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ROTHE'S METHOD FOR SOLVING SEMI-LINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. We consider a semi-linear differential equation of parabolic type with deviating arguments in a Banach space with uniformly convex dual, and apply Rothe's method to establish the existence and uniqueness of a strong solution. We also include an example as an application of the main result.

1. Introduction

In differential equations with deviating arguments the unknown function and its derivative are evaluated at different values of their arguments. They are considered as one of the most important and frequently used differential equations and hence the study of these equations has been rapidly increasing. They are widely used in various branches of science and technology such as self-oscillating systems, automatic control, problems related with combustion in rocket motion, long-term planning in economics, biological problems, and many other areas of science and technology [11, 13]. The very familiar hot shower problem is closely related to these differential equations. For an extensive reading on differential equations with deviating arguments, we refer the reader to [8, 9, 10, 12, 14, 17].

Rothe's method was introduced by Rothe [25] in 1930 to solve a scalar parabolic initial value problem of second order. Rothe used time discretization to develop his method so the method is also known as the method of semidiscretization or the method of lines. Later on many authors have used and developed this method, see [1, 2, 5, 7, 16, 24]. Rothe's method is effectively used to establish the existence and uniqueness of solution of equations such as linear, nonlinear, parabolic and hyperbolic equations with higher orders. The method is also used to study the diffusion problems [6, 18, 20, 22, 23]. Recently Rothe's method is also applied to study variational-hemivariational inequalities with applications to contact mechanics [3, 4, 19],. Thus the application of this method is not limited to mathematics but also applicable to physics and biology. The method becomes a strong and efficient tool to analyze the existence and uniqueness of solution to differential equations.

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Raheem and Bahuguna [23] applied Rothe's method to study the fractional integral diffusion equation in a Banach space X,

$$\frac{\partial u(t)}{\partial t} + Au(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}\, ds + f(t), \quad t \in (0,T],$$
$$u(0) = u_0,$$

where $0 < \alpha < 1$, -A is the infinitesimal generator of a C_0-semigroup of contractions, f is a given map from $[0,T]$ to X, and the initial point $u_0 \in D(A) \subset X$, the domain of A.

Dubey [7] used the method for the nonlinear nonlocal functional differential equation in a Banach space X,

$$u'(t) + Au(t) = f(t, u(t), u_t), \quad t \in (0,T],$$
$$h(u_0) = \phi \quad \text{on } [-\tau, 0],$$

where $0 < T < \infty$, $\phi \in C_0 := C([-\tau,0];X), \tau > 0$, the nonlinear operator A is single-valued and m-accretive defined from the domain $D(A) \subset X$ to X, the nonlinear map f is defined from $[0,T] \times X \times C_0$ to X, the map h is defined from C_0 to C_0. For $u \in C_T := C([-\tau,T];X)$, the map $u_t \in C_0$ is defined by $u_t(s) = u(t+s)$ for $s \in [-\tau, 0]$. Here, $C_t := C([-\tau,t];X)$ for $t \in [0,T]$ is the Banach space of all continuous functions from $[-\tau,t]$ into X endowed with the supremum norm,

$$\|\phi\|_t = \sup_{\tau \le \eta \le t} \|\phi(\eta)\|, \quad \phi \in C_t,$$

where $\|\cdot\|$ is the norm in X.

In this article, we consider the following semi-linear differential equation with deviating arguments. Let $(X, \|\cdot\|)$ be a Banach space with a uniformly convex dual X^*.

$$\begin{gathered} \frac{\partial u(t)}{\partial t} + Au(t) = f(t, u(t), u(h(u(t),t))), \quad t \in (0,T], \\ u(0) = u_0, \quad u_0 \in X. \end{gathered} \tag{1.1}$$

We assume that for each $t \in (0,T]$, $-A$ is the infinitesimal generator of a C_0-semigroup of contractions, the non-linear continuous maps $f : [0,T] \times X \times X \to X$ and $h : X \times [0,T] \to [0,T]$ satisfy suitable growth conditions in its arguments stated in next section.

2. Preliminaries and main result

In this section we briefly state some definitions and results need for proving the main result. At the end of this section, we state our main result.

Definition 2.1. Let X be a Banach space and X^* be its dual. For every $x \in X$ we define the duality map P as

$$P(x) = \{x^* \in X^* : (x^*, x) = \|x\|^2 = \|x^*\|^2\},$$

where (x^*, x) denotes the value of x^* at x.

