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# EXISTENCE AND UNIQUENESS FOR A GINZBURG-LANDAU SYSTEM FOR SUPERCONDUCTIVITY

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Abstract. We prove the existence of a unique solution for a time-dependent Ginzburg-Landau model in superconductivity under the Coulomb gauge. Also we prove the uniform-in-$\epsilon$ well-posedness of the solution, where $\epsilon$ is the coefficient of the double-well potential energy.

## 1. Introduction

This article concerns the Ginzburg-Landau model in superconductivity,

$$\eta\partial_t\psi + i\eta k\phi\psi + \Big(\frac{i}{k}\nabla + A\Big)^2\psi + \epsilon^a(|\psi|^2-1)\psi = 0, \tag{1.1}$$

$$\partial_t A + \nabla\phi + \operatorname{curl}^2 A + \operatorname{Re}\Big\{\Big(\frac{i}{k}\nabla\psi + \psi A\Big)\overline{\psi}\Big\} = 0 \tag{1.2}$$

in $Q_T := (0,T)\times\Omega$, with boundary and initial conditions

$$\nabla\psi\cdot\nu = 0, \quad A\cdot\nu = 0, \quad \operatorname{curl} A\times\nu = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{1.3}$$

$$(\psi, A)(x,0) = (\psi_0, A_0)(x) \quad \text{in } \Omega. \tag{1.4}$$

Here $\Omega\subset\mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$, $\nu$ is the outward normal to $\partial\Omega$, and $T$ is any given positive constant. The unknowns $\psi, A$, and $\phi$ are $\mathbb{C}$-valued, $\mathbb{R}^d$-valued, and $\mathbb{R}$-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. $\eta$ and $k$ are Ginzburg-Landau positive constants. $\overline{\psi}$ denotes the complex conjugate of $\psi$, $\operatorname{Re}\psi := (\psi+\overline{\psi})/2$, $|\psi|^2 := \psi\overline{\psi}$ is the density of superconducting carriers, and $i := \sqrt{-1}$. $\epsilon$ is a positive constant. We will assume $a = 1$.

It is well known that the Ginzburg-Landau equations are gauge invariant, namely if $(\psi, A, \phi)$ is a solution of (1.1)-(1.4), then for any real-valued smooth function $\chi$, $(\psi e^{ik\chi}, A+\nabla\chi, \phi-\partial_t\chi)$ is also a solution of (1.1)-(1.4). So, in order to obtain the well-posedness of the problem, we need to impose suitable gauge condition. From the physical point of view, one usually has four types of the gauge conditions:

- Coulomb gauge: $\operatorname{div} A = 0$ in $\Omega$ and $\int_\Omega \phi dx = 0$.
- Lorentz gauge: $\phi = -\operatorname{div} A$ in $\Omega$.
- Lorenz gauge: $\partial_t\phi = -\operatorname{div} A$ in $\Omega$.
- Temporal gauge(Weyl gauge): $\phi = 0$ in $\Omega$.

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For the initial data $\psi_0 \in H^1(\Omega), |\psi_0| \le 1, A_0 \in H^1(\Omega)$, Chen, Elliott and Tang [3], Chen, Hoffmann and Liang [4], Du [5] and Tang [11] proved the existence and uniqueness of a global strong solution to (1.1)-(1.4), in the case of the Coulomb, Lorentz, and temporal gauges. For the initial data $\psi_0 \in H^1(\Omega)$, $A_0 \in H^1(\Omega)$, Tang and Wang [12] obtained the existence and uniqueness of global strong solutions, while Fan and Jiang [8] showed the existence of global weak solutions when $\psi_0, A_0 \in L^2$. Fan and Ozawa [9] (2-D) and Fan, Gao and Guo [7, 6] (3-D) prove the uniqueness of a weak solution for $\psi_0, A_0 \in L^d$ with $d = 2, 3$, which is critical. This comes from a scaling argument for (1.1) and (1.2). Move precisely, if $(\psi(t,x), A(t,x), \phi(t,x))$ is a solution of (1.1) and (1.2) associated with the initial data $(\psi_0(x), A_0(x))$ without linear lower order term $\psi$, then

$$(\lambda\psi(\lambda^2 t, \lambda x), \lambda A(\lambda^2 t, \lambda x), \lambda^2\phi(\lambda^2 t, \lambda x)) =: (\psi_\lambda, A_\lambda, \phi_\lambda) \tag{1.5}$$

is also a solution for any $\lambda > 0$.

