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MAXIMAL REGULARITY FOR NON-AUTONOMOUS CAUCHY PROBLEMS IN WEIGHTED SPACES

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Abstract. We consider the regularity for the non-autonomous Cauchy problem

$$u'(t) + A(t)u(t) = f(t) \quad (t \in [0, \tau]), \quad u(0) = u_0.$$

The time dependent operator $A(t)$ is associated with (time dependent) sesquilinear forms on a Hilbert space $\mathcal{H}$. We prove the maximal regularity result in temporally weighted L^2-spaces and other regularity properties for the solution of the problem under minimal regularity assumptions on the forms and the initial value u_0. Our results are motivated by boundary value problems.

1. Introduction

The aim of this article is to study autonomous and non-autonomous evolution equation governed by time dependent sesquilinear forms. Let $(\mathcal{H}, (\cdot,\cdot), \|\cdot\|)$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. We consider another Hilbert space $\mathcal{V}$ which is densely and continuously embedded in $\mathcal{H}$. We denote by $\mathcal{V}'$ the (anti-) dual space of $\mathcal{V}$, so that

$$\mathcal{V} \hookrightarrow_d \mathcal{H} \hookrightarrow_d \mathcal{V}'.$$

i.e. $\mathcal{V}$ is a dense subspace of $\mathcal{H}$ such that for some constant $C_{\mathcal{H}} > 0$,

$$\|u\| \le C_{\mathcal{H}} \|u\|_{\mathcal{V}} \quad (u \in \mathcal{V}).$$

We denote by $\langle , \rangle$ the duality $\mathcal{V}' - \mathcal{V}$ and note that $\langle \psi, v\rangle = (\psi, v)$ if $\psi, v \in \mathcal{H}$. We consider a family of sesquilinear forms

$$\mathfrak{a} : [0, \tau] \times \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

such that

- (H1) $D(\mathfrak{a}(t)) = \mathcal{V}$ (constant form domain),
- (H2) $|\mathfrak{a}(t, u, v)| \le M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ (uniform boundedness),
- (H3) $\operatorname{Re} \mathfrak{a}(t, u, u) + \nu\|u\|^2 \ge \delta\|u\|^2_{\mathcal{V}}$ for all $u \in \mathcal{V}$, for some $\delta > 0$ and some $\nu \in \mathbb{R}$ (uniform quasi-coercivity).

We denote by $A(t), \mathcal{A}(t)$ the usual associated operators with $\mathfrak{a}(t)$ (as operators on $\mathcal{H}$ and $\mathcal{V}'$).

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In 1961 J. L. Lions proved that the non-autonomous Cauchy problem

$$\begin{gathered} \dot{u}(t)+\mathcal{A}(t)u(t)=f(t)\\ u(0)=u_0 \end{gathered} \tag{1.1}$$

has L^2-maximal regularity in $\mathcal{V}'$.

Theorem 1.1 (Lions' theorem). *Given $f \in L^2(0,\tau;\mathcal{V}')$ and $u_0 \in \mathcal{H}$, there is a unique solution $u \in MR(\mathcal{V},\mathcal{V}') := H^1(0,\tau;\mathcal{V}') \cap L^2(0,\tau;\mathcal{V})$ of problem* (1.1).

Note that $MR(\mathcal{V},\mathcal{V}') \hookrightarrow C([0,\tau];\mathcal{H})$ so that the initial condition makes sense. In Theorem 1.1 only measurability of $t \to \mathfrak{a}(t,\cdot,\cdot)$ with respect to the time variable is required to have a solution $u \in MR(\mathcal{V},\mathcal{V}')$. However, considering boundary valued problems one is interested in strong solution, i.e. solution $u \in H^1(0,\tau;\mathcal{H})$ and not only in $H^1(0,\tau;\mathcal{V}')$ (note that $\mathcal{H} \hookrightarrow \mathcal{V}'$ by the natural embedding).

In the recent decades, the maximal regularity approach has become very useful in application to parabolic partial differential equations. The question of maximal regularity in $\mathcal{H}$ (autonomous or non-autonomous cases) is so important for several reasons. First of all, if Robin boundary conditions are considered, only the operator $A(t)$ realizes these boundary conditions. The main reason for studying this problem is its importance for non-linear problems. They are mainly solved by applying Banach or Schauder fixed point theorems.

Problem 1.2. Let $f \in L^2(0,\tau;\mathcal{H})$. Under which conditions on the forms $\mathfrak{a}(\cdot)$ the solution $u \in MR(\mathcal{V},\mathcal{V}')$ of (1.1) satisfies $u \in H^1(0,\tau;\mathcal{H})$.

Lions asked this question on maximal regularity for several conditions on the form and on the initial value. He also gave partial positive answers in [17, XVIII Chapter 3, p. 513]. More recently, this problem has been studied with a lot of progress. See the recent papers [3] or [4] for more details and references. The main focus of this work is the presence of the temporal weights. The choice of the weighted spaces has a big advantages. One of them is to reduce the necessary regularity for initial conditions of evolution equations. Time-weights can be used also to exploit parabolic regularization which is typical for quasilinear parabolic problems.

This paper focuses on proving the maximal regularity in the non-autonomous case, i.e. we prove the existence and the uniqueness of solution to Problem (1.1). We shall allow considerably less restrictive assumptions on f and the initial data u_0. Here, f belongs to the weighted Hilbert space $L^2(0,\tau,t^\beta dt;\mathcal{H})$, with $\beta \in [0,1[$ and the initial data u_0 takes its values in a certain interpolation space $(\mathcal{H},D(A(0)))_{\frac{1-\beta}{2},2}$ between $\mathcal{H}$ and $D(A(0))$.

The maximal regularity for the autonomous case in weighted spaces was the subject of treatment of many authors, see for instance [5]. In the non-autonomous case (Section 5) we prove that if $f \in L^2(0,\tau,t^\beta dt;\mathcal{H})$ and $u_0 \in (\mathcal{H},D(A(0)))_{\frac{1-\beta}{2},2}$ for arbitrary $\beta \geq 0$ with the assumption that the operator $A(\cdot)$ belongs to the space $W^{1/2,2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}')) \cap C^\varepsilon([0,\tau],\mathcal{L}(\mathcal{V},\mathcal{V}'))$ for some $\varepsilon > 0$, then problem (5.2) has a unique solution u such that $\dot{u}, A(\cdot)u \in L^2(0,\tau,t^\beta dt;\mathcal{H})$. Throughout this paper we assume that the Kato square root property (3.3) is satisfied. This property plays an important role in the questions of (non-autonomous) maximal regularity and optimal control. To prove our results we appeal to classical tools from harmonic analysis such as square function estimate or functional calculus and from functional analysis such as interpolation theory or operator theory.

This work is structured as follows. In Section 2 we present basic definitions and properties used throughout this paper, in particular those of weighted spaces. In Section 3, we prove some preparatory results. Section 4 uses this result to show the maximal regularity for the autonomous equations, while in Section 5 we prove our result on maximal regularity to the considered non-autonomous Cauchy problems in the weighted space $L^2(0,\tau,t^\beta dt;\mathcal{H})$ and other regularity properties for the solution. We illustrate our abstract results by two applications in the final section. One of them concerns the heat equation with Robin boundary conditions on a bounded Lipschitz domain Ω.

Notation. We denote by $\mathcal{L}(E,F)$ (or $\mathcal{L}(E)$) the space of bounded linear operators from E to F (from E to E). The spaces $L^p(a,b;E)$ and $W^{1,p}(a,b;E)$ denote respectively the Lebesgue and Sobolev spaces of function on (a,b) with values in E. $C^\alpha(a,b;E)$ denote the space of Hölder continuous functions of order α. Recall that the norms of $\mathcal{H}$ and $\mathcal{V}$ are denoted by $\|\cdot\|$ and $\|\cdot\|_{\mathcal{V}}$. The scalar product of $\mathcal{H}$ is $(\cdot,\cdot)$.

We denote by C, C' or c. all inessential positive constants. Their values may change from line to line. In some cases we will use the notation $a \lesssim b$ to signify that there exists an inessential positive constant C such that $a \le Cb$.

2. Properties of weighted spaces

In this section we briefly recall the definitions and we give the basic properties of vector-valued function spaces with temporal weights. Let $(X,\|\cdot\|_X)$ be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. For $-1<\beta<1$ we set $L^2_\beta(0,\tau;X)=L^2(0,\tau,t^\beta dt;X)$, endowed with the norm

$$\|u\|^2_{L^2_\beta(0,\tau,X)} := \int_0^\tau \|u(t)\|_X^2 t^\beta\,dt.$$

It known that $L^2_\beta(0,\tau;X)\hookrightarrow L^1_{loc}(0,\tau;X)$. Indeed, for $u\in L^2_\beta(0,\tau;X)$ we find by Hölder's inequality

$$\int_0^\tau \|u(t)\|_X\,dt \le \Big(\int_0^\tau t^{-\beta}\,dt\Big)^{1/2}\|u\|_{L^2_\beta(0,\tau;X)}.$$

It clearly holds that $L^2(0,\tau;X)\hookrightarrow L^2_\beta(0,\tau;X)$ for $\beta>0$ and $L^2_\beta(0,\tau;X)\hookrightarrow L^2(0,\tau;X)$ for $\beta<0$.

We define the corresponding weighted Sobolev spaces

$$W^{1,2}_\beta(0,\tau;X) := \{u\in W^{1,1}(0,\tau;X) \text{ s.t. } u,\dot u\in L^2_\beta(0,\tau;X)\},$$
$$W^{1,2}_{\beta,0}(0,\tau;X) := \{u\in W^{1,2}_\beta(0,\tau;X), \text{ s.t. } u(0)=0\},$$

which are Banach spaces for the norms, respectively,

$$\|u\|^2_{W^{1,2}_\beta(0,\tau;X)} := \|u\|^2_{L^2_\beta(0,\tau;X)} + \|\dot u\|^2_{L^2_\beta(0,\tau;X)},$$
$$\|u\|^2_{W^{1,2}_{\beta,0}(0,\tau;X)} := \|\dot u\|^2_{L^2_\beta(0,\tau;X)}.$$

We set also

$$L^\infty_\beta(0,\tau;X) := \{u\in L^1(0,\tau;X), \text{ s.t. } s\to s^{\beta/2}u(s)\in L^\infty(0,\tau;X)\},$$

endowed with the norm $\|u\|_{L^\infty_\beta(0,\tau;X)} := \|s \mapsto s^{\beta/2}u(s)\|_{L^\infty(0,\tau;X)}$. For $s \in (0,1)$ we define the fractional weighted Sobolev space $W^{s,2}_\beta(0,\tau;X)$ by

$$W^{s,2}_\beta(0,\tau;X) = (L^2_\beta(0,\tau;X); W^{1,2}_\beta(0,\tau;X))_{s,2},$$

endowed with the norm

$$\|u\|^2_{W^{s,2}_\beta(0,\tau;X)} := \|u\|^2_{L^2_\beta(0,\tau;X)} + \int_0^\tau \int_0^t \frac{\|u(t)-u(s)\|^2_X}{|t-s|^{1+2s}} s^\beta \, ds \, dt.$$

Here, $(\cdot;\cdot)_{s,2}$ is the real interpolation space. For more details we refer the reader to [20, (2.6)].

Lemma 2.1 (Weighted Hardy inequality)**.** *For every $f \in L^2_\beta(0,\tau,X)$, we have*

$$\int_0^\tau \Big(\frac{1}{t}\int_0^t \|f(s)\|_X \, ds\Big)^2 t^\beta \, dt \lesssim \|f\|_{L^2_\beta(0,\tau;X)}.$$

This lemma was proved in [23, Lemma 6].

