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GLOBAL ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO QUASILINEAR SCHRÖDINGER EQUATIONS

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Abstract. We are concerned with the existence and blowup of solutions for a class of quasilinear Schrödinger equations. In particular, we examine the combined effect of local type nonlinearity and Hartree type ones, and depending upon different parameter regimes, we find the dominant roles exhibited by these nonlinear effects. We also consider the asymptotic behavior for the global solution and lower bound for the blowup rate of the blowup solution by using pseudo-conformal conservation laws.

1. Introduction

In this article, we consider the quasilinear schrödinger equation:

$$\begin{gathered} iu_t = \Delta u + 2\alpha u|u|^{2\alpha-2}\Delta(|u|^{2\alpha}) - V(x)u + A|u|^{p-1}u + (W * |u|^2)u, \quad t > 0 \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \quad N \geq 3. \end{gathered} \tag{1.1}$$

We assume the following set of conditions:

(C1) $\alpha, p \in Z^+$, $V(x)$ and $W(x)$ are real functions, $V(x) \geq 0$, $V(x) \in \mathfrak{B}^\infty(\mathbb{R}^N)$, $W(x)$ is even, $\partial^K W(x) \in L^1(\mathbb{R}^N)$ for any $K \in Z^+$, and $W(x) = W_1(x) + W_2(x)$, $W_1(x) \in L^{q_1}(\mathbb{R}^N)$, $q_1 > 1$, $W_2(x) \in L^\infty(\mathbb{R}^N)$ and

$$u_0 \in \Lambda := \{v \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} |\nabla(|v|^{2\alpha})|^2 < \infty\},$$

where $\mathfrak{B}^\infty(\mathbb{R}^N)$ denotes the space of all functions in $C^\infty(\mathbb{R}^N)$ such that all partial derivatives are bounded in $\mathbb{R}^N$.

Equations of the form (1.1) have appeared in mathematical physics, in models of superfluid in plasma physics and quantum mechanics; see for example [1, 2, 3, 6, 10, 12, 14, 15, 18, 19, 20, 22]. From the physics point of view, (1.1) obeys the following mass and energy conservation laws, which will be proved in the appendix:

(i) Mass:

$$m(u) = \Big(\int_{\mathbb{R}^N} |u(\cdot,t)|^2 dx\Big)^{1/2} = \Big(\int_{\mathbb{R}^N} |u_0(x)|^2 dx\Big)^{1/2} = m(u_0); \tag{1.2}$$

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(ii) Energy:

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla u|^2 + |\nabla(|u|^{2\alpha})|^2 - G(|u|^2)]dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla u_0|^2 + |\nabla(|u_0|^{2\alpha})|^2 - G(|u_0|^2)]dx = E(u_0), \end{aligned} \tag{1.3}$$

where

$$G(|u|^2) = -V(x)|u|^2 + \frac{2A}{p+1}|u|^{p+1} + \frac{1}{2}(W * |u|^2)|u|^2.$$

There are many interesting topics about (1.1), such as local well-posedness, global well-posedness, and asymptotic behavior for the solution. About the local well-posedness of the solution of (1.1), we have [5, 7, 8, 13, 17] and the references therein. We will analyze the interaction between the local type power nonlinear term and the nonlocal type Hartree nonlinearity. also we examine the individual and combined roles played by these nonlinearities for the global existence and blowup in finite time. First we give a definition.

Definition 1.1. Let $u(x,t)$ be a solution of (1.1). We call $u(x,t)$ global solution if its maximum interval of existence for t is $[0,+\infty)$; while we call $u(x,t)$ the blowup solution if there exists a time $0 < T < +\infty$ such that

$$\lim_{t\to T^-}\int_{\mathbb{R}^N}[|\nabla u(x,t)|^2 + |\nabla(|u|^{2\alpha})|^2)]dx = +\infty. \tag{1.4}$$

There are many results on the existence of global solutions and in blowup phenomena of semilinear Schrödinger equation; we can refer to [4, 9] and the references therein. However, there are only a few works about this topic on quasilinear Schödinger equation. In [11], the authors studied the problem

$$\begin{gathered} i\varphi_t + \Delta\varphi + 2(\Delta|\varphi|^2)\varphi + |\varphi|^{q-2}\varphi = 0, \quad x \in \mathbb{R}^N, \quad t > 0 \\ \varphi(x,0) = \bar{u}_0(x), \quad x \in \mathbb{R}^N \end{gathered} \tag{1.5}$$

and found that the solution of (1.5) will blow up in finite time if $4 + \frac{4}{N} \le q < 2 \cdot 2^*$ under certain assumptions. More general equations like (1.1) were given in [5, 8]. Recently, Song and Wang [21] established results on the global existence and blowup phenomena of quasilinear Schrödinger equation. Our first result gives sufficient conditions for the blowup in finite time.

