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POINTWISE ESTIMATES OF SOLUTIONS TO CONSERVATION LAWS WITH NONLOCAL DISSIPATION-TYPE TERMS

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Abstract. This article concerns the Cauchy problem of conservation laws with nonlocal dissipation-type terms in $\mathbb{R}^3$. By using Green's function and the time-frequency decomposition method, we study global classical solutions and their long time behavior including pointwise estimates for large initial data, for solutions near the nontrivial equilibrium state.

1. Introduction

In this article, we study the existence of global solutions and their decay estimates to conservation laws with nonlocal dissipation-type terms in $\mathbb{R}^3$. We consider the solution, near the equilibrium state $u^* \neq 0$, of the equation

$$\partial_t u - \gamma_1 \triangle u_t - \gamma_2 \triangle u = \operatorname{div} h(u), \tag{1.1}$$

subject to the initial value

$$u(x,0) = u_0(x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3. \tag{1.2}$$

Here, $\gamma_1, \gamma_2 > 0$, and $h(u) = (h_1(u), h_2(u), h_3(u))$ satisfies the condition

$$h_k(u) = O(u^2), \quad k = 1, 2, 3.$$

This model is motivated by physical considerations from fluid dynamics. It describes a variety of important physical processes, such as the aggregation of populations in which u describes the population density [12]. It can also be used in the analysis of a seepage of homogeneous fluids through a fissured rock and the unidirectional propagation of nonlinear dispersive long waves [1, 14]. There are many equations similar to problem (1.1). For instance, the equation

$$\partial_t u - k \triangle u_t - \triangle u = \operatorname{div} h(u),$$

which is (1.1) with $\gamma_2 = 1$, can be used to describe the nonstationary processes in semiconductors in the presence of sources [9, 10]. Here $k\Delta u_t - u_t$ indicates the rate of change in free charge's density, Δu indicates the electric current of linear dissipative free charge and $\operatorname{div} h(u)$ describes a source of free electron current [9]. Ting, Showalter and Golpala Rao studied the initial-boundary value problem and the Cauchy problem for the linear equation when $\gamma_2 = 1$ and proved the existence and uniqueness of the solution [6, 13, 15]. Afterwards, considerable attention has

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been paid to the study of the nonlinear equation, and the large time behavior of the solutions [7, 8, 21].

Cao, Yin and Wang [2] studied the equation

$$\partial_t u - k\triangle u_t - \triangle u = u^p.$$

They focused on the parameter p and sufficiently small initial data. They obtained two critical indices about p and then considered the global existence or the blow up of the solutions.

In [16], the authors considered the following equation under the condition $0 \leq s \leq 1/2$:

$$u_t - \frac{\Delta u}{(1-\Delta)^s} = \operatorname{div} h(u). \tag{1.3}$$

They got the global classical solution and long-time behavior for arbitrary large initial data. In [17], Wang proved that in the case $s = 1$, the solution is global with sufficiently small initial data, but will blow up with some large initial data.

Wang and Zhang [20] investigated the problem

$$\partial_t u - \gamma_1 \triangle u_t - \gamma_2 \triangle u + \gamma_3 \Delta^2 u = \sum_{j=1}^{n} h_j(u)_{x_j}, \tag{1.4}$$

which is what we call the Benjamin-Bona-Mahony (BBM) equation. They obtained ta global solution and the optimal decay estimates with large initial data when $\gamma_3 \neq 0$.

In our case, i.e. $\gamma_3 = 0$, there are some new difficulties. First of all, the condition $\gamma_3 \neq 0$ in (1.4) gives a dissipation term $\Delta^2 u$, which allows using the energy method to get the L^∞ estimate, and it also gives a better decay. Without this term, we need to use some new methods such as the time-frequency decomposition. Moreover, there is no maximum principle like for the viscous Burgers equation, which increases the difficulty.

Another motivation for us to consider the solution near the nontrivial equilibrium state is that the equation in our paper actually has some hyperbolic properties which however will vanish if we just consider the trivial equilibrium state. From the main results, we will see from the solution that the phenomenon of a propagating wave appears, and the propagation direction and speed of the wave are decided by the initial data. Obviously, if we only consider the solution near zero, the speed and direction will both be zero and will not be observed. So here, to get the optimal decay estimate and the hyperbolic property of the solution, we first do the pointwise estimate of the Green's function, then combining it with the variable substitution and Green's method to obtain the L^1 and L^∞ bounded estimate. Based on this, we obtain the pointwise estimate of the solution.

Since we consider the solution near the equilibrium state u^*, using $u - u^*$ as the new u, we can rewrite equation (1.1) as

$$\partial_t u - \gamma_1 \triangle u_t - \gamma_2 \triangle u + \sum_{j=1}^{3} b_j u_{x_j} = \operatorname{div} f(u), \tag{1.5}$$

with initial condition

$$u(x,0) = u_0(x) - u^*. \tag{1.6}$$

Here, $\gamma_1, \gamma_2 > 0$, $b = (C_1 u^*, C_2 u^*, C_3 u^*)$ and $f(u) = (f_1(u), f_2(u), f_3(u))$ satisfies the condition

$$f_k(u) = O(u^2), \quad (k = 1, 2, 3.$$

The main results of this article read as follows.

Theorem 1.1. *Suppose that the initial data $v_0 = u_0 - u^* \in L^1(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$, with $s > 1 + [\frac{3}{2}]$. Then, the Cauchy problem* (1.1)-(1.2) *has a global solution*

$$(u - u^*) \in L^\infty(0, \infty; H^s(\mathbb{R}^3)).$$

Theorem 1.2. *Assume that the initial data u_0 satisfies the conditions in Theorem 1.1, and u is a solution of* (1.1)-(1.2) *in $L^\infty([1, \infty), H^s(\mathbb{R}^3))$ with initial data u_0. Then*

$$\|(u - u^*)\|_{\dot{H}^s(\mathbb{R}^3)} \le C(1+t)^{-\frac{3}{4} - \frac{s}{2}},$$

for any $t \ge 1$, where C only depends on the initial data.

Theorem 1.3. *Assume that the initial data u_0 satisfies the conditions in Theorem 1.1, and v_0 satisfies*

$$|D_x^\beta v_0| = |D_x^\beta (u_0 - u^*)| \le C(1 + |x|^2)^{-r}, \quad r > \frac{3}{2},$$

with $|\beta| \le s - 1$. Then the solution satisfies the pointwise estimate

$$|D_x^\alpha (u - u^*)| \le C(1+t)^{-\frac{3+|\alpha|}{2}} B_r(|x - bt|, t).$$

Where $B_r(|x|, t)$ is defined by (2.1)*, $t \ge 1$, and $|\alpha| \le s - 1$.*

From the above results, we see that the solution mainly follows the decay of heat equation and, near the equilibrium state $u^* \ne 0$, the equation has some hyperbolic properties with speed b.

This article is organized as follows. In Section 2, we first recall some important lemmas that will be used in this paper. In Section 3, we introduce the properties of the Green's function, mainly focusing on the pointwise estimate and the L^p estimate of the Green's function. Then in Section 4, using Green's function and the variable substitution, we obtain the L^p boundedness of the solution u to equation (1.5) with initial value (1.6). In Section 5, we obtain the optimal decay estimates of the solution in $\dot{H}^s$ by using the time-frequency decomposition method. And, as a corollary of Theorem 1.2, the optimal L^∞ convergence rate can be obtained easily. Finally, in Section 6, based on the L^p boundedness and the decay of the solution, the pointwise estimate of the solution is established.

2. Preliminaries

In this section, we first introduce some notation, and then give some lemmas which will be used later.

For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we write $D_x^\alpha = \partial_{x_n}^{\alpha_n} \cdots \partial_{x_2}^{\alpha_2} \partial_{x_1}^{\alpha_1}$ and $|\alpha| = \sum_{i=1}^n |\alpha_i|$. As usual, the Fourier transform of $f(t, x)$ with respect to the variable $x \in \mathbb{R}^n$ is

$$\hat{f}(t, \xi) = (\mathcal{F}(f))(t, \xi) = \int_{\mathbb{R}^n} f(t, x) e^{-ix\cdot\xi} \mathrm{d}x,$$

and the inverse Fourier transform is

$$f(t, x) = (\mathcal{F}^{-1}\hat{f})(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(t, \xi) e^{ix\cdot\xi} \mathrm{d}\xi.$$

Here, n denotes the space dimension. The space $W^{s,p}(\mathbb{R}^n)$, with s a positive integer and $p \in [1, +\infty]$, is the usual Sobolev space with the norm

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \sum_{|\alpha|=0}^{s} \|D_x^\alpha f\|_{L^p(\mathbb{R}^n)}.$$

In particular, when $p = 2$, $W^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

We denote by $\dot{W}^{s,p}(\mathbb{R}^n)$ the Sobolev space equipped with the norm

$$\|\cdot\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = \|\Lambda^s \cdot\|_{L^p(\mathbb{R}^n)},$$

where Λ^s is defined by

$$\widehat{\Lambda^s f(\xi)} = |\xi|^s \hat{f}(\xi).$$

When $p = 2$, $\dot{W}^{s,2}(\mathbb{R}^n) = \dot{H}^s(\mathbb{R}^n)$ is the homogeneous Sobolev space. Next, we recall some useful lemmas. The first lemma gives some estimate of $f(u)$, its proof can be found in [11].

Lemma 2.1. *Assume that $f(u)$ is smooth enough and $f(u) = O(|u|^{1+\alpha_0})$ when $|u| \le \nu_0$, where $\alpha_0 \ge 1$ is an integer. For each integer $s \ge 0$, if u satisfies $\|u\|_{L^\infty(\mathbb{R}^n)} \le \nu_0$, then*

$$\|f(u)\|_{\dot{W}^{s,r}(\mathbb{R}^n)} \le C\|u\|_{\dot{W}^{s,q}(\mathbb{R}^n)}\|u\|_{L^p(\mathbb{R}^n)}\|u\|^{\alpha_0-1}_{L^\infty(\mathbb{R}^n)},$$
$$\|f(u)\|_{L^1(\mathbb{R}^n)} \le C\|u\|^2_{L^2(\mathbb{R}^n)}\|u\|^{\alpha_0-1}_{L^\infty(\mathbb{R}^n)},$$

where C is a constant and only depends on s and ν_0, and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, with $1 \le p, q, r \le \infty$.

The following two lemmas are very useful for obtaining the pointwise estimate of the solutions. Their proofs can be found in [19].