Proposition 2.2. *We have the following properties*

(1) *(a) For $p > 2$ and $\beta > \frac{2}{p} - 1$, $L^p(0,\tau;X) \hookrightarrow L^2_\beta(0,\tau,X)$,*
(b) For $p < 2$ and $\beta < \frac{2}{p} - 1$, $L^2_\beta(0,\tau,X) \hookrightarrow L^p(0,\tau;X)$.

(2) *For all $u \in L^2_\beta(0,\tau,X)$, we have $t \to v(t) = \frac{1}{t}\int_0^t u(s)\,ds \in L^2_\beta(0,\tau,X)$.*

(3) *We define the operator $\Phi : L^2_\beta(0,\tau;X) \to L^2(0,\tau;X)$, such that $(\Phi f)(t) = t^{\beta/2}f(t)$ for $f \in L^2_\beta(0,\tau;X)$ and $t \in [0,\tau]$. Then Φ is an isometric isomorphism. We note also that $\Phi \in \mathcal{L}(L^2(0,\tau;X), L^2_{-\beta}(0,\tau;X))$ and $\Phi \in \mathcal{L}(W^{1,2}_{\beta,0}(0,\tau;X), W^{1,2}_0(0,\tau;X))$.*

(4) *$W^{1,2}_{\beta,0}(0,\tau;X) \hookrightarrow L^2_{\beta-2}(0,\tau;X) \cap L^\infty_{\beta-1}(0,\tau;X)$.*

(5) *$L^2_{-\beta}(0,\tau;\mathcal{V}')$ is the dual space of $L^2_\beta(0,\tau;\mathcal{H})$ by the duality defined for the space $L^2(0,\tau;\mathcal{H})$.*

(6) *If $u \in W^{1,2}_\beta(0,\tau;X)$, we obtain that u has a continuous extension on X and $W^{1,2}_\beta(0,\tau;X) \hookrightarrow C([0,\tau];X)$.*

(7) *$C^\infty_c((0,\tau);X)$ and $C^\infty([0,\tau];X)$ are dense in $L^2_\beta(0,\tau;X)$ and $W^{s,2}_\beta(0,\tau;X)$ respectively, for all $s \in [0,1]$.*

Proof. (1a) Let $p > 2$ and $\beta > \frac{2}{p} - 1$, we set $p' = \frac{p}{2} > 1$, $\frac{1}{p'} + \frac{1}{q} = 1$. This implies that $q = \frac{p}{p-2}$ and by using Hölder's inequality we obtain

$$\begin{aligned}\|u\|^2_{L^2_\beta(0,\tau;X)} &= \int_0^\tau \|u(t)\|^2_X t^\beta \, dt \\ &\le \Big(\int_0^\tau \|u(t)\|^p_X \, dt\Big)^{2/p}\Big(\int_0^\tau t^{\beta q}\, dt\Big)^{1/q} \\ &= \Big(\frac{1}{\beta q + 1}\tau^{\beta q+1}\Big)^{1/q}\|u\|^2_{L^p(0,\tau;X)}.\end{aligned}$$

(1b) Similarly, for $p < 2$ and $\beta < \frac{2}{p} - 1$ we have

$$\begin{aligned}\|u\|^p_{L^p(0,\tau;X)} &= \int_0^\tau \|u(t)\|^p_X t^{-\frac{\beta p}{2}} t^{\frac{\beta p}{2}} \, dt \\ &\le \Big(\int_0^\tau \|u(t)\|^2_X t^\beta \, dt\Big)^{p/2}\Big(\int_0^\tau t^{\frac{\beta p}{p-2}}\, dt\Big)^{\frac{2-p}{2}}\end{aligned}$$

$$= C\|u\|^p_{L^2_\beta(0,\tau;X)}.$$

(2) Lemma 2.1 shows that

$$\|v\|^2_{L^2_\beta(0,\tau;X)} = \int_0^\tau \|\frac{1}{t}\int_0^t u(t)\,ds\|^2_X t^\beta\,dt \lesssim \|u\|^2_{L^2_\beta(0,\tau;X)}.$$

We obtain the result since $u \in L^2_\beta(0,\tau;X)$.

(3) Note that $\|\Phi f\|_{L^2(0,\tau;X)} = \|f\|_{L^2_\beta(0,\tau;X)}$ and $\Phi^{-1} : L^2(0,\tau;X) \to L^2_\beta(0,\tau;X)$ where $(\Phi^{-1}g)(t) = t^{-\beta/2}g(t)$ for all $g \in L^2(0,\tau;X)$.

(4) Let $u \in W^{1,2}_{\beta,0}(0,\tau;X)$. We write $u(t) = \int_0^t \dot u(l)\,dl$. Then

$$\|u(t)\|^2_X t^{\beta-2} = \|\int_0^t \dot u(l)\,dl\|^2_X t^{\beta-2}.$$

This implies

$$\begin{aligned}
\|u\|^2_{L^2_{\beta-2}(0,\tau;X)} &= \int_0^\tau \|u(t)\|^2_X t^{\beta-2}\,dt\\
&= \int_0^\tau \frac{1}{t^2}\|\int_0^t \dot u(s)\,ds\|^2_X t^\beta\,dt\\
&\le \int_0^\tau \Big(\frac{1}{t}\int_0^t \|\dot u(s)\|_X\,ds\Big)^2 t^\beta\,dt\\
&\lesssim \|\dot u\|_{L^2_\beta(0,\tau;X)} \le \|u\|_{W^{1,2}_\beta(0,\tau;X)},
\end{aligned}$$

where we used Lemma 2.1. For $t \in [0,\tau]$, by Hölder's inequality we have

$$\|u(t)\|_X t^{\frac{\beta-1}{2}} \le \int_0^t \|\dot u(s)\|_X\,ds\, t^{\frac{\beta-1}{2}} \le \|u\|_{W^{1,2}_{\beta,0}(0,\tau;X)}.$$

It follows that $W^{1,2}_{\beta,0}(0,\tau;X) \hookrightarrow L^2_{\beta-2}(0,\tau;X) \cap L^\infty_{\beta-1}(0,\tau;X)$.

(5) For this proof we use the simple functions in $L^2_{-\beta}(0,\tau;\mathcal{V}')$ and the Cauchy-Schwartz inequality (the proof is analogous to the non-weighted case, for more details see [11, p.98].

(6) For $u \in W^{1,2}_\beta(0,\tau;X)$ and $(t,s) \in [0,\tau]^2$, we obtain

$$\begin{aligned}
\|u(t)-u(s)\|_X &= \|\int_s^t \dot u(l)\,dl\|_X\\
&\le \Big(\int_s^t l^{-\beta}\,dl\Big)^{1/2}\|\dot u\|_{L^2_\beta(0,\tau;X)}\\
&= \frac{1}{\sqrt{1-\beta}}\big(t^{-\beta+1} - s^{-\beta+1}\big)^{1/2}\|\dot u\|_{L^2_\beta(0,\tau;X)}.
\end{aligned}$$

Letting $s \to t$ we obtain $u(s) \to u(t)$ in X. Therefore u has a continuous extension on X. Thus we can always identify a function in $W^{1,2}_\beta(0,\tau;X)$ by its continuous representative.

(7) First we note that $C^\infty_c((0,\tau);X)$ is dense $L^2(0,\tau;X)$. Then for all $f \in L^2_\beta(0,\tau;X)$ and for any given $\varepsilon > 0$ there exists a function $\psi \in C^\infty_c((0,\tau);X)$ such that

$$\|(\Phi f) - \psi\|^2_{L^2(0,\tau;X)} \le \varepsilon.$$

It follows that

$$\|f-(\Phi^{-1}\psi)\|^2_{L^2_\beta(0,\tau;X)} \le \|\Phi\|_{\mathcal{L}(L^2_\beta(0,\tau;X);L^2(0,\tau;X))}\|(\Phi f)-\psi\|^2_{L^2(0,\tau;X)} \le \varepsilon.$$

Thus $C_c^\infty((0,\tau);X)$ is dense in $L^2_\beta(0,\tau;X)$.

As in [24, Theorem 2.9.1] for the scalar-valued case, one sees that the space of all function f in $C^\infty([0,\tau];X)$ such that $f(0)=0$ is dense in $W^{1,2}_0(0,\tau;X)$. Then for all $g\in W^{1,2}_{\beta,0}(0,\tau;X)$ and $\epsilon>0$ there exists $\phi\in C^\infty([0,\tau];X)$ with $\phi(0)=0$ such that

$$\|\phi-\Phi g\|^2_{W^{1,2}(0,\tau;X)} \le \varepsilon.$$

Then $\|\Phi^{-1}\phi-g\|^2_{W^{1,2}_\beta(0,\tau;X)} \le \varepsilon$. This shows that the space of all function f in $C^\infty([0,\tau];X)$ such that $f(0)=0$, is dense in $W^{1,2}_{\beta,0}(0,\tau;X)$. Let $f\in W^{1,2}_\beta(0,\tau;X)$ and $\phi\in C^\infty([0,\tau];X)$ such that $\phi(0)=f(0)$. Then $f-\phi\in W^{1,2}_{\beta,0}(0,\tau;X)$ and there is $\xi\in C^\infty([0,\tau];X)$ with $\xi(0)=0$, such that $\|f-\xi-\phi\|^2_{W^{1,2}_\beta(0,\tau;X)}\le\varepsilon$. Since $\xi+\phi\in C^\infty([0,\tau];X)$, then $C^\infty([0,\tau];X)$ is dense in $W^{1,2}_\beta(0,\tau;X)$.

Since $C^\infty([0,\tau];X)$ is dense in $W^{1,2}_\beta(0,\tau;X)$ and

$$W^{s,2}_\beta(0,\tau;X) = (L^2_\beta(0,\tau;X);W^{1,2}_\beta(0,\tau;X))_{s,2},$$

we obtain that $C^\infty([0,\tau];X)$ is also dense in $W^{s,2}_\beta(0,\tau;X)$ by [24, p.39]. □

3. Preliminaries

In this section we prove several estimates which will play an important role in the proof of our results. From now we assume without loss of generality that the forms are coercive, that is (H3) holds with $\nu=0$. The reason is that by replacing $A(t)$ by $A(t)+\nu$, the solution v of (1.1) is $v(t)=e^{-\nu t}u(t)$ and it is clear that $u\in W^{1,2}_\beta(0,\tau;\mathcal{H})\cap L^2_\beta(0,\tau;\mathcal{V})$ if and only if $v\in W^{1,2}_\beta(0,\tau;\mathcal{H})\cap L^2_\beta(0,\tau;\mathcal{V})$.