Theorem 1.2. *Let u be the solution of* (1.1). *Assume* (C1) *and the following conditions hold: $A \ge 0$,*

$$\begin{gathered} 4\alpha - 1 + \frac{4}{N} \le p \le 2\alpha 2^* - 1, \\ [(2\alpha-1)N + 2]W + (x\cdot\nabla W) \le 0, \\ [(2\alpha-1)N + 2]V + x\cdot\nabla V \ge 0, \end{gathered}$$

$E(u_0) \le 0$, $u_0 \in \Lambda$, $xu_0 \in L^2(\mathbb{R}^N)$ and $\Im\int_{\mathbb{R}^N}\bar{u}_0(x\cdot\nabla u_0)dx > 0$. Then u will blow up in finite time.

Our second result is about the sufficient conditions for the existence of a global solution.

Theorem 1.3. *Assume that $u(x,t)$ is the solution of* (1.1) *and the conditions* (C1) *hold. If $\alpha, p \in Z^+$, then u is global solution in each of the following cases:*

(1) $1 < p < 4\alpha - 1 + \frac{4}{N}$ *and* $q_1 > 1$;
(2) $p = 4\alpha - 1 + \frac{4}{N}$, $q_1 > 1$, *and*

$$\frac{2|A|}{p+1}\|u_0\|_{L^2(\mathbb{R}^N)}C_s^{1/2} < 1.$$

From Theorems 1.2 and 1.3, we can say that the power term $|u|^{p-1}u$ helps global solutions if $1 < p < \frac{4}{N} + 4\alpha - 1$. On the other hand, we can see that the power term $|u|^{p-1}u$ helps blowup if $\frac{4}{N} + 4\alpha - 1 < p < 2\alpha 2^* - 1$.

Naturally, an interesting question arises: If one of the power terms and the Hartree term helps the existence of global solutions, and the other one helps blowup in finite time, which one plays the dominant role in the combined effect? To give an answer, we state our third main result.

Theorem 1.4. *Assume that $u(x,t)$ is the solution of (1.1) and the conditions (C1) hold. Moreover, suppose that $4\alpha - 1 + \frac{4}{N} < p < 2\alpha 2^*$, $\frac{4q_1}{2q_1-1} < p+1$, $q_1 > 1$ and there exist a constant $(2\alpha-1)N + 2 < K < \frac{N(p-1)}{2}$ such that*

$$KV(x) + x\cdot\nabla V \geq 0,$$
$$0 \leq KW + x\cdot\nabla W \leq C_1 W \quad \text{for some } C_1.$$

Then u will blow up in finite time if $xu_0 \in L^2(\mathbb{R}^N)$, $\Im\int_{\mathbb{R}^N}\bar{u}_0(x\cdot\nabla u_0)dx > 0$, $\|u_0\|_{L^2}$ small enough, and $E(u_0) < 0$.

Remark 1.5. The assumptions in Theorem 1.4 imply that the power term helps blowup in finite time and the Hartree type term does not help blowup. Theorem 1.4 shows that the term which helps blowup plays the dominant role.

The organization of this article is as follows. In Section 2, we prove the mass and energy conservation laws and some equalities, and prove Theorem 1.2. In Section 3, we prove Theorem 1.3. In Section 4, we prove Theorem 1.4. Section 5 is devoted to asymptotic behavior and blowup rate of solutions. For completeness, in the appendix, we prove the mass and energy conservation laws for this class of quasilinear equations.

