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EXISTENCE AND BOUNDEDNESS OF SOLUTIONS FOR A KELLER-SEGEL SYSTEM WITH GRADIENT DEPENDENT CHEMOTACTIC SENSITIVITY

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ABSTRACT. We consider the Keller-Segel system with gradient dependent chemotactic sensitivity

$$\begin{gathered} u_t = \Delta u - \nabla \cdot (u|\nabla v|^{p-2}\nabla v), \quad x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, \quad x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega \end{gathered}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. We shown that for all reasonably regular initial data $u_0 \geq 0$ and $v_0 \geq 0$, the corresponding Neumann initial-boundary value problem possesses a global weak solution which is uniformly bounded provided that $1 < p < n/(n-1)$.

1. Introduction

In this article, we consider the chemotaxis system with gradient dependent chemotactic sensitivity

$$\begin{gathered} u_t = \Delta u - \nabla \cdot (u|\nabla v|^{p-2}\nabla v), \quad x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, \quad x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary and $1 < p < n/(n-1)$.

Keller and Segel [9] introduced a mathematical model to describe chemotactic aggregation of cellular slime molds. The classical Keller-Segel system is

$$\begin{gathered} u_t = \Delta u - \nabla(u\nabla v), \\ v_t = \Delta v - v + u, \end{gathered} \tag{1.2}$$

where u denotes the cell density and v describes the concentration of the chemical signal secreted by cells. This parabolic-parabolic Keller-Segel system has been studied extensively in literature, see the review paper [2, 6, 7] for details. Here we

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point out that the authors in [11] proved that (1.2) has global bounded solutions under the condition $\int_\Omega u_0(x) < 4\pi$ in $\mathbb{R}^2$ or under the condition $\int_\Omega u_0(x) < 8\pi$ for radial solutions on a disk. Winkler[20] proved that finite-time blow-up occurs for radially symmetric initial data when $\int_\Omega u_0$ is arbitrary prescribed number.

The chemotactic sensitivity can depend nonlinearly on the cell density. Some authors studied the system

$$\begin{aligned} u_t &= \nabla(D(u)\nabla u) - \nabla(S(u)\nabla v), \\ v_t &= \Delta v - v + u \end{aligned} \tag{1.3}$$

in the past decades. Horstmann and Winkler [8] determined the critical blow-up exponent for (1.3), where $D(u) = 1$ and the chemotactic sensitivity equals some nonlinear function of the particle density. In [18], it is proved that if $S(u)/D(u)$ grows faster than $u^{2/n}$ as $u \to \infty$ and $D(u)$ satisfies some technical conditions, then there exist solutions that blow up in either finite or infinite time. In [14], Tao and Winkler showed that if $S(u)/D(u) \le cu^\alpha$ with $\alpha < 2/n$ and $D(u)$ satisfies algebraic upper and lower growth, then the classical solutions to (1.3) are uniformly bounded.

By the Weber-Fechner law, the classical Keller-Segel system has been modified to the Keller-Segel system with a singular sensitivity

$$\begin{aligned} u_t &= \Delta u - \chi\nabla\left(\frac{u}{v}\nabla v\right), \\ v_t &= \Delta v - v + u. \end{aligned} \tag{1.4}$$

Winkler [19] proved that if $0 < \chi < \sqrt{2/n}$, (1.4) has a global-in-time classical solution. Furthermore, relaxing the solution concept, the global existence of weak solutions is established whenever $0 < \chi < \sqrt{(n+2)/(3n-4)}$. In [13], Stinner and Winkler introduced a generalized solution concept, and then proved that such generalized solution for any $\chi > 0$. In [10], the authors introduced another generalized solution concept, which exists for the some range of χ.

Recently, Bellomo and Winkler posed a model where the chemotactic sensitivity depends on ∇v. In [3] the authors deduced the existence of a unique radial classical solution to the system

$$\begin{aligned} u_t &= \nabla\cdot\Big(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\Big) - \chi\nabla\cdot\Big(\frac{u\nabla v}{\sqrt{1+|\nabla v|^2}}\Big), \\ 0 &= \Delta v - M + u, \end{aligned} \tag{1.5}$$

where $M = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$, $n \ge 2$ and $\chi < 1$. In [4], it is showed that for some $T > 0$, (1.5) possesses a uniquely determined classical solution blowing up at time T. [22] concerns the null controllability of a control system governed by coupled degenerate parabolic equations with lower order terms.

Negreanu and Tello [12] proposed the model

$$\begin{aligned} u_t &= \Delta u - \nabla\cdot(\chi u|\nabla v|^{p-2}\nabla v), \\ 0 &= \Delta v - M + u, \end{aligned} \tag{1.6}$$

where $M = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$. The authors obtained uniform bounds in $L^\infty(\Omega)$ provided that $1 < p < n/(n-1)$ $(n > 1)$. In the one-dimensional case, they proved that for any positive constants χ and M, if $p \in (1,2)$, then the model (1.6) has infinitely many non-constant solutions.

In this article, we study the global existence and boundedness of (1.1), the parabolic-parabolic version of (1.6). Now we state the main results of this article. We assume that the initial data u_0 and v_0 satisfy

$$\begin{gathered} u_0 \in C^0(\bar{\Omega}) \quad \text{with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) \quad \text{with } v_0 \geq 0 \text{ in } \bar{\Omega}. \end{gathered} \tag{1.7}$$

Our main results read as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with smooth boundary. Then for all u_0 and v_0 satisfying (1.7), system (1.1) with $1 < p < n/(n-1)$ possesses at least one global weak solution in the sense of Definition 2.1.*

Theorem 1.2. *Under the assumption of Theorem 1.1, there exists a constant $C = C(u_0, p, \Omega) > 0$, such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0.$$

The rest of this article is organized as follows. In Section 2, we introduce the conception of the weak solution. Section 3 is devoted to showing the existence of the weak solution. Finally, we give the proof of the boundedness in Section 4.

