \quad \text{when } t \in (0, t_{\max}), \\
\frac{d^2}{dt^2}I_{\lambda,\mu}(tu,tv) < 0, \quad \text{when } t \in (t_{\max}, \infty).
\end{gathered}$$

Thus, we have

$$I_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le t_{\max}} I_{\lambda,\mu}(tu,tv), \quad I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge t_{\max}} I_{\lambda,\mu}(tu,tv).$$

(iii) For $(u,v) \in \mathcal{H}$ with $Q_{\lambda,\mu}(u,v) > 0$ and $\int_\Omega h(x)|u|^\beta|v|^\gamma dx \le 0$, define

$$\overline{\xi}_{(u,v)}(t) = t^{2-\beta-\gamma}\|(u,v)\|^2 - t^{r-\beta-\gamma}Q_{\lambda,\mu}(u,v), \quad \text{for } t > 0.$$

We have $\overline{\xi}_{(u,v)}(t) \to -\infty$ as $t \to 0$, $\overline{\xi}_{(u,v)}(t) \to 0$ as $t \to \infty$. Since

$$\overline{\xi}'_{(u,v)}(t) = (2-\beta-\gamma)t^{1-\beta-\gamma}\|(u,v)\|^2 - (r-\beta-\gamma)t^{r-\beta-\gamma-1}Q_{\lambda,\mu}(u,v),$$

we obtain $\overline{\xi}'_{(u,v)}(t) = 0$ at $t = \overline{t}_{\max}$, $\overline{\xi}'_{(u,v)}(t) > 0$ for $t \in (0, \overline{t}_{\max})$ and $\overline{\xi}'_{(u,v)}(t) < 0$ for $t \in (\overline{t}_{\max}, \infty)$. So $\overline{\xi}_{(u,v)}(t)$ attains its maximum at $\overline{t}_{\max}$. $\overline{\xi}_{(u,v)}(t)$ is increasing function for $t \in (0, \overline{t}_{\max})$ and decreasing for $t \in (\overline{t}_{\max}, \infty)$. Now, using the same argument used in previous parts, there exists a unique $0 < t^+ < \overline{t}_{\max}$ such that $\overline{\xi}_{(u,v)}(t^+) = \int_\Omega h(x)|u|^\beta|v|^\gamma dx \le 0$, $\overline{\xi}'_{(u,v)}(t^+) > 0$. Also, $\langle I'_{\lambda,\mu}(t^+u, t^+v), (t^+u, t^+v)\rangle = 0$. Thus, $(t^+u, t^+v) \in \mathcal{N}_{\lambda,\mu}$. Further $\Psi''_{(u,v)}(t^+) > 0$ so $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$. Since $0 < t^+ < \overline{t}_{\max}$, then $\Psi''_{(u,v)}(t) > 0$. Moreover, for $t = t^+$, $\frac{d^2}{dt^2}I_{\lambda,\mu}(tu,tv) > 0$ and $\frac{d}{dt}I_{\lambda,\mu}(tu,tv) = 0$. Hence

$$I_{\lambda,\mu}(t^+u, t^+v) = \inf_{t\ge 0} I_{\lambda,\mu}(tu,tv).$$

(iv) If $Q_{\lambda,\mu}(u,v) < 0$ and $\int_\Omega h(x)|u|^\beta|v|^\gamma dx < 0$, then $\Psi_{(u,v)}(0) = 0$, $\Psi'_{(u,v)}(t) > 0$ for all $t > 0$. This implies $\Psi_{(u,v)}$ is strictly increasing function and does not have critical point. This completes the proof. □

Lemma 4.5. *If* $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, *then* $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.

Proof. On contrary, assume that there exists $(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}$ with $0<(\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}<\Lambda_1$, such that $\mathcal{N}^0_{\lambda,\mu}\neq\emptyset$. Then for $(u,v)\in\mathcal{N}^0_{\lambda,\mu}$, using (4.3), we obtain

$$\|(u,v)\|^2=\frac{\beta+\gamma-r}{2-r}\int_\Omega h(x)|u|^\beta|v|^\gamma dx,\quad \|(u,v)\|^2=\frac{\beta+\gamma-r}{\beta+\gamma-2}Q_{\lambda,\mu}(u,v). \tag{4.4}$$

Now, by Young's inequality and Sobolev embedding theorem, we have

$$\begin{aligned}\int_\Omega h(x)|u|^\beta|v|^\gamma dx&\le |h|_\infty\Big(\frac{\beta}{\beta+\gamma}\int_\Omega|u|^{\beta+\gamma}dx+\frac{\gamma}{\beta+\gamma}\int_\Omega|v|^{\beta+\gamma}dx\Big)\\&\le |h|_\infty S^{-\frac{\beta+\gamma}{2}}\|(u,v)\|^{\beta+\gamma}.\end{aligned}\tag{4.5}$$

Similarly, by Hölder's inequality and Sobolev embedding theorem, we obtain

$$Q_{\lambda,\mu}(u,v)\le S^{-r/2}\big((\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}\big)^{\frac{2-r}{2}}\|(u,v)\|^r. \tag{4.6}$$

Thus, by (4.4), (4.5) and (4.6), we obtain

$$\|(u,v)\|\ge\Big(\frac{2-r}{\beta+\gamma-r}\frac{S^{\frac{\beta+\gamma}{2}}}{|h|_\infty}\Big)^{\frac{1}{\beta+\gamma-2}} \tag{4.7}$$

and

$$\|(u,v)\|\le\Big(\frac{\beta+\gamma-r}{\beta+\gamma-2}\Big)^{\frac{1}{2-r}}S^{-\frac{r}{2(2-r)}}\big((\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}\big)^{\frac12}. \tag{4.8}$$

On combining (4.7) and (4.8), we have

$$\begin{aligned}&(\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}\\&\ge\Lambda_1:=\Big(\frac{2-r}{(\beta+\gamma-r)|h|_\infty}\Big)^{\frac{2}{\beta+\gamma-2}}\Big(\frac{\beta+\gamma-r}{\beta+\gamma-2}\Big)^{-\frac{2}{2-r}}S^{\frac{2(\beta+\gamma-r)}{(2-r)(\beta+\gamma-2)}},\end{aligned}$$

which is a contradiction. This completes the proof. □

Note that from Lemma 4.5, if $0<(\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}<\Lambda_1$, then $\mathcal{N}_{\lambda,\mu}=\mathcal{N}^+_{\lambda,\mu}\cup\mathcal{N}^-_{\lambda,\mu}$. Now we define

$$\theta_{\lambda,\mu}=\inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}}I_{\lambda,\mu}(u,v),\quad \theta^\pm_{\lambda,\mu}=\inf_{(u,v)\in\mathcal{N}^\pm_{\lambda,\mu}}I_{\lambda,\mu}(u,v).$$

We end this section with the following result.