2. Preliminaries and the proof of Theorem 1.2

Lemma 2.1. *If u is a solution of (1.1) that exists on the time interval $[0,t]$, then u satisfies*

$$\frac{d}{dt}|u|^2 = \nabla\cdot 2\Im(\bar{u}\nabla u), \quad m(u) = m(u_0); \tag{2.1}$$
$$E(u) = E(u_0); \tag{2.2}$$
$$\frac{d}{dt}\int_{\mathbb{R}^N}|x|^2|u|^2dx = -4\Im\int_{\mathbb{R}^N}\bar{u}(x\cdot\nabla u)dx; \tag{2.3}$$

and

$$\begin{aligned}
&\frac{d}{dt}\Im\int_{\mathbb{R}^N}\bar{u}(x\cdot\nabla u)dx\\
&= -2\int_{\mathbb{R}^N}|\nabla u|^2dx - [(2\alpha-1)N+2]\int_{\mathbb{R}^N}|\nabla(|u|^{2\alpha})|^2dx + \int_{\mathbb{R}^N}(x\cdot\nabla V)|u|^2dx\\
&\quad + \frac{NA(p-1)}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx - \frac{1}{2}\int_{\mathbb{R}^N}[(x\cdot\nabla W)*|u|^2]|u|^2dx.
\end{aligned}$$

The proof of this lemma is given in the appendix.

Proof of Theorem 1.2. Let

$$y(t) = \Im \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx.$$

From Lemma 2.1, we have

$$\begin{aligned}
&\frac{d}{dt} y(t) \\
&= -2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - [(2\alpha - 1)N + 2] \int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx + \int_{\mathbb{R}^N} (x \cdot \nabla V)|u|^2 dx \\
&\quad + \int_{\mathbb{R}^N} \frac{NA(p-1)}{p+1} |u|^{p+1} dx - \frac{1}{2} \int_{\mathbb{R}^N} [(x \cdot \nabla W) * |u|^2]|u|^2 dx \\
&= (2\alpha - 1)N \int_{\mathbb{R}^N} |\nabla u|^2 dx - 2[(2\alpha - 1)N + 2]E(u_0) \\
&\quad + \int_{\mathbb{R}^N} [[(2\alpha - 1)N + 2]V + (x \cdot \nabla V)]|u|^2 dx \\
&\quad + \frac{2A}{p+1} (\frac{N(p-1)}{2} - [(2\alpha - 1)N + 2]) \int_{\mathbb{R}^N} |u|^{p+1} dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} \Big([[(2\alpha - 1)N + 2]W + (x \cdot \nabla W)] * |u|^2\Big)|u|^2 dx.
\end{aligned}$$

Under the assumptions on $V(x)$ and $W(x)$, we have $y'(t) \geq 0$. Consequently, $y(t) > 0$ because $y(0) = \Im \int_{\mathbb{R}^N} \bar{u}_0(x \cdot \nabla u_0) dx > 0$, and

$$\frac{d}{dt} \int_{\mathbb{R}^N} |x|^2 |u|^2 dx = -4\Im \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u) dx = -4y(t) < 0,$$

i.e.

$$\int_{\mathbb{R}^N} |x|^2 |u|^2 dx \leq \int_{\mathbb{R}^N} |x|^2 |u_0|^2 dx := m_0^2 < +\infty.$$

Setting

$$J(t) = \int_{\mathbb{R}^N} |x|^2 |u|^2 dx,$$

we obtain $J'(t) = -4y(t) < -4y(0) < 0$. Consequently,

$$0 \leq J(t) = J(0) + \int_0^t J'(\tau) d\tau < J(0) - 4y(0)t,$$

which implies that the maximum existence interval of time for u is finite, and u will blow up before $\frac{J(0)}{4y(0)}$.

Especially, since $\alpha \in Z^+$, i.e. $\alpha \geq 1$, using the Schwarz's inequality to $y(t)$, we obtain

$$y(t) \leq \Big(\int_{\mathbb{R}^N} |x|^2 |u^2| dx\Big)^{1/2} \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^{1/2} \leq m_0 \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^{1/2}.$$

Therefore,

$$y'(t) \geq (2\alpha - 1)N \int_{\mathbb{R}^N} |\nabla u|^2 dx \geq (2\alpha - 1)N \frac{y^2(t)}{m_0^2}.$$

Integrating, we obtain

$$y(t) \geq \frac{y(0)m_0^2}{m_0^2 - y(0)(2\alpha-1)Nt}, \quad 0 \leq t < \frac{m_0^2}{y(0)(2\alpha-1)N}.$$