Proposition 3.1. *The solution of problem* (1.1) *is unique.*

Proof. We suppose that there are two solutions u_1,u_2 to Problem (1.1). Obviously, $v=u_1-u_2$ satisfies

$$\begin{gathered}\dot v(t)+\mathcal{A}(t)v(t)=0\\ v(0)=0.\end{gathered}\tag{3.1}$$

Then for all $t\in[0,\tau]$ we have

$$2\operatorname{Re}\int_0^t(\dot v(s),v(s))s^\beta\,ds+2\operatorname{Re}\int_0^t(\mathcal{A}(s)v(s),v(s))s^\beta\,ds=0.$$

Integration by parts gives

$$t^\beta\|v(t)\|^2-\beta\int_0^t\|v(s)\|^2s^{\beta-1}\,ds+2\delta\int_0^t\|v(s)\|^2_{\mathcal{V}}s^\beta\,ds\le 0.$$

It is clear that for the case $\beta\le 0$ we obtain $v(t)=0$ for all $t\in[0,\tau]$. Therefore $u_1=u_2$ and then the solution of Problem (1.1) is unique. For the case $\beta\ge 0$ we have

$$t^\beta\|v(t)\|^2+\int_0^t\|v(s)\|^2(2\delta C^2_{\mathcal{H}}s^\beta-\beta s^{\beta-1})\,ds\le 0.$$

So for the case $t\le\frac{2\delta C^2_{\mathcal{H}}}{\beta}$ we have $v(t)=0$ for all $t\in[0,\frac{\delta C^2_{\mathcal{H}}}{\beta}]$. Now we proceed inductively to obtain $v=0$ on $[0,\tau]$. □

We denote by S_θ the open sector $S_\theta = \{z \in \mathbb{C}^* : |arg(z)| < \theta\}$ with vertex 0. It is known that $-A(t)$ is sectorial operator and generates a bounded holomorphic semigroup on $\mathcal{H}$. The same is true for $-\mathcal{A}(t)$ on $\mathcal{V}'$. From [14] (Proposition 2.1), we have the following lemma which point out that the constants involved in the estimates are uniform with respect to t.

Lemma 3.2. *For any $t \in [0,\tau]$, the operators $-A(t)$ and $-\mathcal{A}(t)$ generate strongly continuous analytic semigroups of angle $\gamma = \frac{\pi}{2} - \arctan(\frac{M}{\delta})$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. In addition, there exist real constants $C > 0$, $C_\theta > 0$ independent of t, such that*

1. $\|e^{-zA(t)}\|_{\mathcal{L}(\mathcal{H})} \le 1$ *and* $\|e^{-z\mathcal{A}(t)}\|_{\mathcal{L}(\mathcal{V}')} \le C$ *for all* $z \in S_\gamma$.
2. $\|A(t)e^{-sA(t)}\|_{\mathcal{L}(\mathcal{H})} \le \frac{C}{s}$ *and* $\|\mathcal{A}(t)e^{-s\mathcal{A}(t)}\|_{\mathcal{L}(\mathcal{V}')} \le \frac{C}{s}$ *for all* $s \in (0,\infty)$.
3. $\|e^{-sA(t)}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \le \frac{C}{\sqrt{s}}$ *for all* $s \in (0,\infty)$.
4. $\|(z-A(t))^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \le \frac{C_\theta}{\sqrt{|z|}}$ *and* $\|(z-\mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}',\mathcal{H})} \le \frac{C_\theta}{\sqrt{|z|}}$ *for all* $z \notin S_\theta$ *with fixed* $\theta > \gamma$.

The following lemma is proved in [19, Corollary 4.3.12]

Lemma 3.3. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces, with $\mathcal{H}_2 \subset \mathcal{H}_1$, and $\mathcal{H}_2$ dense in $\mathcal{H}_1$. Then for every $\theta \in (0,1)$,*

$$[\mathcal{H}_1, \mathcal{H}_2]_\theta = (\mathcal{H}_1, \mathcal{H}_2)_{\theta,2},$$

with $\|u\|_{[\mathcal{H}_1,\mathcal{H}_2]_\theta} = C\|u\|_{(\mathcal{H}_1,\mathcal{H}_2)_{\theta,2}}$, where C is a positive constant independent of $\mathcal{H}_1$ and $\mathcal{H}_2$.

As a consequence from the previous lemma and [19, Theorem 4.2.6] we have that for all $\gamma \in (0,1), t \in [0,\tau]$,

$$(\mathcal{H}, D(A(t)))_{\gamma,2} = [\mathcal{H}, D(A(t))]_\gamma = D(A(t)^\gamma).$$

Lemma 3.4. *For all $x \in (\mathcal{H}, D(A(t)))_{1/2,2}$ one has*

$$\int_0^\infty \|A(t)e^{-sA(t)}x\|^2\, ds \le C\|x\|^2_{(\mathcal{H},D(A(t)))_{\frac{1}{2},2}},$$

where $C > 0$ is independent of t.

Proof. Note that $\|e^{-sA(t)}\|_{\mathcal{L}(\mathcal{H})} \le 1$ and $\|sA(t)e^{-sA(t)}\|_{\mathcal{L}(\mathcal{H})} \le M_1$, where M_1 is independent of t. Let $x \in (\mathcal{H}, D(A(t)))_{\frac{1}{2},2}$. We write $x = a + b$, where $a \in \mathcal{H}$ and $b \in D(A(t))$ to obtain

$$\begin{aligned}
s^{1/2}\|A(t)e^{-sA(t)}x\| &\le \inf_{x=a+b;\ a\in\mathcal{H},\ b\in D(A(t))} M_1 s^{-1/2}\|a\| + s^{1/2}\|b\|_{D(A(t))} \\
&\le \max\{M_1,1\} \inf_{x=a+b;\ a\in\mathcal{H},\ b\in D(A(t))} s^{-1/2}\{\|a\| + s\|b\|_{D(A(t))}\} \\
&\le \max\{M_1,1\} \inf_{x=a+b;\ a\in\mathcal{H},\ b\in D(A(t))} s^{-1/2}K(s,x;\mathcal{H},D(A(t))).
\end{aligned}$$

So $\|A(t)e^{-sA(t)}x\| \le \max\{M_1,1\}s^{-1}K(s,x;\mathcal{H},D(A(t)))$, where

$$K(s,x;\mathcal{H},D(A(t))) = \inf_{x=a+b;\ a\in\mathcal{H},\ b\in D(A(t))} \Big(\|a\| + s\|b\|_{D(A(t))}\Big).$$

Since $\|x\|^2_{(\mathcal{H},D(A(t)))_{\frac{1}{2},2}} = \int_0^\infty |K(s,x;\mathcal{H},D(A(t)))|^2 \frac{ds}{s^2}$ [19, Definition 1.1.1],

$$\int_0^\infty \|A(t)e^{-sA(t)}x\|^2\, ds \le \max\{M_1,1\}\|x\|^2_{(\mathcal{H},D(A(t)))_{\frac{1}{2},2}}.$$

This completes the proof. □

In the next lemma we prove the quadratic estimate that was proved in [3] under assumption (3.3). Here we prove it without the assumption.

Lemma 3.5. *Let $x \in \mathcal{H}$ and $t \in [0, \tau]$. We have*

$$\int_0^\tau \|A(t)^{1/2} e^{-sA(t)} x\|^2 \, ds \le c\|x\|^2, \tag{3.2}$$

where c is a positive constant independent of t.

Proof. Note that by [16, (A1) p. 269],

$$A(t)^{-\beta} = \frac{1}{\pi} \int_0^\infty \mu^{-\beta} (\mu + A(t))^{-1} \, d\mu.$$

Then Lemma 3.2 gives $\|A(t)^{-1/2}\|_{\mathcal{L}(\mathcal{H})} \le C'$, where C' is a positive constant independent of t. Let $x \in \mathcal{H}$ and $t \in [0, \tau]$. By Lemma 3.4 we have

$$\begin{aligned}
\int_0^1 \|A(t)^{1/2} e^{-sA(t)} x\|^2 \, ds &= \int_0^1 \|A(t) e^{-sA(t)} A(t)^{-1/2} x\|^2 \, ds \\
&\le \|A(t)^{-1/2} x\|^2_{(\mathcal{H}; D(A(t)))_{\frac{1}{2},2}} \\
&= \|x\|^2 + \|A(t)^{-1/2} x\|^2 \\
&\le (C'^2 + 1)\|x\|^2.
\end{aligned}$$

This completes the proof. □

In the sequel, we assume that $D(A(t)^{1/2}) = \mathcal{V}$ for all $t \in [0, \tau]$ and there exist $c_1, c^1 > 0$ such that for all $v \in \mathcal{V}$

$$c_1 \|v\|_{\mathcal{V}} \le \|A(t)^{1/2} v\| \le c^1 \|v\|_{\mathcal{V}}, \tag{3.3}$$

this also holds for adjoint-operators and we find

$$c_1 \|v\|_{\mathcal{V}} \le \|A^*(t)^{1/2} v\| \le c^1 \|v\|_{\mathcal{V}}.$$

Note that this assumption is always true for symmetric forms such that $c_1 = \sqrt{\delta}$ and $c^1 = \sqrt{M}$.

Lemma 3.6. *For all $t \in [0, \tau]$ we have $D(\mathcal{A}(t)^{1/2}) = \mathcal{H}$ and $D(A(t)^{*\frac{1}{2}}) = \mathcal{V}$.*

Proof. We write

$$\mathcal{A}(t)^{1/2} u = \mathcal{A}(t) A(t)^{-1/2} u.$$

Therefore

$$\frac{\alpha}{c^1} \|u\| \le \|\mathcal{A}(t)^{1/2} u\|_{\mathcal{V}'} \le \frac{M}{c_1} \|u\|.$$

So $\mathcal{A}(t)^{1/2} \in \mathcal{L}(\mathcal{H}, \mathcal{V}')$ and by duality we find $A(t)^{*\frac{1}{2}} \in \mathcal{L}(\mathcal{V}, \mathcal{H})$. □

Let $t \in [0, \tau]$. For $f \in L^2(0, t; \mathcal{H})$, we define the operator

$$(R(t)f) := \int_0^t e^{-(t-s)A(t)} f(s) \, ds.$$

The next lemma shows that $R(t)$ is bounded in $\mathcal{L}(L^2(0, t; \mathcal{H}), \mathcal{V})$, and it was proved in [3, Lemma 4.1].

Lemma 3.7. *We have $R(t) \in \mathcal{L}(L^2(0, t; \mathcal{H}), \mathcal{V})$ for all $t \in [0, \tau]$.*

Lemma 3.8. *Assume that $A(\cdot) \in C^{\varepsilon}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}')), \varepsilon > 0$. Then for all $\lambda \in (0,\infty)$, we obtain $(\lambda + A(\cdot))^{-1} \in C^{\varepsilon}([0,\tau];\mathcal{L}(\mathcal{H}))$ and*

$$\|(\lambda + A(\cdot))^{-1}\|_{C^{\varepsilon}([0,\tau];\mathcal{L}(\mathcal{H}))} \leq \frac{C}{\lambda}.$$

Proof. Let $\lambda \in (0,\infty), t, s \in [0,\tau]$. We obtain

$$(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1} = (\lambda + A(t))^{-1}(\mathcal{A}(t) - \mathcal{A}(s))(\lambda + A(s))^{-1}.$$

Therefore by Lemma 3.2 we have

$$\begin{aligned} &\|(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1}\|_{\mathcal{L}(\mathcal{H})} \\ &\leq \|(\lambda + A(t))^{-1}\|_{\mathcal{L}(\mathcal{V}',\mathcal{H})}\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}',\mathcal{V})}\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \\ &\leq C\frac{|t-s|^{\varepsilon}}{|\lambda|}. \end{aligned}$$

□

We denote by $L^2_{\beta}(0,\tau;D(A(\cdot)))$ the space of all measurable functions $f:[0,\tau] \to \mathcal{H}$ for which $f(t) \in D(A(t))$ for almost all $t \in [0,\tau]$ and $A(\cdot)f \in L^2_{\beta}(0,\tau;\mathcal{H})$. Then the following density result holds.