Lemma 2.2. *If $\hat{f}(\xi, t)$ has a compact support about ξ, and there exists $b > 0$ such that*

$$|D_\xi^\beta \xi^\alpha \hat{f}(\xi, t)| \le C(|\xi|^{(|\alpha|+k-|\beta|)_+} + |\xi|^{(|\alpha|+k)} t^{\frac{|\beta|}{2}})(1 + t|\xi|^2)^m e^{-b|\xi|^2 t},$$

for all $|\beta| \le 2N$, with N a positive integer, then

$$|D_x^\alpha f(x, t)| \le C_N(1 + t)^{-(|\alpha|+k+n)/2} B_N(t, |x|),$$

where k and m are positive integers, and

$$B_N(t, |x|) = \Big(1 + \frac{|x|^2}{1 + t}\Big)^{-N}, \quad (a)_+ = \begin{cases} a, & a \ge 0, \\ 0, & a < 0. \end{cases} \tag{2.1}$$

Lemma 2.3. *If $\operatorname{supp} \hat{f}(\xi) \subset O_R =: \{\xi, |\xi| > R\}$, and $\hat{f}(\xi)$ satisfies*

$$|D_\xi^\beta \hat{f}(\xi)| \le C|\xi|^{-1-|\beta|},$$

then there exist distributions $f_1(x)$, $f_2(x)$ and a constant C_0 such that

$$f(x) = f_1(x) + f_2(x) + C_0\delta(x),$$

where $\delta(x)$ is the Dirac function. Furthermore, for positive integers N with $2N > 3 + |\alpha|$, we have $|D_x^\alpha f_1(x)| \le C(1 + |x|^2)^{-N}$, $\|f_2\|_{L_1} \le C$ and $\operatorname{supp} f_2(x) \subset \{x; |x| < 2\epsilon_0\}$, with ϵ_0 sufficiently small.

Lemma 2.4 ([18]). *For $|y| \le M$, $t \ge 4M^2$ and $N > 0$, there exists $C_N > 0$ such that*
$$\Big(1 + \frac{|y-x|^2}{1+t}\Big)^{-N} \le C_N \Big(1 + \frac{|x|^2}{1+t}\Big)^{-N}.$$

Finally, note that for any $g(u) \in L^p(\mathbb{R}^n)$ and $p \ge 1$, we have
$$(1-\gamma_1\Delta)^{-1} g(u) = K_{\gamma_1}(\cdot) * g(u(\cdot,t)),$$
where
$$K_{\gamma_1}(x) = (4\pi)^{-n/2} \int_0^t e^{-\gamma_1 s - \frac{|x|^2}{4s}} s^{-n/2} \mathrm{d}s,$$
$$\|K_{\gamma_1}\|_{L^1(\mathbb{R}^n)} = \gamma_1.$$
Thus, we have
$$\begin{aligned} \|(1-\gamma_1\Delta)^{-1} g(u)\|_{L^p(\mathbb{R}^n)} &\le C \|K_{\gamma_1}\|_{L^1(\mathbb{R}^n)} \|g(u(\cdot))\|_{L^p(\mathbb{R}^n)} \\ &\le C \|g(u(\cdot))\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{2.2}$$

3. Existence of a local solution

In this section, we construct a convergent Cauchy sequence to obtain the local solution. Let $v^m(x,t)$ be the solution of the Cauchy problem
$$\begin{gathered} \partial_t v^m - \gamma_1 \triangle v_t^m - \gamma_2 \triangle v^m = \operatorname{div} h(v^{m-1}), \\ v^m(x,0) = v_0(x), \end{gathered} \tag{3.1}$$
where $m \ge 1$, $v^0(x,t) = 0$ and $v_0(x) \in H^s(\mathbb{R}^3)$.

For any given integer $s \ge 1 + [3/2]$, define the function space
$$\mathbb{X} = \{v(x,t) : \|v\|_{\mathbb{X}} < E\},$$
where $\|v\|_{\mathbb{X}} = \sup_{0\le t\le T_0} \|v\|_{H^s}$, T_0 is an undetermined constant and $E = C_0 v_0 \|v\|_{H^s}$, with $C_0 > 2$. One readily checks that $\mathbb{X}$ is a complete metric space. Next we prove that the function sequence $v^m(x,t)$ converges in the space $\mathbb{X}$.

Lemma 3.1. *For any $m \ge 1$, there exists a sufficiently small constant T_0, such that $v^m(x,t) \in \mathbb{X}$.*

Proof. We use induction. For $m = 1$, we know that
$$\partial_t v^1 - \gamma_1 \Delta v_t^1 - \gamma_2 \Delta v^1 = \operatorname{div} h(v^0).$$
Taking derivatives with respect to x, α times, on both sides of the above equation, where $|\alpha| \le s-1$, we have
$$D_x^\alpha v_t^1 - \gamma_1 D_x^\alpha \Delta v_t^1 - \gamma_2 D_x^\alpha \Delta v^1 = 0.$$
Multiplying $D_x^\alpha v^1$ on both sides, and integrating with respect to x and t on $\mathbb{R}^3 \times [0,t]$, we have
$$\|v^1\|_{H^{s-1}}^2 + \gamma_1 \|\nabla v^1\|_{H^{s-1}}^2 + 2\gamma_2 \int_0^t \|\nabla v^1\|_{H^{s-1}}^2 \mathrm{d}\tau = \|v_0\|_{H^{s-1}}^2 + \gamma_1 \|\nabla v_0\|_{H^{s-1}}^2.$$
Since $\gamma_1 > 0$ is a constant, it implies that
$$\sup_{0\le t\le T_0} \|v^1\|_{H^{s-1}} \le \|v_0\|_{H^s}.$$
Thus for $T_0 > 0$, we have $v^1(x,t) \in \mathbb{X}$.

Assume that there is a sufficiently small T_0 such that for each $j \le m$, $v^j(x,t) \in \mathbb{X}$. We shall show that $v^{m+1}(x,t) \in \mathbb{X}$. In fact, from (3.1) we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D_x^\alpha v^{m+1}\|_{L^2}^2 + \frac{\gamma_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(D_x^\alpha v^{m+1})\|_{L^2}^2 + \gamma_2\|\nabla(D_x^\alpha v^{m+1})\|_{L^2}^2\\
&= \int (D_x^\alpha v^{m+1})\dot{D}_x^\alpha \operatorname{div} h(v^m)\mathrm{d}x\\
&\le C\|\nabla(D_x^\alpha v^{m+1})\|_{L^2}^2\|v^m\|_{L^\infty}\|D_x^\alpha v^m\|_{L^2}^2\\
&\le C\|\nabla(D_x^\alpha v^{m+1})\|_{L^2}^2\|v^m\|_{H^s}\|D_x^\alpha v^m\|_{L^2}^2\\
&\le C\|v^m|_{H^s}^2\|D_x^\alpha v^m\|_{L^2}^2 + \epsilon\|\nabla(D_x^\alpha v^{m+1})\|_{L^2}^2.
\end{aligned}
\tag{3.2}
$$

Here, we have used the inequalities $\|v^m\|_{L^\infty} \le C\|v^m\|_{H^s}$ and $s \ge 1 + [\frac{2}{3}]$. Integrating with respect to t and taking the summation of $|\alpha|$ from 0 to $s-1$, we have

$$
\begin{aligned}
&\|v^{m+1}\|_{H^{s-1}}^2 + \gamma_1\|\nabla v^{m+1}\|_{H^{s-1}}^2 + \gamma_2\int_0^t \|\nabla v^{m+1}\|_{H^{s-1}}^2\mathrm{d}\tau\\
&\le \int_0^t \|v^m|_{H^s}^2\|v^m\|_{H^{s-1}}^2\mathrm{d}\tau + \|v_0\|_{H^{s-1}}^2 + \gamma_1\|\nabla v_0\|_{H^{s-1}}^2.
\end{aligned}
\tag{3.3}
$$

By (3.3), we have

$$
\|v^{m+1}\|_{H^{s-1}}^2 + \gamma_1\|\nabla v^{m+1}\|_{H^{s-1}}^2 \le CE^4T_0 + \|v_0\|_{H^{s-1}}^2 + \gamma_1\|\nabla v_0\|_{H^{s-1}}^2.
$$

Since $\gamma_1 > 0$ is an arbitrary constant,

$$
\|v^{m+1}\|_{H^s}^2 \le CE^4T_0 + \|v_0\|_{H^s}^2.
$$

Thus for T_0 small enough, we obtain $v^{m+1}(x,t) \in \mathbb{X}$. □

Lemma 3.2. *There is a sufficiently small constant T_0, such that $v^m(x,t)$ is a Cauchy sequence in the complete metric space $\mathbb{X}$.*

Proof. We are going to show that for some $0 < \kappa < 1$,

$$
\|v^{m+1} - v^m\|_{\mathbb{X}} \le \kappa\|v^m - v^{m+1}\|_{\mathbb{X}}.
$$

Similar to the proof of (3.2), we have

$$
\begin{aligned}
&\|v^{m+1} - v^m\|_{\mathbb{X}} + \gamma_1\|\nabla(v^{m+1} - v^m)\|_{H^s}^2 + \gamma_2\int_0^t \|\nabla(v^{m+1} - v^m)\|_{H^s}^2\mathrm{d}\tau\\
&\le \int_0^t \|v^{m+1} - v^m\|_{H^s}^2(\|v^m\|_{H^{s-1}}^2 + \|v^m\|_{H^s}^2)\mathrm{d}\tau
\end{aligned}
$$

Since $v^m(x,t) \in \mathbb{X}$, we have $\sup_{0\le t\le T_0}\|v^m\|_{H^s} \le E$. Thus,

$$
\sup_{0\le t\le T_0}\|v^{m+1} - v^m\|_{H^s} \le (CE^2T_0)^{1/2}\sup_{0\le t\le T_0}\|v^m - v^{m-1}\|_{H^s}.
$$

Choosing sufficiently small T_0 such that $CE^2T_0 \le \frac{1}{4}$, we obtain

$$
\sup_{0\le t\le T_0}\|v^{m+1} - v^m\|_{H^s} \le \frac{1}{2}\sup_{0\le t\le T_0}\|v^m - v^{m-1}\|_{H^s}.
$$

Thus $v^m(x,t)$ is a Cauchy sequence in $\mathbb{X}$. □

Since $\mathbb{X}$ is a complete metric space, combining Lemmas 3.1 and 3.2, the limit function $v(x,t) \in \mathbb{X}$ satisfies the Cauchy problem (3.1). Hence we obtain the existence of a local solution.

4. L^p decay estimate of Green's function

In this section, we consider the behavior of the solution to the linear form of (1.5). Consider the following Cauchy problem

$$\begin{gathered}\partial_t G - \gamma_1 \triangle G_t - \gamma_2 \triangle G + 2\sum_{j=1}^{3} b_j \cdot G_{x_j} = 0, \quad x \in \mathbb{R}^3,\ t > 0, \\ G\big|_{t=0} = \delta(x).\end{gathered} \tag{4.1}$$

Here, $b = (b_1, b_2, b_3)$, $b_i = c_i u^*$, $\delta(x)$ is the Dirac function. G is called Green's function.