Theorem 4.6. *The following facts hold:*

- (i) *If $0<(\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}<\Lambda_1$, then $\theta_{\lambda,\mu}\le\theta^+_{\lambda,\mu}<0$.*
- (ii) *If $0<(\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}<(\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, then $\theta^-_{\lambda,\mu}>d_0$, where d_0 is a positive constant depending on λ, μ, r, N, S, $|f|_\alpha$, $|g|_\alpha$ and $|h|_\infty$.*

Proof. (i) Assume $(u,v)\in\mathcal{N}^+_{\lambda,\mu}$. Then by (4.3), we have

$$\frac{2-r}{\beta+\gamma-r}\|(u,v)\|^2>\int_\Omega h(x)|u|^\beta|v|^\gamma dx. \tag{4.9}$$

Using (4.1) and (4.9), we obtain

$$\begin{aligned}I_{\lambda,\mu}(u,v)&=\big(\frac12-\frac1r\big)\|(u,v)\|^2+\big(\frac1r-\frac{1}{\beta+\gamma}\big)\int_\Omega h(x)|u|^\beta|v|^\gamma dx\\&<\big[\big(\frac12-\frac1r\big)+\big(\frac1r-\frac{1}{\beta+\gamma}\big)\frac{2-r}{\beta+\gamma-r}\big]\|(u,v)\|^2\end{aligned}$$

$$= -\frac{(2-r)(\beta+\gamma-2)}{2r(\beta+\gamma)}\|(u,v)\|^2 < 0.$$

So, from the definitions of $\theta_{\lambda,\mu}$, $\theta^+_{\lambda,\mu}$, we can deduce that $\theta_{\lambda,\mu} \le \theta^+_{\lambda,\mu} < 0$.

(ii) Let $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$. Then from (4.3),

$$\frac{2-r}{\beta+\gamma-r}\|(u,v)\|^2 < \int_\Omega h(x)|u|^\beta|v|^\gamma dx. \tag{4.10}$$

Hölder's inequality and Sobolev embedding theorem imply that

$$\|(u,v)\| > \Big(\frac{2-r}{(\beta+\gamma-r)|h|_\infty}\Big)^{\frac{1}{\beta+\gamma-2}} S^{\frac{\beta+\gamma}{2(\beta+\gamma-2)}} \quad \text{for all } (u,v) \in \mathcal{N}^-_{\lambda,\mu}. \tag{4.11}$$

By (4.2) and (4.11), it follows that

$$\begin{aligned}
&I_{\lambda,\mu}(u,v)\\
&\ge \|(u,v)\|^r\Big[\frac{\beta+\gamma-2}{2(\beta+\gamma)}\|(u,v)\|^{2-r} - \frac{\beta+\gamma-r}{r(\beta+\gamma)}S^{-r/2}\big((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}}\big)^{\frac{2-r}{2}}\Big]\\
&> \big(\frac{2-r}{\beta+\gamma-r}\big)^{\frac{r}{\beta+\gamma-2}} S^{\frac{r(\beta+\gamma)}{2(\beta+\gamma-2)}}\Big[\frac{\beta+\gamma-2}{2(\beta+\gamma)}\Big(\frac{2-r}{(\beta+\gamma-r)|h|_\infty}\Big)^{\frac{2-r}{\beta+\gamma-2}} S^{\frac{(2-r)(\beta+\gamma)}{2(\beta+\gamma-2)}}\\
&\quad - \frac{\beta+\gamma-r}{r(\beta+\gamma)}S^{-r/2}((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}})^{\frac{2-r}{2}}\Big].
\end{aligned}$$

Thus, if $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, then $I_{\lambda,\mu}(u,v) > d_0$ for all $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$, for some positive constant $d_0 = d_0(\lambda,\mu,r,N,S,|f|_{L^\alpha},|g|_{L^\alpha},|h|_\infty)$. □

5. Proof of Theorems 1.1 and 1.2

In this section, we show the existence of Palais-Smale sequence in $\mathcal{N}^\pm_{\lambda,\mu}$ and give the proof of Theorems 1.1 and 1.2.

Lemma 5.1. *Suppose $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, where Λ_1 is same as given in (1.4). Then for every $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, there exist $\epsilon > 0$ and a differentiable mapping $\zeta : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ such that $\zeta(0) = 1$, $\zeta(w)(z-w) \in \mathcal{N}_{\lambda,\mu}$ and for all $w = (w_1,w_2) \in \mathcal{H}$*

$$\langle \zeta'(0), w\rangle = \frac{2\mathcal{B}(z,w) - r\mathcal{Q}_{\lambda,\mu}(z,w) - 2\mathcal{P}(z,w)}{(2-r)\|(u,v)\|^2 - (\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx}, \tag{5.1}$$

where

$$\begin{gathered}
\mathcal{B}(z,w) = \int_\Omega D^m u \cdot D^m w_1 dx + \int_\Omega D^m v \cdot D^m w_2 dx,\\
\mathcal{Q}_{\lambda,\mu}(z,w) = \lambda\int_\Omega f(x)|u|^{r-2}uw_1 dx + \mu\int_\Omega g(x)|v|^{r-2}vw_2 dx,\\
\mathcal{P}(z,w) = \int_\Omega \beta|u|^{\beta-2}|v|^\gamma uw_1 dx + \int_\Omega \gamma|u|^\beta|v|^{\gamma-2}vw_2 dx.
\end{gathered}$$

Proof. For $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, define a map $\vartheta_z : \mathbb{R}\times\mathcal{H} \to \mathbb{R}$ such that

$$\begin{aligned}
\vartheta_z(\zeta,w) = \langle I'_{\lambda,\mu}(\zeta(z-w)), \zeta(z-w)\rangle &= \zeta^2\|(u-w_1, v-w_2)\|^2\\
&- \zeta^r\int_\Omega(\lambda f(x)|u-w_1|^r + \mu g(x)|v-w_2|^r)dx
\end{aligned}$$

$$-\zeta^{\beta+\gamma}\int_\Omega h(x)|u-w_1|^\beta|v-w_2|^\gamma dx$$

Then $\vartheta_z(1,(0,0))=\langle I'_{\lambda,\mu}(z),z\rangle=0$ and

$$\begin{aligned}&\frac{d}{d\zeta}\vartheta_z(1,(0,0))\\&=2\|(u,v)\|^2-r\int_\Omega(\lambda f(x)|u|^r+\mu g(x)|v|^r)dx-(\beta+\gamma)\int_\Omega h(x)|u|^\beta|v|^\gamma dx\\&=(2-r)\|(u,v)\|^2-(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx\neq 0.\end{aligned}$$