2. A weak solution concept and approximate problems

Let us firstly introduce a natural concept of weak solutions to (1.1).

Definition 2.1. Assume that u_0 and v_0 satisfy (1.7). For all $T > 0$, a pair (u, v) of functions

$$u \in L^\infty(\bar{\Omega} \times [0,T)), \quad v \in L^\infty(\bar{\Omega} \times [0,T)) \cap L^2([0,T); W^{1,2}(\Omega)) \tag{2.1}$$

with

$$u \geq 0 \text{ a.e. in } \Omega \times (0,T) \text{ and } v \geq 0 \text{ a.e. in } \Omega \times (0,T), \tag{2.2}$$

and

$$|\nabla v|^{p-2}\nabla v \in L^2(\bar{\Omega} \times [0,T)), \tag{2.3}$$

will be called a *weak solution* of (1.4) if u has the mass conservation property

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x) \quad \text{for a.e. } t > 0, \tag{2.4}$$

and the following two identities

$$-\int_\Omega u_0\varphi(\cdot,0) - \int_0^T\int_\Omega u\varphi_t = \int_0^T\int_\Omega u\cdot\Delta\varphi + \int_0^T\int_\Omega u|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi \tag{2.5}$$

and

$$\int_0^T\int_\Omega v\psi_t + \int_\Omega v_0\psi(\cdot,0) = \int_0^T\int_\Omega \nabla v\cdot\nabla\psi + \int_0^T\int_\Omega v\psi - \int_0^T\int_\Omega u\psi \tag{2.6}$$

hold for non-negative φ, $\psi \in C_0^\infty(\bar{\Omega} \times [0,T))$.

We intend to construct a solution of (1.1) as the limit of a sequence of solutions to the approximate problems

$$\begin{gathered} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla\cdot\Big(u_\varepsilon(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon\Big), \quad x\in\Omega,\ t>0,\\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, \quad x\in\Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_0(x), \quad v_\varepsilon(x,0)=v_0(x), \quad x\in\Omega, \end{gathered} \tag{2.7}$$

where $\varepsilon\in(0,1)$ is a positive parameter. We construct a suitable fixed point framework to prove the existence of classical solutions to (2.7).

Lemma 2.2. *Assume that* (1.7) *holds, and let* $\varepsilon\in(0,1)$. *Then there exists* $T_{\max,\varepsilon}\le\infty$, *such that* (2.7) *possesses a classical solution* $(u_\varepsilon,v_\varepsilon)$,

$$\begin{gathered} u_\varepsilon\in C^0(\bar\Omega\times[0,T_{max,\varepsilon}))\cap C^{2,1}(\bar\Omega\times(0,T_{max,\varepsilon}))\\ v_\varepsilon\in C^0(\bar\Omega\times[0,T_{max,\varepsilon}))\cap C^{2,1}(\bar\Omega\times(0,T_{max,\varepsilon}))\cap L^\infty_{\rm loc}([0,T_{max,\varepsilon});W^{1,\vartheta}(\Omega)) \end{gathered}$$

for each $\vartheta>n$, *which satisfies* $u_\varepsilon>0$ *in* $\bar\Omega\times(0,\infty)$ *and*

$$\int_\Omega u_\varepsilon(x,t)dx = \int_\Omega u_0(x)dx \quad \text{for all } t\in(0,T_{\max,\varepsilon}), \tag{2.8}$$

as well as

$$\int_\Omega v_\varepsilon(t) = \int_\Omega u_0 + \Big(\int_\Omega v_0 - \int_\Omega u_0\Big)e^{-t} \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{2.9}$$

Proof. Let us prove the existence of solutions by a standard contraction argument referring to [8]. For $T\in(0,1)$, we define a Banach space

$$X := C^0(\bar\Omega\times[0,T])\times L^\infty((0,T);W^{1,\vartheta}(\Omega)).$$

Consider the closed set

$$S := \big\{(u_\varepsilon,v_\varepsilon)\in X : \|(u_\varepsilon,v_\varepsilon)\|_X\le R\big\} \quad \text{with } R=\|(u_0,v_0)\|_X+1.$$

We claim that for T sufficiently small, the map

$$\begin{aligned} \Psi(u_\varepsilon,v_\varepsilon)(t) &:= \begin{pmatrix}\Psi_1(u_\varepsilon,v_\varepsilon)(t)\\ \Psi_2(u_\varepsilon,v_\varepsilon)(t)\end{pmatrix}\\ &:= \begin{pmatrix} e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon(s))ds\\ e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u_\varepsilon(s)ds \end{pmatrix} \end{aligned}$$

is a contraction from S to S. We fix $\beta\in(\frac{n}{2\vartheta},\frac12)$ and $\delta\in(0,\frac12-\beta)$. Then for all $t\in[0,T]$ we have

$$\begin{aligned} &\|\Psi_1(u_\varepsilon,v_\varepsilon)(t)\|_{C^0(\bar\Omega)}\\ &\le \|e^{t\Delta}u_0\|_{C^0(\bar\Omega)}\\ &\quad + C\int_0^t \|(-\Delta+1)^\beta e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon(s))\|_{L^\vartheta(\Omega)}ds\\ &\le \|u_0\|_{C^0(\bar\Omega)} + C\int_0^t (t-s)^{-\beta-\frac12-\delta}\|u_\varepsilon(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon(s)\|_{L^\vartheta(\Omega)}ds\\ &\le \|u_0\|_{C^0(\bar\Omega)} + CR^pT^{\frac12-\beta-\delta}, \end{aligned} \tag{2.10}$$

where we have used the estimate

$$\begin{aligned}\|u_\varepsilon(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon\|_{L^\vartheta(\Omega)} &\le R\||\nabla v_\varepsilon|^{p-1}\|_{L^\vartheta(\Omega)}\\ &\le R\|\nabla v_\varepsilon\|^{p-1}_{L^{\vartheta(p-1)}(\Omega)}\\ &\le CR\|\nabla v_\varepsilon\|^{p-1}_{L^\vartheta(\Omega)}.\end{aligned}$$