Let $G(t,x)$ be the inverse Fourier transform corresponding to $\hat{G}(t,\xi)$. Taking the Fourier transform with respect to x and solving the ordinary differential equation directly, we obtain

$$\hat{G}(t,\xi) = e^{\mu(\xi)t + \eta(\xi)t},$$

with

$$\mu(\xi) = -\frac{\gamma_2|\xi|^2}{1+\gamma_1|\xi|^2}, \quad \eta(\xi) = -\frac{\sqrt{-1}b\cdot\xi}{1+\gamma_1|\xi|^2}.$$

Note that the solution of the Cauchy problem (1.5) with initial data (1.6) has the integral representation

$$u = G * u_0 + \int_0^t G(t-\tau,\cdot) * \frac{\operatorname{div}}{1-\gamma_1\Delta} f(u)(\tau,\cdot)\mathrm{d}\tau.$$

Next, we use the frequency decomposition to get an estimate for the Green's function G. We divide the solution into the low frequency part and the high frequency part and discuss them separately to obtain the decay property of the solution. Let

$$\chi_1(\xi) = \begin{cases} 1, & |\xi| \le \varepsilon, \\ 0, & |\xi| > 2\varepsilon, \end{cases}$$

$$\chi_3(\xi) = \begin{cases} 1, & |\xi| \ge R, \\ 0, & |\xi| < R-1, \end{cases}$$

$$\chi_2(\xi) = 1 - \chi_1(\xi) - \chi_3(\xi)$$

be smooth cut-off functions in $C^\infty(\mathbb{R}^3)$, where ε and R are positive constants satisfying $2\varepsilon < R-1$. We define three pseudo-differential operators $\chi_1(D), \chi_2(D), \chi_3(D)$ with the symbols $\chi_1(\xi)$, $\chi_2(\xi)$, $\chi_3(\xi)$ respectively and set

$$\hat{G}_i(t,\xi) = \chi_i(\xi)\hat{G}(t,\xi), \quad i = 1,2,3.$$

Now we estimate the three parts separately. Using Lemma 2.2, we have the following estimate on $G_1(t,x)$.

Proposition 4.1. *For sufficiently small ϵ, there exists a constant $C_N > 0$ such that*

$$|D_x^\alpha G_1(t,x)| \le C_N t^{-\frac{3+|\alpha|}{2}} B_N(t, |x-bt|).$$

Proof. For ξ sufficiently small, the Taylor expansion gives

$$\mu(\xi) + \eta(\xi) = -(\gamma_2|\xi|^2 + \sqrt{-1}b\cdot\xi)\frac{1}{1+\gamma_1|\xi|^2} = -(\gamma_2|\xi|^2 + \sqrt{-1}b\cdot\xi) + O(|\xi|^3).$$

Therefore,

$$\hat{G}(t,\xi) = e^{-\gamma_2|\xi|^2 t}\cdot e^{-\sqrt{-1}b\cdot\xi t}(1 + O(|\xi|^3)t).$$

Now we take the Fourier transform and set

$$\hat{G}_1(t,\xi) = \chi_1(\xi)e^{-\gamma_2|\xi|^2 t}(1+O(|\xi|^3)t)e^{-\sqrt{-1}b\cdot\xi} =: \hat{H}(t,\xi)e^{-\sqrt{-1}b\cdot\xi}.$$

By properties of the Fourier transform, we have

$$\begin{gathered} G_1(t,\xi) = H(t,x) * \mathcal{F}^{-1}(e^{-\sqrt{-1}b\cdot\xi}) = H(t,x-bt), \\ D_x^\alpha G_1(t,\xi) = H(t,x) * D_x^\alpha \mathcal{F}^{-1}(e^{-\sqrt{-1}b\cdot\xi}) = D_x^\alpha H(t,x-bt). \end{gathered}$$

Thus it suffices to show that

$$|D_x^\alpha H(t,x)| \le C_N t^{-\frac{3+|\alpha|}{2}} B_N(t,|x|).$$

Since $\hat{H}(t,\xi)$ is smooth in the variable ξ near $|\xi|=0$, for $|\beta| \le 2N$, we obtain

$$|D_\xi^\beta(\xi^\alpha \hat{H}(t,\xi))| \le C(|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|} t^{\frac{|\beta|}{2}})(1+t|\xi|^2)^{\frac{|\beta|}{2}+1} e^{-\gamma_2|\xi|^2 t}.$$

Thus, by Lemma 2.2, we have

$$|D_x^\alpha H(t,x)| \le C_N(1+t)^{-\frac{3+|\alpha|}{2}} B_N(t,|x|).$$

This completes the proof. □

Next, we consider $G_2(t,x)$.

Proposition 4.2. *For fixed ε and R, there exist positive numbers m_0 and C such that*

$$|D_x^\alpha G_2(t,x)| \le C e^{-\frac{t}{2m_0}} B_N(t,|x|).$$

Proof. Choose $m > 1/(2\varepsilon)$. When $\varepsilon \le |\xi| \le R$, we have

$$\operatorname{Re}(\mu(\xi)+\eta(\xi)) \le -\frac{1}{2m}.$$

This implies

$$|\hat{G}_2| = |\chi_2(\xi)e^{\mu(\xi)t}| \le Ce^{-\frac{t}{2m}}. \tag{4.2}$$

Thus,

$$|G_2(t,x)| \le C\Big|\int_{\mathbb{R}^3} e^{ix\cdot\xi}\hat{G}_2(t,\xi)\mathrm{d}\xi\Big| \le Ce^{-\frac{t}{2m}}. \tag{4.3}$$

Next, we give an estimate for $x^\beta G_2(t,x)$ using induction on β. Assume that for any $|\beta| \le l-1$, it holds

$$|D_\xi^\beta \hat{G}_2(t,\xi)| \le Ct^{|\beta|}e^{-\frac{t}{2m}}, \tag{4.4}$$

for $|\beta|=0$ by (4.2).

Taking the Fourier transform of (4.1) with respect to x and then multiplying by $\chi_2(\xi)$ we have

$$\begin{gathered} \partial_t \hat{G}_2(t,\xi) - (\mu(\xi)+\eta(\xi))\hat{G}_2(t,\xi) = 0, \\ \hat{G}_2(0,\xi) = \chi_2(\xi). \end{gathered} \tag{4.5}$$

Applying the operator D_ξ^β on (4.5), we obtain

$$\begin{gathered} \partial_t D_\xi^\beta \hat{G}_2(t,\xi) - (\mu(\xi)+\eta(\xi))D_\xi^\beta \hat{G}_2(t,\xi) = F(\xi), \\ \hat{G}_2(0,\xi) = a_0, \end{gathered}$$

where a_0 is a polynomial of $|\xi|$, and

$$F(\xi)=\sum_{\beta_1+\beta_2=\beta,|\beta_1|\neq 0}\frac{\beta!}{\beta_1!\beta_2!}(D_\xi^{\beta_1}(\mu(\xi)+\eta(\xi)))D_\xi^{\beta_2}\hat{G}_2(t,\xi).$$

Then for $|\beta|=l$, by the ODE theory, we have

$$D_\xi^{\beta}\hat{G}_2(t,\xi)=a_0\hat{G}(t,\xi)+\int_0^t\hat{G}(t-s,\xi)F(\xi)\mathrm{d}s.$$

Using the induction hypothesis, we obtain

$$|D_\xi^{\beta}\hat{G}_2(t,\xi)|_{\epsilon\le|\xi|\le R}\le Ce^{-\frac{t}{2m}}+C\int_0^t e^{-\frac{t-s}{2m}}t^{|\beta|-1}e^{-\frac{s}{2m}}\mathrm{d}s\le Ct^{|\beta|}e^{-\frac{t}{2m}},$$

which implies that (4.4) is also valid for $|\beta|=l$. Then for $1\le|\beta|\le l$, we have

$$\begin{aligned}|x^\beta D_x^\alpha G_2(t,x)|&\le C\Big|\int_{\mathbb{R}^3}e^{\sqrt{-1}x\xi}D_\xi^\beta(\xi^\alpha\hat{G}_2(t,\xi))\mathrm{d}\xi\Big|\\&\le Ce^{-\frac{t}{2m}}\int_{\epsilon\le|\xi|\le R}(|\xi|^{|\alpha|}+|\xi|^{(|\alpha|-|\beta|)_+})t^{|\beta|}\mathrm{d}\xi\\&\le Ct^{|\beta|}e^{-\frac{t}{2m}}\\&\le Ct^{\frac{|\beta|}{2}}e^{-\frac{t}{2m_0}},\quad m_0<m.\end{aligned}\tag{4.6}$$

Using (4.3) when $|x|^2\le t$ and using (4.6) when $|x|^2>t$, we obtain ($|\beta|=2N$)

$$|D_x^\alpha G_2(t,x)|\le Ce^{-\frac{t}{2m_0}}\min\{1,\frac{t^N}{|x|^{2N}}\}.$$

Since

$$1+\frac{|x|^2}{t}\le\begin{cases}2, & |x|^2\le t,\\ 2\frac{|x|^2}{t}, & |x|^2\ge t,\end{cases}$$

we have

$$|D_x^\alpha G_2(t,x)|\le Ce^{-\frac{t}{2m_0}}B_N(t,|x|),$$

which completes the proof. □

Finally, we consider $G_3(t,x)$ by using Lemma 2.3.

Proposition 4.3. *For sufficiently large R, there exist positive numbers C_0 and C such that*

$$|D_x^\alpha(G_3(t,x)-Gf_\alpha)|\le Ce^{-C_0t}B_N(t,|x|).$$

Here Gf_α is the distribution of the form

$$Gf_\alpha=\chi_3(D)\Big[e^{-\gamma_2t/\gamma_1}\Big(\delta(x)+\sum_{j=1}^{[\frac{|\alpha|+3}{2}]}p_j(t)\Delta^{-j+\frac{|\alpha|}{2}}\Big)\Big],\tag{4.7}$$

where $p_j(t)s$ are polynomials of degree j.

Proof. Since R is large enough,

$$|D_x^\alpha G_3|\le Ce^{-C_1t}e^{\mu(\xi)t}.$$

We set

$$\rho=\frac{1}{|\xi|^2},\quad h(\rho)=e^{-\frac{\gamma_2t}{|\rho|+\gamma_1}}.$$

Then, it is easy to see that

$$h(\rho) = e^{-\gamma_2 t/\gamma_1}(1 + O(|\rho|)), \quad |\rho| \to 0.$$

Thus,

$$e^{\mu(\xi)t} = \chi_3(\xi)e^{-\gamma_2 t/\gamma_1}\Big(1 + \sum_{j=1}^{[\frac{|\alpha|+3}{2}]} p_j(t)(|\xi|^{-2})^j\Big) + \hat{R}(t,\xi).$$

Using Lemma 2.3, we obtain

$$||\xi|^\alpha \hat{R}(t,\xi)| \le Ce^{-\gamma_2 T/\gamma_1} \sum_{j=[\frac{|\alpha|+3}{2}]+1}^{+\infty} p_j(t)(|\xi|^{-2j}),$$

$$|D_x^\alpha R(t,x)| \le Ce^{-\gamma_2 t/\gamma_1}(1 + |x|^2)^{-N},$$

which implies

$$|x^\beta D_x^\alpha R(t,x)| \le Ce^{-\gamma_2 t/\gamma_1} \int_{|\xi|>R} |\xi|^{-2(\frac{|\alpha|+3}{2})-|\beta|} \mathrm{d}\xi \le Ce^{-\gamma_2 t/\gamma_1}.$$

Hence,

$$|D_x^\alpha(G_3 - Gf_\alpha)| \le Ce^{-\gamma_2 t/\gamma_1} B_N(t,|x|).$$

□

In conclusion, we obtain the following estimate for the regular part of G.