Now, by the Implicit Function Theorem, there exists $\epsilon>0$ and a differentiable mapping $\zeta: B(0,\epsilon)\subset\mathcal{H}\to\mathbb{R}^+$ such that $\zeta(0)=1$,

$$\langle\zeta'(0),w\rangle=\frac{2\mathcal{B}(z,w)-r\mathcal{Q}_{\lambda,\mu}(z,w)-2\mathcal{P}(z,w)}{(2-r)\|(u,v)\|^2-(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx},$$

$\vartheta_z(\zeta(w),w)=0$ for all $w\in B(0,\epsilon)$. Thus,

$$\langle I'_{\lambda,\mu}(\zeta(w)(z-w)),\zeta(w)(z-w)\rangle=0\ \forall\ w\in B(0,\epsilon).$$

Therefore $\zeta(w)(z-w)\in\mathcal{N}_{\lambda,\mu}$. □

Lemma 5.2. *Suppose* $0<(\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}<\Lambda_1$, *where* Λ_1 *is same as given in* (1.4). *Then for every* $z=(u,v)\in\mathcal{N}^-_{\lambda,\mu}$, *there exist* $\epsilon>0$ *and a differentiable map* $\zeta^-: B(0,\epsilon)\subset\mathcal{H}\to\mathbb{R}^+$ *such that* $\zeta^-(0)=1$ *and* $\zeta^-(w)(z-w)\in\mathcal{N}^-_{\lambda,\mu}$. *Moreover, for all* $(w_1,w_2)\in\mathcal{H}$

$$\langle(\zeta^-)'(0),w\rangle=\frac{2\mathcal{B}(z,w)-r\mathcal{Q}_{\lambda,\mu}(z,w)-2\mathcal{P}(z,w)}{(2-r)\|(u,v)\|^2-(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx},$$

where $\mathcal{B}$, $\mathcal{Q}_{\lambda,\mu}$ *and* $\mathcal{P}$ *are defined same as in Lemma 5.1.*

Proof. By argument used in Lemma 5.1, there exists $\epsilon>0$ and a differentiable function $\zeta^-: B(0,\epsilon)\subset\mathcal{H}\to\mathbb{R}^+$ such that $\zeta^-(0)=1$ and $\zeta^-(w)(z-w)\in\mathcal{N}^-_{\lambda,\mu}$. Since

$$\Psi''_{(u,v)}(1)=(2-r)\|(u,v)\|^2-(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx<0.$$

By the continuity of Ψ'' and ζ^-, we have

$$\begin{aligned}\Psi''_{\zeta^-(w)(z-w)}(1)&=(2-r)\|\zeta^-(w)(z-w)\|^2\\&\quad-(\beta+\gamma-r)\int_\Omega h(x)|\zeta^-(w)(z-w)|^\beta|\zeta^-(w)(z-w)|^\gamma<0,\end{aligned}$$

for $\epsilon>0$ is sufficiently small. Thus, $\zeta^-(w)(z-w)\in\mathcal{N}^-_{\lambda,\mu}$. □

Lemma 5.3. *Let* $1\le r<2<N/m$ *and* $2<\beta+\gamma\le 2^*_m$, *then the following results hold:*

- (i) *If* $0<(\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}<\Lambda_1$, *then there exists a* $(PS)_{\theta_{\lambda,\mu}}$*-sequence* $\{(u_n,v_n)\}\subset\mathcal{N}_{\lambda,\mu}$ *in* $\mathcal{H}$ *for* $I_{\lambda,\mu}$.
- (ii) *If* $0<(\lambda|f|_\alpha)^{\frac{2}{2-r}}+(\mu|g|_\alpha)^{\frac{2}{2-r}}<(\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, *then there exists a* $(PS)_{\theta^-_{\lambda,\mu}}$*-sequence*
  $\{(u_n,v_n)\}\subset\mathcal{N}^-_{\lambda,\mu}$ *in* $\mathcal{H}$ *for* $I_{\lambda,\mu}$, *where* Λ_1 *is same as given in* (1.4).

Proof. (i) By Lemma 4.1 and Ekeland Variational Principle [16], there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$\begin{gathered} I_{\lambda,\mu}(u_n, v_n) < \theta_{\lambda,\mu} + \frac{1}{n}, \\ I_{\lambda,\mu}(u_n, v_n) < I_{\lambda,\mu}(u, v) + \frac{1}{n}\|(u, v) - (u_n, v_n)\|, \quad \text{for each } (u, v) \in \mathcal{N}_{\lambda,\mu}. \end{gathered} \tag{5.2}$$

Since $\theta_{\lambda,\mu} < 0$ and taking n large, we obtain

$$\begin{aligned} &I_{\lambda,\mu}(u_n, v_n) \\ &= \Big(\frac{1}{2} - \frac{1}{\beta+\gamma}\Big)\|(u_n, v_n)\|^2 - \Big(\frac{1}{r} - \frac{1}{\beta+\gamma}\Big)\int_\Omega (\lambda f(x)|u_n|^r + \mu g(x)|v_n|^r)dx \\ &< \theta_{\lambda,\mu} + \frac{1}{n} < \frac{\theta_{\lambda,\mu}}{2}. \end{aligned} \tag{5.3}$$

Thus, we have

$$\begin{aligned} 0 < -\frac{r(\beta+\gamma)\theta_{\lambda,\mu}}{2(\beta+\gamma-r)} &< \int_\Omega (\lambda f(x)|u_n|^r + \mu g(x)|v_n|^r)dx \\ &\le S^{-r/2}((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}})^{\frac{2-r}{2}}\|(u_n, v_n)\|^r. \end{aligned} \tag{5.4}$$

Consequently, $(u_n, v_n) \neq (0, 0)$. Also, (5.3), (5.4) and Hölder's inequality assert that

$$\|(u_n, v_n)\| \le \Big[\frac{2(\beta+\gamma-r)}{r(\beta+\gamma-2)} S^{-r/2}((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}})^{\frac{2-r}{2}}\Big]^{\frac{1}{2-r}}, \tag{5.5}$$

and

$$\|(u_n, v_n)\| \ge \Big[-\frac{r(\beta+\gamma)}{2(\beta+\gamma-r)}\theta_{\lambda,\mu} S^{\frac{r}{2}}((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}})^{\frac{r-2}{2}}\Big]^{1/r}.$$

Now, we show that

$$\|I'_{\lambda,\mu}(u_n, v_n)\|_{\mathcal{H}^{-1}} \to 0, \quad \text{as } n \to \infty.$$

