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NULL CONTROLLABILITY FROM THE EXTERIOR OF FRACTIONAL PARABOLIC-ELLIPTIC COUPLED SYSTEMS

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Abstract. We analyze the null controllability properties from the exterior of two parabolic-elliptic coupled systems governed by the fractional Laplacian $(-d_x^2)^s$, $s \in (0,1)$, in one space dimension. In each system, the control is located on a non-empty open set of $\mathbb{R} \setminus (0,1)$. Using the spectral theory of the fractional Laplacian and a unique continuation principle for the dual equation, we show that the problem is null controllable if and only if $1/2 < s < 1$.

1. Introduction

Let ω be a non-empty open set of $\mathbb{R} \setminus (0,1)$. We will denote by 1_ω the characteristic function of ω. We consider the following linear parabolic-elliptic one-dimensional coupled systems

$$
\begin{gathered}
\partial_t u + (-d_x^2)^s u = au + bv \text{ in } (0,1) \times (0,T), \\
(-d_x^2)^s v = cu + dv \text{ in } (0,1) \times (0,T), \\
u = g 1_\omega, \quad v = 0 \quad \text{in } [\mathbb{R} \setminus (0,1)] \times (0,T), \\
u(\cdot, 0) = u^0 \quad \text{in } (0,1),
\end{gathered} \tag{1.1}
$$

and

$$
\begin{gathered}
\partial_t u + (-d_x^2)^s u = au + bv \quad \text{ in } (0,1) \times (0,T), \\
(-d_x^2)^s v = cu + dv \quad \text{in } (0,1) \times (0,T), \\
u = 0, \quad v = h 1_\omega \quad \text{in } [\mathbb{R} \setminus (0,1)] \times (0,T), \\
u(\cdot, 0) = u^0 \quad \text{in } (0,1),
\end{gathered} \tag{1.2}
$$

where $s \in (0,1)$ and a, b, c, d are real numbers. In (1.1) and (1.2), we have that $u = u(x,t)$, $v = v(x,t)$, u^0 is the initial state, g and h are the controls acting on the system through $\omega \times (0,T)$. The operator $(-d_x^2)^s$ denotes the one-dimensional fractional Laplace operator which is defined, for smooth functions u that will be

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specified in Section 3, by

$$\begin{aligned}(-d_x^2)^s u(x) &= \lim_{\varepsilon\to 0^+} c_{1,s}\int_{\{y\in\mathbb{R}:|x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{1+2s}}dy \\ &= c_{1,s}\text{P.V.}\int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}}dy, \quad x\in\mathbb{R},\end{aligned} \tag{1.3}$$

provided that the limit exists. The constant $c_{1,s}$ is

$$c_{1,s} = \frac{2^{2s}s\Gamma\left(\frac{1+2s}{2}\right)}{\sqrt{\pi}\Gamma(1-s)}, \tag{1.4}$$

Γ being the Euler Gamma function.

In this article, we study the controllability of the parabolic-elliptic systems (1.1) and (1.2) from the exterior. As far as we know, the null controllability of such fractional parabolic-elliptic systems involving the fractional Laplacian has not been studied yet.

The null controllability of the systems (1.1) and (1.2) is the purpose of [6] in the semilinear case. Both equations considered in [6] are governed by the Laplacian and to reach the result, the authors proved some Carleman estimates for the solution of the adjoint system. They also extended these results in [7] to a semilinear system of two parabolic PDEs and one elliptic PDE. In the fractional case, the authors of [13] showed the null controllability of a parabolic equation governed by the fractional power of the Dirichlet Laplacian. Thus they generalized the classical null controllability results for the heat equation established for example in [5, 22, 2, 14]. In [15] the author analyzed controllability properties of the fractional diffusion equation involving the spectral fractional Laplacian when the control is localized in the domain under consideration. Then the authors of [1] focused on the null controllability of a similar parabolic problem governed by the one-dimensional fractional Laplacian $(-d_x^2)^s$ defined by (1.3) for all $s\in(0,1)$. They proved that the interior approximate controllability holds for every $s\in(0,1)$ and that the interior null controllability of the equation holds if and only if $s>1/2$. The latter result follows from the spectral method developed in [9, 10]. The null controllability from the exterior of the interval $(0,1)$ of the fractional heat equation was subsequently investigated in [20]. The null controllability from the exterior means that the control is located in $\mathbb{R}\setminus(0,1)$, precisely in a non-empty open subset of $\mathbb{R}\setminus(0,1)$. This type of controllability problem was introduced first by Warma for space-time fractional diffusion equations associated with the fractional Laplacian and the Caputo derivative of order $\alpha\in(0,1]$ (see [19]). The approximate controllability by means of a unique continuation property for the dual equation is the purpose of Warma's paper. The same work was extended in [12] to the fractional wave equation with Dirichlet or Robin type exterior conditions and in [21] to the strong damping nonlocal wave equation.

Systems such as (1.1) (resp. (1.2)) can arise in chemistry to describe the behavior in systems of interacting components. The fractional Laplacian is a nonlocal operator modelling the multi-scale behavior. Applications with models containing fractional diffusion operators such as phase field models, are possible.

The rest of the paper is organized as follows. In Section 2 we state the main result dealing with null controllability properties of (1.1) (resp. (1.2)). The main result is based on a unique continuation property for the realization of $(-d_x^2)^s$ in $L^2(0,1)$ with the homogeneous exterior Dirichlet condition (see Lemma 3.2). Then we give in Section 3 some preliminary results including the spectral properties of the fractional Laplacian. In Section 4 a representation in the form of convergent series of the solution of (1.1) (resp. (1.2)) and of the dual equation, is proved. Finally Section 5 is devoted to the proof of the main result stated in Section 2.

2. Main result

In this section the main result of this work is stated. Let s be a real number which will be fixed in $(0,1)$ until the end of the article.

First we give the dual system associated with (1.1) (resp. (1.2)). Using the integration by parts (3.3) (see Lemma 3.1), we have the following dual system

$$\begin{gathered} -\partial_t\varphi + (-d_x^2)^s\varphi = a\varphi + c\sigma \quad \text{in } (0,1)\times(0,T), \\ (-d_x^2)^s\sigma = b\varphi + d\sigma \quad \text{in } (0,1)\times(0,T), \\ \varphi = \sigma = 0 \quad \text{in } [\mathbb{R}\setminus(0,1)]\times(0,T), \\ \varphi(\cdot,T) = \varphi^T \quad \text{in } (0,1), \end{gathered} \tag{2.1}$$

where $\varphi^T \in L^2(0,1)$.

Then we define the notion of weak solution of (1.1) (resp. (1.2)). Let $\langle\cdot,\cdot\rangle$ be the duality pair between the fractional order Sobolev space $H^s(\mathbb{R})$ and its dual $H^{-s}(\mathbb{R})$. We shall give the definition of these spaces in Section 3.

Definition 2.1. Let a, b, c, d be real constants, $g \in L^2(\mathbb{R}\setminus(0,1))$ (resp. $h \in L^2(\mathbb{R}\setminus(0,1))$) and $u^0 \in L^2(0,1)$. A function (u,v) is said to be a weak solution of (1.1) (resp. (1.2)) if the following properties hold.