That is,

$$\|\nabla u\|_{L^2} \geq \frac{y(0)m_0}{m_0^2 - y(0)(2\alpha N - N)t},$$

and $T = \frac{m_0^2}{y(0)(2\alpha N - N)}$ is the singular point and the solution will blowup before T. □

3. Proof of Theorem 1.3

Recall interpolation inequality

$$\|u\|_{L^r(\mathbb{R}^N)} \leq \|u\|^{\theta}_{L^s(\mathbb{R}^N)} \|u\|^{1-\theta}_{L^t(\mathbb{R}^N)},$$

where

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}.$$

In particular for $r = 1$, we have

$$\theta = \frac{s(t-1)}{t-s}, \quad 1-\theta = \frac{t(1-s)}{t-s}.$$

By the energy conservation law, using Hölder's, Young's, Sobolev's inequalities, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} [|\nabla u|^2 + |\nabla(|u|^{2\alpha})|^2 + V(x)|u|^2]dx \\
&= 2E(u_0) + \int_{\mathbb{R}^N} [\frac{2|A|}{p+1}|u|^{p+1} + \frac{1}{2}(W * |u|^2)|u|^2]dx \\
&\leq 2E(u_0) + \frac{2|A|}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}dx + \frac{1}{2}\|W_2\|_{L^\infty}\|u_0\|^4_{L^2(\mathbb{R}^N)} \\
&\quad + \frac{1}{2}\|W_1\|_{L^{q_1}}\Big(\int_{\mathbb{R}^N} |u|^{\frac{4q_1}{2q_1-1}}dx\Big)^{\frac{2q_1-1}{q_1}} \\
&\leq C + \frac{2|A|}{p+1}\Big(\int_{\mathbb{R}^N}(|u|^{p+1})^{s_1}dx\Big)^{1/\tau_1}\Big(\int_{\mathbb{R}^N}(|u|^{p+1})^{t_1}dx\Big)^{1/\tau_1'} \\
&\quad + \frac{1}{2}\|W_1\|_{L^{q_1}}\Big(\int_{\mathbb{R}^N}(|u|^{\frac{4q_1}{2q_1-1}})^{s_2}dx\Big)^{\frac{2q_1-1}{\tau_2 q_1}}\Big(\int_{\mathbb{R}^N}(|u|^{\frac{4q_1}{2q_1-1}})^{t_2}dx\Big)^{\frac{2q_1-1}{\tau_2' q_1}} \\
&\leq C + \frac{2|A|}{p+1}\|u_0\|^{2/\tau_1}_{L^2} C_s^{1/\tau_1'}\Big\{\int_{\mathbb{R}^N}|\nabla(|u|^{2\alpha})|^2dx\Big\}^{\frac{2^*}{2\tau_1'}} \\
&\quad + \frac{1}{2}\|W_1\|_{L^{q_1}}\|u_0\|^{\frac{4q_1-2}{\tau_2 q_1}}_{L^2} C_s^{\frac{2q_1-1}{\tau_2' q_1}}\Big\{\int_{\mathbb{R}^N}|\nabla(|u|^{2\alpha})|^2dx\Big\}^{\frac{2^*(2q_1-1)}{2\tau_2' q_1}}.
\end{aligned} \tag{3.1}$$

Here

$$s_1 = \frac{2}{p+1}, \quad t_1 = \frac{2\alpha 2^*}{p+1}, \quad s_2 = \frac{2q_1-1}{2q_1}, \quad t_2 = \frac{(2q_1-1)\alpha 2^*}{2q_1},$$
$$\frac{1}{\tau_j} = \frac{t_j-1}{t_j-s_j}, \quad \frac{1}{\tau_j'} = \frac{1-s_j}{t_j-s_j}, \quad j=1,2.$$

While C_s is the best constant in the Sobolev embedding inequality

$$\int_{\mathbb{R}^N} |u|^{2^*} dx \le C_s \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx \Big)^{2^*/2}.$$

Now we discuss inequality (3.1) in two cases when $\alpha, p \in Z^+$.