Using Lemma 5.1 for each $z_n = (u_n, v_n)$ to obtain the mapping $\zeta_n : B(0, \epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$ such that $\zeta_n(w)(z_n - w) \in \mathcal{N}_{\lambda,\mu}$. Choose $0 < \eta < \epsilon_n$. Let $z = (u, v) \in \mathcal{H}$ with $z \neq 0$ and take $w^*_\eta = \frac{\eta z}{\|z\|}$. We set $w_\eta = \zeta_n(w^*_\eta)(z_n - w^*_\eta)$. Since $w_\eta \in \mathcal{N}_{\lambda,\mu}$, from (5.2), we obtain

$$I_{\lambda,\mu}(w_\eta) - I_{\lambda,\mu}(z_n) \ge -\frac{1}{n}\|w_\eta - z_n\|.$$

Using Mean Value Theorem, we obtain

$$\langle I'_{\lambda,\mu}(z_n), w_\eta - z_n\rangle + o(\|w_\eta - z_n\|) \ge -\frac{1}{n}\|w_\eta - z_n\|.$$

Therefore

$$\begin{aligned} &\langle I'_{\lambda,\mu}(z_n), -w^*_\eta\rangle + (\zeta_n(w^*_\eta) - 1)\langle I'_{\lambda,\mu}(z_n), z_n - w^*_\eta\rangle \\ &\ge -\frac{1}{n}\|w_\eta - z_n\| + o(\|w_\eta - z_n\|). \end{aligned} \tag{5.6}$$

Since $\zeta_n(w^*_\eta)(z_n - w^*_\eta) \in \mathcal{N}_{\lambda,\mu}$ and from (5.6), we obtain

$$\begin{aligned} &-\eta\langle I'_{\lambda,\mu}(z_n), \frac{z}{\|z\|}\rangle + (\zeta_n(w^*_\eta) - 1)\langle I'_{\lambda,\mu}(z_n - w_\eta), z_n - w^*_\eta\rangle \\ &\ge -\frac{1}{n}\|w_\eta - z_n\| + o(\|w_\eta - z_n\|). \end{aligned}$$

Thus, we have

$$\begin{aligned} &\langle I'_{\lambda,\mu}(z_n), \frac{z}{\|z\|}\rangle \\ &\leq \frac{1}{n\eta}\|w_\eta - z_n\| + \frac{1}{\eta}o(\|w_\eta - z_n\|) + \frac{(\zeta_n(w_\eta^*) - 1)}{\eta}\langle I'_{\lambda,\mu}(z_n - w_\eta), z_n - w_\eta^*\rangle \end{aligned} \tag{5.7}$$

Since $\|w_\eta - z_n\| \leq \eta|\zeta_n(w_\eta^*)| + |\zeta_n(w_\eta^*) - 1|\|z_n\|$ and

$$\lim_{\eta\to 0} \frac{|\zeta_n(w_\eta^*) - 1|}{\eta} \leq \|\zeta_n'(0)\|,$$

if we take $\eta \to 0$ in (5.7) for a fixed $n \in \mathbb{N}$ and using (5.5) we can find a constant $M > 0$, free from η such that

$$\langle I'_{\lambda,\mu}(z_n), \frac{z}{\|z\|}\rangle \leq \frac{M}{n}(1 + \|\zeta_n'(0)\|).$$

Now, we show that $\|\zeta_n'(0)\|$ is uniformly bounded. From (5.1), (5.7) and by Hölder's inequality, we have

$$|\langle \zeta_n'(0)\rangle| \leq \frac{M_1\|(w_1, w_2)\|}{|(2-r)\|(u_n, v_n)\|^2 - (\beta+\gamma-r)\int_\Omega h(x)|u_n|^\beta|v_n|^\gamma dx|},$$

for some $M_1 > 0$.

Next we show that

$$|(2-r)\|(u_n, v_n)\|^2 - (\beta+\gamma-r)\int_\Omega h(x)|u_n|^\beta|v_n|^\gamma dx| \geq M_2,$$

for some $M_2 > 0$ and n is taking large enough. On the contrary, suppose there exists a subsequence $\{(u_n, v_n)\}$ such that

$$(2-r)\|(u_n, v_n)\|^2 - (\beta+\gamma-r)\int_\Omega h(x)|u_n|^\beta|v_n|^\gamma dx = o_n(1). \tag{5.8}$$

From (5.8) and using $(u_n, v_n) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\begin{gathered} \|(u_n, v_n)\|^2 = \frac{\beta+\gamma-r}{2-r}\int_\Omega h(x)|u_n|^\beta|v_n|^\gamma dx + o_n(1), \\ \|(u_n, v_n)\|^2 = \frac{\beta+\gamma-r}{\beta+\gamma-2}Q_{\lambda,\mu}(u_n, v_n) + o_n(1). \end{gathered}$$

By Hölder's inequality and the Sobolev embedding theorem, we obtain

$$\begin{gathered} \|(u_n, v_n)\| \geq \Big(\frac{2-r}{\beta+\gamma-r}\frac{S^{\frac{\beta+\gamma}{2}}}{|h|_\infty}\Big)^{\frac{1}{\beta+\gamma-2}} + o_n(1), \\ \|(u_n, v_n)\| \leq \Big(\frac{\beta+\gamma-r}{\beta+\gamma-2}\Big)^{\frac{1}{2-r}} S^{-\frac{r}{2(2-r)}}\big((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}}\big)^{\frac{1}{2}} + o_n(1). \end{gathered}$$

This implies that $(\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} \geq \Lambda_1$, which is a contradiction to the fact that $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$. Hence

$$\langle I'_{\lambda,\mu}(u_n, v_n), \frac{(u,v)}{\|(u,v)\|}\rangle \leq \frac{M}{n}.$$

This completes the proof of (i).

(ii) By Lemma 5.2, part (ii) can be shown in similar way as above. ☐

Now, we show the existence of a local minimum for $I_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$.

Theorem 5.4. *Let Λ_1 be the same defined as in* (1.4). *If $1 \leq r < 2 < \frac{N}{m}$, $2 < \beta+\gamma \leq 2^*_m$, and $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, then $I_{\lambda,\mu}$ has a minimizer $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ in $\mathcal{N}^+_{\lambda,\mu}$ and it satisfies the following:*

(i) $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = \theta_{\lambda,\mu} = \theta^+_{\lambda,\mu} < 0$.
(ii) $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ *is a nontrivial solution of* (1.1).
(iii) $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \to (0,0)$ *as* $\lambda \to 0^+$, $\mu \to 0^+$.