Let $\gamma\in(1/2,1)$; for for all $t\in[0,T]$ we have

$$\begin{aligned}&\|\Psi_2(u_\varepsilon,v_\varepsilon)(t)\|_{w^{1,q}(\Omega)}\\ &\le \|e^{t(\Delta-1)}v_0\|_{W^{1,\vartheta}(\Omega)}+C\int_0^t\|(-\Delta+1)^\gamma e^{(t-s)(\Delta-1)}u_\varepsilon(s)\|_{L^\vartheta(\Omega)}ds\\ &\le \|v_0\|_{W^{1,\vartheta}(\Omega)}+C\int_0^t(t-s)^\gamma\|u_\varepsilon(s)\|_{L^\vartheta(\Omega)}ds\\ &\le \|v_0\|_{W^{1,\vartheta}(\Omega)}+CRT^{1-\gamma}.\end{aligned}\tag{2.11}$$

From (2.10) and (2.11), it follows that $\Psi S\subset S$ if we choose T small. For all $(u_\varepsilon,v_\varepsilon),(\bar u_\varepsilon,\bar v_\varepsilon)\in S$, we have

$$\begin{aligned}&\|\Psi_1(u_\varepsilon,v_\varepsilon)(t)-\Psi_1(\bar u_\varepsilon,\bar v_\varepsilon)(t)\|_{C^0(\bar\Omega)}\\ &\le C\int_0^t\Big\|(-\Delta+1)^\beta e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon(s)\\ &\quad-\bar u_\varepsilon(|\nabla\bar v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla\bar v_\varepsilon(s))\Big\|_{L^\vartheta(\Omega)}ds\\ &\le C\int_0^t(t-s)^{-\beta-\frac12-\delta}\Big\|u_\varepsilon(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon(s)\\ &\quad-\bar u_\varepsilon(|\nabla\bar v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla\bar v_\varepsilon(s)\Big\|_{L^\vartheta(\Omega)}ds\\ &\le C(R+R^{p-1})T^{\frac12-\beta-\delta}\|(u_\varepsilon,v_\varepsilon)-(\bar u_\varepsilon,\bar v_\varepsilon)\|_X\end{aligned}$$

and

$$\begin{aligned}&\|\Psi_2(u_\varepsilon,v_\varepsilon)(t)-\Psi_2(\bar u_\varepsilon,\bar v_\varepsilon)(t)\|_{W^{1,\vartheta}(\Omega)}\\ &\le C\int_0^t\|(\Delta+1)^\gamma e^{(t-s)(\Delta-1)}(u_\varepsilon(s)-\bar u_\varepsilon)\|_{L^\vartheta(\Omega)}ds\\ &\le C\int_0^t(t-s)^{-\gamma}\|u_\varepsilon(s)-\bar u_\varepsilon\|_{L^\vartheta(\Omega)}ds\\ &\le CT^{1-\gamma}\|(u_\varepsilon,v_\varepsilon)-(\bar u_\varepsilon,\bar v_\varepsilon)\|_X,\end{aligned}$$

so Ψ is shown to be a contraction if T is sufficiently small. By the Banach's fixed point theorem, we obtain that the existence of $(u,v)\in X$ satisfies $(u,v)=\Psi(u,v)$.

Properties (2.8) and (2.9) follow by integrating the PDEs in (2.7) in space. □

3. Existence of the weak solutions

The construction of a global weak solution is based on a limit procedure of solutions to suitably regularized problems. The Aubin-Lions lemma is very helpful. We collect some ε-independent a priori estimates of the solutions to (2.7). For the second equation in (2.7), using the parabolic theory, we obtain the following lemma.

Lemma 3.1 ([19, Lemma 2.4]). *Let $T>0$ and $1\le\theta,\mu<\infty$.*

(i) *If* $\frac{n}{2}(\frac{1}{\theta}-\frac{1}{\mu})<1$ *then there exists* $C>0$ *such that*

$$\|v_\varepsilon(\cdot,t)\|_{L^\mu(\Omega)} \le C\Big(1+\sup_{s\in(0,t)}\|u_\varepsilon(\cdot,s)\|_{L^\theta(\Omega)}\Big) \tag{3.1}$$

for all $t\in(0,T)$ *and* $\varepsilon\in(0,1)$.

(ii) *If* $\frac{1}{2}+\frac{n}{2}(\frac{1}{\theta}-\frac{1}{\mu})<1$ *then*

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\mu(\Omega)} \le C\Big(1+\sup_{s\in(0,t)}\|u_\varepsilon(\cdot,s)\|_{L^\theta(\Omega)}\Big) \tag{3.2}$$

for all $t\in(0,T)$ *and* $\varepsilon\in(0,1)$ *is valid with* $C>0$.

Proof. For convenience, we give the proof.

(i) We represent v_ε by

$$v_\varepsilon(\cdot,t)=e^{t(\Delta-1)}v_0+\int_0^t e^{(t-s)(\Delta-1)}u_\varepsilon(\cdot,s)ds, \tag{3.3}$$

where $(e^{t\Delta})_{t\ge 0}$ denotes the Neumann heat semigroup. By standard smoothing estimates, we find that if $\mu\ge\theta$ then

$$\|v_\varepsilon(\cdot,t)\|_{L^\mu(\Omega)} \le C\Big(\|v_0\|_{L^\infty(\Omega)}+\int_0^t (t-s)^{-\frac{n}{2}-(\frac{1}{\theta}-\frac{1}{\mu})}\|u_\varepsilon(\cdot,s)\|_{L^\mu(\Omega)}ds\Big) \tag{3.4}$$

for a constant $C>0$. By (3.4) and Hölder's inequality, we obtain (3.1) for $\mu<\theta$.