Case 1: $1 < p < 4\alpha - 1 + \frac{4}{N}$ and $q_1 > 1$. Since $\alpha \ge 1$, we have $\frac{N}{2N\alpha - N + 2} \le 1$, $\frac{2^*}{2\tau_1'} < 1$ and $\frac{2^*(2q_1-1)}{2\tau_2' q_1} < 1$. Then (3.1) can be written as

$$\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla(|u|^{2\alpha})|^2 + V(x)|u|^2 dx \le C(u_0, p, q_1, W_1, W_2, A). \tag{3.2}$$

Case 2: $p = 4\alpha - 1 + \frac{4}{N}$ and $q_1 > 1$. In this case, $\frac{2^*}{2\tau_1'} = 1$, $\frac{2^*(2q_1-1)}{2\tau_2' q_1} < 1$. Note that $\alpha, p \in Z^+$, it needs $N = 4$, i.e., $p = 4\alpha$, then

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla(|u|^{2\alpha})|^2 + V(x)|u|^2 dx \\
&\le C + \frac{2|A|}{p+1} \|u_0\|_{L^2}^{\frac{4}{N}} C_s^{1-\frac{2}{N}} \int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx \\
&= C + \frac{2|A|}{p+1} \|u_0\|_{L^2} C_s^{1/2} \int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx.
\end{aligned}$$

If

$$\frac{2|A|}{p+1} \|u_0\|_{L^2} C_s^{1/2} < 1,$$

we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla(|u|^{2\alpha})|^2 + V(x)|u|^2 dx \le C(u_0, q_1, W_1, W_2, A). \tag{3.3}$$

The proof of Theorem 1.3 is complete.

4. Proof of Theorem 1.4

Besides proving Theorem 1.4, we give an answer to which one plays the dominant role: the power term helps the existence of global solutions, or the Hartree term that helps blowup in finite time.

Proof of Theorem 1.4. Note that $A > 0$, $4\alpha - 1 + \frac{4}{N} < p < 2\alpha 2^*$, $q_1 > 1$, and there exist positive constants $(2\alpha - 1)N + 2 < K < \frac{N(p-1)}{2}$ and $C_1 > 0$ such that

$$0 \le KW + x \cdot \nabla W \le C_1 W.$$

We know that $\frac{N(p-1)}{2} - (2\alpha - 1)N - 2 \ge 0$ and that for any $\epsilon > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big((KW + x \cdot \nabla W) * |u|^2 \Big) |u|^2 dx \\
&\le C_1 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx \\
&\le C_1 \|W_1\|_{L^{q_1}} \Big(\int_{\mathbb{R}^N} |u|^{\frac{4q_1}{2q_1-1}} dx \Big)^{\frac{2q_1-1}{q_1}} + C_1 \|W_2\|_{L^\infty} \|u_0\|_{L^2(\mathbb{R}^N)}^4 \\
&\le C_1 \|W_1\|_{L^{q_1}} \|u_0\|_{L^2}^{\frac{4q_1-2}{\tau q_1}} C_s^{\frac{2q_1-1}{\tau' q_1}} \Big\{ \int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx \Big\}^{\frac{2^*(2q_1-1)}{2\tau' q_1}} \\
&\quad + C_1 \|W_2\|_{L^\infty} \|u_0\|_{L^2(\mathbb{R}^N)}^4
\end{aligned}$$

$$
\begin{aligned}
&\le \epsilon C_1\|W_1\|_{L^{q_1}}\int_{\mathbb{R}^N}|\nabla(|u|^{2\alpha})|^2dx + C(C_1,C_s,\epsilon,\alpha,q_1,N)\|u_0\|_{L^2}^{M_2(q_1,N,\alpha)}\\
&\quad + C_1\|W_2\|_{L^\infty}\|u_0\|^4_{L^2(\mathbb{R}^N)}.
\end{aligned}
$$