Theorem 4.4. *Assume that G is the solution of the linear form of equation* (1.5)*. Then*

$$|D_x^\alpha(G - Gf_\alpha)| \le Ct^{-\frac{3+|\alpha|}{2}} B_N(t,|x - bt|),$$

with Gf_α defined by (4.7)*.*

Proof. Since

$$|D_x^\alpha(G - Gf_\alpha)(t,x)| \le |D_x^\alpha G_1(t,x)| + |D_x^\alpha G_2(t,x)| + |D_x^\alpha(G_3 - Gf_\alpha)(t,x)|,$$

the desired estimate follows from Propositions 4.1, 4.2 and 4.3. □

Noting that

$$\int_{\mathbb{R}^n} B_N(t,|x|)\mathrm{d}x \le C(1+t)^{3/2},$$

we can use the Hausdorff-Young's inequality to obtain the following theorems about the Green's function, which are essential for the proof of the existence and the decay estimate of the Cauchy problem (1.5)-(1.6).

Theorem 4.5. *For any multi-index α and $p \in [1,\infty]$, we have*

$$\|D_x^\alpha(G - Gf_\alpha)\|_{L^p(\mathbb{R}^3)} \le Ct^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}.$$

Theorem 4.6. *For all φ and $N \ge 0$ integer, we have*

$$\|(G - Gf_\alpha) * \varphi\|_{W^{N,\infty}(\mathbb{R}^3)} \le C_0(1+t)^{-3/2}\|\varphi\|_{W^{N+4,1}(\mathbb{R}^3)},$$

$$\|(G - Gf_\alpha) * \varphi\|_{W^{N,1}(\mathbb{R}^3)} \le \|\varphi\|_{W^{N,1}(\mathbb{R}^3)}.$$

Theorem 4.7. *For $f \in L^p(\mathbb{R}^3)$ with $p \in [1,\infty]$, we have*

$$\|G * f\|_{L^p(\mathbb{R}^3)} \le C_p\|f\|_{L^p(\mathbb{R}^3)}.$$

Here C_p depends only on p.

5. L^p BOUND ESTIMATE OF SOLUTIONS

Now we prove an L^p ($p \in [0, \infty]$) estimate of the solution u of (1.5)-(1.6). First, by the fundamental energy estimate, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2) + 2\gamma_2\|\nabla u\|_{L^2}^2 = 0. \tag{5.1}$$

Integrating with respect to t, we obtain

$$(\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2) + 2\gamma_2 \int_0^t \|\nabla u\|_{L^2}^2 \mathrm{d}\tau = \|u_0\|_{L^2}^2 + \gamma_1\|\nabla u_0\|_{L^2}^2.$$

Therefore, we have the following basic bounds for u:

$$\|u\|_{L^2} \le C, \quad \|\nabla u\|_{L^2} \le C, \quad \int_0^t \|\nabla u(\tau, \cdot)\|_{L^2}^2 \mathrm{d}\tau \le C.$$

5.1. L^1 bounds.

Let

$$\chi_0(\eta) = \begin{cases} 1, & |\eta| \le 1, \\ 0, & |\eta| > 2, \end{cases}$$

be a smooth cut-off function in $C^\infty(\mathbb{R}^3)$. Define the time-frequency cut-off operator $\chi(t, D)$ with symbol $\chi(t, \xi) = \chi_0\big(\frac{1+t}{\mu}|\xi|^2\big)$, where μ is a constant and $\mu > \max\{\frac{3}{2\gamma_1}, \frac{3}{2\gamma_2}\}$. A solution u of (1.5) can be decomposed it into two parts: the low frequency part u_L and the high frequency part u_H, where $u_L(t, x) = \chi(t, D)u(t, x)$ and $u_H(t, x) = (1 - \chi(t, D))u(t, x)$.

Lemma 5.1. *Assume that $u_0 \in L^1(\mathbb{R}^3)$. Then*

$$\|u_L\|_{L^2} \le C(1 + t)^{-3/4},$$

where C is a positive constant depending only on u_0.

Proof. Using Green's function, we have

$$u_L = \chi(t, D)(G * u_0) + \int_0^t \chi(t, D)\Big(\frac{G(t - \tau, \cdot)}{1 - \gamma_1\Delta} * \operatorname{div} f(u)\Big)\mathrm{d}\tau.$$

Using Minkowski's inequality, we obtain

$$\begin{aligned} &\|u_L\|_{L^2} \\ &= \|\chi(t, D)(G * u_0)\|_{L^2} + \Big(\int_0^t \big\|\chi(t, D)\big(\frac{G(t - \tau, \cdot)}{1 - \gamma_1\Delta} * \operatorname{div} f(u)\big)\big\|_{L^2}^2 \mathrm{d}\tau\Big)^{1/2} \\ &= S_1 + S_2. \end{aligned} \tag{5.2}$$

For S_1, by Hausdorff-Young's inequality and Theorem 4.4, we have

$$S_1 \le \|\chi(t, D)G\|_{L^2} \cdot \|u_0\|_{L^1} \le C(1 + t)^{-3/4}. \tag{5.3}$$

For S_2, by Plancherel's theorem, we obtain

$$
\begin{aligned}
|S_2|^2 &= \int_0^t \big\|\chi(t,D)\Big(\frac{G(t-\tau,\cdot)}{1-\gamma_1\Delta} * \operatorname{div} f(u)\Big)\big\|_{L^2}^2 \mathrm{d}\tau \\
&= \int_0^t \big\|\chi(t,\xi)\frac{\hat{G}}{1+\gamma_1|\xi|^2}|\xi|\hat{f}(u)\big\|_{L^2}^2 \mathrm{d}\tau \\
&\le C\int_0^t \int_{\mathbb{R}^3_\xi} \chi^2(t,\xi)|\xi|^2 \frac{|\hat{f}(u)|^2}{(1+\gamma_1|\xi|^2)} \mathrm{d}\xi \mathrm{d}\tau \\
&\le C\int_0^t \|\frac{\hat{f}(u)}{1+\gamma_1|\xi|^2}\|_{L^\infty}^2 \int_{\mathbb{R}^3_\xi} \chi^2(t,\xi)|\xi|^2 \mathrm{d}\xi \mathrm{d}\tau \\
&\le C\int_0^t \|\frac{f(u)}{1-\gamma_1\Delta}\|_{L^1}^2 (1+t)^{-\frac{3}{2}-1} \mathrm{d}\tau \\
&\le C\int_0^t \|f(u)\|_{L^1}^2 (1+t)^{-\frac{3}{2}-1} \mathrm{d}\tau \\
&\le C(1+t)^{-3/2}.
\end{aligned} \tag{5.4}
$$

Hence, by (5.2), (5.3) and (5.4), we obtain the desired estimate. □

From the decay estimate of $\|u_L\|_{L^2}$, we have the following interesting relation between the decay rates of u_L and u.

Lemma 5.2. *If $u_0 \in H^1(\mathbb{R}^3)$ and $\|u_L\|_{L^2} \le C(1+t)^{-\sigma}$, then*

$$
\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2 \le C(1+t)^{-2\sigma},
$$

where C is a positive constant depending only on u_0.

Proof. We already have the basic energy estimate (5.1), that is

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2) + 2\gamma_2\|\nabla u\|_{L^2}^2 = 0.
$$

Set $\varepsilon^2(t) = \mu/(1+t)$. Then

$$
\begin{aligned}
\|\nabla u\|_{L^2} &\ge \int_{|\xi|>\varepsilon(t)} |\xi|^2|\hat{u}|^2 \mathrm{d}\xi \\
&\ge \varepsilon^2(t) \int_{|\xi|>\varepsilon(t)} |\hat{u}|^2 \mathrm{d}\xi \\
&\ge \varepsilon^2(t)\Big(\|u\|_{L^2}^2 - \int_{|\xi|\le\varepsilon(t)} |\hat{u}|^2 \mathrm{d}\xi\Big).
\end{aligned}
$$

From the above inequality, we deduce that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2) + \frac{\gamma_2\mu}{1+t}\|u\|_{L^2}^2 + \gamma_2\|\nabla u\|_{L^2}^2 &\le \varepsilon^2(t)\int_{|\xi|\le\varepsilon(t)} \gamma_2|\hat{u}|^2 \mathrm{d}\xi \\
&\le C\varepsilon^2(t)\|u_L\|_{L^2}^2.
\end{aligned} \tag{5.5}
$$

By the assumption, (5.5) becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2) + C_0\frac{\mu}{1+t}(\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2) \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2) + \frac{\gamma_2\mu}{1+t}\|u\|_{L^2}^2 + \gamma_2\|\nabla u\|_{L^2}^2
\end{aligned}
$$

$$\le C(1+t)^{-2\sigma-1}.$$

Multiplying the above inequality by $e^{\int_0^t \frac{C_0\mu}{1+\tau}\mathrm{d}\tau} = (1+t)^{C_0\mu}$ and integrating from 0 to t, we obtain

$$(1+t)^{C_0\mu}(\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2) \le \|u_0\|_{L^2}^2 + \gamma_1\|\nabla u_0\|_{L^2}^2 + C(1+t)^{C_0\mu-2\sigma}.$$

Thus

$$\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2 \le (\|u_0\|_{L^2} + \gamma_1\|\nabla u_0\|_{L^2}^2)(1+t)^{-C_0\mu} + C(1+t)^{-2\sigma}.$$

Since $C_0\mu > 2\sigma$, we have

$$\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2 \le C(1+t)^{-2\sigma}.$$

□

Remark 5.3. Combining Lemmas 5.2 with 5.1, we obtain

$$\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2 \le C(1+t)^{-3/2}.$$

Proposition 5.4. *If $u_0 \in L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, then we have $\|u\|_{L^1} \le C$, where C is a positive constant depending only on u_0.*

Proof. Using Green's function, we have

$$\begin{aligned} \|u\|_{L^1} &\le \|G * u_0\|_{L^1} + \Big\|\int_0^t \frac{G(t-\tau,\cdot)}{1-\gamma_1\Delta} * \operatorname{div} f(u)\mathrm{d}\tau\Big\|_{L^1} \\ &\le \|G * u_0\|_{L^1} + \int_0^t \|G(t-\tau,\cdot) * \operatorname{div} f(u)\|_{L^1}\mathrm{d}\tau \\ &= Q_1 + Q_2. \end{aligned} \tag{5.6}$$