(ii) Applying ∇ to both sides in (3.3) and invoking corresponding smoothing properties involving gradient [16], we similarly find that

$$\|\nabla v_\varepsilon(\cdot.t)\|_{L^\mu(\Omega)} \le C\Big(\|\nabla v_0\|_{L^\infty(\Omega)}+\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}-(\frac{1}{\theta}-\frac{1}{\mu})}\|u_\varepsilon(\cdot,s)\|_{L^\mu(\Omega)}ds\Big)$$

with a certain $C>0$. So we conclude using the similar method of proving (i). □

With Lemma 3.1 in hand, using the Gagliardo-Nirenberg inequality, we can prove the boundedness in the L^2-norm of u_ε.

Lemma 3.2. *Let* $1<p<n/(n-1)$. *For all* $T>0$, *there exists* $C>0$ *such that for any* $\varepsilon\in(0,1)$,

$$\int_0^T\int_\Omega u_\varepsilon^2 \le C(T+1). \tag{3.5}$$

Proof. We multiply the first equation in (2.7) by u_ε, and integrate by parts to find that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2=-\int_\Omega|\nabla u_\varepsilon|^2+\int_\Omega u_\varepsilon\left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{\frac{p-2}{2}}\nabla v_\varepsilon\cdot\nabla u_\varepsilon.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{d}{dt}\int_\Omega u_\varepsilon^2+\int_\Omega|\nabla u_\varepsilon|^2 \le \int_\Omega u_\varepsilon^2\left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{p-2}|\nabla v_\varepsilon|^2.$$

We can find μ satisfying $2(p-1)<\mu<n/(n-1)$. Using Lemma 3.1 and Hölder's inequality, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon^2+\int_\Omega|\nabla u_\varepsilon|^2 &\le \int_\Omega u_\varepsilon^2\Big(|\nabla v_\varepsilon|^2+\varepsilon\Big)^{p-1}\\ &\le \Big(\int_\Omega u_\varepsilon^{\frac{2\mu}{\mu-2(p-1)}}\Big)^{\frac{\mu-2p-1}{\mu}}\Big(\int_\Omega\big(|\nabla v_\varepsilon|^2+\varepsilon\big)^{\frac{\mu}{2}}\Big)^{\frac{2(p-1)}{\mu}}\\ &\le C\Big(\int_\Omega u_\varepsilon^{\frac{2\mu}{\mu-2(p-1)}}\Big)^{\frac{\mu-2p-1}{\mu}}\Big[\Big(\int_\Omega|\nabla v_\varepsilon|^\mu\Big)^{\frac{2(p-1)}{\mu}}+1\Big]\\ &\le C\Big(\int_\Omega u_\varepsilon^{\frac{2\mu}{\mu-2(p-1)}}\Big)^{\frac{\mu-2p-1}{\mu}}. \end{aligned} \tag{3.6}$$

Using the Gagliardo-Nirenberg inequality, we can find a positive constant $C>0$ such that

$$\|u_\varepsilon\|_{L^{\frac{2\mu}{\mu-2(p-1)}}(\Omega)}\le C\|\nabla u_\varepsilon\|^a_{L^2(\Omega)}\|u_\varepsilon\|^{1-a}_{L^1(\Omega)}+C\|u_\varepsilon\|_{L^1(\Omega)}, \tag{3.7}$$

where

$$a=\frac{1-\frac{\mu-2(p-1)}{2\mu}}{\frac12+\frac1n}.$$

Thanks to $1<p<n/(n-1)$, we have $a\in(0,1)$. We now apply inequality (3.7) to (3.6), and obtain

$$\begin{aligned} \Big(\int_\Omega u_\varepsilon^{\frac{2\mu}{\mu-2(p-1)}}\Big)^{\frac{\mu-2p-1}{\mu}} &\le C\Big(\|\nabla u_\varepsilon\|^a_{L^2(\Omega)}\|u_\varepsilon\|^{1-a}_{L^1(\Omega)}+\|u_\varepsilon\|_{L^1(\Omega)}\Big)^2\\ &\le C(\|\nabla u_\varepsilon\|^{2a}_{L^2(\Omega)}+1). \end{aligned}$$

By Young's inequality for a positive constant $\delta\in(0,1)$, we have

$$\frac{d}{dt}\int_\Omega u_\varepsilon^2+\int_\Omega|\nabla u_\varepsilon|^2\le C(\|\nabla u_\varepsilon\|^{2a}_{L^2(\Omega)}+1)\le\delta\int_\Omega|\nabla u_\varepsilon|^2+C(\delta),$$

which is equivalent to

$$\frac{d}{dt}\int_\Omega u_\varepsilon^2+(1-\delta)\int_\Omega|\nabla u_\varepsilon|^2\le C.$$

By the Poincaré-Wirtinger inequality, we obtain

$$\int_\Omega|\nabla u_\varepsilon|^2\ge C\int_\Omega\Big(u_\varepsilon-\frac{1}{|\Omega|}\int_\Omega u_\varepsilon\Big)^2=C\Big(\int_\Omega u_\varepsilon^2-\frac{1}{|\Omega|}\Big|\int_\Omega u_\varepsilon\Big|^2\Big),$$

which implies

$$\frac{d}{dt}\int_\Omega u_\varepsilon^2+\int_\Omega u_\varepsilon^2\le C.$$

Finally using the standard ODE argument, we obtain (3.5). □

Next, we prove the almost everywhere convergence of u_{ε_k} by referring to the method in [21].