A Banach space $\mathbf{B}$ of distributions on $\mathbb{R} \times \mathbb{R}^d$ is a critical space if its norm verifies for any $\lambda$ and any $u \in \mathbf{B}$,

$$\|u\|_{\mathbf{B}} = \|\lambda u(\lambda^2 \cdot, \lambda \cdot)\|_{\mathbf{B}}.$$

If we choose $\mathbf{B}$ as $L^r(0,\infty; L^p(\mathbb{R}^d))$, then $(r,p)$ should satisfy

$$\frac{2}{r} + \frac{d}{p} = 1.$$

In this article, we will choose the Coulomb gauge. First, we will prove the following theorem.

**Theorem 1.1.** *Let $d = 3$ and $0 < \epsilon < 1$. Let $\psi_0 \in H^1, |\psi_0| \le 1$ and $A_0 \in H^1$. Then the solution $(\psi, A, \phi)$ satisfies*

$$\begin{gathered}
|\psi| \le 1, \quad \|\psi\|_{L^\infty(0,T;H^1)} + \|\psi\|_{L^2(0,T;H^2)} + \|\partial_t\psi\|_{L^2(0,T;L^2)} \le C, \\
\|A\|_{L^\infty(0,T;H^1)} + \|A\|_{L^2(0,T;H^2)} + \|\partial_t A\|_{L^2(0,T;L^2)} \le C, \\
\|\phi\|_{L^2(0,T;H^1)} \le C
\end{gathered} \tag{1.6}$$

*for any $0 < T < \infty$. Here and later $C$ will denote a positive constant independent of $\epsilon$.*

**Theorem 1.2.** *Let $d = 3$ and $0 \le \epsilon \le 1$ and $\psi_0, A_0 \in L^3(\Omega)$. Then the problem (1.1)-(1.4) has a unique solution $(\psi, A, \phi)$ satisfying*

$$\begin{gathered}
\|\psi\|_{L^\infty(0,T;L^3)} + \|\psi\|_{L^5(0,T;L^5)} + \|\psi\|_{L^2(0,T;H^1)} + \||\psi|^{3/2}\|_{L^2(0,T;H^1)} \le C, \\
\|\partial_t\psi\|_{L^2(0,T;H^{-1})} \le C, \\
\|A\|_{L^\infty(0,T;L^3)} + \|A\|_{L^5(0,T;L^5)} + \|A\|_{L^2(0,T;H^1)} + \||A|^{3/2}\|_{L^2(0,T;H^1)} \le C, \\
\|\partial_t A\|_{L^2(0,T;H^{-1})} \le C, \quad \|\nabla\phi\|_{L^{5/3}(0,T;L^{5/3})} \le C
\end{gathered} \tag{1.7}$$

*for any $T > 0$.*

**Remark 1.3.** When $a = -1$, we are unable to prove a similar result at present. Our results also hold true with the choice of Lorentz gauge.

In our proofs, we will use the following lemmas.

**Lemma 1.4** ([1, 10])**.** *Let $\Omega$ be a smooth and bounded open set in $\mathbb{R}^3$. Then there exists $C > 0$ such that*

$$\|f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\Omega)}^{1-\frac{1}{p}} \|f\|_{W^{1,p}(\Omega)}^{1/p} \tag{1.8}$$

*for any* $1 < p < \infty$ *and* $f : \Omega \to \mathbb{R}^3$ *be in* $W^{1,p}(\Omega)$.