Taking

$$
\epsilon = \frac{2[K-(2\alpha-1)N-2]}{C_1\|W_1\|_{L^{q_1}}},
$$

we have

$$
\begin{aligned}
&\frac{d}{dt}y(t)\\
&= (K-2)\int_{\mathbb{R}^N}|\nabla u|^2dx - 2KE(u_0) + [K-(2\alpha-1)N-2]\int_{\mathbb{R}^N}|\nabla(|u|^{2\alpha})|^2dx\\
&\quad + \frac{A[N(p-1)-2K]}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx - \frac{1}{2}\int_{\mathbb{R}^N}\Big[(KW+x\cdot\nabla W)*|u|^2\Big]|u|^2dx\\
&\quad + \int_{\mathbb{R}^N}(KV+x\cdot\nabla V)|u|^2dx\\
&\ge (K-2)\int_{\mathbb{R}^N}|\nabla u|^2dx - 2KE(u_0) + [K-(2\alpha-1)N-2]\int_{\mathbb{R}^N}|\nabla(|u|^{2\alpha})|^2dx\\
&\quad + \frac{A[N(p-1)-2K]}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx - \frac{1}{2}C_1\|W_2\|_{L^\infty}\|u_0\|^4_{L^2(\mathbb{R}^N)}\\
&\quad + \int_{\mathbb{R}^N}(KV+x\cdot\nabla V)|u|^2dx - C(C_1,C_s,\epsilon,\alpha,q_1,N)\|u_0\|_{L^2}^{M_2(q_1,N,\alpha)}\\
&\quad - \frac{\epsilon}{2}C_1\|W_1\|_{L^{q_1}}\int_{\mathbb{R}^N}|\nabla(|u|^{2\alpha})|^2dx\\
&\ge (K-2)\int_{\mathbb{R}^N}|\nabla u|^2dx - 2KE(u_0) + \frac{A[N(p-1)-2K]}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx\\
&\quad + \Big[K-(2\alpha-1)N-2-\frac{\epsilon}{2}C_1\|W_1\|_{L^{q_1}}\Big]\int_{\mathbb{R}^N}|\nabla(|u|^{2\alpha})|^2dx\\
&\quad - \frac{1}{2}C_1\|W_2\|_{L^\infty}\|u_0\|^4_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N}(KV+x\cdot\nabla V)|u|^2dx\\
&\quad - C(C_1,C_s,\epsilon,\alpha,q_1,N)\|u_0\|_{L^2}^{M_2(q_1,N,\alpha)}\\
&= (K-2)\int_{\mathbb{R}^N}|\nabla u|^2dx - 2KE(u_0) + \frac{A[N(p-1)-2K]}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx\\
&\quad - \frac{1}{2}C_1\|W_2\|_{L^\infty}\|u_0\|^4_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N}(KV+x\cdot\nabla V)|u|^2dx\\
&\quad - C(C_1,C_s,\epsilon,\alpha,q_1,N)\|u_0\|_{L^2}^{M_2(q_1,N,\alpha)}.
\end{aligned}
$$

If $E(u_0)<0$ and $\|u_0\|_{L^2}$ are small enough, then

$$
\frac{d}{dt}y(t) \ge (K-2)\int_{\mathbb{R}^N}|\nabla u|^2dx \ge (2\alpha-1)N\int_{\mathbb{R}^N}|\nabla u|^2dx \ge 0.
$$

As in the proof of Theorem 1.2, we can show that the solution will blow up in finite time. □

5. Asymptotic behavior of solutions

In this section, we establish a pseudo-conformal conservation law and consider the asymptotic behavior for the solution.

5.1. Pseudo-conformal conservation law.

Theorem 5.1. *1. Assume that u is the global solution of* (1.1)*, the conditions in* (C1) *hold, and $xu_0 \in L^2(\mathbb{R}^N)$. Then*

$$\begin{aligned} P(t) &= \int_{\mathbb{R}^N} |(x-2it\nabla)u|^2dx + 4t^2\int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2dx + 4t^2\int_{\mathbb{R}^N} V(x)|u|^2dx \\ &\quad - \frac{8t^2A}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}dx - 2t^2\int_{\mathbb{R}^N} (W*|u|^2)|u|^2dx \\ &= \int_{\mathbb{R}^N} |xu_0|^2dx + 4\int_0^t \tau\theta(\tau)d\tau. \end{aligned}$$

2. Assume that u is a blowup solution of (1.1) *with blowup time T, the conditions in* (C1) *hold, and $xu_0 \in L^2(\mathbb{R}^N)$. Then*

$$\begin{aligned} B(t) &:= \int_{\mathbb{R}^N} |(x+2i(T-t)\nabla)u|^2dx + 4(T-t)^2\int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2dx \\ &\quad + 4(T-t)^2\int_{\mathbb{R}^N} V(x)|u|^2dx - \frac{8A(T-t)^2}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}dx \\ &\quad - 2(T-t)^2\int_{\mathbb{R}^N} (W*|u|^2)|u|^2dx \\ &= \int_{\mathbb{R}^N} |(x+2iT\nabla)u_0|^2dx + 4T^2\int_{\mathbb{R}^N} |\nabla(|u_0|^{2\alpha})|^2dx \\ &\quad + 4T^2\int_{\mathbb{R}^N} V(x)|u_0|^2dx - \frac{8AT^2}{p+1}\int_{\mathbb{R}^N} |u_0|^{p+1}dx \\ &\quad - 2T^2\int_{\mathbb{R}^N} (W*|u_0|^2)|u_0|^2dx - 4\int_0^t (T-\tau)\theta(\tau)d\tau. \end{aligned} \tag{5.1}$$