Lemma 3.3. *Let $1<p<n/(n-1)$. For all $T>0$, there exists $C>0$ such that for any $\varepsilon\in(0,1)$, we have*

$$\int_0^T\int_\Omega|\nabla\ln(u_\varepsilon+1)|^2\le C(T+1). \tag{3.8}$$

Proof. We multiply the first equation in (2.7) by $\frac{1}{u_\varepsilon+1}$, and integrate by parts to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \ln(u_\varepsilon+1)\\
&=\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}-\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\Big(\nabla u_\varepsilon\cdot\big(|\nabla v_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}\nabla v_\varepsilon\Big)\\
&=\int_\Omega |\nabla \ln(u+1)|^2-\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}\Big(\nabla \ln(u_\varepsilon+1)\cdot\big(|\nabla v_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}\nabla v_\varepsilon\Big).
\end{aligned}$$

By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}\Big(\nabla \ln(u_\varepsilon+1)\cdot\big(|\nabla v_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}\nabla v_\varepsilon\Big)\\
&\le \frac12\int_\Omega |\nabla \ln(u_\varepsilon+1)|^2+\frac12\int_\Omega \frac{u_\varepsilon^2}{(u_\varepsilon+1)^2}\big(|\nabla v_\varepsilon|^2+\varepsilon\big)^{p-2}|\nabla v_\varepsilon|^2\\
&\le \frac12\int_\Omega |\nabla \ln(u_\varepsilon+1)|^2+\frac12\int_\Omega \frac{u_\varepsilon^2}{(u_\varepsilon+1)^2}\big(|\nabla v_\varepsilon|^2+\varepsilon\big)^{p-1}\\
&\le \frac12\int_\Omega |\nabla \ln(u_\varepsilon+1)|^2+\frac12\int_\Omega \big(|\nabla v_\varepsilon|^2+\varepsilon\big)^{p-1}.
\end{aligned}$$

Then, we have

$$\frac{d}{dt}\int_\Omega \ln(u_\varepsilon+1)\ge \int_\Omega |\nabla \ln(u+1)|^2-\frac12\int_\Omega |\nabla \ln(u_\varepsilon+1)|^2-\frac12\int_\Omega \big(|\nabla v_\varepsilon|^2+\varepsilon\big)^{p-1}.$$

By integrating with respect to time we obtain

$$\begin{aligned}
&\frac12\int_0^T\int_\Omega |\nabla \ln(u_\varepsilon+1)|^2\\
&\le \int_\Omega \ln(u_\varepsilon(\cdot,T)+1)-\int_\Omega \ln(u_0+1)+\frac12\int_0^T\int_\Omega (|\nabla v_\varepsilon|^2+\varepsilon)^{p-1}\\
&\le \int_\Omega u_\varepsilon+\frac12\int_0^T\int_\Omega (|\nabla v_\varepsilon|^2+\varepsilon)^{p-1}\\
&\le m+\frac12\int_0^T\int_\Omega |\nabla v_\varepsilon|^{2(p-1)}+C,
\end{aligned}$$

where $m:=\int_\Omega u_0$. From $2(p-1)<n/(n-1)$, we obtain (3.8) by Lemma 3.1. □

Lemma 3.4. *Let $1<p<n/(n-1)$. For all $T>0$, there exists $C>0$ such that for any $\varepsilon\in(0,1)$,*

$$\int_0^T \|\partial_t \ln(u_\varepsilon+1)\|_{(W^{n,2}(\Omega))^*}dt\le C(T+1). \tag{3.9}$$

Proof. Testing the first equation in (2.7) by $\frac{\psi}{u_\varepsilon+1}$ for fixed $t>0$ and arbitrary $\psi\in C^\infty(\bar\Omega)$, we obtain

$$\begin{aligned}
\int_\Omega \partial_t \ln(u_\varepsilon+1)\cdot\psi&=\int_\Omega |\nabla \ln(u_\varepsilon+1)|^2\psi-\int_\Omega \nabla \ln(u_\varepsilon+1)\cdot\nabla\psi\\
&\quad-\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}\Big(\nabla \ln(u_\varepsilon+1)\cdot\big(|\nabla v_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}\nabla v_\varepsilon\Big)\psi
\end{aligned}$$

$$+\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}\left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{\frac{p-2}{2}}\nabla v_\varepsilon\cdot\nabla\psi.$$

By the Cauchy-Schwarz inequality and Young's inequality, we have

$$\begin{aligned}
&\Big|\int_\Omega \partial_t \ln(u_\varepsilon+1)\cdot\psi\Big| \\
&\le \int_\Omega |\nabla\ln(u_\varepsilon+1)|^2\|\psi\|_{L^\infty(\Omega)} + \Big(\int_\Omega |\ln(u_\varepsilon+1)|^2\Big)^{1/2}\|\nabla\psi\|_{L^2(\Omega)} \\
&\quad + \Big(\frac12\int_\Omega \frac{u_\varepsilon^2}{(u_\varepsilon+1)^2}|\nabla\ln(u_\varepsilon+1)|^2 + \frac12\int_\Omega \left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{p-2}|\nabla v_\varepsilon|^2\Big)\|\psi\|_{L^\infty(\Omega)} \\
&\quad + \Big(\int_\Omega \frac{u_\varepsilon^2}{(u_\varepsilon+1)^2}\left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{p-2}|\nabla v_\varepsilon|^2\Big)^{1/2}\|\nabla\psi\|_{L^2(\Omega)} \\
&\le \Big(\int_\Omega |\nabla\ln(u_\varepsilon+1)|^2 + \frac12\int_\Omega |\nabla\ln(u_\varepsilon+1)|^2 + \frac12\int_\Omega \left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{p-1}\Big)\|\psi\|_{L^\infty(\Omega)} \\
&\quad + \Big(\Big(\int_\Omega |\nabla\ln(u_\varepsilon+1)|^2\Big)^{1/2} + \Big(\int_\Omega \left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{p-1}\Big)^{1/2}\Big)\|\nabla\psi\|_{L^2(\Omega)} \\
&\le \Big(2\int_\Omega |\nabla\ln(u_\varepsilon+1)|^2 + \int_\Omega \left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{p-1}+1\Big)\Big(\|\psi\|_{L^\infty(\Omega)}+\|\nabla\psi\|_{L^2(\Omega)}\Big).
\end{aligned}$$

Since in view of the fact that $W^{n,2}(\Omega)\hookrightarrow L^\infty(\Omega)$ we can fix $C>0$ such that

$$\|\nabla\psi\|_{L^2(\Omega)}+\|\psi\|_{L^\infty(\Omega)}\le C\|\psi\|_{W^{n,2}(\Omega)}$$

for any such ψ, this entails

$$\begin{aligned}
&\|\partial_t\ln(u_\varepsilon(\cdot,t)+1)\|_{(W^{n,2}(\Omega))^*} \\
&\le C\Big(2\int_\Omega |\nabla\ln(u_\varepsilon+1)|^2+\int_\Omega \left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{p-1}+1\Big).
\end{aligned}$$

After an integration with respect to time, by Lemmas 3.1 and 3.3, this implies (3.9). ☐

On the basis of previous three lemmas, we can extract a subsequence of the approximate solutions of (2.7). By the compactness arguments, the limit function can be shown to be a weak solution of (1.1).