**Lemma 1.5** ([2]). *Let* $\Omega$ *be a regular bounded domain in* $\mathbb{R}^3$, *let* $f : \Omega \to \mathbb{R}^3$ *be a smooth enough vector field, and let* $1 < p < \infty$. *Then*

$$\begin{aligned} &-\int_\Omega \Delta f \cdot f|f|^{p-2}\,\mathrm{d}x \\ &= \int_\Omega |f|^{p-2}|\nabla f|^2\,\mathrm{d}x + \frac{4(p-2)}{p^2}\int_\Omega |\nabla |f|^{\frac{p}{2}}|^2\,\mathrm{d}x - \int_{\partial\Omega} |f|^{p-2}(\nu\cdot\nabla)f\cdot f\,\mathrm{d}S. \end{aligned} \tag{1.9}$$

**Lemma 1.6** ([8]). $\nabla\phi \in L^{5/3}(0,T;L^{5/3})$ *satisfies*

$$-\Delta\phi = \operatorname{div}\operatorname{Re}\Big\{\Big(\frac{i}{k}\nabla\psi + \psi A\Big)\overline{\psi}\Big\} \quad \text{in } \Omega\times(0,T), \tag{1.10}$$

$$\nabla\phi\cdot\nu = 0 \quad \text{on } (0,T)\times\partial\Omega. \tag{1.11}$$

## 2. Proof of Theorem 1.1

We only need to show the a priori estimates (1.6). It is easy to show that (see [3, 4, 5, 11])

$$|\psi| \le 1 \quad \text{in } \Omega\times(0,T). \tag{2.1}$$

Testing (1.1) by $\overline{\psi}$ and taking the real parts, we see that

$$\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\psi|^2\,\mathrm{d}x + \int \Big|\frac{i}{k}\nabla\psi + \psi A\Big|^2\,\mathrm{d}x + \epsilon\int |\psi|^4\,\mathrm{d}x = \epsilon\int|\psi|^2\,\mathrm{d}x,$$

which gives

$$\int_0^T\int \Big|\frac{i}{k}\nabla\psi + \psi A\Big|^2\,\mathrm{d}x\,\mathrm{d}t \le C. \tag{2.2}$$

Testing (1.2) by $\partial_t A + \operatorname{curl}^2 A$, using (2.1), (2.2) and (1.11), we find that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int |\operatorname{curl} A|^2\,\mathrm{d}x + \int (|\partial_t A|^2 + |\operatorname{curl}^2 A|^2)\,\mathrm{d}x \\ &\le \int \Big|\frac{i}{k}\nabla\psi + \psi A\Big|\,|\partial_t A + \operatorname{curl}^2 A|\,\mathrm{d}x \\ &\le \frac{1}{2}\int (|\partial_t A|^2 + |\operatorname{curl}^2 A|^2)\,\mathrm{d}x + C\int \Big|\frac{i}{k}\nabla\psi + \psi A\Big|^2\,\mathrm{d}x, \end{aligned}$$

which leads to

$$\|A\|_{L^\infty(0,T;H^1)} + \|A\|_{L^2(0,T;H^2)} + \|\partial_t A\|_{L^2(0,T;L^2)} \le C, \tag{2.3}$$

whence

$$\|\phi\|_{L^2(0,T;H^1)} \le C. \tag{2.4}$$

Multiplying (1.1) by $-\Delta\overline{\psi}$, integrating by parts and taking the real part, using (2.1), (2.3) and (2.4), we obtain

$$\begin{aligned} &\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\psi|^2\,\mathrm{d}x + \frac{1}{k^2}\int|\Delta\psi|^2\,\mathrm{d}x \\ &\le \Big|\operatorname{Re}\int i\eta k\phi\psi\cdot\Delta\overline{\psi}\,\mathrm{d}x\Big| + 2\Big|\operatorname{Re}\frac{1}{k}\int iA\nabla\psi\cdot\Delta\overline{\psi}\,\mathrm{d}x\Big| \\ &\quad + \operatorname{Re}\int A^2\psi\Delta\overline{\psi}\,\mathrm{d}x + \epsilon\operatorname{Re}\int(|\psi|^2-1)\psi\cdot\Delta\overline{\psi}\,\mathrm{d}x \\ &\le \frac{1}{2}\frac{1}{k^2}\int|\Delta\psi|^2\,\mathrm{d}x + C\int|\nabla\phi||\nabla\psi|\,\mathrm{d}x \end{aligned}$$