For Q_1, by Theorem 4.7, we have

$$Q_1 \le C\|u_0\|_{L^1} \le C. \tag{5.7}$$

For Q_2, notice that

$$\|f(u)\|_{L^1} \le C\|u\|_{L^2}^2 \le C(1+t)^{-3/2}.$$

Again, by Theorem 4.7, we have

$$\begin{aligned} Q_2 &= \int_0^t \|G(t-\tau,\cdot) * \operatorname{div} f(u)\|_{L^1}\mathrm{d}\tau \\ &= \int_0^t \|(G - Gf_1 + Gf_1) * \operatorname{div} f(u)\|_{L^1}\mathrm{d}\tau \\ &\le \int_0^t \|(G - Gf_1) * \operatorname{div} f(u)\|_{L^1}\mathrm{d}\tau + \int_0^t \|Gf_1 * \operatorname{div} f(u)\|_{L^1}\mathrm{d}\tau \\ &= Q_{2,1} + Q_{2,2}. \end{aligned} \tag{5.8}$$

For $Q_{2,1}$, by Young's inequality, we have

$$
\begin{aligned}
Q_{2,1} &= \int_0^t \|(G - Gf_1) * \operatorname{div} f(u)\|_{L^1} \mathrm{d}\tau \\
&= \int_0^t \|\nabla(G - Gf_1) * f(u)\|_{L^1} \mathrm{d}\tau \\
&\leq C \int_0^t \|\nabla(G - Gf_1)\|_{L^1} \|f(u)\|_{L^1} \mathrm{d}\tau \\
&\leq C \int_0^t (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-3/2} \mathrm{d}\tau \leq C.
\end{aligned}
\tag{5.9}
$$

For $Q_{2,2}$, we have

$$
\begin{aligned}
Q_{2,2} &= \int_0^t \|Gf_1 * \operatorname{div} f(u)\|_{L^1} \mathrm{d}\tau \\
&\leq \int_0^t e^{-\frac{\gamma_2}{\gamma_1}(t-\tau)} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \mathrm{d}\tau \\
&\leq \int_0^t e^{-\frac{\gamma_2}{\gamma_1}(t-\tau)} (1 + \tau)^{-3/2} \mathrm{d}\tau \leq C.
\end{aligned}
\tag{5.10}
$$

Now (5.6), (5.7), (5.8), (5.9) and (5.10) imply $\|u\|_{L^1} \leq C$, as desired. □

5.2. L^∞ **bounds.** We now estimate $\|u\|_{L^\infty}$. Unlike the method in [20], the energy method is not enough. However, using the variable substitution and Green's function method, we can first obtain the bounded estimate of $\|u\|_{H^2}$ and then apply the embedding theorem to get the L^∞ bounded estimate. Set

$$
w = (1 - \gamma_1 \Delta) u, \quad w_0 = (1 - \gamma_1 \Delta) u_0.
$$

Then (1.5) can be rewritten as

$$
\partial_t w - \frac{\gamma_2 \Delta}{1 - \gamma_1 \Delta} w + b \frac{\gamma_2 \nabla}{1 - \gamma_1 \Delta} w = \operatorname{div} f(u).
$$

By Duhamel Principle, we know that the solution will be of the form

$$
w = G * w_0 + \int_0^t G(t - \tau, \cdot) * \operatorname{div} f(u)(\tau, \cdot) \mathrm{d}\tau. \tag{5.11}
$$

Now we use Green's function to estimate w.

Proposition 5.5. *Suppose that $u_0 \in L^1(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$, $s \geq 2$. Then we have*

$$
\|w\|_{L^2} \leq C,
$$

where C is a positive constant depending only on u_0.

Proof. Applying Theorem 4.7 to (5.11) and using the Hölder inequality, we obtain

$$\begin{aligned}
\|w\|_{L^{3/2}} &= \|G * w_0\|_{L^{3/2}} + \| \int_0^t G(t-\tau,\cdot) * \operatorname{div} f(u)(\tau,\cdot) d\tau \|_{L^{3/2}} \\
&\le \|G * w_0\|_{L^{3/2}} + \int_0^t \|G(t-\tau,\cdot) * \operatorname{div} f(u)\|_{L^{3/2}} d\tau \\
&\le C\|w_0\|_{L^{3/2}} + C\int_0^t \|u\cdot\nabla u\|_{L^{3/2}} d\tau \\
&\le C + C\int_0^t \|u\|_{L^6}\|\nabla u\|_{L^2} d\tau \\
&\le C + C\int_0^t \|\nabla u\|_{L^2}^2 d\tau \le C.
\end{aligned} \tag{5.12}$$

Note that we have used the Sobolev inequality $\|u\|_{L^6(\mathbb{R}^3)} \le C\|\nabla u\|_{L^2(\mathbb{R}^3)}$ above.

Combining (5.12) with (2.2), we obtain

$$\|u\|_{L^{3/2}} = \|(1-\gamma_1\Delta)^{-1} w\|_{L^{3/2}} \le C.$$

Thus

$$\|\Delta u\|_{L^{3/2}} = \|u + w\|_{L^{3/2}} \le \|u\|_{L^{3/2}} + \|w\|_{L^{3/2}} \le C.$$

Combining these with the Green's function again, we obtain

$$\begin{aligned}
\|w\|_{L^2} &= \|G * w_0\|_{L^2} + \| \int_0^t G(t-\tau,\cdot) * \operatorname{div} f(u)(\tau,\cdot) d\tau \|_{L^2} \\
&\le C\|w_0\|_{L^2} + \int_0^t \|(G - Gf_1 + Gf_1) * \operatorname{div} f(u)(\tau,\cdot)\|_{L^2} d\tau \\
&\le C + \int_0^t (\|(G - Gf_1) * \operatorname{div} f(u)\|_{L^2} + \|Gf_1 * \operatorname{div} f(u)\|_{L^2}) d\tau \\
&\le C + I_1 + I_2.
\end{aligned} \tag{5.13}$$

For I_1, we have

$$\begin{aligned}
I_1 &= \int_0^t \|(G - Gf_1) * \operatorname{div} f(u)\|_{L^2} d\tau \\
&\le \int_0^t \|\operatorname{div}(G - Gf_1)\|_{L^2}\|f(u)\|_{L^1} d\tau \\
&\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}(1-\frac{1}{2})-\frac{1}{2}} \|u\|_{L^2}^2 d\tau \le C.
\end{aligned} \tag{5.14}$$

For I_2, we have

$$\begin{aligned}
I_2 &= \int_0^t \|Gf_1 * \operatorname{div} f(u)\|_{L^2} d\tau \\
&\le \int_0^t e^{-\frac{\gamma_2}{\gamma_1}(t-\tau)} \|u\|_{L^6}\|\nabla u\|_{L^3} d\tau \\
&\le \int_0^t e^{-\frac{\gamma_2}{\gamma_1}(t-\tau)} \|\nabla u\|_{L^2}\|\Delta u\|_{L^{\frac{3}{2}}} d\tau \le C.
\end{aligned} \tag{5.15}$$

In the last inequality we used that $\|\nabla u\|_{L^3(\mathbb{R}^3)} \le C\|\Delta u\|_{L^{3/2}(\mathbb{R}^3)}$.

From (5.13), (5.14) and (5.15), we obtain $\|w\|_{L^2} \le C + I_1 + I_2 \le C$. $\square$

Remark 5.6. By Proposition 5.5, we can easily obtain

$$\|u\|_{H^2(\mathbb{R})^3} \leq C.$$

Then by the Sobolev embedding theorem $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we can get the L^∞ boundedness of u,

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq C.$$

5.3. L^p **bounds.** Since we already have L^1 and L^∞ bounds, by the interpolation inequality we easily get

$$\|u\|_{L^p(\mathbb{R}^3)} \leq C, \quad p \in [1, \infty].$$

Here C is a positive constant depending only on the initial data u_0.

5.4. H^s **bounds.** With the L^∞ boundedness, we can improve the regularity of u as follows.

Lemma 5.7. *If $u_0 \in H^s(\mathbb{R}^3)$, $3 \leq s \in \mathbb{Z}$, then $\|u\|_{H^s} \leq C$, where C is a positive constant depending only on u_0.*

Proof. Multiplying $\Lambda^{2j} u$ to both sides of (1.5) and integrating with respect to x, we obtain

$$\frac{d}{dt}(\|u\|^2_{\dot{H}^j} + \gamma_1 \|u\|^2_{\dot{H}^{j+1}}) + 2\gamma_2 \|u\|^2_{\dot{H}^j} = -2 \int_{\mathbb{R}^3} \Lambda^{2j} u \operatorname{div} f(u) dx.$$

By Hölder's inequality, we have

$$\big|2 \int_{\mathbb{R}^3} \Lambda^{2j} u \operatorname{div} f(u) dx\big| \leq \gamma_2 \|u\|^2_{\dot{H}^j} + C\|f(u)\|^2_{\dot{H}^j} \leq \gamma_2 \|u\|^2_{\dot{H}^j} + C_j \gamma_2 \|u\|^2_{\dot{H}^j}.$$

Thus, for any $j \geq 1$, we obtain

$$\frac{d}{dt}(\|u\|^2_{\dot{H}^j} + \gamma_1 \|u\|^2_{\dot{H}^{j+1}}) + \gamma_2 \|u\|^2_{\dot{H}^j} \leq C_j \gamma_2 \|u\|^2_{\dot{H}^j}.$$

By summation, we have

$$\frac{d}{dt} \sum_{j=0}^{s} C^{s-j}(\|u\|^2_{\dot{H}^j} + \gamma_1 \|u\|^2_{\dot{H}^{j+1}}) + \sum_{j=0}^{s} C^{s-j} \gamma_2 \|u\|^2_{\dot{H}^j} \leq \sum_{j=0}^{s} (j+1) C^{s-j+1} C_j \gamma_2 \|u\|^2_{\dot{H}^j}.$$

One readily checks that this implies

$$\frac{d}{dt} \sum_{j=0}^{s} C^{s-j}(\|u\|^2_{\dot{H}^j} + \gamma_1 \|u\|^2_{\dot{H}^{j+1}}) \leq 0.$$

Integrating the above inequality with respect to t, we obtain

$$\sum_{j=0}^{s} C^{s-j}(\|u\|^2_{\dot{H}^j} + \gamma_1 \|u\|^2_{\dot{H}^{j+1}}) \leq \sum_{j=0}^{s} C^{s-j}(\|u_0\|^2_{\dot{H}^j} + \gamma_1 \|u_0\|^2_{\dot{H}^{j+1}}).$$

Then, it follows that $\|u\|_{H^s} \leq C$. □

From the H^s boundedness and the existence of local solutions, we obtain the existence of global solutions to (1.5), by the continuity method. Moreover,

$$(u - u^*) \in L^\infty(0, \infty; H^s(\mathbb{R}^3)), \quad s \geq 1 + [\frac{3}{2}].$$

6. Decay estimate in $\dot H^s$ and L^∞

In this section, we establish the optimal decay rates of the solution u in the homogeneous Sobolev spaces.