Lemma 3.5. *Let $1<p<n/(n-1)$. There exist non-negative functions u,v defined on $\Omega\times(0,\infty)$ as well as a sequence $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,1)$, and such that as $\varepsilon=\varepsilon_k\searrow 0$,*

$$u_\varepsilon\to u \quad \text{a.e. in } \Omega\times(0,T), \tag{3.10}$$

$$u_\varepsilon\rightharpoonup u \quad \text{in } L^2(\Omega\times(0,T)), \tag{3.11}$$

$$v_\varepsilon\to v \quad \text{in } L^2((0,T);W^{1,2}(\Omega)), \tag{3.12}$$

$$\nabla v_\varepsilon\to\nabla v \quad \text{a.e. in } \Omega\times(0,T), \tag{3.13}$$

$$|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon\rightharpoonup|\nabla v|^{p-2}\nabla v \quad \text{in } L^{p'}(\Omega\times(0,T)), \tag{3.14}$$

where $\frac1p+\frac1{p'}=1$.

Proof. By Lemmas 3.3, 3.4 and the Aubin-Lions lemma([15]), we choose a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,1)$ such that $\ln(u_\varepsilon+1)\to\ln(u+1)$ in $L^2(\Omega\times(0,T))$ as $\varepsilon=\varepsilon_k\searrow 0$, $k\to\infty$. Then we have $\ln(u_\varepsilon+1)\to\ln(u+1)$ a.e. in $\Omega\times(0,T)$ and

(3.10) is deduced. By Lemma 3.2 and (3.10), we obtain (3.11). It follows from the parabolic regularity theory [5, Theorem 3.1] and Lemma 3.2 that

$$\|v_\varepsilon\|_{L^2((0,T);W^{2,2}(\Omega))} + \|v_{\varepsilon t}\|_{L^2(\Omega\times(0,T))} \le C(T+1).$$

Choosing an appropriate subsequence again and applying the Aubin-Lions lemma [15], we obtain (3.12). Then (3.13) results from (3.12). Since

$$\begin{aligned} \int_0^T \int_\Omega \left(|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon\right)^{p'} &\le \int_0^T \int_\Omega |\nabla v_\varepsilon|^{p'(p-1)} \\ &= \int_0^T \int_\Omega |\nabla v_\varepsilon|^p \\ &\le C\int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \le C(T+1), \end{aligned} \tag{3.15}$$

we obtain (3.14) by (3.13) and (3.15). □

Now we are ready to prove the main result of this section.

Proof of Theorem 1.1. For arbitrary non-negative $\varphi \in C_0^\infty(\bar\Omega \times [0,T))$, multiplying the first equation in (2.7) by φ, and integrating by parts, we have

$$\begin{aligned} &-\int_\Omega u_0(x)\varphi(\cdot,0) - \int_0^T \int_\Omega u_\varepsilon \varphi_t \\ &= \int_0^T \int_\Omega u_\varepsilon \cdot \Delta\varphi + \int_0^T \int_\Omega u_\varepsilon(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon \cdot \nabla\varphi \end{aligned} \tag{3.16}$$

for all $\varepsilon \in (0,1)$. Choosing $T > 0$ large enough such that $\varphi \equiv 0$ in $\Omega \times (T,\infty)$. Since $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega\times(0,T))$ as $\varepsilon = \varepsilon_k \searrow 0$ by (3.11), we have

$$\int_0^T \int_\Omega u_\varepsilon \varphi_t \to \int_0^T \int_\Omega u\varphi_t \quad \text{and} \quad \int_0^T \int_\Omega u_\varepsilon \cdot \Delta\varphi \to \int_0^T \int_\Omega u \cdot \Delta\varphi \tag{3.17}$$

as $\varepsilon = \varepsilon_k \searrow 0$. Moreover, because $|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \rightharpoonup |\nabla v|^{p-2}\nabla v$ in $L^{p'}(\Omega\times(0,T))$ as $\varepsilon = \varepsilon_k \searrow 0$ by (3.14), we can choose a subsequence which is also written as v_{ε_k} such that $|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \to |\nabla v|^{p-2}\nabla v$ in $L^2(\Omega\times(0,T))$ as $\varepsilon = \varepsilon_k \searrow 0$. Then we have

$$\int_0^T \int_\Omega u_\varepsilon(|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla v_\varepsilon \cdot \nabla\varphi \to \int_0^T \int_\Omega u|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi \tag{3.18}$$

as $\varepsilon = \varepsilon_k \searrow 0$. Then (2.5) follows from (3.16)-(3.18).

Finally, for arbitrary non-negative $\psi \in C_0^\infty(\bar\Omega \times [0,\infty))$, multiplying the second equation in (2.7) by ψ, and integrating by parts, we have

$$\int_\Omega v_0\psi(\cdot,0) + \int_0^T \int_\Omega v_\varepsilon\psi_t = \int_0^T \int_\Omega \nabla v_\varepsilon \cdot \nabla\psi + \int_0^T \int_\Omega v_\varepsilon\psi - \int_0^T \int_\Omega u_\varepsilon\psi \tag{3.19}$$

for all $\varepsilon \in (0,1)$. Thanks to (3.12), We can find that each of the terms in (3.19) converges to its expected limits as $\varepsilon = \varepsilon_k \searrow 0$. So (2.6) results from (3.19). □

4. Boundedness

In this section, our goal is to prove Theorem 1.2. Firstly, by means of a Moser-Alikakos iteration, we can achieve the following boundedness results.