Definition 2.2. A nonlinear operator $A : D(A) \subset X \to X$ is called m-accretive if

(i) $(Ax - Ay, P(x-y)) \ge 0$, for all $x, y \in D(A)$,
(ii) $R(I + A) = X$, where $R(\cdot)$ is the range of an operator.

Lemma 2.3 ([21, Theorem 1.4.3])**.** *If -A is the infinitesimal generator of a C_0 semigroup of contractions, then A is m-accretive, i.e.,*

(i) $(Au - Av, P(u - v)) \geq 0$, *for all* $u, v \in D(A)$, *where* P *is the duality map.*
(ii) *For each* $\lambda > 0$, *we have* $R(I + \lambda A) = X$, *where* I *is the identity operator on* X *and* $R(\cdot)$ *denotes range of an operator.*

Lemma 2.4 ([15, Lemma 2.5])**.** *Let* X *be a Banach space and* X^* *be its uniformly convex dual. Let -*A *be the infinitesimal generator of a* C_0 *semigroup of contractions. Consider the sequence* $X^n \in D(A)$, $n = 1, 2, 3, \dots$ *such that* $X^n \to u \in X$ *and if* $\|AX^n\|$ *are bounded, then* $u \in D(A)$ *and* $AX^n \rightharpoonup Au$.

Lemma 2.5 ([23])**.** *Let* $\alpha_1, \alpha_2, \dots, \alpha_j$ *be non-negative numbers satisfying*
(i) $\alpha_1 \leq B$,
(ii) $\alpha_i \leq B + C\lambda \sum_{k=1}^{i-1} \alpha_k$, *where* B, C *and* λ *are positive constants.*
Then for each $i = 1, 2, \dots, j$ *we have* $\alpha_i \leq Be^{C(i-1)\lambda}$.

We use the following assumptions for proving our main result.

(A1) Suppose there exists a constant $L_f > 0$ such that for each $x, y, x', y' \in X$ and $t, s \in [0, T]$, the function $f : [0, T] \times X \times X \to X$ satisfies
$$\|f(t, x, x') - f(s, y, y')\| \leq L_f(|t - s| + \|x - y\| + \|x' - y'\|).$$
(A2) Let there exists a constant $L_1 > 0$ such that for each $x, y \in X$ and $t, s \in [0, T]$ the map $h : X \times [0, T] \to [0, T]$ satisfies
$$|h(x, t) - h(y, s)| \leq L_1(\|x - y\| + |t - s|).$$

Remark 2.6. For each $x, y \in X$ and $t, s \in [0, T]$, we have
$$\begin{aligned}\|x(h(x, t)) - y(h(y, s))\| &\leq L_2(|h(x, t) - h(y, s)|)\\ &\leq L_2[L_1(\|x - y\| + |t - s|)] = L_h(\|x - y\| + |t - s|),\end{aligned}$$
for some constants $L_1, L_2, L_h > 0$.

Theorem 2.7. *Let* (A1) *and* (A2) *be satisfied. Then the initial value problem* (1.1) *has a unique strong solution* u *on the interval* $[0, T]$. *More precisely we have,* $u \in C([0, T]; X)$ *such that* $u(t) \in D(A)$, u *is differentiable a.e. on* $[0, T]$ *and* u *satisfies* (1.1).

3. Approximation

In this section, we use time discretization to approximate the system (1.1) by corresponding parabolic problems and construct an approximate solution to the original problem. Also we prove the convergence of this approximate solution to the solution of (1.1) with the help of analogues results for the approximate equations of the original system.

To apply the Rothe's Method, we consider the interval $[0, T]$ and divide it into the subintervals of length $\lambda_n = \frac{T}{n}$. We use the following approximate equations to replace the system (1.1). For $i = 1$, we have
$$\begin{gathered}\frac{u_1^n - u_0^n}{\lambda_n} + Au_1^n = f_0,\\ u_0^n = u_0,\end{gathered} \tag{3.1}$$
and for $2 \leq i \leq n$, we use the equations,
$$\frac{u_i^n - u_{i-1}^n}{\lambda_n} + Au_i^n = f_{i-1}^n, \tag{3.2}$$

where,

$$f_i^n = f(t_i^n, u_i^n, {u'}_i^n), \quad u_i^n = u(t_i^n), \quad {u'}_i^n = u(h(u_i^n, t_i^n)),$$

and $f_0 = f^n(0, u_0, u(h(u_0, 0)))$.