Proof. By Lemma 5.3 (i), there exists a minimizing sequence $\{(u_n, v_n)\}$ for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu} + o_n(1), \quad I'_{\lambda,\mu}(u_n, v_n) = o_n(1) \text{ in } \mathcal{H}^{-1}. \tag{5.9}$$

By coercivity of $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, we obtain that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. Therefore up to a subsequence still denoted by $\{(u_n, v_n)\}$ converges weakly to $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{H}$. This implies

$$\begin{gathered} u_n \rightharpoonup u^1_{\lambda,\mu},\ v_n \rightharpoonup v^1_{\lambda,\mu} \quad \text{weakly in } H^m_0(\Omega), \\ u_n \rightharpoonup u^1_{\lambda,\mu},\ v_n \rightharpoonup v^1_{\lambda,\mu} \quad \text{a.e. } \Omega, \\ u_n \rightharpoonup u^1_{\lambda,\mu},\ v_n \rightharpoonup v^1_{\lambda,\mu} \quad \text{strongly in } L^s(\Omega)\ \forall 1 \leq s < 2^*_m. \end{gathered} \tag{5.10}$$

It is easy to see that as $n \to \infty$

$$Q_{\lambda,\mu}(u_n, v_n) = Q_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) + o_n(1). \tag{5.11}$$

First we claim that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a nontrivial solution of (1.1). From (5.9) and (5.10), one can easily verify that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a weak solution of the system (1.1). Since $(u_n, v_n) \in \mathcal{N}_{\lambda,\mu}$ and by the definition of $I_{\lambda,\mu}$, we have

$$Q_{\lambda,\mu}(u_n, v_n) = \frac{r(\beta+\gamma-2)}{2(\beta+\gamma-r)}\|(u_n, v_n)\|^2 - \frac{r(\beta+\gamma)}{(\beta+\gamma-r)} I_{\lambda,\mu}(u_n, v_n). \tag{5.12}$$

Then letting $n \to \infty$ in (5.12) and using (5.9), (5.11) with $\theta_{\lambda,\mu} < 0$, we obtain

$$Q_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \geq -\frac{r(\beta+\gamma)}{(\beta+\gamma-r)}\theta_{\lambda,\mu} > 0.$$

Thus, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial solution of (1.1).

Now, we show that $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathcal{H}$ and $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = \theta_{\lambda,\mu}$. If $(u,v) \in \mathcal{N}_{\lambda,\mu}$, then

$$I_{\lambda,\mu}(u,v) = \frac{\beta+\gamma-2}{2(\beta+\gamma)}\|(u,v)\|^2 - \frac{\beta+\gamma-r}{r(\beta+\gamma)} Q_{\lambda,\mu}(u,v). \tag{5.13}$$

To prove that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = \theta_{\lambda,\mu}$, it is sufficient to recall that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}_{\lambda,\mu}$, (5.13) and apply Fatou's lemma to obtain

$$\begin{aligned} \theta_{\lambda,\mu} &\leq I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \\ &= \frac{\beta+\gamma-2}{2(\beta+\gamma)}\|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2 - \frac{\beta+\gamma-r}{r(\beta+\gamma)} Q_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \\ &\leq \liminf_{n\to\infty}\Big(\frac{\beta+\gamma-2}{2(\beta+\gamma)}\|(u_n, v_n)\|^2 - \frac{(\beta+\gamma-r)}{r(\beta+\gamma)} Q_{\lambda,\mu}(u_n, v_n)\Big) \\ &\leq \liminf_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu}. \end{aligned} \tag{5.14}$$

This implies that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = \theta_{\lambda,\mu}$ and $\lim_{n\to\infty} \|(u_n, v_n)\|^2 = \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2$. Let $(\overline{u}_n, \overline{v}_n) = (u_n - u^1_{\lambda,\mu}, v_n - v^1_{\lambda,\mu})$, then by Brézis and Lieb lemma [7] gives

$$\|(\overline{u}_n, \overline{v}_n)\|^2 = \|(u_n, v_n)\|^2 - \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2 + o_n(1).$$

Therefore, $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathcal{H}$. Moreover, we have $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. Thus, $\theta_{\lambda,\mu} = \theta^+_{\lambda,\mu}$. On the contrary, if $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$, then using (4.10) and (5.14), we have that $\int_\Omega h(x)|u^1_{\lambda,\mu}|^\beta |v^1_{\lambda,\mu}|^\gamma > 0$ and $Q_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > 0$. Thus, from Lemma 4.4, there exist unique t_1^+ and t_1^- such that $(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and $(t_1^- u^1_{\lambda,\mu}, t_1^- v^1_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. In particular, we have $t_1^+ < t_1^- = 1$. Since

$$\frac{d}{dt} I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) = 0, \quad \frac{d^2}{dt^2} I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) > 0,$$

there exists $t_1^+ < \bar{t} \le t_1^-$ such that $I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t} u^1_{\lambda,\mu}, \bar{t} v^1_{\lambda,\mu})$. By Lemma 4.4, we obtain

$$\begin{aligned} I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) &< I_{\lambda,\mu}(\bar{t} u^1_{\lambda,\mu}, \bar{t} v^1_{\lambda,\mu}) \\ &\le I_{\lambda,\mu}(t_1^- u^1_{\lambda,\mu}, t_1^- v^1_{\lambda,\mu}) \\ &= I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = \theta_{\lambda,\mu}, \end{aligned}$$

which is a contradiction. Therefore, using Lemma 4.2, we conclude that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a nontrivial solution of (1.1).

(iii) Further, from Theorem 4.6 (i) and (4.2), we have

$$\begin{aligned} 0 > \theta^+_{\lambda,\mu} \ge \theta_{\lambda,\mu} &= I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \\ &> -\frac{\beta+\gamma-r}{r(\beta+\gamma)} S^{-r/2} ((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}})^{\frac{2-r}{2}} \|(u,v)\|^r, \end{aligned}$$

which implies that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \to (0,0)$ as $\lambda \to 0^+$, $\mu \to 0^+$. This completes the proof. □

Theorem 5.5. *If* $1 \le r < 2 < \frac{N}{m}$, $2 < \beta + \gamma < 2^*_m$ *and* $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}} \Lambda_1$, *then* $I_{\lambda,\mu}$ *has a minimizer* $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ *in* $\mathcal{N}^-_{\lambda,\mu}$ *and satisfies the following:*

(i) $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = \theta^-_{\lambda,\mu}$;

(ii) $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ *is a solution of the system* (1.1).