Lemma 4.1. *Let $1 < p < n/(n-1)$. For all $t > 0$, there exists $C > 0$ such that for any $\varepsilon \in (0,1)$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C. \tag{4.1}$$

Proof. We multiply the first equation in (2.7) by u_ε^{q-1} (for $q > 1$), and integrate by parts to find that

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u_\varepsilon^q = -(q-1)\int_\Omega u_\varepsilon^{q-2}|\nabla u_\varepsilon|^2 + (q-1)\int_\Omega u_\varepsilon^{q-1}\left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{\frac{p-2}{2}}\nabla v_\varepsilon\cdot\nabla u_\varepsilon.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega u_\varepsilon^q + \frac{2(q-1)}{q^2}\int_\Omega |\nabla u_\varepsilon^{q/2}|^2 &\le \frac{q-1}{2}\int_\Omega u_\varepsilon^q\left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{p-2}|\nabla v_\varepsilon|^2\\ &\le \frac{q-1}{2}\int_\Omega u_\varepsilon^q\left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{p-1}\end{aligned}$$

We can find a positive constant μ satisfying $2(p-1) < \mu < n/(n-1)$. Using Lemma 3.1 and Hölder's inequality, we have

$$\begin{aligned}&\frac{1}{q}\frac{d}{dt}\int_\Omega u_\varepsilon^q + \frac{2(q-1)}{q^2}\int_\Omega |\nabla u_\varepsilon^{q/2}|^2\\ &\le \frac{q-1}{2}\Big(\int_\Omega u_\varepsilon^{\frac{q}{2}\frac{2\mu}{\mu-2(p-2)}}\Big)^{\frac{\mu-2(p-1)}{\mu}}\Big(\int_\Omega\left(|\nabla v_\varepsilon|^2+\varepsilon\right)^{\frac{\mu}{2}}\Big)^{\frac{2(p-1)}{\mu}}\\ &\le C\cdot\frac{q-1}{2}\Big(\int_\Omega u_\varepsilon^{\frac{q}{2}\cdot\frac{2\mu}{\mu-2(p-1)}}\Big)^{\frac{\mu-2p-1}{\mu}}\Big[\Big(\int_\Omega|\nabla v_\varepsilon|^\mu\Big)^{\frac{2(p-1)}{\mu}}+1\Big]\\ &\le C\cdot\frac{q-1}{2}\Big(\int_\Omega u_\varepsilon^{\frac{q}{2}\cdot\frac{2\mu}{\mu-2(p-1)}}\Big)^{\frac{\mu-2p-1}{\mu}}.\end{aligned} \tag{4.2}$$

By the Gagliardo-Nirenberg inequality, we can find a positive constant $C > 0$ such that

$$\|u_\varepsilon^{q/2}\|_{L^{\frac{2\mu}{\mu-2(p-1)}}(\Omega)} \le C\|\nabla u_\varepsilon^{q/2}\|_{L^2(\Omega)}^a\|u_\varepsilon^{q/2}\|_{L^1(\Omega)}^{1-a} + C\|u_\varepsilon^{q/2}\|_{L^1(\Omega)}, \tag{4.3}$$

where

$$a = \frac{1-\frac{\mu-2(p-1)}{2\mu}}{\frac{1}{2}+\frac{1}{n}}.$$

Since $1 < p < n/(n-1)$, we have $a \in (0,1)$. We apply inequality (4.3) to (4.2) and use Young's inequality to obtain

$$\begin{aligned}&\Big(\int_\Omega u_\varepsilon^{\frac{q}{2}\cdot\frac{2\mu}{\mu-2(p-1)}}\Big)^{\frac{\mu-2p-1}{\mu}}\\ &= \|u_\varepsilon^{q/2}\|^2_{L^{\frac{2\mu}{\mu-2(p-1)}}(\Omega)}\\ &\le C\|\nabla u_\varepsilon^{q/2}\|_{L^2(\Omega)}^{2a}\|u_\varepsilon^{q/2}\|_{L^1(\Omega)}^{2(1-a)} + C\|u_\varepsilon^{q/2}\|^2_{L^1(\Omega)}\\ &\le \frac{2}{Cq^2}\int_\Omega|\nabla u_\varepsilon^{\frac{a}{2}}|^2 + (1-a)[Caq^2]^{\frac{a}{1-a}}\Big(\int_\Omega u_\varepsilon^{q/2}\Big)^2 + C\Big(\int_\Omega u_\varepsilon^{q/2}\Big)^2.\end{aligned}$$

Then we have

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u_\varepsilon^q + \frac{q-1}{q^2}\int_\Omega |\nabla u_\varepsilon^{q/2}|^2 \le C(q-1)q^{\frac{2a}{1-a}}\Big(\int_\Omega u_\varepsilon^{q/2}\Big)^2,$$

which is equivalent to

$$\frac{q}{q-1}\frac{d}{dt}\int_\Omega u_\varepsilon^q + \int_\Omega |\nabla u_\varepsilon^{q/2}|^2 \le Cq^{\frac{2}{1-a}}\Big(\int_\Omega u_\varepsilon^{q/2}\Big)^2.$$

By the Poincaré-Wirtinger inequality, we obtain

$$\int_\Omega |\nabla u_\varepsilon^{q/2}|^2 \ge C\int_\Omega \Big(u_\varepsilon^{q/2} - \frac{1}{|\Omega|}\int_\Omega u_\varepsilon^{q/2}\Big)^2 = C\Big(\int_\Omega u_\varepsilon^q - \frac{1}{|\Omega|}\Big|\int_\Omega u_\varepsilon^{q/2}\Big|^2\Big),$$

which implies

$$\frac{q}{q-1}\frac{d}{dt}\int_\Omega u_\varepsilon^q + C\int_\Omega u_\varepsilon^q \le Cq^{\frac{2}{1-a}}\Big(\int_\Omega u_\varepsilon^{q/2}\Big)^2 \le Cq^{\frac{2}{1-a}}\Big(\sup_{t\ge 0}\int_\Omega u_\varepsilon^{q/2}\Big)^2.$$