Next we successively establish the existence and uniqueness of solution of the approximate equations

$$\frac{u_1^n - u_0^n}{\lambda_n} + Au_1^n = f_0, \quad u_0^n = u_0, \tag{3.3}$$

$$\frac{u_i^n - u_{i-1}^n}{\lambda_n} + Au_i^n = f_{i-1}^n, \quad i = 2, 3, \dots, n. \tag{3.4}$$

The existence of a unique solution $u_i^n \in D(A)$ to the system (3.2) is a consequence of Lemma 2.3.

We now define the Rothe's sequence as

$$U^n(t) = \begin{cases} u_0, & \text{if } t = 0, \\ u_{i-1}^n + \frac{1}{\lambda_n}(t - t_{i-1}^n)(u_i^n - u_{i-1}^n), & \text{if } t \in (t_{i-1}^n, t_i^n]. \end{cases}$$

Before proving the main result, we now present some results which are needed.

Lemma 3.1. *For each $n \in \mathbb{N}$ and $i = 1, 2, \dots, n$, the estimate*

$$\|u_i^n\| \le C$$

holds for some constant $C > 0$ and the constant is independent of n, i and λ_n.

Proof. From (3.3), we have

$$u_1^n + \lambda_n Au_1^n = u_0^n + \lambda_n f_0.$$

Applying $P(u_1^n)$ on both sides and using the definition of accretivity of A, we obtain

$$\|u_1^n\| \le \|u_0^n\| + \lambda_n\|f_0\| \le \|u_0\| + T\|f_0\| = C_1(\text{say}).$$

From (3.4) and for $2 \le i \le n$, we have

$$u_i^n + \lambda_n Au_i^n = u_{i-1}^n + \lambda_n f_{i-1}^n.$$

We apply $P(u_i^n)$ on both sides and use the definition of accretivity of A to obtain

$$\|u_i^n\| \le \|u_{i-1}^n\| + \lambda_n\|f_{i-1}^n\|.$$

By using the hypotheses (A1) and (A2) in the above equation, we obtain

$$\begin{aligned} \|u_i^n\| &\le \|u_{i-1}^n\| + \lambda_n\big[L_f\big\{|t_{i-1}^n| + \|u_{i-1}^n - u_0\| + L_h\big(\|u_{i-1}^n - u_0\| + |t_{i-1}^n|\big)\big\} + \|f_0\|\big] \\ &= \|u_{i-1}^n\| + \lambda_n\big[L_f|t_{i-1}^n|(1 + L_h) + \|f_0\|\big] + \lambda_n L_f\|u_{i-1}^n - u_0\|(1 + L_h). \end{aligned}$$

Repeating above process, we obtain

$$\begin{aligned} \|u_i^n\| &\le \|u_0\| + i\lambda_n\big[L_f|t_{i-1}^n|(1 + L_h) + \|f_0\|\big] + i\lambda_n L_f\|u_0\|(1 + L_h) \\ &\quad + \lambda_n L_f(1 + L_h)\sum_{j=1}^{i-1}\|u_j^n\| \\ &\le \|u_0\|\big\{1 + TL_f(1 + L_h)\big\} + T\big\{L_f T(1 + L_h) + \|f_0\|\big\} \\ &\quad + \lambda_n L_f(1 + L_h)\sum_{j=1}^{i-1}\|u_j^n\|. \end{aligned}$$

Applying Lemma 2.5 on the above inequality, we obtain

$$\begin{aligned}\|u_i^n\| &\le \big[\|u_0\|\{1+TL_f(1+L_h)\}+T\{L_fT(1+L_h)+\|f_0\|\}\big]e^{L_f(1+L_h)(i-1)\lambda_n}\\ &\le \Big[\|u_0\|\{1+TL_f(1+L_h)\}+T\{L_fT(1+L_h)+\|f_0\|\}\Big]e^{L_f(1+L_h)T}\\ &= C.\end{aligned}$$