Lemma 3.9. *Suppose that $A(\cdot) \in C^{\varepsilon}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))$. Then $L^2_{\beta}(0,\tau;D(A(\cdot)))$ is dense in $L^2_{\beta}(0,\tau;\mathcal{H})$.*

Proof. Let $f \in L^2_{\beta}(0,\tau;\mathcal{H})$ and set $f_n(t) = n(n + A(t))^{-1}f(t)$ for $n \in \mathbb{N}$. Since the map $t \mapsto (n + A(t))^{-1} \in C^{\epsilon}([0,\tau];\mathcal{L}(\mathcal{H}))$, then for all $n \in \mathbb{N}$ the function $f_n:[0,\tau] \to \mathcal{H}$ is measurable and satisfies $f_n(t) \in D(A(t))$ almost everywhere as well as $\|A(t)f_n(t)\| \leq Cn\|f(t)\|$. Moreover

$$\|f_n(t) - f(t)\| = \|(n(n + A(t))^{-1} - I)f(t)\|.$$

Hence, the convergence $f_n \to f$ in $L^2_{\beta}(0,\tau;\mathcal{H})$ holds by the dominated convergence theorem. □

Proposition 3.10. *Assume that $A(\cdot) \in C^{\varepsilon}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))$, for some $\varepsilon > 0$. Then for all $f \in L^2_{\beta}(0,\tau;\mathcal{H})$, with $\beta < 1$ the operator L defined by*

$$(Lf)(t) := A(t)\int_0^t e^{-(t-s)A(t)}f(s)\,ds$$

is bounded on $L^2_{\beta}(0,\tau;\mathcal{H})$.

Proof. Let $f \in L^2_{\beta}(0,\tau;D(A(\cdot)))$. We split the integral into two parts to obtain

$$\begin{aligned} (Lf)(t) &= A(t)\int_0^{t/2} e^{-(t-s)A(t)}f(s)\,ds + A(t)\int_{t/2}^t e^{-(t-s)A(t)}f(s)\,ds \\ &:= I_1(t) + I_2(t). \end{aligned}$$

We begin by estimating the first integral

$$\begin{aligned} \|I_1(t)\| = \|A(t)\int_0^{t/2} e^{-(t-s)A(t)}f(s)\,ds\| &\lesssim \int_0^{t/2}\frac{1}{t-s}\|f(s)\|\,ds \\ &\lesssim \frac{2}{t}\int_0^{t/2}\|f(s)\|\,ds. \end{aligned}$$

Lemma 2.1 gives

$$\begin{aligned}
&\int_0^\tau \|A(t)\int_0^{t/2} e^{-(t-s)A(t)}f(s)\,ds\|^2 t^\beta\,dt\\
&\lesssim \int_0^\tau (\frac{2}{t}\int_0^{t/2}\|f(s)\|\,ds)^2 t^\beta\,dt\\
&\lesssim \|f\|^2_{L^2_\beta(0,\tau;\mathcal{H})}.
\end{aligned}$$

Similarly, we estimate the second integral. For $x\in\mathcal{H}$ we obtain

$$\begin{aligned}
&|(I_2(t),x)|\\
&=|\int_{t/2}^t (A(t)^{1/2}e^{-\frac12(t-s)A(t)}f(s), A(t)^{\frac12 *}e^{-\frac12(t-s)A(t)^*}x)\,ds|\\
&\le \Big(\int_{t/2}^t\|A(t)^{1/2}e^{-\frac12(t-s)A(t)}f(s)\|^2\,ds\Big)^{1/2}\Big(\int_{t/2}^t\|A(t)^{*\frac12}e^{-\frac12(t-s)A(t)^*}x\|^2\,ds\Big)^{1/2}\\
&\lesssim \Big(\int_{t/2}^t\|A(t)^{1/2}e^{-\frac12(t-s)A(t)}f(s)\|^2\,ds\Big)^{1/2}\|x\|.
\end{aligned}$$

In the above inequality we used the quadratic estimate (3.5). Taking the supremum over all $x\in\mathcal{H}$, we obtain

$$\begin{aligned}
\int_0^\tau t^\beta\|I_2(t)\|\,dt &= \int_0^\tau t^\beta\|A(t)\int_{t/2}^t e^{-(t-s)A(t)}f(s)\,ds\|^2\,dt\\
&\lesssim \int_0^\tau t^\beta\int_{t/2}^t\|A(t)^{1/2}e^{-\frac12(t-s)A(t)}f(s)\|^2\,ds\,dt\\
&\lesssim \int_0^\tau\int_{t/2}^t\|A(t)^{1/2}e^{-\frac12(t-s)A(t)}\big(s^{\beta/2}f(s)\big)\|^2\,ds\,dt.
\end{aligned}$$

Let g be the function defined by $g=(\Phi f)$. Using Fubini's theorem and the basic inequality $(x+y)^2\le 2x^2+2y^2$, we obtain

$$\begin{aligned}
&\int_0^\tau\int_{t/2}^t\|A(t)^{1/2}e^{-\frac12(t-s)A(t)}[s^{\beta/2}f(s)]\|^2\,ds\,dt\\
&\le 2\int_0^\tau\int_{t/2}^t\|A(s)^{1/2}e^{-\frac12(t-s)A(s)}g(s)\|^2\,ds\,dt\\
&\quad+2\int_0^\tau\int_{t/2}^t\|(A(s)^{1/2}e^{-\frac12(t-s)A(s)}-A(t)^{1/2}e^{-\frac12(t-s)A(t)})g(s)\|^2\,ds\,dt\\
&\le 2\int_0^\tau\int_s^{2s}\|A(s)^{1/2}e^{-\frac12(t-s)A(s)}g(s)\|^2\,dt\,ds\\
&\quad+2\int_0^\tau\int_{t/2}^t\|(A(s)^{1/2}e^{-\frac12(t-s)A(s)}-A(t)^{1/2}e^{-\frac12(t-s)A(t)})g(s)\|^2\,ds\,dt\\
&\lesssim \|g\|^2_{L^2(0,\tau;\mathcal{H})}+\int_0^\tau\int_{t/2}^t\|(A(s)^{1/2}e^{-\frac12(t-s)A(s)}-A(t)^{1/2}e^{-\frac12(t-s)A(t)})g(s)\|^2\,ds\,dt.
\end{aligned}$$

The functional calculus for the sectorial operators $A(t),A(s)$ gives

$$A(s)^{1/2}e^{-\frac12(t-s)A(s)}-A(t)^{1/2}e^{-\frac12(t-s)A(t)}$$

$$= \int_\Gamma \lambda^{1/2} e^{-\frac{1}{2}(t-s)\lambda} (\lambda - \mathcal{A}(t))^{-1} (\mathcal{A}(t) - \mathcal{A}(s)) (\lambda - \mathcal{A}(s))^{-1} \, d\lambda.$$

Hence,

$$\begin{aligned} &\|A(s)^{1/2} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{1/2} e^{-\frac{1}{2}(t-s)A(t)}\|_{\mathcal{L}(\mathcal{H})} \\ &\le \int_\Gamma |\lambda|^{1/2} e^{-\frac{1}{2}(t-s)\operatorname{Re}\lambda} \|(\lambda - \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}',\mathcal{H})} \\ &\quad \times \|(\mathcal{A}(t) - \mathcal{A}(s))\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \|(\lambda - \mathcal{A}(s))^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \, |d\lambda|. \end{aligned}$$

Thus

$$\begin{aligned} &\|A(s)^{1/2} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{1/2} e^{-\frac{1}{2}(t-s)A(t)}\|_{\mathcal{L}(\mathcal{H})} \\ &\le \int_0^\infty |\lambda|^{-1/2} e^{-\frac{1}{2}(t-s)cos(\gamma)|\lambda|} \, d|\lambda| \|(\mathcal{A}(t) - \mathcal{A}(s))\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}. \end{aligned}$$

where γ is the angle mentioned in Lemma 3.2. Then

$$\|A(s)^{1/2} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{1/2} e^{-\frac{1}{2}(t-s)A(t)}\|_{\mathcal{L}(\mathcal{H})} \lesssim \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{(t-s)^{1/2}}.$$

Therefore

$$\begin{aligned} &\int_0^\tau \int_{t/2}^t \|(A(s)^{1/2} e^{-(t-s)A(s)} - A(t)^{1/2} e^{-(t-s)A(t)}) g(s)\|^2 \, ds \, dt \\ &\lesssim \int_0^\tau \int_{t/2}^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-s} \|g(s)\|^2 \, ds \, dt \\ &\lesssim \sup_{s \in [0,\tau]} \int_s^\tau \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-s} \, dt \|g\|^2_{L^2(0,\tau;\mathcal{H})} \\ &\lesssim \tau^{2\epsilon} \|\mathcal{A}\|^2_{C^\epsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|f\|^2_{L^2_\beta(0,\tau;\mathcal{H})}. \end{aligned}$$

This completes the proof. □

Proposition 3.11. *For $\beta \ge 1$ the operator L is not bounded on $L^2_\beta(0,\tau;\mathcal{H})$ in general.*

Proof. Let $u \in \mathcal{H}$ and $g \in L^2_{-\beta}(0,\tau;\mathcal{H})$. Noting that

$$(L^*g)(t) = \int_t^\tau A(s)^* e^{-(s-t)A(s)^*} g(s) \, ds, t \in (0,\tau)$$

and $L \in \mathcal{L}(L^2_\beta(0,\tau;\mathcal{H}))$ if and only if $L^* \in \mathcal{L}(L^2_{-\beta}(0,\tau;\mathcal{H}))$. If $A(s)^* = A(0)^*$ for all $s \in [0,\tau]$, then $(L^*g)(t) = \int_t^\tau A(0)^* e^{-(s-t)A(0)^*} g(s) \, ds$. Assume now that $t < 1 < \tau$ and take $g(s) = \mathbf{1}_{[1,\tau]}(s) u$, so

$$(L^*g)(t) = e^{-(1-t)A(0)^*} u - e^{-(\tau-t)A(0)^*} u,$$

which converges to $e^{-A(0)^*} u - e^{-\tau A(0)^*} u$ as $t \to 0$. We claim that

$$e^{-A(0)^*} u - e^{-\tau A(0)^*} u \neq 0,$$

then

$$\begin{aligned} \|L^*g\|^2_{L^2_{-\beta}(0,\tau;\mathcal{H})} &\ge \|L^*g\|^2_{L^2_{-\beta}(0,1;\mathcal{H})} \\ &= \int_0^1 \|e^{-(1-t)A(0)^*} u - e^{-(\tau-t)A(0)^*} u\|^2 \frac{dt}{t^\beta} = \infty. \end{aligned}$$

Now, suppose that $e^{-A(0)^*}u - e^{-\tau A(0)^*}u = 0$, thus

$$e^{-A(0)^*}u = e^{-(2\tau-1)A(0)^*}u.$$

Using induction, for all $n \in \mathbb{N}$ we obtain

$$e^{-A(0)^*}u - e^{-(n(\tau-1)+1)A(0)^*}u = 0.$$

Since $\|A(0)^* e^{-(n(\tau-1)+1)A(0)^*}A(0)^{*-1}u\| \lesssim \frac{1}{(n(\tau-1)+1)}\|A(0)^{*-1}u\|$, by letting $n \to \infty$ it follows that $e^{-A(0)^*}u = 0$. Hence $e^{-tA(0)^*}u = 0$ for all $t \geq 1$, and we deduce that $u = 0$ by an application of the isolated point theorem and the analyticity of the semigroup. □