6.1. Decay rate of the low frequency part in $\dot H^s$.

Based on the boundedness of $\|u\|_{L^1}$, Plancherel's theorem, and Hausdorff-Young's inequality, we can directly obtain the following estimate for the low frequency part,

$$
\begin{aligned}
\|\Lambda^s u_L\|_{L^2} &= \|\chi(t,D)\Lambda^s u\|_{L^2}\\
&= \|\chi(t,\xi)|\xi|^s\hat u\|_{L^2}\\
&\le C\Big\|\chi_0\Big(\frac{1+t}{\mu}|\xi|^2\Big)|\xi|^s\Big\|_{L^2}\|\hat u\|_{L^\infty}\\
&\le C\|u\|_{L^1}\Big\|\chi_0\Big(\frac{1+t}{\mu}|\xi|^2\Big)|\xi|^s\Big\|_{L^2}\\
&\le C(1+t)^{-\frac34-\frac s2}.
\end{aligned} \tag{6.1}
$$

Here, the constant C depends on the choice of μ, which is the same as in the last section.

6.2. Decay rate of the high frequency part in $\dot H^s$.

According to the definition of u_H, u_H satisfies the equation

$$
\begin{aligned}
&(u_H)_t-\gamma_1\Delta(u_H)_t-\gamma_2\Delta u_H+2b\cdot\nabla u_H\\
&=\operatorname{div}(f(u))_H-\chi_t(t,D)u+\gamma_1\chi_t(t,D)\Delta u.
\end{aligned} \tag{6.2}
$$

Lemma 6.1. *If $u_0\in H^1(\mathbb{R}^3)$, then we have the estimate*

$$
\int_0^t(1+t)^{C_0\mu}\|u\|_{\dot H^1}\mathrm{d}\tau\le C(1+t)^{C_0\mu-\frac32}.
$$

Proof. As in Lemma 5.2, using (5.1), we obtain

$$
\begin{aligned}
\frac{\mathrm d}{\mathrm dt}(\|u\|_{L^2}^2+\gamma_1\|\nabla u\|_{L^2}^2)+\gamma_2\|\nabla u\|_{L^2}^2 &= -\gamma_2\|\nabla u\|_{L^2}^2\\
&\le -\varepsilon^2(t)\gamma_2\Big(\|u\|_{L^2}^2-\int_{|\xi|\le\varepsilon(t)}|\hat u|^2\mathrm d\xi\Big).
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\frac{\mathrm d}{\mathrm dt}(\|u\|_{L^2}^2+\gamma_1\|\nabla u\|_{L^2}^2)+\frac{\gamma_2\mu}{1+t}\|u\|_{L^2}^2+\gamma_2\|\nabla u\|_{L^2}^2 &\le \varepsilon^2(t)\int_{|\xi|\le\varepsilon(t)}\gamma_1\hat u^2\mathrm d\xi\\
&\le C\varepsilon^2(t)\|u_L\|_{L^2}^2\\
&\le C(1+t)^{-\frac32-1}.
\end{aligned}
$$

The last inequality uses (6.1). Therefore,

$$
\begin{aligned}
&\frac{\mathrm d}{\mathrm dt}(\|u\|_{L^2}^2+\gamma_1\|\nabla u\|_{L^2}^2)+\frac{C_0\mu}{1+t}(\|u\|_{L^2}^2+\gamma_1\|\nabla u\|_{L^2}^2)+C_1\|\nabla u\|_{L^2}^2\\
&\le \frac{\mathrm d}{\mathrm dt}(\|u\|_{L^2}^2+\gamma_1\|\nabla u\|_{L^2}^2)+\frac{\gamma_2\mu}{1+t}\|u\|_{L^2}^2+\gamma_2\|\nabla u\|_{L^2}^2\\
&\le C(1+t)^{-\frac32-1}.
\end{aligned}
$$

Multiplying the inequality above by $e^{\int_0^t \frac{C_0\mu}{1+\tau}\mathrm{d}\tau} = (1+t)^{C_0\mu}$ and integrating from 0 to t, we obtain

$$
\begin{aligned}
&(1+t)^{C_0\mu}(\|u\|_{L^2}^2 + \gamma_1\|\nabla u\|_{L^2}^2) + C_1\int_0^t (1+t)^{C_0\mu}\|\nabla u\|_{L^2}^2\mathrm{d}\tau\\
&\le \|u_0\|_{L^2}^2 + \gamma_1\|\nabla u_0\|_{L^2}^2 + C\int_0^t (1+\tau)^{C_0\mu-\frac{3}{2}-1}\mathrm{d}\tau\\
&\le C(1+t)^{C_0\mu-\frac{3}{2}}.
\end{aligned}
$$

This completes the proof. □

The next result gives the optimal decay estimate for $\|u_H\|_{\dot{H}^s}$.

Theorem 6.2. *If $u_0 \in L^1(\mathbb{R}^3)\cap H^s(\mathbb{R}^3)$ with $s \ge 1+[\frac{3}{2}]$ a given integer, and u is a solution of (1.5) in $L^\infty(0,\infty;H^s(\mathbb{R}^3))$ with initial data u_0, then*

$$
\|u_H\|_{\dot{H}^s} \le C(1+t)^{-\frac{3}{4}-\frac{s}{2}},
$$

for any $t \ge 1$, where C only depends on the initial data u_0.

Proof. Multiplying (6.2) by $\Lambda^{2s}u_H$ and integrating it in x, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u_H\|_{\dot{H}^s}^2 + \gamma_1\|u_H\|_{\dot{H}^{s+1}}^2) + \gamma_2\|u_H\|_{\dot{H}^{s+1}}^2 - 2\int_{\mathbb{R}^3}\Lambda^{2s}b\cdot u_H\nabla u_H\mathrm{d}x\\
&= \int_{\mathbb{R}^3}\Lambda^{2s}u_H\,\mathrm{div}(f(u))_H\mathrm{d}x - \int_{\mathbb{R}^3}\hat{u}\hat{u_H}|\xi|^{2s}\partial_t\chi(t,\xi)\mathrm{d}\xi\\
&\quad + \gamma_1\int_{\mathbb{R}^3}\hat{u}\hat{u_H}|\xi|^{2s+2}\partial_t\chi(t,\xi)\mathrm{d}\xi\\
&= R_1 + R_2 + R_3.
\end{aligned}
\tag{6.3}
$$

For R_1, by Hölder's inequality and Lemma 2.1, we have

$$
\begin{aligned}
|R_1| &= \big|\int_{\mathbb{R}^3} D(u^2)_H D^{2s}u_H\mathrm{d}x\big|\\
&= \big|\int_{\mathbb{R}^3}(i\xi)(\hat{u^2})_H|\xi|^{2s}\hat{u_H}\mathrm{d}\xi\big|\\
&\le \big\||\xi|^s\big|\widehat{f(u)}_H\big|\cdot|\xi|^{s+1}\hat{u}\big\|_{L^1}\\
&\le C\|f(u)_H\|_{\dot{H}^s}\|u\|_{\dot{H}^{s+1}}\\
&\le C(\|u\|_{L^\infty}\|u\|_{\dot{H}^s})\|u\|_{\dot{H}^{s+1}}\\
&\le C\|u\|_{\dot{H}^s}\|u\|_{\dot{H}^{s+1}}.
\end{aligned}
$$

By the Galgliardo-Nirenberg-Sobolev inequality, we have

$$
\|u\|_{\dot{H}^s} \le C\|u\|_{\dot{H}^{s+1}}^{\frac{s-1}{s}}\|u\|_{\dot{H}^1}^{1/s}.
$$

Then, using the Young's inequality, we obtain

$$
\begin{aligned}
|R_1| &\le C\|u\|_{\dot{H}^{s+1}}^{\frac{s-1}{s}+1}\|u\|_{\dot{H}^1}^{1/s}\\
&\le \frac{\gamma_2}{4}\|u\|_{\dot{H}^{s+1}}^2 + \frac{C}{2}\|u\|_{\dot{H}^1}^2\\
&\le \frac{\gamma_2}{4}\|u_L\|_{\dot{H}^{s+1}}^2 + \frac{\gamma_2}{4}\|u_H\|_{\dot{H}^{s+1}}^2 + \frac{C}{2}\|u\|_{\dot{H}^1}^2.
\end{aligned}
\tag{6.4}
$$

The same calculation can be done to $\int_{\mathbb{R}^3} \Lambda^{2s} u_H \nabla u_H \mathrm{d}x$.

$$\begin{aligned}
\big|2\int_{\mathbb{R}^3} \Lambda^{2s} b\cdot u_H \nabla u_H \mathrm{d}x\big| &= 2b\big|\int_{\mathbb{R}^3} (i\xi)(\hat{u})_H |\xi|^{2s} \hat{u_H} \mathrm{d}\xi\big| \\
&\le C\| |\xi|^s |\hat{u}_H| \cdot |\xi|^{s+1} \hat{u}\|_{L^1} \\
&\le C\|u_H\|_{\dot{H}^s} \|u\|_{\dot{H}^{s+1}} \\
&\le \frac{\gamma_2}{4}\|u_L\|^2_{\dot{H}^{s+1}} + \frac{\gamma_2}{4}\|u_H\|^2_{\dot{H}^{s+1}} + \frac{C}{2}\|u\|^2_{\dot{H}^1}.
\end{aligned} \tag{6.5}$$

For R_2, by Proposition 5.4, we have

$$\begin{aligned}
|R_2| &= \int_{\mathbb{R}^3} \hat{u}\hat{u_H} |\xi|^{2s} \partial_t \chi(t,\xi) \mathrm{d}\xi \\
&\le \int_{\mathbb{R}^3} |\hat{u}\hat{u}| |\xi|^{2s+2} \frac{1}{\mu} \chi_0'\Big(\frac{1+t}{\mu}|\xi|^2\Big)\big|1-\chi_0\Big(\frac{1+t}{\mu}|\xi|^2\Big)\big| \mathrm{d}\xi \\
&\le C\|\hat{u}^2\|_{L^\infty} \int_{\mathbb{R}^3} (1+t)^{-(s+1)} |\eta|^{2s+2} \chi_0'(|\eta|^2)|1-\chi_0(|\eta|^2)|(1+t)^{-3/2} \mathrm{d}\eta \\
&\le C\|u^2\|_{L^1}(1+t)^{-\frac{3}{2}-s-1} \\
&\le C(1+t)^{-\frac{3}{2}-s-1}.
\end{aligned} \tag{6.6}$$