$$+ C\|A\|_{L^\infty}^2 \|\nabla\psi\|_{L^2}^2 + C\|A\|_{L^\infty}\|\nabla A\|_{L^2}\|\nabla\psi\|_{L^2} + C\|\nabla\psi\|_{L^2}^2,$$

which yields

$$\|\psi\|_{L^\infty(0,T;H^1)} + \|\psi\|_{L^2(0,T;H^2)} \le C. \tag{2.5}$$

Whence

$$\|\partial_t\psi\|_{L^2(0,T;L^2)} \le C. \tag{2.6}$$

This completes the proof.

## 3. Proof of Theorem 1.2

To prove the existence, we only need to prove (1.7). First, we still have (2.2). Multiplying (1.1) by $|\psi|\overline{\psi}$, integrating by parts, and then taking the real part, we obtain

$$\frac{\eta}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int |\psi|^3\,\mathrm{d}x + \int \big|\frac{i}{k}\nabla\psi + \psi A\big|^2|\psi|\,\mathrm{d}x + \epsilon\int |\psi|^5\,\mathrm{d}x = \epsilon\int |\psi|^3\,\mathrm{d}x,$$

which gives

$$\sup_{0\le t\le T}\int |\psi|^3\,\mathrm{d}x + \int_0^T\int \big|\frac{i}{k}\nabla\psi + \psi A\big|^2|\psi|\,\mathrm{d}x\,\mathrm{d}t \le C. \tag{3.1}$$

Using the diamagnetic inequality

$$\big|\frac{1}{k}\nabla|\psi|\big| \le \big|\frac{i}{k}\nabla\psi + \psi A\big|, \tag{3.2}$$

and the Gagliardo-Nirenberg inequality

$$\|w\|_{L^p} \le C\|w\|_{L^2}^{\frac{3}{p}-\frac{1}{2}}\|\nabla w\|_{L^2}^{\frac{3}{2}-\frac{3}{p}} + C\|w\|_{L^2} \tag{3.3}$$

for $w := |\psi|^{3/2}$ and $p := \frac{10}{3}$, we find that

$$\|\psi\|_{L^5(0,T;L^5)} \le C. \tag{3.4}$$

Testing (1.2) by $A$ and using (3.1), we observe that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |A|^2\,\mathrm{d}x + \int |\operatorname{curl} A|^2\,\mathrm{d}x\\
&\le \int \big|\frac{i}{k}\nabla\psi + \psi A\big|\,|\psi|\,|A|\,\mathrm{d}x\\
&\le \int \big|\frac{i}{k}\nabla\psi + \psi A\big|\,|\psi|^{1/2}\,|\psi|^{1/2}\,|A|\,\mathrm{d}x\\
&\le \big\|\,\big|\frac{i}{k}\nabla\psi + \psi A\big|\,|\psi|^{1/2}\big\|_{L^2}\|\,|\psi|^{1/2}\|_{L^6}\|A\|_{L^3}\\
&\le \frac{1}{2}\|\operatorname{curl} A\|_{L^2}^2 + C\big\|\,\big|\frac{i}{k}\nabla\psi + \psi A\big|\,|\psi|^{1/2}\big\|_{L^2}^2,
\end{aligned}$$

which leads to

$$\|A\|_{L^\infty(0,T;L^2)} + \|A\|_{L^2(0,T;H^1)} \le C. \tag{3.5}$$

Here we have used the estimate $\|A\|_{L^3} \le C\|\operatorname{curl} A\|_{L^2}$. Since

$$\int_0^T\int |\psi A|^2\,\mathrm{d}x\,\mathrm{d}t \le \|\psi\|_{L^3}^2\int_0^T \|A\|_{L^6}^2\,\mathrm{d}t \le C,$$

it follows from (2.2) that

$$\|\psi\|_{L^2(0,T;H^1)} \le C. \tag{3.6}$$

Testing (1.2) by $|A|A$ and letting $u := |A|^{3/2}$, using (1.3), (1.8), (1.9), (1.10), (1.11), (3.1) and the vector identities