- Regularity: $(u,v) \in \big(C([0,T];L^2(0,1))\big)^2$.
- Variational identity: for every $(\varphi,\sigma) \in (H^s(\mathbb{R}))^2$ and a.e. $t \in (0,T)$,
$$\begin{gathered} \langle\partial_t u,\varphi\rangle + \langle(-d_x^2)^s u,\varphi\rangle = a\langle u,\varphi\rangle + b\langle v,\varphi\rangle, \\ \langle(-d_x^2)^s v,\sigma\rangle = c\langle u,\sigma\rangle + d\langle v,\sigma\rangle. \end{gathered}$$
- Initial and exterior conditions: $u = g$ and $v = 0$ in $[\mathbb{R}\setminus(0,1)]\times(0,T)$ (resp. $u = 0$ and $v = h$ in $[\mathbb{R}\setminus(0,1)]\times(0,T)$), and $u(\cdot,0) = u^0$ in $(0,1)$.

System (1.1) (resp. (1.2)) is well posed in the sense that for each $u^0 \in L^2(0,1)$ and each $g \in \mathcal{D}([\mathbb{R}\setminus(0,1)]\times(0,T))$ (resp. $h \in \mathcal{D}([\mathbb{R}\setminus(0,1)]\times(0,T))$), it possesses exactly one solution (u,v) such that

$$(u,v) \in \big(C([0,T];L^2(0,1))\big)^2 \tag{2.2}$$

and there is a constant $C > 0$ such that for any $t \in [0,T)$,

$$\begin{gathered} \|(u,v)(\cdot,t)\|^2_{(L^2(0,1))^2} \leqslant C\Big(\|u^0\|^2_{L^2(0,1)} + T^3\|g_t\|^2_{L^\infty(0,T;L^2(\mathbb{R}\setminus(0,1)))}\Big) \\ \Big(\text{resp. } \|(u,v)(\cdot,t)\|^2_{(L^2(0,1))^2} \leqslant C\Big(\|u^0\|^2_{L^2(0,1)} + T^3\|h_t\|^2_{L^\infty(0,T;L^2(\mathbb{R}\setminus(0,1)))}\Big)\Big). \end{gathered} \tag{2.3}$$

The assertions (2.2) and (2.3) are proved in Theorems 4.4 and 4.5.

The notion of null controllability of the system (1.1) (resp. (1.2)) is the following.

Definition 2.2. System (1.1) (resp. (1.2)) is null controllable at time $T > 0$ if for any given $u^0 \in L^2(0,1)$, there exists a control $g \in L^2(\omega\times(0,T))$ (resp. $h \in$

$L^2(\omega \times (0,T)))$ such that the corresponding solution (u,v) satisfies

$$\begin{gathered} (u,v) \in \big(C([0,T];L^2(0,1))\big)^2 \\ u(x,T)=0 \text{ in } (0,1), \quad \limsup_{t\to T^-} \|v(\cdot,t)\|_{L^2(0,1)} = 0. \end{gathered} \tag{2.4}$$

Let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of eigenvalues of the fractional Laplacian $(-d_x^2)^s$. We shall recall their properties in Section 3. The main result of this article reads as follows.

Theorem 2.3. *Let ω be a non-empty open set of $\mathbb{R}\setminus(0,1)$. Assume that*

$$c=b, \quad b>0, \quad \max\{a,d\} \leqslant -b \quad \text{and} \quad b^2 < (\lambda_1 - a)(\lambda_1 - d). \tag{2.5}$$

1. *If $1/2 < s < 1$ then system* (1.1) *(resp.* (1.2)*) is null controllable at any time $T>0$ for any $g \in L^2(\omega\times(0,T))$ (resp. $h \in L^2(\omega\times(0,T))$).*
2. *If $0 < s \leqslant \frac{1}{2}$ then system* (1.1) *(resp.* (1.2)*) is not null controllable at time $T>0$.*

3. Preliminary results

In this section, we define the functional framework, we present the spectral theory of the fractional Laplace operator, and we give some preliminary results that will be useful in the rest of the paper. First we define the space

$$\mathcal{L}_s^1(\mathbb{R}) = \Big\{u:\mathbb{R}\to\mathbb{R}, \int_{\mathbb{R}} \frac{|u(x)|}{(1+|x|)^{1+2s}}dx < \infty\Big\}.$$

Then Definition (1.3) is valid for any $u \in \mathcal{L}_s^1(\mathbb{R})$ because the integral

$$\int_{\{y\in\mathbb{R}:|x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{1+2s}}dy$$

exists for any $\varepsilon > 0$.

The fractional Sobolev space $H^s(0,1)$ is defined for $s\in(0,1)$, as

$$H^s(0,1) = \Big\{u \in L^2(0,1) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{1}{2}+s}} \in L^2((0,1)\times(0,1))\Big\}.$$

Endowed with the norm

$$\|u\|_{H^s(0,1)} = \Big(\int_0^1 |u|^2\,dx + \int_0^1\int_0^1 \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,dx\,dy\Big)^{1/2}.$$

Then $H^s(0,1)$ is a Hilbert space. The dual of $H^s(0,1)$ is denoted by $H^{-s}(0,1)$. The space of test functions on $(0,1)$, that is C^∞ functions compactly supported in $(0,1)$, is denoted by $\mathcal{D}(0,1)$. The fractional space $H_0^s(0,1)$ is defined as the closure of $\mathcal{D}(0,1)$ in $H^s(0,1)$ with respect to the norm $\|\cdot\|_{H^s(0,1)}$ (i.e. $H_0^s(0,1) = \overline{\mathcal{D}(0,1)}^{\|\cdot\|_{H^s(0,1)}}$). We refer the reader to [3] for a precise definition of these spaces.

If $E\subset\mathbb{R}$, the scalar product in $L^2(E)$ is denoted by $(\cdot,\cdot)_{L^2(E)}$. Let $(\psi_k)_{k\in\mathbb{N}}$ be the orthonormal basis of eigenfunctions of the fractional Laplace operator associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$. Then $(\psi_k)_{k\in\mathbb{N}}$ is total in $L^2(0,1)$ and is solution of the system

$$\begin{gathered} (-d_x^2)^s\psi_k = \lambda_k\psi_k, x\in(0,1), \\ \psi_k = 0 \quad \text{in } \mathbb{R}\setminus(0,1). \end{gathered} \tag{3.1}$$

We set

$$H_0^s(\overline{(0,1)}) = \{u \in H^s(\mathbb{R}) : u = 0 \text{ in } \mathbb{R}\setminus(0,1)\}.$$

Let $\mathcal{L}$ be the bilinear form

$$\mathcal{L}(u,v)=\frac{c_{1,s}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2s}}\,dx\,dy,\quad u,v\in H_0^s(\overline{(0,1)})$$

and let $(-d_x^2)_D^s$ be the selfadjoint operator on $L^2(0,1)$ associated with $\mathcal{L}$ in the sense that

$$D((-d_x^2)_D^s)=\{u\in H_0^s(\overline{(0,1)}):\exists f\in L^2(0,1),\mathcal{L}(u,v)=(f,v)_{L^2(0,1)}\,\forall v\in H_0^s(\overline{(0,1)})\}$$

and $(-d_x^2)_D^s u=f$. We have that for $u\in H_0^s(\overline{(0,1)})$,

$$\|u\|^2_{H_0^s(\overline{(0,1)})}=\sum_{n\in\mathbb{N}}\big|\lambda_n^{1/2}(u,\psi_n)_{L^2(0,1)}\big|^2$$

defines an equivalent norm on $H_0^s(\overline{(0,1)})$. We also have that

$$D((-d_x^2)_D^s)=\{u\in L^2(0,1):\sum_{n\in\mathbb{N}}|\lambda_n(u,\psi_n)_{L^2(0,1)}|^2<\infty\}.$$