For R_3, similar to R_2, we obtain

$$|R_3| \le C(1+t)^{-\frac{3}{2}-s-2}. \tag{6.7}$$

Combining (6.3)-(6.7), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|u_H\|^2_{\dot{H}^s} + \gamma_1\|u_H\|^2_{\dot{H}^{s+1}}) + \gamma_2\|u_H\|^2_{\dot{H}^{s+1}} \\
&\le \frac{\gamma_2}{2}\|u_L\|^2_{\dot{H}^{s+1}} + C\|u\|^2_{\dot{H}^1} + C(1+t)^{-\frac{3}{2}-s-1} + C(1+t)^{-\frac{3}{2}-s-2} \\
&\le C\|u\|^2_{\dot{H}^1} + C(1+t)^{-\frac{3}{2}-s-1}.
\end{aligned}$$

We used (6.1) in the last inequality above. Also, according to the definition of the cut-off operator, we find that

$$\|u_H\|^2_{\dot{H}^{s+1}} \ge \frac{\mu}{1+t}\|u_H\|^2_{\dot{H}^s}.$$

Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|u_H\|^2_{\dot{H}^s} + \gamma_1\|u_H\|^2_{\dot{H}^{s+1}}) + C_0\frac{\mu}{1+t}(\|u_H\|^2_{\dot{H}^s} + \gamma_1\|u_H\|^2_{\dot{H}^{s+1}}) \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}(\|u_H\|^2_{\dot{H}^s} + \gamma_1\|u_H\|^2_{\dot{H}^{s+1}}) + \frac{\gamma_2}{2}\|u_H\|^2_{\dot{H}^{s+1}} \\
&\le C\|u\|^2_{\dot{H}^1} + C(1+t)^{-\frac{3}{2}-s-1}.
\end{aligned}$$

Using Lemma 6.1 and multiplying the inequality above by $e^{\int_0^t \frac{C_0\mu}{1+\tau}\mathrm{d}\tau} = (1+t)^{C_0\mu}$ and integrating from 0 to t, we have

$$\begin{aligned}
&(1+t)^{C_0\mu}(\|u_H\|^2_{\dot{H}^s} + \gamma_1\|u_H\|^2_{\dot{H}^{s+1}}) \\
&\le \|u_{0H}\|^2_{\dot{H}^s} + \gamma_1\|u_{0H}\|^2_{\dot{H}^{s+1}} + C\int_0^t (1+\tau)^{C_0\mu}\|u\|^2_{\dot{H}^1}\mathrm{d}\tau \\
&\quad + C\int_0^t (1+\tau)^{C_0\mu-\frac{3}{2}-s-1}\mathrm{d}\tau
\end{aligned}$$

$$\le \|u_{0H}\|^2_{\dot H^s} + \gamma_1\|u_{0H}\|^2_{\dot H^{s+1}} + C(1+t)^{C_0\mu-\frac{3}{2}} + C(1+t)^{C_0\mu-\frac{3}{2}-s}.$$

Since $C_0\mu > 3/2$, we have

$$\|u_H\|^2_{\dot H^s} + \gamma_1\|u_H\|^2_{\dot H^{s+1}} \le C(1+t)^{-3/2}.$$

Combining the above with (6.1) and setting $s = 3$, we obtain

$$\|u\|^2_{\dot H^3} \le C(1+t)^{-3/4}.$$

Therefore, by the Gagliardo-Nirenberg-Sobolev inequality, we have

$$\|u\|_{L^\infty} \le C\|u\|^{1/2}_{\dot H^3}\|u\|^{1/2}_{L^2} \le C(1+t)^{-3/4}.$$

Hence, we can use the estimate of $\|u\|_{L^\infty}$ to get a better decay as follows.

$$\begin{aligned} |R_1| &\le C(\|u\|_{L^\infty}\|u\|_{\dot H^s})\|u\|_{\dot H^{s+1}} \\ &\le \frac{\gamma_2}{2}\|u\|^2_{\dot H^{s+1}} + C\|u\|^2_{\dot H^s}\|u\|^2_{L^\infty} \\ &\le \frac{\gamma_2}{2}\|u\|^2_{\dot H^{s+1}} + C(1+t)^{-3/2}(\|u_L\|^2_{\dot H^s} + \|u_H\|^2_{\dot H^s}). \end{aligned} \tag{6.8}$$

By (6.3) and (6.6)-(6.8), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|u_H\|^2_{\dot H^s} + \gamma_1\|u_H\|^2_{\dot H^{s+1}}) + \gamma_2\|u_H\|^2_{\dot H^{s+1}} \\ &\le C(1+t)^{-3/2}(\|u_H\|^2_{\dot H^s}) + C(1+t)^{-\frac{3}{2}-s-1}. \end{aligned}$$

Similar to the proof above and note that $C_0\mu > \frac{3}{2}$, we have

$$\begin{aligned} &(1+t)^{C_0\mu}(\|u_H\|^2_{\dot H^s} + \gamma_1\|u_H\|^2_{\dot H^{s+1}}) \\ &\le \|u_{0H}\|^2_{\dot H^s} + \gamma_1\|u_{0H}\|^2_{\dot H^{s+1}} + C\int_0^t (1+\tau)^{C_0\mu-\frac{3}{2}}(\|u_H\|^2_{\dot H^s})\mathrm{d}\tau \\ &\quad + C\int_0^t (1+\tau)^{C_0\mu-\frac{3}{2}-s-1}\mathrm{d}\tau \\ &\le \|u_{0H}\|^2_{\dot H^s} + \gamma_1\|u_{0H}\|^2_{\dot H^{s+1}} + C(1+t)^{C_0\mu-\frac{3}{2}-s} + C\int_0^t (1+\tau)^{C_0\mu-\frac{3}{2}}(\|u_H\|^2_{\dot H^s})\mathrm{d}\tau. \end{aligned}$$

By Gronwall's inequality [3], we have

$$\begin{aligned} (1+t)^{C_0\mu}\|u_H\|^2_{\dot H^s} &\le C(1 + (1+t)^{C_0\mu-\frac{3}{2}-s})e^{\int_0^t (1+\tau)^{-3/2}\mathrm{d}\tau} \\ &\le C + C(1+t)^{C_0\mu-\frac{3}{2}-s}. \end{aligned}$$

Since $C_0\mu > 3/2$, we obtain

$$\|u_H\|_{\dot H^s} \le C(1+t)^{-\frac{3}{4}-\frac{s}{2}},$$

which is the optimal decay estimates of u_H. This completes the proof. □

Corollary 6.3. *Suppose that u_0 satisfies the same assumptions as in Theorem 6.1 and u is a solution of* (1.5) *in $L^\infty([1,\infty), H^s(\mathbb{R}^3))$. Then*

$$\|u\|_{L^\infty} \le C(1+t)^{-3/2},$$

for $t \ge 1$, where C only depends on $\|u_0\|_{L^1}$.

Proof. By Theorem 1.2, we have

$$\|u\|_{\dot{H}^s} \le C(1+t)^{-\frac{3}{4}-\frac{s}{2}}.$$

In particular, for $s=3$, we have $\|u\|_{\dot{H}^3} \le C(1+t)^{-\frac{9}{4}}$. Therefore, by the Gagliardo-Nirenberg-Sobolev inequality, we immediately obtain

$$\|u\|_{L^\infty} \le C\|u\|_{\dot{H}^3}^{1/2}\|u\|_{L^2}^{1/2} \le C(1+t)^{-3/2},$$

which completes the proof. □

7. Pointwise estimate of the solution

Finally, we come to the pointwise estimate of the solution u. Our goal is to prove the following theorem.

Theorem 7.1. *For Cauchy problem* (1.1), *if the initial data* $v_0 = u_0 - u^* \in L^1(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$, *and satisfies*

$$|D_x^\beta v_0| = |D_x^\beta (u_0 - u^*)| \le C(1+|x|^2)^{-r}, \quad r > \frac{3}{2},$$

then the solution has the pointwise estimate

$$|D_x^{\tilde{\alpha}}(u-u^*)| \le C(1+t)^{-\frac{3+|\tilde{\alpha}|}{2}} B_r(t,|x-bt|),$$

where $|\beta| \le s-1$, $|\tilde{\alpha}| \le s-1$, $t \ge 1$ *and* $B_r(t,|x|)$ *is given in* (2.1).

Proof. From (5.11), we have

$$D_x^{\tilde{\alpha}} u = D_x^{\tilde{\alpha}} G * u_0 + \int_0^t D_x^{\tilde{\alpha}} G(t-\tau,\cdot) * \frac{\operatorname{div}}{1-\gamma_1\Delta} f(u)\mathrm{d}\tau = \Pi_1 + \Pi_2. \tag{7.1}$$

Obviously Π_1 means the initial term while Π_2 represents the nonlinear term.

7.1. Estimate of the initial term.

For $\Pi_1 = D_x^{\tilde{\alpha}} G * u_0$, since G has a distribution $Gf_{\tilde{\alpha}}$, we need to separate it into two parts:

$$\begin{aligned}
|\Pi_1| &= |D_x^{\tilde{\alpha}} G * u_0| \\
&\le |D_x^{\tilde{\alpha}}(G - Gf_{\tilde{\alpha}}) * u_0| + |Gf_{\tilde{\alpha}} * D_x^{\alpha} u_0| \\
&= \Pi_{1,1} + \Pi_{1,2}.
\end{aligned}$$

For $\Pi_{1,1}$, since $|u_0| \le (1+|y|^2)^{-N}$ and supp $u_0 \subset \{|y| \le M\}$, if t is large enough, then applying Lemma 2.4 we obtain

$$\begin{aligned}
|D_x^{\tilde{\alpha}}(G - Gf_{\tilde{\alpha}}) * u_0| &\le Ct^{-\frac{3}{2}-\frac{|\tilde{\alpha}|}{2}} \int_{\mathbb{R}^n} B_r(t,|x-bt-y|)(1+|y|^2)^{-r}\mathrm{d}y \\
&\le Ct^{-\frac{3}{2}-\frac{|\tilde{\alpha}|}{2}} B_r(t,|x-bt|).
\end{aligned} \tag{7.2}$$

For $\Pi_{1,2}$, by the assumption of the initial data and [18], we have

$$\begin{aligned}
|Gf_{\tilde{\alpha}} * D_x^{\tilde{\alpha}} u_0| &\le C\int_{\mathbb{R}^n} e^{-\frac{\gamma_2}{\gamma_1}t}(1+|y|^2)^{-r}\mathrm{d}y \\
&\le Ct^{-\frac{3}{2}-\frac{|\tilde{\alpha}|}{2}} B_r(t,|x-bt|).
\end{aligned} \tag{7.3}$$

Combining (7.2) and (7.3), we obtain

$$|D_x^{\tilde{\alpha}} G * u_0| \le Ct^{-\frac{3}{2}-\frac{|\tilde{\alpha}|}{2}} B_r(t,|x-bt|). \tag{7.4}$$

7.2. **The estimate of the nonlinear term.** Set

$$\Phi = (1+t)^{-3/2} B_r(t, |x-bt|),$$
$$\Phi_{\tilde{\alpha}} = (1+t)^{-\frac{3}{2}-\frac{|\tilde{\alpha}|}{2}} B_r(t, |x-bt|),$$
$$M(t) = \sup_{0\le s\le t, x\in R} |D_x^{\tilde{\alpha}} u(x,s)| \Phi_{\tilde{\alpha}}^{-1}(x,s).$$