Proof. Let $\{(u_n, v_n)\}$ be a minimizing sequence for $I_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$. Then by $I_{\lambda,\mu}$ coercive on $\mathcal{N}_{\lambda,\mu}$ and the compact imbedding theorem, there exist a subsequence $\{(u_n, v_n)\}$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{H}$ such that $u_n \rightharpoonup u^2_{\lambda,\mu}$ and $v_n \rightharpoonup v^2_{\lambda,\mu}$ weakly in $H^m_0(\Omega)$, $u_n \to u^2_{\lambda,\mu}$ and $v_n \to v^2_{\lambda,\mu}$ strongly in $L^r(\Omega), L^{\beta+\gamma}(\Omega)$. This implies

$$\begin{gathered} Q_{\lambda,\mu}(u_n, v_n) = Q_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) + o_n(1), \\ \int_\Omega h(x)|u_n|^\beta |v_n|^\gamma = \int_\Omega h(x)|u^2_{\lambda,\mu}|^\beta |v^2_{\lambda,\mu}|^\gamma + o_n(1). \end{gathered}$$

Using (4.10) and (4.11), there exists $M_3 > 0$ such that $\int_\Omega h(x)|u_n|^\beta |v_n|^\gamma dx > M_3$. This implies that

$$\int_\Omega h(x)|u^2_{\lambda,\mu}|^\beta |v^2_{\lambda,\mu}|^\gamma dx \ge M_3.$$

Now, we prove that $(u_n, v_n) \to (u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ strongly in $\mathcal{H}$. On contrary, we assume that $\|(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})\| < \liminf_{n\to\infty} \|(u_n, v_n)\|$. Then using Lemma 4.4, there exists a unique t_2^- such that $(t_2^- u^2_{\lambda,\mu}, t_2^- v^2_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. Since $(u_n, v_n) \in \mathcal{N}^-_{\lambda,\mu}$, $I_{\lambda,\mu}(u_n, v_n) \geq I_{\lambda,\mu}(tu_n, tv_n)$ for all $t \geq 0$, we have

$$\theta^-_{\lambda,\mu} \leq I_{\lambda,\mu}(t^- u^2_{\lambda,\mu}, t^- v^2_{\lambda,\mu}) < \lim_{n\to\infty} I_{\lambda,\mu}(t^- u_n, t^- v_n) \leq \lim_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = \theta^-_{\lambda,\mu}.$$

Hence, $(u_n, v_n) \to (u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ strongly in $\mathcal{H}$. This implies

$$I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = \lim_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = \theta^-_{\lambda,\mu}.$$

By Lemma 4.2 and (5.14), we say that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is a nontrivial solution of the system (1.1). Finally, by using the same arguments as in the proof of Theorem 5.4, for all $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, we have that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is a solution of the system (1.1). □

Theorems 1.1 and 1.2 follow from Theorems 5.4 and 5.5 respectively. Also from Theorem 5.4 and 5.5, we obtain that for all $1 < r < 2 < \frac{N}{m}$, $2 < \beta+\gamma < 2^*_m$, $\lambda, \mu > 0$ and $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, the system (1.1) has two nontrivial solutions $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. Since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \phi$, we can conclude that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ are distinct.

6. Proof of Theorem 1.3

In this section, we show the existence of a second weak solution in the critical case $\beta+\gamma = 2^*_m$ as a limit of Palais-Smale sequence which is obtained by minimizing sequence for $I_{\lambda,\mu}$ in $\mathcal{N}^-_{\lambda,\mu}$.

For this, taking $\rho > 0$ small enough such that $B_\rho(0) \subset \Omega$ and define the function $u_\epsilon(x) = \phi(x)U_\epsilon(x)$, where $\phi(x) \in C_0^\infty(B_\rho(0))$ is a cut-off function such that $\phi(x) \equiv 1$ in $B_{\rho/2}(0)$ and $U_\epsilon(x)$ is same as mentioned in (2.2). Then, we have the following estimates (see [18, 19, 32]).

Lemma 6.1. *Suppose $N \geq 2m+1$. Then the following estimates hold when $\epsilon \to 0$:*

$$\|u_\epsilon\|^2 = S^{\frac{N}{2m}} + O(\epsilon^{N-2m}), \tag{6.1}$$

$$\int_\Omega |u_\epsilon|^{2^*_m} dx = S^{\frac{N}{2m}} + O(\epsilon^N), \tag{6.2}$$

$$\int_\Omega |u_\epsilon|^r dx = \begin{cases} O_1(\epsilon^{\frac{(N-2m)r}{2}}) & \text{if } 1 < r < \frac{N}{N-2m}, \\ O_1(\epsilon^{N-\frac{(N-2m)r}{2}}|\ln\epsilon|) & \text{if } r = \frac{N}{N-2m}, \\ O_1(\epsilon^{N-\frac{(N-2m)r}{2}}) & \text{if } \frac{N}{N-2m} < r < 2^*_m. \end{cases} \tag{6.3}$$

Lemma 6.2. *Suppose that* (A1)–(A4) *hold with $\delta_0 > N-2m$ and $\frac{N}{N-2m} \leq r < 2$. Then there exists $\overline{\Lambda} > 0$ such that for all $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \overline{\Lambda}$ there exists $(u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathcal{H}\setminus\{(0,0)\}$ such that*

$$\sup_{t\geq 0} I_{\lambda,\mu}(tu_{\lambda,\mu}, tv_{\lambda,\mu}) < c_\infty,$$

where c_∞ is the constant given in Lemma 3.3. In particular, $\theta^-_{\lambda,\mu} < c_\infty$ for all $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \overline{\Lambda}$.