If $u\in D((-d_x^2)_D^s)$, then

$$\|u\|^2_{D((-d_x^2)_D^s)}=\|(-d_x^2)_D^s u\|^2_{L^2(0,1)}=\sum_{n\in\mathbb{N}}|\lambda_n(u,\psi_n)_{L^2(0,1)}|^2.$$

We know that $\psi_k\in D((-d_x^2)_D^s)$ and, using [18], that the operator $(-d_x^2)_D^s$ has a compact resolvent and its eigenvalues are real numbers such that

$$0<\lambda_1\leqslant\lambda_2\leqslant\dots\leqslant\lambda_k\leqslant\cdots\quad\text{and}\quad\lim_{n\to\infty}\lambda_n=\infty.$$

Moreover, the eigenvalues of $(-d_x^2)_D^s$ possess the following asymptotic behavior (see [10, Theorem 1])

$$\lambda_k=\Big(\frac{k\pi}{2}-\frac{(2-2s)\pi}{8}\Big)^{2s}+O\Big(\frac{1}{k}\Big)\quad\text{as }k\to\infty.\tag{3.2}$$

For $u\in H^s(\mathbb{R})$, the nonlocal normal derivative is

$$\mathcal{N}_s u(x)=c_{1,s}\int_0^1\frac{u(x)-u(y)}{|x-y|^{1+2s}}dy,\quad x\in\mathbb{R}\setminus\overline{(0,1)},$$

where $c_{1,s}$ is the positive constant given by (1.4). From [8, Lemma 3.2] we have that for every $u\in H^s(\mathbb{R})$, $\mathcal{N}_s u\in L^2(\mathbb{R}\setminus(0,1))$.

The following integration by parts formula will be useful in the remainder of the article (see [20, Lemma 2.2], [19, Proposition 13], [4, Lemma 3.3]).

Lemma 3.1. *Let $u\in H_0^s(\overline{(0,1)})$ be such that $(-d_x^2)^s u\in L^2(0,1)$. Then for every $v\in H^s(\mathbb{R})$ we have*

$$\begin{aligned}&\frac{c_{1,s}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2s}}\,dx\,dy\\&=\int_0^1 v(x)(-d_x^2)^s u(x)\,dx+\int_{\mathbb{R}\setminus(0,1)}v(x)\mathcal{N}_s u(x)\,dx\,.\end{aligned}\tag{3.3}$$

The following unique continuation property which has been proved in [19, Theorem 16], is one of the main tool in this work.

Lemma 3.2. *Let $\omega \subset [\mathbb{R} \setminus (0,1)]$ an arbitrary non-empty open set and $\lambda > 0$ be a real number. If $\varphi \in D((-d_x^2)_D^s)$ satisfies*

$$(-d_x^2)_D^s = \lambda\varphi \text{ in } (0,1) \text{ and } \mathcal{N}_s\varphi = 0 \text{ in } \omega$$

then $\varphi = 0$ in $\mathbb{R}$.

4. Well-posedness results

This section is devoted to the proof of the well-posedness of system (1.1) (resp. (1.2)) and its dual system (2.1). Also an explicit representation of their solution is given. Let $(\psi_n)_{n\in\mathbb{N}}$ be the sequence of eigenfunctions of the fractional Laplacian operator associated with the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$.

4.1. **Well-posedness of systems** (1.1) **and** (1.2)**.** We begin with system (1.1). First of all, we consider the following elliptic Dirichlet problem for the fractional Laplacian

$$\begin{aligned} &(-d_x^2)^s\alpha = a\alpha + b\beta \quad \text{in } (0,1),\\ &(-d_x^2)^s\beta = c\alpha + d\beta \quad \text{in } (0,1),\\ &\alpha = g, \quad \beta = 0 \quad \text{in } \mathbb{R} \setminus (0,1). \end{aligned} \tag{4.1}$$

Definition 4.1. Let $g \in H^s(\mathbb{R} \setminus (0,1))$, $a, b, c, d \in \mathbb{R}$ and let $G \in H^s(\mathbb{R})$ be such that $G|_{\mathbb{R}\setminus(0,1)} = g$. A function (α, β) is said to be a weak solution of (4.1) if $(\alpha - G, \beta) \in \big(H_0^s(\overline{(0,1)})\big)^2$ and for every $(\varphi, \sigma) \in \big(H_0^s(\overline{(0,1)})\big)^2$ it holds

$$\frac{c_{1,s}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(U(x) - U(y)) \cdot (V(x) - V(y))}{|x-y|^{1+2s}}\,dx\,dy = \int_0^1 AU \cdot V\,dx\,, \tag{4.2}$$

where $U = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, $V = \begin{pmatrix}\varphi\\ \sigma\end{pmatrix}$, and $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$.

We prove the existence and the uniqueness of a solution of (4.1) in the following proposition.

Proposition 4.2. *Let $g \in H^s(\mathbb{R} \setminus (0,1))$. Assume that the real numbers a, b, c, d satisfy*

$$c = b, \quad b > 0, \quad \max\{a, d\} \leqslant -b, \quad b^2 < (\lambda_1 - a)(\lambda_1 - d). \tag{4.3}$$

Then there exists a unique solution (α, β) in $\big(H^s(\mathbb{R})\big)^2$ of (4.1) in the sense of Definition 4.1, given by

$$\alpha(x) = -\sum_{n\in\mathbb{N}} \frac{1}{\lambda_n - a - \frac{bc}{\lambda_n - d}} (g, \mathcal{N}_s\psi_n)_{L^2(\mathbb{R}\setminus(0,1))}\psi_n(x), \tag{4.4}$$

$$\beta(x) = -\sum_{n\in\mathbb{N}} \frac{c}{\lambda_n - d}\frac{1}{\lambda_n - a - \frac{bc}{\lambda_n - d}} (g, \mathcal{N}_s\psi_n)_{L^2(\mathbb{R}\setminus(0,1))}\psi_n(x). \tag{4.5}$$

Moreover there is a constant $C > 0$ such that

$$\|(\alpha, \beta)\|_{(H^s(\mathbb{R}))^2} \leqslant C\|g\|_{H^s(\mathbb{R}\setminus(0,1))}. \tag{4.6}$$

Remark 4.3. Note that the assumption $\max\{a, d\} \leqslant -b$ with $b \geqslant 0$, implies that $\max\{a, d\} < \lambda_1$ since $\lambda_1 > 0$.

Proof. We prove the proposition in two steps.