Lemma 3.12. *For all* $f \in L^2_\beta(0,\tau;\mathcal{H}), \beta < 1$ *we have* $(L_1 f)(t) \in \mathcal{V}$, *where*

$$(L_1 f)(t) = t^{\beta/2}\int_0^t e^{-(t-s)A(t)}f(s)\,ds,\ t \in [0,\tau].$$

Proof. We write

$$(L_1 f)(t) = t^{\beta/2}\int_0^{t/2} e^{-(t-s)A(t)}f(s)\,ds + t^{\beta/2}\int_{t/2}^t e^{-(t-s)A(t)}f(s)\,ds.$$

A straightforward computation gives

$$\begin{aligned}
\|t^{\beta/2}\int_0^{t/2} e^{-(t-s)A(t)}f(s)\,ds\|_{\mathcal{V}} &\lesssim t^{\beta/2}\int_0^{t/2}\|e^{-(t-s)A(t)}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}\|f(s)\|\,ds\\
&\lesssim t^{\beta/2}(\int_0^{t/2} s^{-\beta-1}\,ds)^{1/2}\|f\|_{L^2_\beta(0,\tau;\mathcal{H})}\\
&\lesssim \|f\|_{L^2_\beta(0,\tau;\mathcal{H})}.
\end{aligned}$$

Using Lemma 3.5 we deduce

$$\begin{aligned}
\|t^{\beta/2}\int_{t/2}^t e^{-(t-s)A(t)}f(s)\,ds\|_{\mathcal{V}} &\lesssim \|\int_{t/2}^t e^{-(t-s)A(t)}(s^{\beta/2}f(s))\,ds\|_{\mathcal{V}}\\
&\lesssim \|f\|_{L^2_\beta(0,\tau;\mathcal{H})}.
\end{aligned}$$

This completes the proof. □

Lemma 3.13. *For all* $u_0 \in (\mathcal{H}; D(A(0)))_{\frac{1-\beta}{2},2}$ *and* $\beta \in [0,1)$, *we have*

$$\int_0^\tau \|t^{\beta/2}A(0)e^{-tA(0)}u_0\|^2\,dt \simeq \|u_0\|^2_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}}.$$

Proof. Note that $(\mathcal{H}; D(A(0)))_{\frac{1-\beta}{2},2} = D(A(0)^{\frac{1-\beta}{2}})$. Let $\beta \in [0,1)$. In light of the quadratic estimate we obtain

$$\begin{aligned}
\int_0^\tau \|t^{\beta/2}A(0)e^{-tA(0)}u_0\|^2\,dt &= \int_0^\tau \|t^{\beta/2}A(0)^{\frac{1+\beta}{2}}e^{-tA(0)}A(0)^{\frac{1-\beta}{2}}u_0\|^2\,dt\\
&\lesssim \int_0^\tau \|A(0)^{1/2}e^{-\frac{t}{2}A(0)}A(0)^{\frac{1-\beta}{2}}u_0\|^2\,dt\\
&\lesssim \|A(0)^{\frac{1-\beta}{2}}u_0\|^2 = \|u_0\|^2_{[\mathcal{H};D(A(0))]_{\frac{1-\beta}{2}}}\\
&\lesssim \|u_0\|^2_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}}.
\end{aligned}$$

Conversely, we know that [19, Definition 1.1.1]

$$\|u_0\|^2_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} = \int_0^1 t^{\beta-2}\|K(t,u_0)\|^2\,dt,$$

where

$$K(t,u_0) = \inf_{u_0=a+b;a\in\mathcal{H},b\in D(A(0))}\big(\|a\| + t\|b\|_{D(A(0))}\big).$$

This allows us to write, for $t\in[0,\tau]$,

$$\begin{aligned} u_0 &= (u_0 - e^{-tA(0)}u_0) + e^{-tA(0)}u_0\\ &= -\int_0^t A(0)e^{-lA(0)}u_0\,dl + e^{-tA(0)}u_0. \end{aligned}$$

Since $e^{-tA(0)}u_0\in D(A(0))$ a.e. $t\in[0,\tau]$ and $(u_0 - e^{-tA(0)}u_0)\in\mathcal{H}$, it follows that

$$\|K(t,u_0)\| \le \int_0^t \|A(0)e^{-lA(0)}u_0\|\,dl + t\|A(0)e^{-tA(0)}u_0\|.$$

Roughly speaking, by Lemma 2.1 we find

$$\|u_0\|^2_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} \lesssim \int_0^\tau \|t^{\beta/2}A(0)e^{-tA(0)}u_0\|^2\,dt.$$

This completes the proof. □

Remark 3.14. From the previous lemma, the orbit the map $t\mapsto e^{-tA(0)}u_0$ belongs to the space $W^{1,2}_\beta(0,\tau;\mathcal{H})\cap L^2_\beta(0,\tau;D(A(0)))$ if and only if $u_0\in(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}$.

We define the space

$$W_\beta(D(A(\cdot)),\mathcal{H}) := \{u\in W^{1,1}(0,\tau;\mathcal{H}),\ \text{s.t. } A(\cdot)u\in L^2_\beta(0,\tau;\mathcal{H}),\ \dot u\in L^2_\beta(0,\tau;\mathcal{H})\},$$

with norm

$$\|u\|_{W_\beta(D(A(\cdot),\mathcal{H})} = \|A(\cdot)u\|_{L^2_\beta(0,\tau;\mathcal{H})} + \|\dot u\|_{L^2_\beta(0,\tau;\mathcal{H})}.$$

It is easy to see that $W_\beta(D(A(\cdot),\mathcal{H})\hookrightarrow W^{1,2}_\beta(0,\tau;\mathcal{H})$.

Lemma 3.15. *For all $\gamma\le 1/2$, we have $(\mathcal{H},D(A(0)))_{\gamma,2} = [\mathcal{H},\mathcal{V}]_{2\gamma}$ and for $\gamma > 1/2$ we have $(\mathcal{H},D(A(0)))_{\gamma,2}\hookrightarrow\mathcal{V}$.*

Proof. As a consequence of the interpolation method [19, Remark 1.3.6], for $\gamma\le 1/2$ we have

$$(\mathcal{H},D(A(0)))_{\gamma,2} = (\mathcal{H},D(A(0)^{1/2}))_{2\gamma,2} = (\mathcal{H},\mathcal{V})_{2\gamma,2}.$$

Since $\mathcal{H}$ and $\mathcal{V}$ are Hilbert spaces we obtain by Lemma 3.3

$$(\mathcal{H},D(A(0)))_{\gamma,2} = (\mathcal{H},\mathcal{V})_{2\gamma,2} = [\mathcal{H},\mathcal{V}]_{2\gamma}.$$

Let $v\in D(A(0))$ and $\gamma>\frac12$. We obtain

$$\begin{aligned} \delta\|v\|^2_{\mathcal{V}} &\le \operatorname{Re}(A(0)v,v)\\ &\lesssim \|A(0)^\gamma v\|\|A(0)^{*(1-\gamma)}v\|\\ &\lesssim \|A(0)^\gamma v\|\|v\|_{[\mathcal{H},\mathcal{V}]_{2(1-\gamma)}}\\ &\lesssim \|A(0)^\gamma v\|\|v\|_{\mathcal{V}}. \end{aligned}$$

Therefore we have that for all $\gamma>\frac12$ and $v\in D(A(0))$,

$$\|v\|_{\mathcal{V}} \lesssim \|v\|_{D(A(0)^\gamma)}.$$

Finally, by the density of $D(A(0))$ in $D(A(0)^\gamma)$ we obtain the desired result. □

4. Maximal regularity for autonomous problems

In this section we are interested in the regularity of the problem

$$\begin{gathered} \dot{u}(t) + \mathcal{A}(0)u(t) = f(t) \\ u(0) = u_0. \end{gathered} \tag{4.1}$$

The following is our main result in this section.

Theorem 4.1. *Let $f \in L^2_\beta(0,\tau,\mathcal{H})$ and $u_0 \in (\mathcal{H}; D(A(0)))_{\frac{1-\beta}{2},2}$ for $\beta \geq 0$ and $u_0 = 0$ if $\beta < 0$. There exists a unique $u \in W_\beta(D(A(0)),\mathcal{H}) \cap L^\infty_\beta(0,\tau;\mathcal{V})$ be the solution to Problem (4.1). Moreover, we have the following embeddings*

$$W_\beta(D(A(0)),\mathcal{H}) \hookrightarrow C([0,\tau]; (\mathcal{H}; D(A(0)))_{\frac{1-\beta}{2},2})$$

$$W_\beta(D(A(0)),\mathcal{H}) \hookrightarrow W^{\frac{1}{2},2}_\beta(0,\tau;\mathcal{V}), \beta \in [0,1[.$$

Proof. Since $A(0)$ is a generator of an analytic semigroup in $\mathcal{H}$, it is well known that by the variation of constants formula the solution of Problem (4.1) is

$$u(t) = e^{-tA(0)}u_0 + \int_0^t e^{-(t-s)A(0)} f(s)\,ds.$$

Thus,

$$\begin{aligned} A(0)u(t) &= A(0)e^{-tA(0)}u_0 + A(0)\int_0^t e^{-(t-s)A(0)} f(s)\,ds \\ &:= (Fu_0)(t) + (Lf)(t). \end{aligned}$$

Lemmas 3.12, 3.13 and Proposition 3.10 gives

$$\begin{aligned} \|A(0)u\|_{L^2_\beta(0,\tau;\mathcal{H})} &\leq \|Fu_0\|_{L^2_\beta(0,\tau;\mathcal{H})} + \|Lf\|_{L^2_\beta(0,\tau;\mathcal{H})} \\ &\leq C\Big(\|u_0\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} + \|f\|_{L^2_\beta(0,\tau;\mathcal{H})}\Big). \end{aligned}$$

Since $\dot{u} = f - A(0)u \in L^2_\beta(0,\tau;\mathcal{H})$, we obtain finally

$$\|u\|_{W_\beta(D(A(0)),\mathcal{H})} \leq C'\Big(\|u_0\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} + \|f\|_{L^2_\beta(0,\tau;\mathcal{H})}\Big). \tag{4.2}$$

Using Proposition 5.1 and (4.2), for all $t \in [0,\tau]$ we obtain

$$\begin{aligned} \|u(t)\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} &\lesssim \|u\|_{W_\beta(D(A(0)),\mathcal{H}) \cap L^\infty_\beta(0,\tau;\mathcal{V})} \\ &\lesssim \|u_0\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} + \|f\|_{L^2_\beta(0,\tau;\mathcal{H})}. \end{aligned} \tag{4.3}$$

For $0 \leq s \leq l \leq t \leq \tau$, we set $v(l) = e^{-(t-l)A(0)}u(l)$. This yields

$$\begin{aligned} u(t) - u(s) &= v(s) - u(s) + \int_s^t \dot{v}(l)\,dl \\ &= (e^{-(t-s)A(0)} - I)u(s) + \int_s^t e^{-(t-l)A(0)} f(l)\,dl. \end{aligned} \tag{4.4}$$

Observe that $e^{-(t-s)A(0)}$ is strongly continuous on $(\mathcal{H}; D(A(0)))_{\frac{1-\beta}{2},2}$. In particular, this ensures that

$$\|(e^{-(t-s)A(0)} - I)u(s)\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} \to 0 \quad \text{as } t \to s.$$

The estimate (4.3) for the case $u_0 = 0$ gives that

$$\|\int_s^t e^{-(t-l)A(0)} f(l)\,dl\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} \lesssim \|f\|_{L^2_\beta(s,t;\mathcal{H})}.$$

It follows that $u(t)$ is right continuous on $(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}$. Now, set $v(l) = e^{-(l-s)A(0)}u(l)$, for $0 \le s \le l \le t$. Then

$$\begin{aligned} u(s) - u(t) &= v(t) - u(t) - \int_s^t \dot v(l)\,dl \\ &= (e^{-(t-s)A(0)} - I)u(t) - \int_s^t e^{-(l-s)A(0)}(f(l) - 2A(0)u(l))\,dl. \end{aligned}$$

The same argument shows that u is left continuous in $(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}$. Thus, $u \in C([0,\tau];(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2})$.