Proof. By assumption (A4), there exists $\delta_0 > N-2m$ such that, for $x \in B(0, 2\rho_0)$ where $0 < \rho_0 \le r_0$
$$h(x) = h(0) + o(|x|^{\delta_0}) \quad \text{as } x \to 0.$$
Define a functional $\tau : \mathcal{H} \to \mathbb{R}$ such that
$$\tau(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{\beta+\gamma}\int_\Omega h(x)|u|^\beta |v|^\gamma dx \quad \forall\, (u,v) \in \mathcal{H}. \tag{6.4}$$
Set $\overline{u}_\epsilon = \sqrt{\beta} u_\epsilon$, $\overline{v}_\epsilon = \sqrt{\gamma} u_\epsilon$ with $(\overline{u}_\epsilon, \overline{v}_\epsilon) \in \mathcal{H}$. The map $\tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon)$ satisfies $\tau(0) = 0$, $\tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon) > 0$ for $t > 0$ small and $\tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon) < 0$ for $t > 0$ large. Moreover, τ attains its maximum at
$$t_0 = \Big(\frac{\|(\overline{u}_\epsilon, \overline{v}_\epsilon)\|^2}{\int_\Omega h(x)|\overline{u}_\epsilon|^\beta |\overline{v}_\epsilon|^\gamma dx}\Big)^{\frac{1}{\beta+\gamma-2}}. \tag{6.5}$$
Thus, from (6.1), (6.2), (6.4), (6.5) and (2.4), we have
$$\begin{aligned}
&\sup_{t\ge 0} \tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon)\\
&= \frac{t_0^2}{2}\|(\overline{u}_\epsilon, \overline{v}_\epsilon)\|^2 - \frac{t_0^{\beta+\gamma}}{\beta+\gamma}\int_\Omega h(x)|\overline{u}_\epsilon|^\beta |\overline{v}_\epsilon|^\gamma dx\\
&= \Big(\frac{1}{2} - \frac{1}{\beta+\gamma}\Big)\frac{\|(\overline{u}_\epsilon, \overline{v}_\epsilon)\|^{\frac{2(\beta+\gamma)}{\beta+\gamma-2}}}{(\int_\Omega h(x)|\overline{u}_\epsilon|^\beta |\overline{v}_\epsilon|^\gamma dx)^{\frac{2}{\beta+\gamma-2}}}\\
&= \frac{m}{N}\Big[\big(\frac{\beta}{\gamma}\big)^{\frac{\gamma}{\beta+\gamma}} + \big(\frac{\gamma}{\beta}\big)^{\frac{\beta}{\beta+\gamma}}\Big]^{\frac{N}{2m}} \Big[\frac{\|u_\epsilon\|^2}{(\int_\Omega h(x)|u_\epsilon|^{2_m^*} dx)^{\frac{2}{2_m^*}}}\Big]^{\frac{N}{2m}}\\
&= \frac{m}{N}\Big[\big(\frac{\beta}{\gamma}\big)^{\frac{\gamma}{\beta+\gamma}} + \big(\frac{\gamma}{\beta}\big)^{\frac{\beta}{\beta+\gamma}}\Big]^{\frac{N}{2m}} \Big[\frac{S^{\frac{N}{2m}} + O(\epsilon^{N-2m})}{(h(0)S^{\frac{N}{2m}} + O(\epsilon^N) + O(\epsilon^{\delta_0}))^{\frac{2}{2_m^*}}}\Big]^{\frac{N}{2m}}\\
&\le \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} + O(\epsilon^{N-2m}) - O(\epsilon^{\delta_0}).
\end{aligned}$$
Therefore
$$\sup_{t\ge 0} \tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon) \le \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} + O(\epsilon^{N-2m}) - O(\epsilon^{\delta_0}). \tag{6.6}$$
Now, we choose $\delta_1 > 0$ such that $c_\infty > 0$ for all $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \delta_1$. Using the definition of $I_{\lambda,\mu}$ and λ, $\mu > 0$, we obtain $I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon) \le \frac{t^2}{2}\|(\overline{u}_\epsilon, \overline{v}_\epsilon)\|^2$ for $t \ge 0$. Thus, there exists $t_0 \in (0,1)$ such that
$$\sup_{0\le t\le t_0} I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon) < c_\infty \quad \text{for all } 0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \delta_1.$$
Using $\beta, \gamma > 1$, (6.6) and (6.3), we obtain
$$\begin{aligned}
&\sup_{t\ge t_0} I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon)\\
&= \sup_{t\ge t_0}\Big(\tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon) - \frac{1}{r}Q_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon)\Big)\\
&\le \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} + O(\epsilon^{N-2m}) - \frac{1}{r}t_0^r \int_\Omega (\lambda f(x)|\overline{u}_\epsilon|^r + \mu g(x)|\overline{v}_\epsilon|^r) dx\\
&\le \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} + O(\epsilon^{N-2m}) - \frac{1}{r}t_0^r (a_0\lambda\beta^{r/2} + b_0\mu\gamma^{r/2}) \int_\Omega |u_\epsilon|^r dx
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} + O(\epsilon^{N-2m}) - \frac{1}{r} t_0^r \eta(\lambda+\mu) \int_\Omega |u_\epsilon|^r dx \\
&\le \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} + O(\epsilon^{N-2m}) \\
&\quad - \frac{1}{r} t_0^r \eta(\lambda+\mu) \begin{cases} O_1(\epsilon^{N-\frac{(N-2m)r}{2}} |\ln \epsilon|) & \text{if } r = \frac{N}{N-2m} \\ O_1(\epsilon^{N-\frac{(N-2m)r}{2}}) & \text{if } \frac{N}{N-2m} < r < 2_m^*, \end{cases}
\end{aligned}
$$

where $\eta = \min\{a_0, b_0\}$. Choose $\delta_2 > 0$ in such a way that $0 \le \epsilon < \delta_2$. Now, take $\epsilon = ((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}})^{\frac{1}{N-2m}}$. Then, we have

$$
\begin{aligned}
&\sup_{t\ge t_0} I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon) \\
&\le \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} + O(\mathcal{A}(\lambda,\mu)) \\
&\quad - \frac{\eta(\lambda+\mu)}{r} \begin{cases} O_1((\mathcal{A}(\lambda,\mu))^{\frac{N}{2(N-2m)}} |\ln(\mathcal{A}(\lambda,\mu))|) & \text{if } r = \frac{N}{N-2m} \\ O_1((\mathcal{A}(\lambda,\mu))^{\frac{N}{N-2m}-\frac{r}{2}}) & \text{if } \frac{N}{N-2m} < r < 2_m^*, \end{cases}
\end{aligned} \tag{6.7}
$$

where $\mathcal{A}(\lambda,\mu) = (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}}$.

Case (i): When $r = \frac{N}{N-2m}$, we can choose $\delta_3 > 0$ with $0 < \mathcal{A}(\lambda,\mu) < \delta_3$ such that

$$
O(\mathcal{A}(\lambda,\mu)) - \frac{\eta(\lambda+\mu)}{r} O_1((\mathcal{A}(\lambda,\mu))^{\frac{N}{2(N-2m)}} |\ln(\mathcal{A}(\lambda,\mu))|) < -P_0(\mathcal{A}(\lambda,\mu)),
$$

as $\lambda, \mu \to 0$, $|\ln(\mathcal{A}(\lambda,\mu))| \to +\infty$.

Case (ii): When $\frac{N}{N-2m} < r < 2_m^*$, we can choose $\delta_4 > 0$ with $0 < \mathcal{A}(\lambda,\mu) < \delta_4$ such that

$$
O(\mathcal{A}(\lambda,\mu)) - \frac{\eta(\lambda+\mu)}{r} O_1((\mathcal{A}(\lambda,\mu))^{\frac{N}{N-2m}-\frac{r}{2}}) < -P_0(\mathcal{A}(\lambda,\mu)),
$$

as $1 + \frac{2}{2-r}(\frac{N}{N-2m} - \frac{r}{2}) < \frac{2}{2-r}$ if and only if $r > \frac{N}{N-2m}$.