Step 1. We show that there is a unique weak solution of (4.1) in $\big(H^s(\mathbb{R})\big)^2$. Let us consider the bilinear form defined for every $U, V \in \big(H^s(\mathbb{R})\big)^2$ by

$$\mathcal{F}(U,V) = \frac{c_{1,s}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(U(x)-U(y))\cdot(V(x)-V(y))}{|x-y|^{1+2s}}\,dx\,dy - \int_0^1 AU\cdot V\ dx.$$

We have that $\mathcal{F}$ is symmetric since the matrix A is symmetric in view of the assumption (4.3). Moreover using the Cauchy-Schwarz inequality, we can write

$$\begin{aligned}
|\mathcal{F}(U,V)| &\leqslant \frac{c_{1,s}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|(U(x)-U(y))\cdot(V(x)-V(y))|}{|x-y|^{\frac12+s}|x-y|^{\frac12+s}}\,dx\,dy + \int_0^1 |AU\cdot V|\,dx \\
&\leqslant \frac{c_{1,s}}{2}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(U(x)-U(y))\cdot(U(x)-U(y))}{|x-y|^{1+2s}}\,dx\,dy\Big)^{1/2} \\
&\quad\times \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(V(x)-V(y))\cdot(V(x)-V(y))}{|x-y|^{1+2s}}\,dx\,dy\Big)^{1/2} \\
&\quad+ (a^2+b^2+c^2+d^2)^{1/2}\Big(\int_0^1 U\cdot U\,dx\Big)^{1/2}\Big(\int_0^1 V\cdot V\,dx\Big)^{1/2} \\
&\leqslant \max\big\{\frac{c_{1,s}}{2}, (a^2+2b^2+d^2)^{1/2}\big\}\|U\|_{(H^s(\mathbb{R}))^2}\|V\|_{(H^s(\mathbb{R}))^2},
\end{aligned}$$

then the bilinear form $\mathcal{F}$ is continuous. In addition, we have

$$\begin{aligned}
\mathcal{F}(U,U) &= \frac{c_{1,s}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(U(x)-U(y))\cdot(U(x)-U(y))}{|x-y|^{1+2s}}\,dx\,dy - \int_0^1 AU\cdot U\,dx \\
&\geqslant \frac{c_{1,s}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(U(x)-U(y))\cdot(U(x)-U(y))}{|x-y|^{1+2s}}\,dx\,dy \\
&\quad+ \min\{-(a+b),-(d+b)\}\int_0^1 U\cdot U\,dx \geqslant 0
\end{aligned} \tag{4.7}$$

in view of assumption (4.3), and $\mathcal{F}$ is coercive. We conclude using the Lax-Milgram Theorem that there exists a unique weak solution $U \in \big(H^s(\mathbb{R})\big)^2$ such that

$$\mathcal{F}(U,V) = 0 \tag{4.8}$$

for every $V \in \big(H^s(\mathbb{R})\big)^2$. Then we show the inequality (4.6). Taking $V = U$ in (4.8) and using (3.3), we obtain

$$\begin{aligned}
&\frac{c_{1,s}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(U(x)-U(y))\cdot(U(x)-U(y))}{|x-y|^{1+2s}}\,dx\,dy \\
&+ \int_{\mathbb{R}\setminus(0,1)} \begin{pmatrix} g\\ 0\end{pmatrix}\cdot\mathcal{N}_sU\,dx - \int_0^1 AU\cdot U\,dx = 0.
\end{aligned}$$

It follows from (4.7) that

$$\begin{aligned}
&\min\big\{\frac{c_{1,s}}{2}, -(a+b), -(d+b)\big\}\|U\|^2_{(H^s(\mathbb{R}))^2} \\
&\leqslant -\int_{\mathbb{R}\setminus(0,1)} \begin{pmatrix} g\\ 0\end{pmatrix}\cdot\mathcal{N}_sU\,dx \\
&\leqslant C\|g\|_{L^2(\mathbb{R}\setminus(0,1))}\|U\|_{(H^s(\mathbb{R}))^2}.
\end{aligned}$$

In the last inequality, we have used the fact that the operator $\mathcal{N}_s : H^s(\mathbb{R}) \to L^2(\mathbb{R}\setminus(0,1))$ is bounded. So we have shown (4.6).

Step 2. We give the series solution of (4.1), that will be useful in the sequel. Let

$$\alpha_n := (\alpha, \psi_n)_{L^2(0,1)}, \quad \beta_n := (\beta, \psi_n)_{L^2(0,1)}. \tag{4.9}$$

Let also $g \in H^s(\mathbb{R} \setminus (0,1))$. Multiplying $(4.1)_1$ and $(4.1)_2$ by ψ_n then integrating by parts over $(0,1)$ while using (3.3), we obtain

$$\begin{gathered}
\int_0^1 \alpha(x)(-d_x^2)^s \psi_n(x)\,dx + \int_{\mathbb{R}\setminus(0,1)} \alpha(x)\mathcal{N}_s\psi_n(x)\,dx \\
= a\int_0^1 \psi_n(x)\alpha(x)\,dx + b\int_0^1 \psi_n(x)\beta(x)\,dx, \\
\int_0^1 \beta(x)(-d_x^2)^s\psi_n(x)\,dx = c\int_0^1 \psi_n(x)\alpha(x)\,dx + d\int_0^1 \psi_n(x)\beta(x)dx.
\end{gathered}$$

Then we find that

$$\begin{gathered}
(\lambda_n - a)\alpha_n - b\beta_n = -(g, \mathcal{N}_s\psi_n)_{L^2(\mathbb{R}\setminus(0,1))} \\
(\lambda_n - d)\beta_n - c\alpha_n = 0
\end{gathered} \tag{4.10}$$

which has a solution

$$\alpha_n = -\frac{1}{\lambda_n - a - \frac{bc}{\lambda_n - d}}(g, \mathcal{N}_s\psi_n)_{L^2(\mathbb{R}\setminus(0,1))}, \tag{4.11}$$

$$\beta_n = -\frac{c}{\lambda_n - d}\frac{1}{\lambda_n - a - \frac{bc}{\lambda_n - d}}(g, \mathcal{N}_s\psi_n)_{L^2(\mathbb{R}\setminus(0,1))}. \tag{4.12}$$

Finally we find (4.4) and (4.5) using (4.9). If g is replaced by $\partial_t^m g := \frac{\partial^m g}{\partial t^m}$, $m \in \mathbb{N}$, in (4.1), then $(\partial_t^m \alpha, \partial_t^m \beta)$ is the unique associated solution. It follows that $(\alpha, \beta) \in \big(C^\infty([0,T]; L^2(0,1))\big)^2$. □

We adopt the following notation: the scalar product $(u^0, \psi_n)_{L^2(0,1)}$ is denoted by u_n^0. Then we have the following existence result.

Theorem 4.4. *Assume that the real numbers a, b, c, d satisfy*

$$c = b, \quad b > 0, \quad \max\{a, d\} \leqslant -b, \quad b^2 < (\lambda_1 - a)(\lambda_1 - d). \tag{4.13}$$

For each $u^0 \in L^2(0,1)$ and $g \in \mathcal{D}([\mathbb{R} \setminus (0,1)] \times (0,T))$, system (1.1) possesses a unique solution (u,v) in $\big(C([0,T]; L^2(0,1))\big)^2$ given by

$$\begin{aligned}
u(x,t) &= \sum_{n\in\mathbb{N}} u_n^0 e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})t}\psi_n(x) \\
&\quad + \sum_{n\in\mathbb{N}} \Big(\int_0^t \big(g(\cdot,\tau), \mathcal{N}_s\psi_n\big)_{L^2(\mathbb{R}\setminus(0,1))} \\
&\quad \times e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)} d\tau\Big)\psi_n(x),
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
v(x,t) &= \sum_{n\in\mathbb{N}} \frac{c}{\lambda_n - d} u_n^0 e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})t}\psi_n(x) \\
&\quad + \sum_{n\in\mathbb{N}} \frac{c}{\lambda_n - d}\Big(\int_0^t \big(g(\cdot,\tau), \mathcal{N}_s\psi_n\big)_{L^2(\mathbb{R}\setminus(0,1))} \\
&\quad \times e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)} d\tau\Big)\psi_n(x).
\end{aligned} \tag{4.15}$$

Proof. Let $g \in \mathcal{D}([\mathbb{R} \setminus (0,1)] \times (0,T))$. We prove the theorem in 3 steps.