Now, we prove that $W_\beta(D(A(0)),\mathcal{H}) \hookrightarrow W_\beta^{\frac12,2}(0,\tau;\mathcal{V})$. Indeed, let $\beta \in [0,1[$ and $u \in C^\infty([0,\tau];D(A(0)))$. We recall that

$$\|u\|^2_{W_\beta^{\frac12,2}(0,\tau;\mathcal{V})} = \|u\|^2_{L^2_\beta(0,\tau;\mathcal{V})} + \int_0^\tau \int_0^t \frac{\|u(t)-u(s)\|^2_{\mathcal{V}}}{|t-s|^2} s^\beta\,ds\,dt.$$

By (4.4) it holds that for all $0 \le s \le t \le \tau$

$$\begin{aligned} u(t) - u(s) &= (e^{-(t-s)A(0)}u(s) - u(s)) + \int_s^t e^{-(t-l)A(0)} f(l)\,dl \\ &:= L_1(t,s) + L_2(t,s), \end{aligned}$$

where $f(l) = A(0)u(l) + \dot u(l)$. So

$$\begin{aligned} \|u\|^2_{W_\beta^{\frac12,2}(0,\tau;\mathcal{V})} &\le \|u\|^2_{L^2_\beta(0,\tau;\mathcal{V})} + 2\int_0^\tau \int_0^t \frac{\|L_1(t,s)\|^2_{\mathcal{V}}}{|t-s|^2} s^\beta\,ds\,dt \\ &\quad + 2\int_0^\tau \int_0^t \frac{\|L_2(t,s)\|^2_{\mathcal{V}}}{|t-s|^2} s^\beta\,ds\,dt. \end{aligned}$$

We write

$$L_1(t,s) = e^{-(t-s)A(0)}u(s) - u(s) = \int_0^{t-s} e^{-lA(0)} A(0)u(s)\,dl.$$

Lemma 2.1 and the quadratic estimate gives

$$\begin{aligned} \int_0^\tau \int_0^t \frac{\|L_1(t,s)\|^2_{\mathcal{V}}}{|t-s|^2} s^\beta\,ds\,dt &\le \int_0^\tau \int_s^\tau \Big(\frac{\int_0^{t-s} \|e^{-lA(0)}A(0)u(s)\|_{\mathcal{V}} dl}{|t-s|}\Big)^2\,dt s^\beta\,ds \\ &\le C\int_0^\tau \int_s^\tau \|e^{-tA(0)}A(0)u(s)\|^2_{\mathcal{V}}\,dt s^\beta\,ds \\ &\le C'\int_0^\tau \|A(0)u(s)\|^2 s^\beta\,ds \\ &= C'\|A(0)u\|^2_{L^2_\beta(0,\tau;\mathcal{H})}. \end{aligned}$$

Similarly, we obtain

$$\int_0^\tau \int_0^t \frac{\|L_2(t,s)\|^2_{\mathcal{V}}}{|t-s|^2} s^\beta\,ds\,dt \le \int_0^\tau \int_0^t \Big(\frac{\int_s^t \|e^{(t-l)A(0)}(\Phi f)(l)\|_{\mathcal{V}}\,dl}{|t-s|}\Big)^2\,ds\,dt$$

$$\begin{aligned}&\le C\int_0^\tau\int_0^t \|e^{(t-s)A(0)}(\Phi f)(s)\|_{\mathcal{V}}^2\,ds\,dt\\&= C\int_0^\tau\int_s^\tau \|e^{(t-s)A(0)}(\Phi f)(s)\|_{\mathcal{V}}^2\,dt\,ds\\&\le C\|\Phi f\|_{L^2(0,\tau;\mathcal{H})}^2 = C\|f\|_{L^2_\beta(0,\tau;\mathcal{H})}^2.\end{aligned}$$

Therefore,

$$\|u\|_{W_\beta^{\frac12,2}(0,\tau;\mathcal{V})} \lesssim \|A(0)u\|_{L^2_\beta(0,\tau;\mathcal{H})} + \|f\|_{L^2_\beta(0,\tau;\mathcal{H})} \lesssim \|u\|_{W_\beta(D(A(0)),\mathcal{H})}.$$

We note that $C^\infty([0,\tau];D(A(0)))$ is dense in $W_\beta(D(A(0)),\mathcal{H})$. This shows that

$$W_\beta(D(A(0)),\mathcal{H}) \hookrightarrow W_\beta^{\frac12,2}(0,\tau;\mathcal{V}).$$

which completes the proof □

Remark 4.2. The following embeddings hold

(1) $W_\beta(D(A(0)),\mathcal{H}) \hookrightarrow C([0,\tau];[\mathcal{H},\mathcal{V}]_{1-\beta})$, for $0\le\beta<1$.
(2) $W_\beta(D(A(0)),\mathcal{H}) \hookrightarrow C([0,\tau];\mathcal{V})$, for $\beta\le 0$.

Theorem 4.3. *For all $f\in W^{1,2}_{\beta,0}(0,\tau,\mathcal{H})$, there exists a unique*

$$u\in C^1([0,\tau];(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2})\cap C([0,\tau];D(A(0))),$$

which satisfies the equation

$$\begin{gathered}\dot u(t)+\mathcal{A}(0)u(t)=f(t)\\ u(0)=0.\end{gathered}\tag{4.5}$$

In addition,

$$\|u\|_{C^1([0,\tau];(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2})\cap C([0,\tau];D(A(0)))} \le C\|f\|_{W_\beta^{1,2}(0,\tau;\mathcal{H})}.$$

Assume now that $\tau=+\infty$ and f is a periodic function with period p. Then u satisfies

$$u(t+p)=e^{-t\mathcal{A}(0)}u(p)+u(t),\ t\in[0,\infty),$$

and it is periodic with the same period p if and only if $u(p)=0$.

Proof. According to Theorem 4.1, there exists a unique solution u to Problem (4.5) and for all $f\in L^2_\beta(0,\tau;\mathcal{H})$

$$u(t)=\int_0^t e^{-(t-s)\mathcal{A}(0)}f(s)\,ds,\quad t\in[0,\tau].\tag{4.6}$$

Moreover $u\in W_\beta(D(A(0)),\mathcal{H})$ and

$$\|u\|_{W_\beta(D(A(0)),\mathcal{H})}\le C\|f\|_{L^2_\beta(0,\tau;\mathcal{H})}.\tag{4.7}$$

Integrating by parts, we obtain for $t\in[0,\tau]$ and $f\in W^{1,2}_{\beta,0}(0,\tau,\mathcal{H})$

$$\begin{aligned}\mathcal{A}(0)u(t)&=\mathcal{A}(0)\int_0^t e^{-(t-s)\mathcal{A}(0)}f(s)\,ds\\&=f(t)-\int_0^t e^{-(t-s)\mathcal{A}(0)}\dot f(s)\,ds\\&=\dot u(t)+\mathcal{A}(0)u(t)-\int_0^t e^{-(t-s)\mathcal{A}(0)}\dot f(s)\,ds.\end{aligned}$$

Hence,

$$\dot{u}(t) = \int_0^t e^{-(t-s)\mathcal{A}(0)} \dot{f}(s)\, ds = (L\dot{f})(t).$$

Theorem 4.1 shows that $u \in C^1([0,\tau]; (\mathcal{H}; D(A(0)))_{\frac{1-\beta}{2},2})$. Since $A(0)u = f - \dot{u}$ we deduce that $A(0)u \in C([0,\tau];\mathcal{H})$. As a consequence, we obtain the final estimate

$$\|u\|_{C^1([0,\tau];(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2})\cap C([0,\tau];D(A(0)))} \le C\|f\|_{W^{1,2}_\beta(0,\tau;\mathcal{H})}.$$

Consider now the case where $\tau = +\infty$ and f is a periodic function with some period $p > 0$, i.e. $f(t+p) = f(t)$ for all $t \in [0,+\infty)$. It is clear that if u is periodic with period p, then $u(p) = u(0) = 0$. Formula (4.6) yields

$$u(t+p) = \int_0^{t+p} e^{-(t+p-s)\mathcal{A}(0)} f(s)\, ds.$$

Hence,

$$\begin{aligned}
u(t+p) &= \int_0^p e^{-(t+p-s)\mathcal{A}(0)} f(s)\, ds + \int_p^{p+t} e^{-(t+p-s)\mathcal{A}(0)} f(s)\, ds \\
&= e^{-t\mathcal{A}(0)} \int_0^p e^{-(p-s)\mathcal{A}(0)} f(s)\, ds + \int_0^t e^{-(t-l)\mathcal{A}(0)} f(l+p)\, dl \\
&= e^{-t\mathcal{A}(0)} u(p) + u(t).
\end{aligned}$$

In the previous equality, we made a change of variables, and in the last equality we used the periodicity of f. Then u is periodic with period p if and only if $e^{-t\mathcal{A}(0)}u(p) = 0$ for all $t \in [0,\infty)$. Therefore, the analyticity of the semigroup shows that $u(p) = 0$ is a necessary condition for u to be periodic. □

5. Maximal regularity for non-autonomous problems

In this section we focus on the maximal regularity for the non-autonomous problem (which is our main aim), i.e. we prove the existence and the uniqueness of the solution to Problem (1.1) in the weighted space $W^{1,2}_\beta(0,\tau;\mathcal{H})$. We start by stating and proving some estimates which we will need in the proof of the main result.

Proposition 5.1. *(1) Assume that*

$$\int_0^\tau \frac{\|\mathcal{A}(t) - \mathcal{A}(0)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t}\, dt < \infty.$$

Then for all $s \in [0,\tau]$,

$$\begin{aligned}
TR_s : W_\beta(D(A(\cdot),\mathcal{H}) \cap L^\infty_\beta(0,\tau;\mathcal{V}) &\longrightarrow (\mathcal{H}; D(A(s)))_{\frac{1-\beta}{2},2} \\
u &\longmapsto u(s)
\end{aligned}$$

is a bounded operator.