This completes the proof. □

Lemma 3.2. *For each $n\in\mathbb{N}$ and $i=1,2,\dots,n$ there exists a positive constant C which is independent of n,i and λ_n such that*

$$\|\frac{u_i^n-u_{i-1}^n}{\lambda_n}\|\le C.$$

Proof. From (3.3), we obtain

$$\frac{u_1^n-u_0}{\lambda_n}+Au_1^n-Au_0=f_0-Au_0.$$

Applying $P(u_1^n-u_0)$ on the above equation and using the definition of accretivity of A, we obtain

$$\|\frac{u_1^n-u_0}{\lambda_n}\|\le\|f_0\|+\|Au_0\|=C_1(\text{say}).\tag{3.5}$$

We rewrite the equation (3.4) for the index $i-1$ and subtract it from (3.4), we obtain

$$\frac{u_i^n-u_{i-1}^n}{\lambda_n}+Au_i^n-Au_{i-1}^n=\frac{u_{i-1}^n-u_{i-2}^n}{\lambda_n}+f_{i-1}^n-f_{i-2}^n.$$

Again we apply $P(u_i^n-u_{i-1}^n)$ on both sides to deduce the following estimates

$$\begin{aligned}&\|\frac{u_i^n-u_{i-1}^n}{\lambda_n}\|\\ &\le\|\frac{u_{i-1}^n-u_{i-2}^n}{\lambda_n}\|+\|f_{i-1}^n-f_{i-2}^n\|\\ &\le\|\frac{u_{i-1}^n-u_{i-2}^n}{\lambda_n}\|+L_f\{|t_{i-1}^n-t_{i-2}^n|+\|u_{i-1}^n-u_{i-2}^n\|\\ &\quad+L_h\Big(\|u_{i-1}^n-u_{i-2}^n\|+|t_{i-1}^n-t_{i-2}^n|\Big)\}\\ &=\|\frac{u_{i-1}^n-u_{i-2}^n}{\lambda_n}\|+L_f(1+L_h)|t_{i-1}^n-t_{i-2}^n|+L_f(1+L_h)\lambda_n\|\frac{u_{i-1}^n-u_{i-2}^n}{\lambda_n}\|\\ &=L_f(1+L_h)\lambda_n+\Big\{1+L_f(1+L_h)\lambda_n\Big\}\|\frac{u_{i-1}^n-u_{i-2}^n}{\lambda_n}\|\,.\end{aligned}$$

We put $C_2=L_f(1+L_h)$ and repeating the above inequality, we obtain

$$\|\frac{u_i^n-u_{i-1}^n}{\lambda_n}\|\le K+(1+C_2\lambda_n)^{i-1}\|\frac{u_1^n-u_0}{\lambda_n}\|,\tag{3.6}$$

where,

$$\begin{aligned}K&=C_2\lambda_n\{1+(1+C_2\lambda_n)+(1+C_2\lambda_n)^2+\dots+(1+C_2\lambda_n)^{i-2}\}\\ &=(1+C_2\lambda_n)^{i-1}-1.\end{aligned}$$

Now, we have

$$(1+C_2\lambda_n)^{i-1}\le e^{C_2\lambda_n(i-1)}\le e^{C_2T}.$$

Hence K is a constant independent of n, i and λ_n. Thus from the estimates (3.5) and (3.6), we obtain

$$\|\frac{u_i^n - u_{i-1}^n}{\lambda_n}\| \leq K + e^{C_2 T} C_1 = C.$$

This completes the proof. □

Next we define a sequence of step functions

$$Y^n(t) = \begin{cases} u_0 & \text{if } t = 0, \\ u_i^n & \text{if } t \in (t_{i-1}^n, t_i^n]. \end{cases}$$

Remark 3.3. From Lemma 3.2, we can conclude that $U^n(t)$ is uniformly Lipschitz continuous and $U^n(t) - Y^n(t) \to 0$ as $n \to \infty$.

We define $f^n(t) = f(t_i^n, u_i^n, {u'}_i^n)$. Then equation (3.3) and (3.4) can be rewritten as

$$\frac{d}{dt} U^n(t) + AY^n(t) = f^n(t), \quad t \in (0, T]. \tag{3.7}$$

where $\frac{d}{dt}$ denotes the left derivative in the interval $(0, T]$. For $t \in (0, T]$, we have

$$\int_0^t AY^n(s) ds = u_0 - U^n(t) + \int_0^t f^n(s) ds. \tag{3.8}$$

Lemma 3.4. *There exists $u \in C([0,T]; X)$ such that $U^n \to u$ in $C([0,T]; X)$ as $n \to \infty$. Moreover, u is Lipschitz continuous on $[0,T]$.*

Proof. From (3.7), we see that

$$\frac{d}{dt} U^n(t) - \frac{d}{dt} U^m(t) + AY^n(t) - AY^m(t) = f^n(t) - f^m(t).$$