Step 1. We prove (4.14) and (4.15). Assume that (u,v) is a solution of (1.1). Setting

$$u_n(t) := (u(\cdot,t), \psi_n)_{L^2(0,1)}, \quad v_n(t) := (v(\cdot,t), \psi_n)_{L^2(0,1)}, \tag{4.16}$$

we can write

$$u(x,t) = \sum_{n\in\mathbb{N}} u_n(t)\psi_n(x), \quad v(x,t) = \sum_{n\in\mathbb{N}} v_n(t)\psi_n(x). \tag{4.17}$$

Multiplying $(1.1)_1$ and $(1.1)_2$ by ψ_n and integrating by parts over $(0,1)$, we obtain

$$\int_0^1 \psi_n(x)(\partial_t u(x,t) + (-d_x^2)^s u(x,t))\,dx = \int_0^1 \psi_n(x)(au(x,t) + bv(x,t))\,dx,$$
$$\int_0^1 \psi_n(x)(-d_x^2)^s v(x,t)\,dx = \int_0^1 \psi_n(x)(cu(x,t) + dv(x,t))\,dx.$$

Using the integration by parts formula (3.3), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_0^1 \psi_n(x)u(x,t)\,dx + \int_0^1 u(x,t)(-d_x^2)^s\psi_n(x)\,dx + \int_{\mathbb{R}\setminus(0,1)} u(x,t)\mathcal{N}_s\psi_n(x)\,dx \\
&= a\int_0^1 \psi_n(x)u(x,t)\,dx + b\int_0^1 \psi_n(x)v(x,t)\,dx,
\end{aligned}$$
$$\int_0^1 v(x,t)(-d_x^2)^s\psi_n(x)\,dx = c\int_0^1 \psi_n(x)u(x,t)\,dx + d\int_0^1 \psi_n(x)v(x,t)\,dx.$$

Hence using the notation (4.16) it follows that

$$u_n'(t) + (\lambda_n - a)u_n(t) - bv_n(t) = -\int_{\mathbb{R}\setminus(0,1)} g(\cdot,t)\mathcal{N}_s\psi_n\,dx, \tag{4.18}$$
$$(\lambda_n - d)v_n(t) - cu_n(t) = 0 \tag{4.19}$$
$$u_n(0) = u_n^0.$$

Then (4.19) gives

$$v_n(t) = \frac{c}{\lambda_n - d}u_n(t). \tag{4.20}$$

Replacing it in (4.18), we find that $u_n(t)$ is solution of the Cauchy problem

$$\begin{gathered}
u_n'(t) + \Big(\lambda_n - a - \frac{bc}{\lambda_n - d}\Big)u_n(t) = -\int_{\mathbb{R}\setminus(0,1)} g(\cdot,t)\mathcal{N}_s\psi_n\,dx \\
u_n(0) = u_n^0.
\end{gathered} \tag{4.21}$$

Now we show that the solution of (4.21) is

$$u_n(t) = u_0^n e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})t} + \int_0^t e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)}\big(g(\cdot,\tau), \mathcal{N}_s\psi_n\big)_{L^2(\mathbb{R}\setminus(0,1))}\,d\tau.$$

Inserting this in (4.20) yields

$$\begin{aligned}
v_n(t) &= \frac{c}{\lambda_n - d}u_0^n e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})t} \\
&\quad + \frac{c}{\lambda_n - d}\int_0^t e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)}(g(\cdot,\tau), \mathcal{N}_s\psi_n)_{L^2(\mathbb{R}\setminus(0,1))}\,d\tau.
\end{aligned} \tag{4.22}$$

Finally we obtain the explicit forms of u and v using (4.17).

Step 2. We prove the existence and the uniqueness solution of an intermediary problem. Let (α, β) be the solution of the Dirichlet problem (4.1). Let also (μ, ν) be the solution of

$$\begin{gathered}
\partial_t \mu + (-d_x^2)^s \mu = -\alpha_t + a\mu + b\nu \text{ in } (0,1) \times (0,T), \\
(-d_x^2)^s \nu = c\mu + d\nu \text{ in } (0,1) \times (0,T), \\
\mu = \nu = 0 \text{ in } [\mathbb{R} \setminus (0,1)] \times (0,T), \\
\mu(x,0) = u^0(x) \text{ in } (0,1).
\end{gathered} \tag{4.23}$$

Note that $\alpha_t \in C^\infty([0,T]; H^s(\mathbb{R}))$ depends on (x,t). We see that $(\alpha, \beta) + (\mu, \nu)$ is solution of (1.1). Thus, using (4.14) and (4.15), the unique weak solution of (4.23) is

$$\begin{aligned}
\mu(x,t) &= -\alpha(x,t) + \sum_{n \in \mathbb{N}} u_n^0 e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})t} \psi_n(x) \\
&\quad + \sum_{n \in \mathbb{N}} \Big(\int_0^t \big(g(\cdot,\tau), \mathcal{N}_s \psi_n\big)_{L^2(\mathbb{R}\setminus(0,1))} e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)} d\tau \Big) \psi_n(x)
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\nu(x,t) &= -\beta(x,t) + \sum_{n \in \mathbb{N}} \frac{c}{\lambda_n - d} u_n^0 e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})t} \psi_n(x) \\
&\quad + \sum_{n \in \mathbb{N}} \frac{c}{\lambda_n - d} \Big(\int_0^t (g(\cdot,\tau), \mathcal{N}_s \psi_n)_{L^2(\mathbb{R}\setminus(0,1))} \\
&\quad \times e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)} d\tau \Big) \psi_n(x).
\end{aligned} \tag{4.25}$$

From (4.11) we obtain

$$\begin{aligned}
&\int_0^t (g(\cdot,\tau), \mathcal{N}_s \psi_n)_{L^2(\mathbb{R}\setminus(0,1))} e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)} d\tau \\
&= -\int_0^t \Big(\lambda_n - a - \frac{bc}{\lambda_n - d} \Big) (\alpha(\cdot,\tau), \psi_n)_{L^2(0,1)} e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)} d\tau \\
&= -(\alpha(x,t), \psi_n)_{L^2(0,1)} + \int_0^t (\alpha_\tau(\cdot,\tau), \psi_n)_{L^2(0,1)} e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)} d\tau.
\end{aligned} \tag{4.26}$$

We obtained the last equality integrating by parts over $[0,t]$. Substituting (4.26) in (4.24) yields

$$\begin{aligned}
\mu(x,t) &= -2\alpha(x,t) + \sum_{n \in \mathbb{N}} u_n^0 e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})t} \psi_n(x) \\
&\quad + \sum_{n \in \mathbb{N}} \Big(\int_0^t (\alpha_\tau(\cdot,\tau), \psi_n)_{L^2(0,1)} e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)} d\tau \Big) \psi_n(x).
\end{aligned} \tag{4.27}$$

Now using (4.12) we obtain

$$
\begin{aligned}
&\int_0^t (g(\cdot,\tau),\mathcal{N}_s\psi_n)_{L^2(\mathbb{R}\setminus(0,1))} e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(t-\tau)}d\tau \\
&= -\frac{\lambda_n-d}{c}\int_0^t \Big(\lambda_n-a-\frac{bc}{\lambda_n-d}\Big)(\beta(\cdot,\tau),\psi_n)_{L^2(0,1)} \\
&\quad \times e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(t-\tau)}d\tau \\
&= \frac{\lambda_n-d}{c}\Big(-(\beta(x,t),\psi_n)_{L^2(0,1)} + \int_0^t (\beta_\tau(\cdot,\tau),\psi_n)_{L^2(0,1)} \\
&\quad \times e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(t-\tau)}d\tau\Big).
\end{aligned}
\tag{4.28}
$$