Now, choose $\overline{\Lambda} = \min\{\delta_1, \delta_2^{N-2m}, \delta_3, \delta_4\} > 0$. Then using this and (6.7), we have

$$
\sup_{t\ge 0} I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon) < \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} - P_0((\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}}) = c_\infty, \tag{6.8}
$$

for $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \overline{\Lambda}$.

Next, we show that $\theta^-_{\lambda,\mu} < c_\infty$ for all $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \overline{\Lambda}$. From (A3), (A4) and the definition of $(\overline{u}_\epsilon, \overline{v}_\epsilon)$, we obtain

$$
\int_\Omega h(x)|\overline{u}_\epsilon|^\beta |\overline{v}_\epsilon|^\gamma dx > 0, \quad Q_{\lambda,\mu}(\overline{u}_\epsilon, \overline{v}_\epsilon) > 0.
$$

Combining this with Lemma 4.4 (ii), definition of $\theta^-_{\lambda,\mu}$ and (6.8), for all $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < \overline{\Lambda}$, we obtain that there exists $t_{\lambda,\mu} > 0$ such that $(t_{\lambda,\mu}\overline{u}_\epsilon, t_{\lambda,\mu}\overline{v}_\epsilon) \in \mathcal{N}^-_{\lambda,\mu}$ with

$$
\theta^-_{\lambda,\mu} \le I_{\lambda,\mu}(t_{\lambda,\mu}\overline{u}_\epsilon, t_{\lambda,\mu}\overline{v}_\epsilon) < \sup_{t\ge 0} I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon) < c_\infty.
$$

On taking $(\overline{u}_\epsilon, \overline{v}_\epsilon) = (u_{\lambda,\mu}, v_{\lambda,\mu})$, we obtain the desired result which completes the proof. □

Theorem 6.3. *Assume that* (A1)–(A4) *hold. Then* $I_{\lambda,\mu}$ *satisfies the* $(PS)_{\theta^-_{\lambda,\mu}}$ *condition for all* $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$. *Moreover,* $I_{\lambda,\mu}$ *has a minimizer* $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ *in* $\mathcal{N}^-_{\lambda,\mu}$ *and satisfies the following conditions:*

(i) $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = \theta^-_{\lambda,\mu} > 0$;
(ii) $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ *is a nontrivial solution of* (1.1), *where* Λ_1 *is same as mentioned in* (1.4).

Proof. By Lemma 5.3 (ii), for $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ in $\mathcal{H}$ for $I_{\lambda,\mu}$. Then, from Lemma 3.2, we find that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. Now, using Lemma 6.2 and Lemma 3.3, $I_{\lambda,\mu}$ satisfies the $(PS)_{\theta^-_{\lambda,\mu}}$-condition. Then, there exists $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{H}$ such that up to subsequence $(u_n, v_n) \to (u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ in $\mathcal{H}$. Moreover, $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = \theta^-_{\lambda,\mu} > 0$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$, since $\mathcal{N}^-_{\lambda,\mu}$ is a closed set. Using the argument as applied in Theorem 5.4, one can easily obtain that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is a nontrivial solution of system (1.1) for $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$. □

Proof of Theorem 1.3. By Theorem 5.4 and Theorem 6.3, we obtain that for all λ, $\mu > 0$ and $0 < (\lambda|f|_\alpha)^{\frac{2}{2-r}} + (\mu|g|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, system (1.1) has two distinct solutions $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$, since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \phi$. □

Next, we show that the solutions $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ are not semi-trivial. Using Theorem 5.4 (i) and Theorem 6.3 (i) respectively, we obtain

$$I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) < 0 \quad \text{and} \quad I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) > 0. \tag{6.9}$$

We observe that, if $(u, 0)$ (or $(0, v)$) is a semi-trivial solution of (1.1), then we have

$$\begin{gathered} (-\Delta)^m u = \lambda f(x)|u|^{r-2}u \quad \text{in } \Omega, \\ D^k u = 0 \quad \text{for all } |k| \le m-1 \quad \text{on } \partial\Omega. \end{gathered} \tag{6.10}$$

Then

$$I_{\lambda,\mu}(u, 0) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{r}\int_\Omega f(x)|u|^r dx = -\frac{2-r}{2r}\|u\|^2 < 0. \tag{6.11}$$

From (6.9) and (6.11), we obtain that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is not semi-trivial. Now, we will prove that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is not semi-trivial. Without loss of generality, we assume that $v^1_{\lambda,\mu} \equiv 0$. Then $u^1_{\lambda,\mu}$ is a non-trivial solution of (6.10) and

$$\|(u^1_{\lambda,\mu}, 0)\|^2 = \|u^1_{\lambda,\mu}\|^2 = \lambda \int_\Omega f(x)|u^1_{\lambda,\mu}|^r dx > 0.$$

We take $w \in H^m_0(\Omega) \setminus \{0\}$ such that

$$\|(0, w)\|^2 = \|w\|^2 = \mu \int_\Omega g(x)|w|^r dx.$$

From Lemma 4.4, there exists a unique $0 < t_1 < t_{\max}(u^1_{\lambda,\mu}, w)$ such that $(t_1 u^1_{\lambda,\mu}, t_1 w) \in \mathcal{N}^+_{\lambda,\mu}$, where

$$t_{\max}(u^1_{\lambda,\mu}, w) = \Big(\frac{(\beta+\gamma-r)\int_\Omega(\lambda f(x)|u^1_{\lambda,\mu}|^r + \mu g(x)|w|^r)dx}{(\beta+\gamma-2)\|(u^1_{\lambda,\mu}, w)\|^2}\Big)^{\frac{1}{2-r}}$$

$$= \Big(\frac{\beta+\gamma-r}{\beta+\gamma-2}\Big)^{\frac{1}{2-r}} > 1.$$

Moreover,

$$I_{\lambda,\mu}(t_1 u^1_{\lambda,\mu}, t_1 w) = \inf_{0\le t\le t_{\max}} I_{\lambda,\mu}(t u^1_{\lambda,\mu}, tw).$$

This and the fact that $(u^1_{\lambda,\mu}, 0) \in \mathcal{N}^+_{\lambda,\mu}$ imply that

$$\theta^+_{\lambda,\mu} \le I_{\lambda,\mu}(t_1 u^1_{\lambda,\mu}, t_1 w) \le I_{\lambda,\mu}(u^1_{\lambda,\mu}, w) < I_{\lambda,\mu}(u^1_{\lambda,\mu}, 0) = \theta^+_{\lambda,\mu},$$

which is a contradiction. Hence, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is not semi-trivial. This completes the proof.

Acknowledgements. S. Goyal was supported by Science and Engineering Research Board, Department of Science and Technology, Government of India, Grant number: ECR/2017/002651.

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