Applying $P(Y^n(t) - Y^m(t))$, using the definition of accretivity of A, we obtain

$$(\frac{d}{dt} U^n(t) - \frac{d}{dt} U^m(t), P(Y^n(t) - Y^m(t)) \leq (f^n(t) - f^m(t), P(Y^n(t) - Y^m(t)).$$

From the above inequality and using that

$$(\frac{d}{dt} U^n(t) - \frac{d}{dt} U^m(t), P(U^n(t) - U^m(t)) = \frac{1}{2} \frac{d}{dt} \|U^n(t) - U^m(t)\|^2,$$

we obtain

$$\begin{aligned}
&(\frac{d}{dt} U^n(t) - \frac{d}{dt} U^m(t), P(U^n(t) - U^m(t)) \\
&\leq (f^n(t) - f^m(t), P(Y^n(t) - Y^m(t)) + (\frac{d}{dt} U^n(t) - \frac{d}{dt} U^m(t), P(U^n(t) - U^m(t)) \\
&\quad - (\frac{d}{dt} U^n(t) - \frac{d}{dt} U^m(t), P(Y^n(t) - Y^m(t)) \\
&= \|\frac{d}{dt}(U^n(t) - U^m(t)\| \Big(\|U^n(t) - U^m(t)\| - \|Y^n(t) - Y^m(t)\| \Big) \\
&\quad + \|f^n(t) - f^m(t)\| \|Y^n(t) - Y^m(t)\| \\
&\leq \|\frac{d}{dt}(U^n(t) - U^m(t)\| \Big(\|U^n(t) - U^m(t) - Y^n(t) + Y^m(t)\| \Big) \\
&\quad + \|f^n(t) - f^m(t)\| \|Y^n(t) - Y^m(t)\|.
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|U^n(t)-U^m(t)\big\|^2\\
&\le \big\|\frac{d}{dt}(U^n(t)-U^m(t)\big\|\Big(\|U^n(t)-Y^n(t)\|+\|U^m(t)-Y^m(t)\|\Big)\\
&\quad + L_f\big\{|t_i^n-t_i^m|+\|u_i^n-u_i^m\|+L_h(\|u_i^n-u_i^m\|+|t_i^n-t_i^m|)\big\}\|Y^n(t)-Y^m(t)\|
\end{aligned}$$

which implies

$$\frac{d}{dt}\big\|U^n(t)-U^m(t)\big\|^2 \le \sigma_{nm}^1(t) \quad \text{(say)},$$

and $\sigma_{nm}^1(t)\to 0$ as $n,m\to\infty$. This implies

$$\|U^n(t)-U^m(t)\|^2 \le \sigma_{nm}^2(t),$$

where

$$\sigma_{nm}^2(t)=\int_0^t \sigma_{nm}^1(s)ds$$

and $\sigma_{nm}^2(t)\to 0$ as $n,m\to\infty$. Taking the supremum, we obtain

$$\sup_{t\epsilon(0,T]}\|U^n(t)-U^m(t)\|^2 \le \sigma_{nm}^2(t).$$

Using the above inequality, we conclude that $U^n\to u$ in $C([0,T],X)$. Since each U^n is uniformly Lipschitz continuous, it follows that u is Lipschitz continuous. □

Remark 3.5. As the sequence $U^n(t)-Y^n(t)\to 0$ as $n\to\infty$, $Y^n(t)\to u(t)$. Furthermore it is clear that $Y^n(t)\in D(A)$ for each $n\in\mathbb{N}$. Also $\|AY^n\|$ are bounded so by Lemma 2.4, we can conclude that $AY^n \rightharpoonup Au$.

Proof of Theorem 2.7. For every $x^*\in X^*$ and $t\in(0,T]$, we have

$$\int_0^t (AY^n(s),x^*)ds=(u_0,x^*)-(U^n(t),x^*)+\int_0^t (f^n(s),x^*)ds. \tag{3.9}$$

From the Lemma 3.4, Remark 3.5 and the bounded convergence theorem, from (3.9) after considering the limit as $n\to\infty$, we obtain

$$\int_0^t (Au(s),x^*)ds=(u_0,x^*)-(u(t),x^*)+\int_0^t (f(s,u(s),u(h(u(s),s))),x^*)ds. \tag{3.10}$$

Since $Au(t)$ is Bochner integrable on $[0,T]$, from equation (3.10), we obtain

$$\frac{d}{dt}u(t)+Au(t)=f(t,u(t),u(h(u(t),t))) \quad \text{a.e. } t\in(0,T]. \tag{3.11}$$

Now it is clear that $u\in C([0,T];X)$ and differentiable on $(0,T]$ with $u(t)\in D(A)$; $u(0)=u_0$ and satisfies the problem (3.11). Therefore it will be a strong solution of the problem (1.1) on $[0,T]$.