Substituting (4.28) in (4.25) gives

$$
\begin{aligned}
\nu(x,t) &= -2\beta(x,t) + \sum_{n\in\mathbb{N}} \frac{c}{\lambda_n-d} u_n^0 e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})t}\psi_n(x) \\
&\quad + \sum_{n\in\mathbb{N}}\Big(\int_0^t (\beta_\tau(\cdot,\tau),\psi_n)_{L^2(0,1)} e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(t-\tau)}d\tau\Big)\psi_n(x).
\end{aligned}
\tag{4.29}
$$

Step 3. We prove that (4.14) and (4.15) are convergent in $C([0,T];L^2(0,1))$. We first observe that there exists a constant $C>0$ such that

$$
\sum_{k=1}^n |u_k^0|^2 e^{-2(\lambda_k-a-\frac{bc}{\lambda_k-d})t} \leqslant C\sum_{k=1}^n |u_k^0|^2. \tag{4.30}
$$

For the second part of the series in (4.14), we get in view of (4.26) that it is sufficient to show the convergence of the series

$$
\sum_{n\in\mathbb{N}}\Big(\int_0^t (\alpha_\tau(\cdot,\tau),\psi_n)_{L^2(0,1)} e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(t-\tau)}d\tau\Big)\psi_n(x)
$$

in $C([0,T];L^2(0,1))$. For any $t\in[0,T]$, we have that there exists a constant $C>0$ such that

$$
\begin{aligned}
&\sum_{k=1}^n \Big(\int_0^t (\alpha_\tau(\cdot,\tau),\psi_k)_{L^2(0,1)} e^{-(\lambda_k-a-\frac{bc}{\lambda_k-d})(t-\tau)}d\tau\Big)^2 \\
&\leqslant t^2\sum_{k=1}^n\Big(\int_0^t |(\alpha_\tau(\cdot,\tau),\psi_k)_{L^2(0,1)}|^2 e^{-2(\lambda_k-a-\frac{bc}{\lambda_k-d})(t-\tau)}d\tau\Big) \\
&\leqslant t^2\int_0^t\Big(\sum_{k=1}^n |(\alpha_\tau(\cdot,\tau),\psi_k)_{L^2(0,1)}|^2 e^{-2(\lambda_k-a-\frac{bc}{\lambda_k-d})(t-\tau)}\Big)d\tau \\
&\leqslant Ct^2\|\alpha_t\|^2_{L^\infty(0,T;L^2(0,1))}\int_0^t d\tau \\
&\leqslant Ct^3\|\alpha_t\|^2_{L^\infty(0,T;H^s(\mathbb{R}))} \\
&\leqslant Ct^3\|g_t\|^2_{L^\infty(0,T;L^2(\mathbb{R}\setminus(0,1)))},
\end{aligned}
\tag{4.31}
$$

where we used (4.6). We deduce that the series in u is uniformly convergent in $L^2(0,1)$ in $[0,T]$. In addition, using (4.30) and (4.31), we deduce that there exists a constant $C > 0$ such that for any $t \in [0, T]$,

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \leqslant C\Big(\|u^0\|^2_{L^2(0,1)} + t^3\|g_t\|^2_{L^\infty(0,T;L^2(\mathbb{R}\setminus(0,1)))}\Big).$$

We use the following inequality to prove the convergence of series (4.15):

$$\|v(\cdot,t)\|^2_{L^2(0,1)} \leqslant \Big|\frac{c}{\lambda_1 - d}\Big|^2 \|u(\cdot,t)\|^2_{L^2(0,1)}. \tag{4.32}$$

We deduce that the series in v is also uniformly convergent in $L^2(0,1)$ in $[0,T]$. We conclude that $(u,v) \in \big(C([0,T];L^2(0,1))\big)^2$. Then the inequality

$$\|v(\cdot,t)\|^2_{L^2(\Omega)} \leqslant C\Big(\|u^0\|^2_{L^2(0,1)} + t^3\|g_t\|^2_{L^\infty(0,T;L^2(\mathbb{R}\setminus(0,1)))}\Big)$$

is a consequence of (4.32). We also have that the solution (μ,ν) of (4.23) given by $(u,v)-(\alpha,\beta)$, is in $\big(C([0,T];L^2(0,1))\big)^2$. The proof is complete.. □

For system (1.2) we have the following existence result.

Theorem 4.5. *Assume that the real numbers a, b, c, d satisfy*

$$c = b, \quad b > 0, \quad \max\{a,d\} \leqslant -b, \quad b^2 < (\lambda_1 - a)(\lambda_1 - d). \tag{4.33}$$

For each $u^0 \in L^2(0,1)$ and $h \in \mathcal{D}([\mathbb{R}\setminus(0,1)] \times (0,T))$, system (1.2) possesses a unique solution (u,v) in $\big(C([0,T];L^2(0,1))\big)^2$, given by

$$\begin{aligned}
u(x,t) &= \sum_{n\in\mathbb{N}} u^0_n e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})t}\psi_n(x)\\
&\quad + \sum_{n\in\mathbb{N}} \Big(\int_0^t (h(\cdot,\tau), \mathcal{N}_s\psi_n)_{L^2(\mathbb{R}\setminus(0,1))} e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)} d\tau\Big)\psi_n(x),\\
v(x,t) &= \sum_{n\in\mathbb{N}} \frac{c}{\lambda_n - d} u^0_n e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})t}\psi_n(x)\\
&\quad + \sum_{n\in\mathbb{N}} \frac{c}{\lambda_n - d}\Big(\int_0^t (h(\cdot,\tau), \mathcal{N}_s\psi_n)_{L^2(\mathbb{R}\setminus(0,1))} e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(t-\tau)} d\tau\Big)\psi_n(x).
\end{aligned}$$

Proof. We consider the elliptic Dirichlet problem

$$\begin{gathered}
(-d_x^2)^s\alpha = a\alpha + b\beta \quad \text{in } (0,1),\\
(-d_x^2)^s\beta = c\alpha + d\beta \quad \text{in } (0,1),\\
\alpha = 0, \quad \beta = h \quad \text{in } \mathbb{R}\setminus(0,1),
\end{gathered} \tag{4.34}$$

and prove this theorem as above. □

4.2. **Well-posedness of the dual system.** Now we prove existence and regularity for the dual system (2.1) associated with (1.1) (resp. (1.2)).