$$(\nu\cdot\nabla)A\cdot A = (A\cdot\nabla)A\cdot\nu + (\operatorname{curl} A\times\nu)\cdot A, \tag{3.7}$$

$$(A\cdot\nabla)A\cdot\nu = -(A\cdot\nabla)\nu\cdot A, \tag{3.8}$$

we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int u^2\,\mathrm{d}x + C_0\int|\nabla u|^2\,\mathrm{d}x + C_0\int|A||\nabla A|^2\,\mathrm{d}x\\
&\le C\int\big|\frac{i}{k}\nabla\psi+\psi A\big|\,|\psi|u^{4/3}\,\mathrm{d}x + C\int|\nabla\phi|u^{4/3}\,\mathrm{d}x + C\int_{\partial\Omega}u^2\,\mathrm{d}S\\
&\le C\big\|\,\big|\frac{i}{k}\nabla\psi+\psi A\big|\,|\psi|^{1/2}\big\|_{L^2}\|\,|\psi|^{1/2}\|_{L^6}\|u^{4/3}\|_{L^3}\\
&\quad + C\|\nabla\phi\|_{L^{3/2}}\|u^{4/3}\|_{L^3} + C\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}\\
&\le C\big\|\,\big|\frac{i}{k}\nabla\psi+\psi A\big|\,|\psi|^{1/2}\big\|_{L^2}\|\,|\psi|^{1/2}\|_{L^6}\|u^{4/3}\|_{L^3} + C\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}\\
&\le C\big\|\,\big|\frac{i}{k}\nabla\psi+\psi A\big|\,|\psi|^{1/2}\big\|_{L^2}\|u\|_{L^2}^{1/3}\|u\|_{H^1} + C\|u\|_{L^2}\|u\|_{H^1}\\
&\le \frac{C_0}{2}\|\nabla u\|_{L^2}^2 + C\big\|\,\big|\frac{i}{k}\nabla\psi+\psi A\big|\,|\psi|^{1/2}\big\|_{L^2}^2\|u\|_{L^2}^{\frac{2}{3}} + C\|u\|_{L^2}^2,
\end{aligned}$$

which implies

$$\|A\|_{L^\infty(0,T;L^3)} + \|\,|A|^{3/2}\|_{L^2(0,T;H^1)} \le C. \tag{3.9}$$

Here we have used the estimate

$$\begin{aligned}
\|\nabla\phi\|_{L^{3/2}} &\le C\big\|\big(\frac{i}{k}\nabla\psi+\psi A\big)\overline{\psi}\big\|_{L^{3/2}}\\
&\le C\big\|\,\big|\frac{i}{k}\nabla\psi+\psi A\big|\,|\psi|^{1/2}\big\|_{L^2}\|\,|\psi|^{1/2}\|_{L^6}\\
&\le C\big\|\,\big|\frac{i}{k}\nabla\psi+\psi A\big|\,|\psi|^{1/2}\big\|_{L^2}.
\end{aligned}\tag{3.10}$$

Using (3.3) for $w = |A|^{3/2}$ and $p = 10/3$ and (3.9), we have

$$\|A\|_{L^5(0,T;L^5)} \le C. \tag{3.11}$$

On the other hand, using (1.1), (1.2), (2.2), (3.1) and (3.9), we easily deduce that

$$\|\partial_t\psi\|_{L^2(0,T;H^{-1})} + \|\partial_t A\|_{L^2(0,T;H^{-1})} \le C. \tag{3.12}$$

This completes the proof of (1.7).

To prove the uniqueness, we use the method considered in [7, 6]. Here we remark the only new estimate: if $(\psi_i, A_i, \phi_i)$ $(i = 1, 2)$ are two weak solutions to the problem (1.1)-(1.4), then the following monotone property holds:

$$\operatorname{Re}\int(|\psi_1|^2\psi_1 - |\psi_2|^2\psi_2)\overline{(\psi_1-\psi_2)}\,\mathrm{d}x \ge 0.$$

The rest of the proof follows as in [7, 6]. This completes the proof.

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