(2) For $u_0 \in (\mathcal{H}; D(A(0)))_{\frac{1-\beta}{2},2}$, we have

$$t \to (Fu_0)(t) = t^{\beta/2} A(t) e^{-tA(t)} u_0 \in L^2(0,\tau;\mathcal{H}).$$

Proof. (1) First we consider the case $s = 0$. We have

$$\begin{aligned}
&\|u(0)\|^2_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} \\
&= \int_0^1 \|t^{\beta/2} A(0) e^{-tA(0)} u(0)\|^2\, dt + \|u(0)\|^2
\end{aligned}$$

$$\begin{aligned}
&\le 2\int_0^1 \|t^{\beta/2}A(0)e^{-tA(0)}(u(0)-u(t))\|^2\,dt + \|u(0)\|^2\\
&\quad + 2\int_0^1 \|t^{\beta/2}A(0)e^{-tA(0)}u(t)\|^2\,dt\\
&\lesssim \int_0^1 t^{\beta}\Big(\frac{1}{t}\int_0^t \|\dot u(l)\|\,ds\Big)^2\,dl + \int_0^\tau t^{\beta}\|A(t)u(t)\|^2\,dt\\
&\quad + \int_0^\tau \|t^{\beta/2}(A(0)e^{-tA(0)} - A(t)e^{-tA(t)})u(t)\|^2\,dt + \|u(0)\|^2\\
&\lesssim \|\dot u\|^2_{L^2_\beta(0,\tau;\mathcal{H})} + \|A(\cdot)u\|^2_{L^2_\beta(0,\tau;\mathcal{H})}\\
&\quad + \int_0^\tau \frac{\|\mathcal{A}(t)-\mathcal{A}(0)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t}\,dt\|u\|_{L^\infty_\beta(0,\tau;\mathcal{V})} + \|u(0)\|^2\\
&\lesssim \|u\|^2_{W_\beta(D(A(\cdot),\mathcal{H})} + \|u\|^2_{L^\infty_\beta(0,\tau;\mathcal{V})} + \|u(0)\|^2,
\end{aligned}$$

where we have used the quadratic estimate, Hardy inequality and the estimate

$$\|A(0)e^{-tA(0)} - A(t)e^{-tA(t)}\|_{\mathcal{L}(\mathcal{V},\mathcal{H})} \lesssim \frac{\|\mathcal{A}(t)-\mathcal{A}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t^{1/2}}.$$

Now, we prove the result for all $s\in]0,\tau]$. Indeed, let $l\in]0,\tau[$ and set

$$v(t) := \begin{cases} u(t+s), & t\in[0,\tau-s].\\ u(\frac{\tau}{s}(\tau-t)), & t\in[\tau-s,\tau].\end{cases}$$

Similarly,

$$B(t) := \begin{cases} A(t+s), & t\in[0,\tau-s].\\ A(\frac{\tau}{s}(\tau-t)), & t\in[\tau-s,\tau].\end{cases}$$

Since $v(t)\in W_\beta(D(B(\cdot),\mathcal{H})$, therefore

$$v(0) = u(s) \in (\mathcal{H}; D(B(0)))_{\frac{1-\beta}{2},2} = (\mathcal{H}; D(A(s)))_{\frac{1-\beta}{2},2}.$$

For the case $s=\tau$, we take $v(t)=u(\tau-t)$ and $B(t)=A(\tau-t)$.

(2) Note that

$$\begin{aligned}
(Fu_0)(t) &= t^{\beta/2}A(t)e^{-tA(t)}u_0\\
&= t^{\beta/2}(A(t)e^{-tA(t)}u_0 - A(0)e^{-tA(0)}u_0) + t^{\beta/2}A(0)e^{-tA(0)}u_0.
\end{aligned}$$

For $\beta>0$ we have by interpolation

$$\|(\lambda - A(0))^{-1}\|_{\mathcal{L}((\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2},\mathcal{V})} \lesssim \frac{1}{|\lambda|^{1-\frac{\beta}{2}}}.$$

Therefore

$$\|(Fu_0)(t)\| \lesssim \frac{\|\mathcal{A}(0)-\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t}\|u_0\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2}}} + \|t^{\beta/2}A(0)e^{-tA(0)}u_0\|.$$

Hence,

$$\begin{aligned}
\|(Fu_0)\|^2_{L^2(0,\tau;\mathcal{H})} &\lesssim \int_0^\tau \frac{\|\mathcal{A}(0)-\mathcal{A}(t)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t}\,dt\|u_0\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}}\\
&\quad + \int_0^\tau \|t^{\beta/2}A(0)e^{-tA(0)}u_0\|^2\,dt
\end{aligned}$$

$$\lesssim \|u_0\|^2_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}}.$$

This shows the second assertion. □

In the sequel we consider only the case $\beta \in [0,1[$.

Proposition 5.2. *Suppose $\mathcal{A} \in C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))$. Then for each $f \in L^2_\beta(0,\tau;\mathcal{H})$, $u_0 \in (\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}$ and for τ small enough, there exists a unique solution u in $L^\infty_\beta(0,\tau;\mathcal{V})$ for* (1.1).

Proof. Let $f \in L^2_\beta(0,\tau;\mathcal{H})$. We set $v(s) = e^{-(t-s)A(t)}u(s)$. Since $u(t) = e^{-tA(t)}u_0 + \int_0^t \dot v(s)\,ds$, therefore

$$\begin{aligned} u(t) &= e^{-tA(t)}u_0 + \int_0^t e^{-(t-s)A(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,ds \\ &\quad + \int_0^t e^{-(t-s)A(t)} f(s)\,ds \\ &:= (Mu_0)(t) + (M_1u)(t) + (L_1f)(t). \end{aligned} \tag{5.1}$$

For $\beta > 0$ and $u_0 \in (\mathcal{H},D(A(0)))_{\frac{1-\beta}{2},2}$ we have by interpolation

$$\|e^{-tA(t)}u_0\|_{\mathcal{V}} \lesssim t^{-\beta/2}\|u_0\|_{(\mathcal{H},D(A(0)))_{\frac{1-\beta}{2},2}}. \tag{5.2}$$

In view of Lemma 3.12 and (5.2), $t^{\beta/2}(Mu_0)(t)$, $t^{\beta/2}(L_1f)(t)$ are bounded in $\mathcal{V}$ for all $t \in [0,\tau]$. Now, we show that $M_1u \in L^\infty_\beta(0,\tau;\mathcal{V})$ for all $u \in L^\infty_\beta(0,\tau;\mathcal{V})$. We write

$$\begin{aligned} (M_1u)(t) &= \int_0^{t/2} e^{-(t-s)A(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,ds \\ &\quad + \int_{t/2}^t e^{-(t-s)A(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,ds \\ &:= (M_{11}u)(t) + (M_{12}u)(t). \end{aligned}$$

By taking $x \in \mathcal{V}'$ we obtain

$$\begin{aligned} &|((M_{12}u)(t),x)_{\mathcal{V}'\times\mathcal{V}}| \\ &= \Big|\int_{t/2}^t (e^{-\frac{(t-s)}{2}A(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s), A(t)^{*\frac12}e^{-\frac{(t-s)}{2}A(t)^*}A(t)^{*-\frac12}x)\,ds\Big| \\ &\le \Big(\int_{t/2}^t \|e^{-\frac{(t-s)}{2}A(t)}\|^2_{\mathcal{L}(\mathcal{V}',\mathcal{H})}\|\mathcal{A}(t)-\mathcal{A}(s)u(s)\|^2_{\mathcal{V}'}\,ds\Big)^{1/2} \\ &\quad \times \Big(\int_{t/2}^t \|A(t)^{*\frac12}e^{-\frac{(t-s)}{2}A(t)^*}A(t)^{*-\frac12}x\|^2\,ds\Big)^{1/2}. \end{aligned}$$

Now, we estimate the norm of $(M_{11}v)(t)$ in $\mathcal{V}$ as follows

$$\begin{aligned} &t^{\beta/2}\|(M_{11}v)(t)\|_{\mathcal{V}} \\ &\lesssim t^{\beta/2}\int_0^{t/2} \|e^{-\frac{(t-s)}{2}A(t)}\|_{\mathcal{L}(\mathcal{V}',\mathcal{V})}\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}s^{-\beta/2}\,ds\|s \\ &\to s^{\beta/2}u(s)\|_{L^\infty(0,\frac{t}{2};\mathcal{V})} \end{aligned}$$

$$\lesssim t^{\beta/2}\int_0^{t/2}\frac{s^{-\beta/2}}{(t-s)^{1-\varepsilon}}\,ds\sup_{s\in[0,t/2]}\frac{\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{(t-s)^{\varepsilon}}\|s$$
$$\to s^{\beta/2}u(s)\|_{L^\infty(0,\frac{t}{2};\mathcal{V})}.$$

Note that

$$t^{\beta/2}\int_0^{t/2}\frac{s^{-\beta/2}}{(t-s)^{1-\varepsilon}}\,ds=t^{\varepsilon}\int_0^{1/2}\frac{l^{-\beta/2}}{(1-l)^{1-\varepsilon}}\,dl.$$

Therefore,

$$\begin{aligned}&t^{\beta/2}\|(M_1v)(t)\|_{\mathcal{V}}\\&\lesssim t^{\epsilon}\|\mathcal{A}\|_{C^{\epsilon}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))}\|s\\&\to s^{\beta/2}u(s)\|_{L^\infty(0,\frac{t}{2};\mathcal{V})}+\Big(\int_{t/2}^t\frac{\|\mathcal{A}(t)-\mathcal{A}(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-s}\,ds)^{1/2}\|u\|_{L^\infty_\beta(\frac{t}{2},t;\mathcal{V})}\\&\lesssim t^{\varepsilon}\|A(\cdot)\|_{C^{\varepsilon}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))}\|u\|_{L^\infty_\beta(0,t;\mathcal{V})}.\end{aligned}$$

Choosing τ small enough, $M_1\in\mathcal{L}(L^\infty_\beta(0,\tau;\mathcal{V}))$, with norm $\|M_1\|_{\mathcal{L}(L^\infty_\beta(0,\tau;\mathcal{V}))}<1$. Therefore $(I-M_1)$ is invertible in $L^\infty_\beta(0,\tau;\mathcal{V})$. Hence,

$$u=(I-M_1)^{-1}(Mu_0+L_1f)\in L^\infty_\beta(0,\tau;\mathcal{V}).$$

This completes the proof. □

Our main result reads as follows.

Theorem 5.3. *Suppose that $\mathcal{A}\in W^{\frac12,2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))\cap C^{\varepsilon}([0,\tau],\mathcal{L}(\mathcal{V},\mathcal{V}'))$ with $\varepsilon>0$, then for all $f\in L^2_\beta(0,\tau;\mathcal{H})$ and $u_0\in(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2}}$, there exists a unique $u\in W_\beta(D(A(\cdot),\mathcal{H})$ be the solution of* (1.1).

Proof. Let τ be small enough and $f\in L^2_\beta(0,\tau;\mathcal{H})$, $u_0\in(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}$. By Proposition 5.2, u belongs to $L^\infty_\beta(0,\tau;\mathcal{V})$, where u is the unique solution to the Cauchy problem (1.1). Using (5.1), for $0\le t\le\tau$, we have

$$\begin{aligned}A(t)u(t)&=A(t)e^{-tA(t)}u_0+A(t)\int_0^t e^{-(t-s)A(t)}(\mathcal{A}(t)-\mathcal{A}(s)u(s)\,ds\\&\quad+A(t)\int_0^t e^{-(t-s)A(t)}f(s)\,ds\\&:=(Fu_0)(t)+(Su)(t)+(Lf)(t).\end{aligned}$$

Thanks to Propositions 3.10, 5.1, Fu_0 and Lf are bounded in $L^2_\beta(0,\tau;\mathcal{H})$. Then to prove that $A(\cdot)u\in L^2_\beta(0,\tau;\mathcal{H})$ it is sufficient to show that Su belongs to $L^2_\beta(0,\tau;\mathcal{H})$. Taking $g\in L^2(0,\tau;\mathcal{H})$ we find that

$$\begin{aligned}&|(.^{\beta/2}Su,g)_{L^2(0,\tau;\mathcal{H})}|\\&=\Big|\int_0^\tau t^{\beta/2}\int_0^t\langle(\mathcal{A}(t)-\mathcal{A}(s))u(s),A(t)^*e^{-(t-s)A(t)^*}g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\,ds\,dt\Big|\\&\le\Big|\int_0^\tau t^{\beta/2}\int_0^{t/2}\langle(\mathcal{A}(t)-\mathcal{A}(s))u(s),A(t)^*e^{-(t-s)A(t)^*}g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\,ds\,dt\Big|\\&\quad+\Big|\int_0^\tau t^{\beta/2}\int_{t/2}^t\langle(\mathcal{A}(t)-\mathcal{A}(s))u(s),A(t)^*e^{-(t-s)A(t)^*}g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\,ds\,dt\Big|\\&:=I_1+I_2.\end{aligned}$$