Theorem 4.6. *Assume that the real numbers a, b, c, d satisfy*

$$c = b, \quad b > 0, \quad \max\{a,d\} \leqslant -b, \quad b^2 < (\lambda_1 - a)(\lambda_1 - d). \tag{4.35}$$

Let $\varphi^T \in L^2(0,1)$, then system (2.1) has a unique solution (φ,σ) in $(C([0,T]; L^2(0,1)))^2$ given by

$$\begin{aligned} \varphi(x,t) &= \sum_{n\in\mathbb{N}} \varphi_n^T e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(T-t)}\psi_n(x), \\ \sigma(x,t) &= \sum_{n\in\mathbb{N}} \varphi_n^T \frac{b}{\lambda_n - d} e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(T-t)}\psi_n(x). \end{aligned} \tag{4.36}$$

Moreover there is a constant $C>0$ such that for all $t\in[0,T]$,

$$\|(\varphi,\sigma)(x,t)\|_{(L^2(0,1))^2} \leqslant C\|\varphi^T(x)\|_{L^2(0,1)}, \tag{4.37}$$

and for any $t\in[0,T)$, there exist $\mathcal{N}_s\varphi(\cdot,t)$ and $\mathcal{N}_s\sigma(\cdot,t)$ in $L^2(\mathbb{R}\setminus(0,1))$, given by

$$\begin{aligned} \mathcal{N}_s\varphi(x,t) &= \sum_{n\in\mathbb{N}} \varphi_n^T e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(T-t)}\mathcal{N}_s\psi_n(x), \\ \mathcal{N}_s\sigma(x,t) &= \sum_{n\in\mathbb{N}} \varphi_n^T \frac{b}{\lambda_n - d} e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(T-t)}\mathcal{N}_s\psi_n(x). \end{aligned} \tag{4.38}$$

Proof. Let

$$\varphi_n(t) := (\varphi(\cdot,t),\psi_n)_{L^2(0,1)}, \quad \sigma_n(t) := (\sigma(\cdot,t),\psi_n)_{L^2(0,1)}, \tag{4.39}$$

and let

$$\varphi(x,t) = \sum_{n\in\mathbb{N}} \varphi_n(t)\psi_n(x), \quad \sigma(x,t) = \sum_{n\in\mathbb{N}} \sigma_n(t)\psi_n(x). \tag{4.40}$$

Multiplying (2.1) by ψ_n and integrating by parts over $(0,1)$, and using (3.3) we obtain

$$\begin{aligned} &-\frac{\partial}{\partial t}\int_0^1 \psi_n(x)\varphi(x,t)\,dx + \int_0^1 \varphi(x,t)(-d_x^2)^s\psi_n(x)\,dx \\ &= a\int_0^1 \psi_n(x)\varphi(x,t)\,dx + c\int_0^1 \psi_n(x)\sigma(x,t)\,dx, \\ &\int_0^1 \sigma(x,t)(-d_x^2)^s\psi_n(x)\,dx = b\int_0^1 \psi_n(x)\varphi(x,t)\,dx + d\int_0^1 \psi_n(x)\sigma(x,t)\,dx. \end{aligned}$$

Hence using (4.39) it follows that $(\varphi_n(t),\sigma_n(t))$ is solution of the eigenvalues problem

$$\begin{gathered} -\varphi_n'(t) + \lambda_n\varphi_n(t) = a\varphi_n(t) + c\sigma_n(t), \\ \lambda_n\sigma_n(t) = b\varphi_n(t) + d\sigma_n(t), \\ \varphi_n(T) = \varphi_n^T, \end{gathered} \tag{4.41}$$

where $\varphi_n^T = (\varphi^T,\psi_n)_{L^2(0,1)}$. Then system (4.41) can be rewritten in the form

$$\begin{gathered} -\varphi_n'(t) + \Big(\lambda_n - a - \frac{bc}{\lambda_n - d}\Big)\varphi_n(t) = 0, \\ \sigma_n(t) = \frac{b}{\lambda_n - d}\varphi_n(t), \\ \varphi_n(T) = \varphi_n^T. \end{gathered} \tag{4.42}$$

We show that $\varphi_n(t)$ is solution of $(4.42)_1$ with

$$\varphi_n(t) = \varphi_n^T e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(T-t)}, \tag{4.43}$$

and using this in $(4.42)_2$, we have

$$\sigma_n(t) = \varphi_n^T \frac{b}{\lambda_n - d} e^{-(\lambda_n - a - \frac{bc}{\lambda_n - d})(T-t)}. \tag{4.44}$$

Using (4.43) and (4.44) in (4.40) gives the series solution of (2.1). The estimation (4.37) follows from the fact that there exists a constant $C > 0$ such that for all $t \in [0,T]$,

$$\sum_{k=1}^{n} |\varphi_k^T|^2 e^{-2(\lambda_k - a - \frac{bc}{\lambda_k - d})(T-t)} \leqslant C \sum_{k=1}^{n} |\varphi_k^T|^2 = C\|\varphi^T\|^2_{L^2(0,1)}$$

and

$$\|\sigma(\cdot,t)\|^2_{L^2(0,1)} \leqslant \Big|\frac{b}{\lambda_1 - d}\Big|^2 \|\varphi(\cdot,t)\|^2_{L^2(0,1)}.$$

The convergence of the series involved in φ and σ follows from the preceding estimates. The estimation (4.38) is easy to prove. The proof of the theorem is complete. □

5. Null controllability of (1.1) and (1.2)

In this section, we prove Theorem 2.3, the main result of the article. It is a consequence of the following lemma which gives a necessary and sufficient condition for system (1.1) (resp. (1.2)) to be null controllable.

Lemma 5.1. *The system* (1.1) *(resp.* (1.2)*) is null controllable at time $T > 0$ if and only if for any given $u^0 \in L^2(0,1)$, there exists a control $g \in L^2(0,T; H^s(\mathbb{R} \setminus (0,1)))$ (resp. $h \in L^2(0,T; H^s(\mathbb{R} \setminus (0,1)))$) such that the solution (φ,σ) of* (2.1) *satisfies*

$$\int_0^1 u^0(x)\varphi(x,0)\,dx = \int_\omega \int_0^T g\mathcal{N}_s\varphi\,dx\,dt, \tag{5.1}$$

$$\Big(\textit{resp.} \int_0^1 u^0(x)\sigma(x,0)\,dx = \int_\omega \int_0^T h\mathcal{N}_s\sigma\,dx\,dt\Big). \tag{5.2}$$

Proof. Let $g \in L^2(0,T; H^s(\mathbb{R}\setminus(0,1)))$. We prove only (5.1), the proof of (5.2) being similar. Let us multiply $(1.1)_1$ and $(1.1)_2$ respectively by φ and σ, where (φ,σ) is the solution of (2.1). Integrating over $(0,1) \times (0,T)$, we obtain

$$\int_0^1 \int_0^T (\partial_t u + (-d_x^2)^s u)\varphi\,dx\,dt = \int_0^1 \int_0^T (au + bv)\varphi\,dx\,dt,$$
$$\int_0^1 \int_0^T \sigma(-d_x^2)^s v\,dx\,dt = \int_0^1 \int_0^T (cu + dv)\sigma\,dx\,dt.$$

Using the integration by parts formula (3.3), we obtain

$$\begin{aligned}
&\int_0^1 (u(x,T)\varphi(x,T) - u(x,0)\varphi(x,0))dx - \int_0^1 \int_0^T u\partial_t\varphi\,dx\,dt \\
&+ \frac{c_{1,s}}{2} \int_{\mathbb{R}^2} \int_0^T \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{1+2s}}\,dx\,dy\,dt - \int_{\mathbb{R}\setminus(0,1)} \int_0^T \varphi\mathcal{N}_s u\,dx\,dt \\
&= a\int_0^1 \int_0^T u\varphi\,dx\,dt + b\int_0^1 \int_0^T v\varphi\,dx\,dt,
\end{aligned}$$

$$
\begin{aligned}
&\frac{c_{1,s}}{2}\int_{\mathbb{R}^2}\int_0^T \frac{(v(x)-v(y))(\sigma(x)-\sigma(y))}{|x-y|^{1+2s}}\,dx\,dy\,dt - \int_{\mathbb{R}\setminus(0,1)}\int_0^T \sigma\mathcal{N}_s v\,dx\,dt \\
&= c\int_0^1\int_0^T u\sigma\,dx\,dt + d\int_0^1\int_0^T v\sigma\,dx\,dt.
\end{aligned}
$$