By the maximum principle, we have

$$\int_\Omega u_\varepsilon^q \le \max\Big\{\int_\Omega u^q(x,0), Cq^{\frac{2}{1-a}}\Big(\sup_{t\ge 0}\int_\Omega u_\varepsilon^{q/2}\Big)^2\Big\}.$$

Then let $q_k := 2^k$, $(k \in \mathbb{N})$, $\delta_k := C2^{\frac{2k}{1-a}}$, and a constant K satisfying

$$K \ge \max\big\{1, \sup \|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)}, \|u(\cdot,0)\|_{L^\infty(\Omega)}\big\}.$$

Using the Moser-Alikakos iteration [1] and assuming, without loss of generality, that $\delta_k \ge 1$, we have

$$\int_\Omega u_\varepsilon^{2^k} \le \max\big\{\delta_k\Big(\sup\int_\Omega u_\varepsilon^{2^{k-1}}\Big)^2, K^{2^k}\big\}.$$

Taking $K \ge 1$, it follows that

$$\int_\Omega u_\varepsilon^{2^k} \le \delta_k\delta_{k-1}^2\delta_{k-2}^{2^2}\cdots\delta_1^{2^{k-1}}K^{2^k},$$

then we have

$$\int_\Omega u_\varepsilon^{2^k} \le C^{2^k-1}2^{\frac{2}{1-a}(-k+2^{k+1}-1)}K^{2^k}. \tag{4.4}$$

Finally by taking the $1/2^k$ power of both sides of (4.4) and by passing to the limit as $k \to \infty$ we obtain

$$\sup_{t\ge 0}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C2^{2\frac{2}{1-a}}K.$$

□

Next, to obtain the limit function u, we need a regularity estimate for $\partial_t u_\varepsilon$.

Lemma 4.2. *Let $1 < p < n/(n-1)$. There exists $C > 0$ such that for any $\varepsilon \in (0,1)$,*

$$\|\partial_t u_\varepsilon(\cdot,t)\|_{(W_0^{2,2}(\Omega))^*} \le C \quad \textit{for all } t > 0. \tag{4.5}$$

In particular,

$$\|u_\varepsilon(\cdot,t) - u_\varepsilon(\cdot,s)\|_{(W_0^{2,2}(\Omega))^*} \le C|t-s| \quad \textit{for all } t \ge 0,\ s \ge 0. \tag{4.6}$$

Proof. We fix $\psi \in C_0^\infty(\Omega)$ and multiply the first equation in (2.7) by ψ. Integrating by parts we find that

$$\int_\Omega \partial_t u_\varepsilon \cdot \psi = \int_\Omega u_\varepsilon \cdot \Delta\psi + \int_\Omega u_\varepsilon (|\nabla v_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla v_\varepsilon \cdot \nabla\psi.$$

Then by Lemmas 3.1 and 4.1, we obtain the inequality

$$\begin{aligned}
\Big|\int_\Omega \partial_t u_\varepsilon \cdot \psi\Big| &\le \|u_\varepsilon\|_{L^\infty(\Omega)} \int_\Omega |\Delta\psi| + \|u_\varepsilon\|_{L^\infty(\Omega)} \int_\Omega \big|(|\nabla v_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla v_\varepsilon \cdot \nabla\psi\big| \\
&\le C \int_\Omega |\Delta\psi| + C \int_\Omega \big|(|\nabla v_\varepsilon|^{p-1} + 1) \cdot \nabla\psi\big| \\
&\le C \int_\Omega |\Delta\psi| + C \int_\Omega |\nabla\psi|.
\end{aligned}$$

This readily establishes (4.5) and thus (4.6). □

Lemma 4.3. *Let u be the function asserted in Lemma 3.5. Then*

$$u_\varepsilon \overset{*}{\rightharpoonup} u \quad in\ L^\infty(\Omega \times (0,\infty)), \tag{4.7}$$

$$u_\varepsilon \to u \quad in\ C^\infty_{\rm loc}\Big([0,\infty); (W_0^{2,2}(\Omega))^*\Big), \tag{4.8}$$

as $\varepsilon = \varepsilon_k \searrow 0$.

Proof. By (4.1) and choosing a subsequence, we can deduce (4.7). Since $L^\infty(\Omega) \hookrightarrow (W_0^{2,2}(\Omega))^*$ is compact, by Lemma 4.3 and Aubin-Lions lemma([15]), we can obtain (4.8) after extracting of an adequate subsequence. □

Finally, we give the proof of Theorem 1.2 by referring to the method in [17].

Proof of Theorem 1.2. From (4.1), it follows that there exists a null set $N \subset [0,\infty)$ such that for all $t \in [0,\infty) \setminus N$, we have $u(\cdot,t) \in L^\infty(\Omega)$. As $[0,\infty) \setminus N$ is dense in $[0,\infty)$, for an arbitrary $t_0 \in [0,\infty)$ we can find $(t_k)_{k\in N} \subset [0,\infty) \setminus N$ such that $t_k \to t_0$ as $k \to \infty$, and extracting a subsequence if necessary we can also achieve that $u(\cdot,t_k) \overset{*}{\rightharpoonup} \widetilde{u}$ in $L^\infty(\Omega)$ as $k \to \infty$ with some $\widetilde{u} \in L^\infty(\Omega)$ satisfying $\|\widetilde{u}\|_{L^\infty(\Omega)} \le C$. Since (4.8) asserts that moreover $u(\cdot,t_k) \to u(\cdot,t_0)$ in $(W_0^{2,2}(\Omega))^*$ as $k \to \infty$, this allows us to identify $\widetilde{u} = u(\cdot,t_0)$ and to conclude that $u(\cdot,t) \in L^\infty(\Omega)$ for all $t \in [0,\infty)$, with $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C$ for all $t \ge 0$. □

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