Next we prove the uniqueness of the solution. For this, if possible we assume that u_1 and u_2 are two strong solutions of (1.1). We put $u=u_1-u_2$, from (3.11), we have

$$\begin{aligned}
&(\frac{du(t)}{dt},P(u(t)))+(Au(t),P(u(t)))\\
&=(f(t,u_1(t),u(h(u_1(t),t))-f(t,u_2(t),u(h(u_2(t),t)),P(u(t))).
\end{aligned}$$

By using the definition of accretivity of A, we obtain

$$\begin{aligned}&\big(\frac{du(t)}{dt}, P(u(t))\big)\\ &\le (f(t,u_1(t),u_1(h(u_1(t),t)) - f(t,u_2(t),u_2(h(u_2(t),t)), P(u(t))).\end{aligned}$$

We used that

$$\big(\frac{du(t)}{dt}, P(u(t))\big) = \frac{1}{2}\frac{d}{dt}\|u(t)\|^2.$$

From an easy calculation we obtain

$$\frac{d}{dt}\|u(t)\|^2 \le C\|u(t)\|^2 \quad \text{a.e. } t \in (0,T],$$

where $C = 2L_f(1+L_h)$. Integrating over the interval $(0,t)$, we obtain

$$\|u(t)\|^2 \le C\int_0^t \|u(s)\|^2 ds.$$

Applying Gronwall's inequality, we obtain $u = 0$ on $[0,T]$. This shows the uniqueness of the strong solution and hence it completes the proof. □

4. Application

We consider the equation with deviating argument,

$$\begin{gathered}\frac{\partial u(t,x)}{\partial t} + \frac{\partial^2 u(t,x)}{\partial x^2} = F(x,u(t,x)) + G(t,x,u(t,x)),\\ u(t,0) = u(t,1), \quad 0 < t \le T,\\ u(0,x) = u_0(x), \quad x \in \Omega.\end{gathered} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$. Here

$$F(x,u(t,x)) = \int_0^x \xi(x,y)u(f(t)|u(t,y)|,y)dy \quad \forall (t,x) \in (0,T]\times\Omega.$$

We assume that $f : [0,T] \to \mathbb{R}_+$ is locally Hölder continuous in t with $f(0) = 0$; $\xi \in C^1(\Omega\times\Omega;\mathbb{R})$; the function $G : [0,T]\times\Omega\times\mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u and uniformly continuous in x.

Let $X = L^2(\Omega;\mathbb{R})$. We define $X_1 = D(A) = H^2(\Omega)\cap H_0^1(\Omega)$ and $Au = \frac{\partial^2 u}{\partial x^2}$. Then $X_{1/2} = D((A)^{1/2}) = H_0^1(\Omega)$.

For $x \in \Omega$, we define the map $f : [0,T]\times H^2(\Omega)\times L^2(\Omega) \to H_0^1(\Omega)$ by

$$g(t,\phi,\psi) = F(x,\psi) + G(t,x,\phi),$$

where $F(x,\psi(x,t)) = \int_0^x \xi(x,y)\psi(y,t)dy$. We also assume that the map $G : [0,T]\times \Omega\times H^2(\Omega) \to H_1^0(\Omega)$ that a $C > 0$,

$$\|G(t,x,u) - G(r,x,w)\| \le C(|t-r| + \|u-v\|).$$

Thus, the map g satisfies assumption (A1) (see [9]) and $h : H^2(\Omega)\times[0,T] \to \mathbb{R}_+$ defined by $h(\phi(x,t),t) = f(t)|\phi(x,t)|$ satisfies assumption (A2) (see [9]).

Then problem (4.1) reduces to the system

$$\begin{gathered}\frac{\partial u(t)}{\partial t} + Au(t) = f(t,u(t),u(h(u(t),t))), \quad t \in (0,T],\\ u(0) = u_0, \quad u_0 \in X,\end{gathered}$$

which is the same as in equation (1.1) and satisfies all the assumptions. By applying Theorem 2.7 we obtain a unique strong solution of (4.1).

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