Using again the integration by parts formula (3.3), we have

$$
\begin{aligned}
&\int_0^1 (u(x,T)\varphi(x,T)-u(x,0)\varphi(x,0))dx - \int_0^1\int_0^T u\partial_t\varphi\,dx\,dt \\
&+\int_0^1\int_0^T u(-d_x^2)^s\varphi\,dx\,dt + \int_{\mathbb{R}\setminus(0,1)}\int_0^T (u\mathcal{N}_s\varphi - \varphi\mathcal{N}_s u)\,dx\,dt \\
&= a\int_0^1\int_0^T u\varphi\,dx\,dt + b\int_0^1\int_0^T v\varphi\,dx\,dt,
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
&\int_0^1\int_0^T v(-d_x^2)^s\sigma\,dx\,dt + \int_{\mathbb{R}\setminus(0,1)}\int_0^T (v\mathcal{N}_s\sigma - \sigma\mathcal{N}_s v)\,dx\,dt \\
&= c\int_0^1\int_0^T u\sigma\,dx\,dt + d\int_0^1\int_0^T v\sigma\,dx\,dt.
\end{aligned} \tag{5.4}
$$

Then (5.3) and (5.4) reduce to

$$
\begin{aligned}
&\int_0^1 (u(x,T)\varphi^T(x) - u^0(x)\varphi(x,0))dx + c\int_0^1\int_0^T u\sigma\,dx\,dt \\
&+\int_\omega\int_0^T g\mathcal{N}_s\varphi\,dx\,dt \\
&= b\int_0^1\int_0^T v\varphi\,dx\,dt,
\end{aligned} \tag{5.5}
$$

$$
b\int_0^1\int_0^T v\varphi\,dx\,dt = c\int_0^1\int_0^T u\sigma\,dx\,dt. \tag{5.6}
$$

Combining (5.5) and (5.6), it follows that

$$
\int_0^1 (u(x,T)\varphi^T(x) - u^0(x)\varphi(x,0))dx + \int_\omega\int_0^T g\mathcal{N}_s\varphi\,dx\,dt = 0.
$$

Then we can check that $u(x,T)=0$ if and only if

$$
\int_0^1 u^0(x)\varphi(x,0)\,dx = \int_\omega\int_0^T g\mathcal{N}_s\varphi\,dx\,dt
$$

which completes the proof. □

We are now able to prove the main result.

Proof of Theorem 2.3. Let (u,v) be the unique solution of (1.1) and (φ,σ) the unique solution of the adjoint problem (2.1). We prove that identity (5.1) is equivalent to the following inequality for the adjoint system: there is a constant $C>0$ such that

$$
\|\varphi(x,0)\|^2_{L^2(0,1)} \leqslant C\int_\omega\int_0^T |\mathcal{N}_s\varphi(x,t)|^2\,dx\,dt. \tag{5.7}
$$

Following Theorem 4.6, the function φ, where (φ,σ) is the solution of (2.1), is given by

$$\varphi(x,t)=\sum_{n\in\mathbb{N}}\varphi_n^T e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(T-t)}\psi_n(x),$$

and its nonlocal normal derivative is

$$\mathcal{N}_s\varphi(x,t)=\sum_{n\in\mathbb{N}}\varphi_n^T e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(T-t)}\mathcal{N}_s\psi_n(x).$$

Then

$$\int_\omega\int_0^T|\mathcal{N}_s\varphi(x,t)|^2\,dx\,dt=\int_0^T\big\|\sum_{n\in\mathbb{N}}\varphi_n^T\mathcal{N}_s\psi_n(x)e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(T-t)}\big\|^2_{L^2(\omega)}dt.$$

Hence to obtain (5.7) it suffices to prove the estimate

$$\begin{aligned}&\int_0^T\Big\|\sum_{n\in\mathbb{N}}\varphi_n^T\mathcal{N}_s\psi_n(x)e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(T-t)}\Big\|^2_{L^2(\omega)}dt\\&\geqslant C\sum_{n\in\mathbb{N}}\|\varphi_n^T\mathcal{N}_s\psi_n(x)\|^2_{L^2(\omega)}.\end{aligned}\tag{5.8}$$

Indeed, if (5.8) holds, then

$$\int_0^T\Big\|\sum_{n\in\mathbb{N}}\varphi_n^T\mathcal{N}_s\psi_n(x)e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(T-t)}\Big\|^2_{L^2(\omega)}dt\geqslant C\sum_{n\in\mathbb{N}}|\varphi_n^T|^2\|\mathcal{N}_s\psi_n(x)\|^2_{L^2(\omega)}.$$

In view of [20, Lemma 4.2], the norm of $\mathcal{N}_s\psi_n$ in $L^2(\omega)$ is uniformly bounded from below by a strictly positive constant η. This results follows from the unique continuation property stated in Lemma 3.2. In addition using (4.37), the estimation

$$\|(\varphi,\sigma)(x,t)\|_{(L^2(0,1))^2}\leqslant C\|\varphi^T(x)\|_{L^2(0,1)}$$

holds. Then it follows from the preceding inequality the existence of a strictly positive constant C_1 such that

$$\begin{aligned}\int_0^T\Big\|\sum_{n\in\mathbb{N}}\varphi_n^T\mathcal{N}_s\psi_n(x)e^{-(\lambda_n-a-\frac{bc}{\lambda_n-d})(T-t)}\Big\|^2_{L^2(\omega)}dt&\geqslant C\eta^2\sum_{n\in\mathbb{N}}|\varphi_n^T|^2\\&=C\eta^2\|\varphi^T(x)\|^2_{L^2(0,1)}\\&\geqslant C_1\|(\varphi,\sigma)(x,0)\|^2_{(L^2(0,1))^2}\\&\geqslant C_1\|\varphi(x,0)\|^2_{L^2(0,1)}.\end{aligned}$$

Then if (5.8) is true, so does (5.7). We must now prove (5.8). Using the Müntz Theorem (see [16, 17]), an inequality of type (5.8) holds if and only if the series

$$\sum_{n\in\mathbb{N}}\frac{1}{\lambda_n-a-\frac{bc}{\lambda_n-d}}\tag{5.9}$$

is convergent. This series is convergent if and only if the series $\sum_{n\in\mathbb{N}}\frac{1}{\lambda_n}$ is convergent. The eigenvalues of the operator $(-d_x^2)^s$ satisfying (3.2), the series (5.9) converges if and only if $s>1/2$. We proceed similarly for the null controllability results of (1.2).

Now it remains to show the third part of (2.4). Let (u,v) be a solution of (1.1). Using (4.32), we get that if $u(x,T)=0$ in $(0,1)$, we have $\lim_{t\to T^-}\|v(\cdot,t)\|_{L^2(0,1)}=0$, which completes the proof of the main result. □

Conclusion. In this work, null controllability results for fractional one-dimensional systems involving the fractional Laplacian, are stated. To extend these results, it would be interesting to study the problem in the semilinear case, the one where the coefficients are not constant. Extending the results to the finite dimensional case might also be considered in a future work.

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