Then

$$|u| \le M\cdot\Phi, \quad |D_x^{\tilde{\alpha}} u| \le M\cdot \Phi_{\tilde{\alpha}}.$$

For the nonlinear term, we again separate it into two parts:

$$\begin{aligned}
\Pi_2 &= \int_0^t D_x^{\tilde{\alpha}} G(t-\tau,\cdot) * \operatorname{div} f(u) d\tau. \\
&\le \int_0^t D_x^{\tilde{\alpha}+1}(G - Gf_{\tilde{\alpha}}) * f(u) d\tau + \int_0^t Gf_{\tilde{\alpha}} * D_x^{\tilde{\alpha}} \operatorname{div} f(u) d\tau \\
&= \Pi_{2,1} + \Pi_{2,2}.
\end{aligned}$$

For $\Pi_{2,1}$, let $\Omega = [0,t]\times\mathbb{R}^n$, $\Omega^1 = \Omega\cap\{\frac{t}{2} < \tau \le t\}$, $\Omega^2 = \Omega \cap \{0\le\tau\le\frac{t}{2}\}$. Then

$$\begin{aligned}
\Pi_{2,1} &= \int_0^t D_x^{\tilde{\alpha}}(G - Gf_{\tilde{\alpha}}) * \operatorname{div} f(u) d\tau \\
&\le C\int_{\Omega^1} (G - Gf_{\tilde{\alpha}}) * D_x^{\tilde{\alpha}} \operatorname{div} f(u) d\tau + C\int_{\Omega^2} D_x^{\tilde{\alpha}+1}(G - Gf_{\tilde{\alpha}}) * f(u) d\tau.
\end{aligned}$$

There are two cases:

Case 1: $|x-bt|^2 < t$. Then

$$\begin{aligned}
\Pi_{2,1} &= C\int_{\Omega^1} (t-\tau)^{-\frac{3}{2}} B_r(t-\tau, |x-y-b(t-\tau)|) |D^{\tilde{\alpha}+1} u^2| dy d\tau \\
&\quad + C\int_{\Omega^2} (t-\tau)^{-\frac{3}{2}-\frac{|\tilde{\alpha}|+1}{2}} B_r(t-\tau, |x-y-b(t-\tau)|) \|u\|_{L^\infty} |u| dy d\tau \\
&\le CM(t)(1+t)^{-\frac{3+|\tilde{\alpha}|}{2}-\frac{1}{8}} \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{3}{2}-\frac{3}{8}} (t-\tau)^{\frac{3}{2}} d\tau \\
&\quad + CM(t) t^{-\frac{4+|\tilde{\alpha}|}{2}} \int_0^{t/2} (1+\tau)^{-3} \tau^{\frac{3}{2}} d\tau \\
&\le CM(t) t^{-\frac{3+|\tilde{\alpha}|}{2}-\frac{1}{8}}.
\end{aligned}$$

When $|x-bt|^2 < t$, we have

$$1 \le 2^r \Big(1 + \frac{|x-bt|^2}{t}\Big)^{-r} = 2^r B_r(t, |x-bt|).$$

Thus, we obtain

$$\Pi_{2,1} \le CM(t) t^{-\frac{3+|\tilde{\alpha}|}{2}-\frac{1}{8}} B_r(t, |x-bt|). \tag{7.5}$$

Case 2: $|x-bt|^2 \ge t$. We set

$$P = \Big(1 + \frac{|x-y-b(t-\tau)|^2}{t-\tau}\Big)^{-r} \Big(1 + \frac{|y-b\tau|^2}{\tau}\Big)^{-r}.$$

Then

$$P \le \begin{cases} C\big(1+\frac{|x-bt|^2}{t-\tau}\big)^{-r}\big(1+\frac{|y-b\tau|^2}{\tau}\big)^{-r}, & |x-bt| \ge \frac{|y-b\tau|}{2},\\ C\big(1+\frac{|x-y-b(t-\tau)|^2}{t-\tau}\big)^{-r}\big(1+\frac{|x-b\tau|^2}{\tau}\big)^{-r}, & |x-bt| \le \frac{|y-b\tau|}{2}. \end{cases}$$

We separate the region into $\{y : |x-bt| \ge \frac{|y-b\tau|}{2}\}$ and $\{y : |x-bt| \le \frac{|y-b\tau|}{2}\}$. Then

$$\begin{aligned}
&\Pi_{2,1}\\
&= C\int_{\Omega^1}(t-\tau)^{-3/2}B_r(t-\tau,|x-y-b(t-\tau)|)(1+\tau)^{-3-\frac{|\tilde\alpha|+1}{2}}B_r(\tau,|y-b\tau|)\mathrm{d}y\mathrm{d}\tau\\
&+ C\int_{\Omega^2}(t-\tau)^{-\frac{3}{2}-\frac{|\tilde\alpha|+1}{2}}B_r(t-\tau,|x-y-b(t-\tau)|)(1+\tau)^{-3}B_r(\tau,|y-b\tau|)\mathrm{d}y\mathrm{d}\tau\\
&\le CM(t)B_r(t,|x-bt|)\int_{\Omega^1}(t-\tau)^{-\frac{4+|\tilde\alpha|}{2}}(1+\tau)^{-3}\frac{\tau^{3/2}}{t^{3/2}}B_r(t-\tau,|x-y-b\tau|)\mathrm{d}y\mathrm{d}\tau\\
&+ CM(t)B_r(t,|x-bt|)\int_{\Omega^1}(t-\tau)^{-3/2}(1+\tau)^{-3-\frac{|\tilde\alpha|+1}{2}}\frac{\tau^{3/2}}{t^{3/2}}B_r(t-\tau,|x-y-b\tau|)\mathrm{d}y\mathrm{d}\tau\\
&+ CM(t)B_r(t,|x-bt|)\int_{\Omega^2}(t-\tau)^{-\frac{3+|\tilde\alpha|+1}{2}}(1+\tau)^{-3}\frac{(t-\tau)^{3/2}}{t^{3/2}}B_r(\tau,|y-b\tau|)\mathrm{d}y\mathrm{d}\tau\\
&+ CM(t)B_r(t,|x-bt|)\int_{\Omega^2}(t-\tau)^{-3/2}(1+\tau)^{-3-\frac{|\tilde\alpha|+1}{2}}\frac{(t-\tau)^{3/2}}{t^{3/2}}B_r(\tau,|y-b\tau|)\mathrm{d}y\mathrm{d}\tau\\
&\le CM(t)B_r(t,|x-bt|)\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{4+|\tilde\alpha|}{2}}(1+\tau)^{-3}\frac{\tau^{3/2}}{t^{3/2}}(t-\tau)^{3/2}\mathrm{d}\tau\\
&+ CM(t)B_r(t,|x-bt|)\int_{\frac{t}{2}}^{t}(t-\tau)^{-3/2}(1+\tau)^{-3-\frac{|\tilde\alpha|+1}{2}}\frac{\tau^{3/2}}{t^{3/2}}(t-\tau)^{3/2}\mathrm{d}\tau\\
&+ CM(t)B_r(t,|x-bt|)\int_{0}^{\frac{t}{2}}(t-\tau)^{-\frac{3+|\tilde\alpha|+1}{2}}(1+\tau)^{-3}\frac{(t-\tau)^{3/2}}{t^{3/2}}\tau^{3/2}\mathrm{d}\tau\\
&+ CM(t)B_r(t,|x-bt|)\int_{0}^{\frac{t}{2}}(t-\tau)^{-3/2}(1+\tau)^{-3-\frac{|\tilde\alpha|+1}{2}}\frac{(t-\tau)^{3/2}}{t^{3/2}}\tau^{3/2}\mathrm{d}\tau\\
&\le CM(t)B_r(t,|x-bt|)t^{-\frac{4+|\tilde\alpha|}{2}}\int_{\frac{t}{2}}^{t}(1+\tau)^{-3/2}\mathrm{d}\tau\\
&+ CM(t)B_r(t,|x-bt|)t^{-\frac{4+|\tilde\alpha|}{2}}\int_{0}^{\frac{t}{2}}(1+\tau)^{-3/2}\mathrm{d}\tau\\
&\le CM(t)t^{-\frac{3+|\tilde\alpha|}{2}-\frac{1}{8}}B_r(t,|x-bt|).
\end{aligned} \tag{7.6}$$

Then, by (7.5) and (7.6), we obtain

$$\Pi_{2,1} \le CM(t)t^{-\frac{3+|\tilde\alpha|}{2}-\frac{1}{8}}B_r(t,|x-bt|) = CM(t)\Phi_{\tilde\alpha}(1+t)^{-1/8}. \tag{7.7}$$

Next, we consider $\Pi_{2,2}$. From [19], we have

$$\begin{aligned}\Pi_{2,2} &= \int_0^t Gf_{\tilde{\alpha}} * D_x^{\tilde{\alpha}} \operatorname{div} f(u) \mathrm{d}\tau \\ &\leq C \int_0^t \int_{\mathbb{R}^n} Gf_{\tilde{\alpha}}(x-y,\tau)\Phi_{\tilde{\alpha}}(y,\tau)\Phi_{\tilde{\alpha}+1} \mathrm{d}y \mathrm{d}\tau \\ &\leq C \int_0^t e^{-\frac{\gamma_2}{\gamma_1}(t-\tau)} \Phi_{\tilde{\alpha}}\Phi_{\tilde{\alpha}+1} \mathrm{d}\tau \\ &\leq C\Phi_{\tilde{\alpha}}.\end{aligned} \tag{7.8}$$

Combining (7.1), (7.4), (7.7) and (7.8), and noticing that $t \geq 1$, we have

$$|D_x^{\tilde{\alpha}} u| \leq C\Phi_{\tilde{\alpha}} + CM(t)\Phi_{\tilde{\alpha}}(1+t)^{-1/8}.$$

Thus we obtain

$$M(t) \leq C + CM(t)(1+t)^{-1/8}.$$

Since there exists $T > 0$ such that when $t > T$,

$$C(1+t)^{-1/8} \leq \frac{1}{2},$$

we conclude that $M(t) \leq 2C,\ t > T$, which implies that

$$\|D_\beta u\|_{L^\infty} \leq C,\ |\beta| = |\alpha| + 2.$$

Then, by the structure of the equation, we have

$$\|\partial_t D^\beta u\|_{L^\infty} \leq C.$$

By the continuity of $M(t)$ on $[1, T]$, $M(1) \leq C_0$ and $M(t) \leq C_0$ for $t > T$, we have $M(t) \leq C_0$ for $t \geq 1$. By the definition of $M(t)$, this gives us

$$|D_x^{\tilde{\alpha}} u| \leq C(1+t)^{-\frac{3+|\tilde{\alpha}|}{2}} B_r(|x - bt|, t).$$

This completes the proof. □

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