For I_2 we find,

$$\begin{aligned}
I_2 &\lesssim \int_0^\tau t^{\beta/2} \int_{t/2}^t \|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \|e^{-\frac{(t-s)}{2}A(t)^*}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \\
&\quad \times \|A(t)^{*\frac12} e^{-\frac{(t-s)}{4}A(t)^*}\|_{\mathcal{L}(\mathcal{H})} \|A(t)^{*\frac12} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\| s^{-\frac{\beta}{2}}\, ds\, dt \|\cdot^{\beta/2} u\|_{L^\infty(0,\tau;\mathcal{V})} \\
&\lesssim \int_0^\tau \int_{t/2}^t \frac{\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-s} \|A(t)^{*\frac12} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\|\, ds\, dt \|\cdot^{\beta/2} u\|_{L^\infty(0,\tau;\mathcal{V})} \\
&\lesssim \|\mathcal{A}\|_{W^{\frac12,2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))} \Big(\int_0^\tau \int_{t/2}^t \|A(t)^{*\frac12} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\|^2\, ds\, dt \Big)^{1/2} \|u\|_{L^\infty_\beta(0,\tau;\mathcal{V})} \\
&\lesssim \|\mathcal{A}\|_{W^{\frac12,2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|g\|_{L^2(0,\tau,\mathcal{H})} \|u\|_{L^\infty_\beta(0,\tau;\mathcal{V})}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_1 &\lesssim \int_0^\tau t^{\beta/2} \int_0^{t/2} \frac{s^{\frac{-\beta}{2}}}{(t-s)^{\frac32-\varepsilon}} \|g(t)\|\, ds\, dt \\
&\quad \times \|\mathcal{A}\|_{C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|\cdot^{\beta/2} u\|_{L^\infty(0,\tau;\mathcal{V})} \\
&\lesssim \|\mathcal{A}\|_{C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|g\|_{L^2(0,\tau,\mathcal{H})} \|u\|_{L^\infty_\beta(0,\tau;\mathcal{V})}.
\end{aligned}$$

Now, we obtain the final estimate

$$\begin{aligned}
\|A(\cdot)u\|_{L^2_\beta(0,\tau;\mathcal{H})} &\lesssim \|Fu_0\|_{L^2_\beta(0,\tau;\mathcal{H})} + \|Su\|_{L^2_\beta(0,\tau;\mathcal{H})} + \|Lf\|_{L^2_\beta(0,\tau;\mathcal{H})} \\
&\lesssim \|u_0\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} + \|u\|_{L^\infty_\beta(0,\tau;\mathcal{V})} + \|f\|_{L^2_\beta(0,\tau;\mathcal{H})} \\
&\lesssim \|u_0\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2},2}} + \|f\|_{L^2_\beta(0,\tau;\mathcal{H})}.
\end{aligned}$$

Therefore $A(\cdot)u \in L^2_\beta(0,\tau;\mathcal{H})$ and since $\dot u = f - Au$, one has $\dot u \in L^2_\beta(0,\tau;\mathcal{H})$. So u belongs to $W_\beta(D(A(\cdot),\mathcal{H})$. Moreover, by Proposition 5.1 we have $u(t) \in (\mathcal{H}; D(A(t)))_{\frac{1-\beta}{2},2}$ for all $t \in [0,\tau]$.

For arbitrary τ we split the interval $[0,\tau]$ into union of small intervals and argue exactly as before to each subinterval. Finally we stick the solutions and we obtain the desired result. □

Proposition 5.4. *For all $g \in L^2(0,\tau;\mathcal{H})$ and $0 \le \beta < 1$ there exists a unique $v \in W_0(D(A(\cdot),\mathcal{H})$ be the solution of the singular equation*

$$\begin{gathered}
\dot v(t) + \mathcal{A}(t)v(t) + \frac{\beta}{2}\frac{v(t)}{t} = g(t) \\
v(0) = 0.
\end{gathered} \tag{5.3}$$

Proof. We set $f(t) = (\Phi g)(t) = t^{\beta/2} g(t)$ with $t \in [0,\tau]$, so that $f \in L^2_\beta(0,\tau;\mathcal{H})$. Let $u \in W_\beta(D(A(\cdot),\mathcal{H})$ be the solution to the problem

$$\begin{gathered}
\dot u(t) + \mathcal{A}(t)u(t) = f(t) \\
u(0) = 0.
\end{gathered} \tag{5.4}$$

Now, set $v = (\Phi^{-1}u)$. Then $v \in W_0(D(A(\cdot),\mathcal{H})$ and v is the unique solution to Problem (5.3). □

6. Applications

This section is devoted to some applications of the results given in the previous sections. We give examples illustrating the theory without seeking for generality.

6.1. **Elliptic operators in the divergence form.** Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$. We set $\mathcal{H} := L^2(\Omega)$ and $\mathcal{V} := H^1(\Omega)$ and we define the sesquilinear forms

$$\mathfrak{a}(t,u,v) := \int_\Omega C(t,x)\nabla u\overline{\nabla v}\,dx$$

where here $u, v \in \mathcal{V}$ and $C : [0,\tau]\times\Omega \to \mathbb{C}^{n\times n}$ is a bounded and measurable function for which there exists $\alpha, M > 0$ such that

$$\alpha|\xi|^2 \le \operatorname{Re}(C(t,x)\xi.\bar{\xi}) \quad \text{and} \quad |C(t,x)\xi.\nu| \le M|\xi||\nu|$$

for all $t \in [0,\tau]$ and a.e $x \in \Omega$, and all $\xi, \nu \in \mathbb{C}^n$. We define the gradient operator $\nabla : \mathcal{V} \to \mathcal{H}$ and $\nabla^* : \mathcal{H} \to \mathcal{V}'$. The non-autonomous form $\mathfrak{a}(t)$ induces the operators

$$\mathcal{A}(t) := -\nabla^* C(t,x)\nabla \in \mathcal{L}(\mathcal{V},\mathcal{V}').$$

The form $\mathfrak{a}(t)$ is $H^1(\Omega)$-bounded and coercive. The part of $\mathcal{A}(t)$ in $\mathcal{H}$ is the operator

$$A(t) := -\operatorname{div}\, C(t,x)\nabla$$

under Neumann boundary conditions.

We note that

$$\|\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \simeq \|C(t,\cdot)\|_{L^\infty(\Omega;\mathbb{C}^{n\times n})} = M.$$

Next, we suppose that $C \in W^{\frac12,2}(0,\tau;L^\infty(\Omega;\mathbb{C}^{n\times n})) \cap C^\varepsilon([0,\tau];L^\infty(\Omega;\mathbb{C}^{n\times n}))$, with $\varepsilon > 0$, which is equivalent to

$$\int_0^\tau\int_0^\tau \sup_{x\in\Omega}\frac{\|C(t,x)-C(s,x)\|^2_{\mathbb{C}^{n\times n}}}{|t-s|^2}\,ds\,dt < \infty,$$
$$\|C(t,x)-C(s,x)\|_{\mathbb{C}^{n\times n}} < C|t-s|^\varepsilon$$

a.e. for $x \in \Omega$ and $t, s \in [0,\tau]$. Note that

$$\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \lesssim \|C(t,.)-C(s,.)\|_{L^\infty(\Omega;\mathbb{C}^{n\times n})}.$$

Hence $\mathcal{A} \in W^{\frac12,2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}')) \cap C^\epsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))$.

Remark 6.1. $D(A(t)^{1/2}) = \mathcal{V} = H^1(\Omega)$ for all $t \in [0,\tau]$ and

$$c_1\|u\|_{H^1(\Omega)} \le \|u\|_{D(A(t)^{1/2})} \le c^1\|u\|_{H^1(\Omega)}$$

where c_1, c^1 are two positive constants independents of t [6, Theorem 1].

In the next proposition we assume that $\beta \in [0,1[$.

Proposition 6.2. *For all $f \in L^2_\beta(0,\tau;L^2(\Omega)), u_0 \in H^{1-\beta}(\Omega)$ there is a unique $u \in W_\beta(D(A(\cdot),L^2(\Omega))$, be the solution of the problem*

$$\begin{gathered}\dot u(t) - \operatorname{div} C(t,x)\nabla u(t) = f(t)\\ \frac{\partial u(t,\sigma)}{\partial n} = 0 \ (\sigma\in\partial\Omega)\\ u(0) = u_0.\end{gathered}\tag{6.1}$$

The above proposition follows by Theorem 5.3.

6.2. **Robin boundary conditions.** Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. We denote by Tr the classical trace operator. Let $\beta : [0,\tau]\times\partial\Omega \to [0,\infty)$ be a bounded function and $\mathcal{H} := L^2(\Omega)$. We define the form

$$\mathfrak{a}(u,v) := \int_\Omega \nabla u.\nabla v\,dx + \int_{\partial\Omega} \beta(\cdot)\mathrm{Tr}(u)\mathrm{Tr}(v)\,d\sigma,$$

for all $u,v \in \mathcal{V} := H^1(\Omega)$.

The form $\mathfrak{a}$ is $H^1(\Omega)$-bounded, symmetric and quasi-coercive. The first statement follows readily from the continuity of the trace operator and the boundedness of β. The second one is a consequence of the inequality

$$\int_{\partial\Omega} |u|^2 d\sigma \le \delta\|u\|^2_{H^1(\Omega)} + C_\delta\|u\|^2_{L^2(\Omega)}$$

which is valid for all $\delta > 0$ (C_δ is a constant depending on δ). Note that this is a consequence of compactness of the trace as an operator from $H^1(\Omega)$ into $L^2(\partial\Omega, d\sigma)$. Formally, the associated operator A is (minus) the Laplacian with the time dependent Robin boundary condition

$$\tfrac{\partial u}{\partial n} + \beta(\cdot)u = 0 \text{ on } \partial\Omega.$$

Here, $\frac{\partial u}{\partial n}$ denotes the normal derivative in the weak sense. For more general boundary conditions with an indefinite weight we refer the reader to the recent paper [10].

Theorems 4.1 combined with Theorem 4.3 yields the following result.

Proposition 6.3. *Let $\beta \in]-1,1[$ and $f \in L^2_\beta(0,\tau;L^2(\Omega))$. There exists a unique $u \in W_\beta(D(A),L^2(\Omega)) \cap C([0,\tau],(L^2(\Omega);D(A))_{\frac{1-\beta}{2},2})$ be the solution to the problem*

$$\begin{gathered} \dot u(t) - \Delta u(t) = f(t) \\ \tfrac{\partial u}{\partial n} + \beta(\cdot)u = 0 \quad \text{on } \partial\Omega \\ u(0) = 0. \end{gathered} \tag{6.2}$$

If we assume moreover that $f \in W^{1,2}_{\beta,0}(0,\tau;L^2(\Omega))$, then the solution u belongs to the space $C^1([0,\tau];(L^2(\Omega);D(A))_{\frac{1-\beta}{2},2}) \cap C([0,\tau];D(A))$.

Remark 6.4. Note that for all $\beta \in [0,1[$ we have

$$(L^2(\Omega);D(A))_{\frac{1-\beta}{2},2} = [L^2(\Omega);D(A)]_{\frac{1-\beta}{2}} = [L^2(\Omega);H^1(\Omega)]_{1-\beta} = H^{1-\beta}(\Omega).$$

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