Here

$$\begin{aligned} \theta(t) &= \int_{\mathbb{R}^N} (1-2\alpha)N|\nabla(|u|^{2\alpha})|^2dx + \int_{\mathbb{R}^N} [2V + (x\cdot\nabla V)]|u|^2dx \\ &\quad + \frac{A[N(p-1)-4]}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}dx \\ &\quad - \int_{\mathbb{R}^N} \Big([W + \frac{(x\cdot\nabla W)}{2}] * |u|^2\Big)|u|^2dx. \end{aligned} \tag{5.2}$$

Proof. 1. Assume that u is the global solution of (1.1), $u_0 \in \Lambda$ and $xu_0 \in L^2(\mathbb{R}^N)$. Since

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla u|^2 + |\nabla(|u|^{2\alpha})|^2 + V(x)|u|^2 - \frac{2A}{p+1}|u|^{p+1} - \frac{1}{2}(W*|u|^2)|u|^2]dx \\ &= E(u_0), \end{aligned}$$

we have

$$
\begin{aligned}
P(t) = &\int_{\mathbb{R}^N} |xu|^2 dx + 4t\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx + 4t^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx \\
&+ 4t^2 \int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx - 4t^2 \int_{\mathbb{R}^N} \Big[\frac{2A}{p+1}|u|^{p+1} \\
&+ \frac{1}{2}(W*|u|^2)|u|^2 - V(x)|u|^2\Big] dx \\
= &\int_{\mathbb{R}^N} |xu|^2 dx + 4t\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx + 8t^2 E(u_0).
\end{aligned} \tag{5.3}
$$

Recalling that

$$
\frac{d}{dt}\int_{\mathbb{R}^N} |x|^2|u|^2 dx = -4\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx,
$$

we obtain

$$
\begin{aligned}
P'(t) &= \frac{d}{dt}\int_{\mathbb{R}^N} |xu|^2 dx + 4\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx \\
&\quad + 4t\frac{d}{dt}\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx + 16tE(u_0) \\
&= 4t\frac{d}{dt}\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx + 16tE(u_0) \\
&= 4t\Big\{-2\int_{\mathbb{R}^N} |\nabla u|^2 dx - (2\alpha N - N + 2)\int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx \\
&\quad + \frac{NA(p-1)}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx - \frac{1}{2}\int_{\mathbb{R}^N} [(x\cdot\nabla W)*|u|^2]|u|^2 dx \\
&\quad + \int_{\mathbb{R}^N} (x\cdot\nabla V)|u|^2 dx\Big\} + 8t\int_{\mathbb{R}^N} \Big[|\nabla u|^2 + |\nabla(|u|^{2\alpha})|^2 - \frac{2A}{p+1}|u|^{p+1} dx \\
&\quad - \frac{1}{2}[(x\cdot\nabla W)*|u|^2]|u|^2 + V(x)|u|^2\Big] dx \\
&= 4t\int_{\mathbb{R}^N} \Big[(1-2\alpha)N|\nabla(|u|^{2\alpha})|^2 + \frac{[N(p-1)-4]A}{p+1}|u|^{p+1} \\
&\quad - [(W + \frac{x\cdot\nabla W}{2})*|u|^2]|u|^2\Big] dx + 4t\int_{\mathbb{R}^N} [2V + (x\cdot\nabla V)]|u|^2 dx.
\end{aligned}
$$

Integrating from 0 to t, we obtain

$$
P(t) = P(0) + 4\int_0^t \tau\theta(\tau)d\tau = \int_{\mathbb{R}^N} |xu_0|^2 dx + 4\int_0^t \tau\theta(\tau)d\tau.
$$

That is,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |(x - 2it\nabla)u|^2 dx + 4t^2\int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx + 4t^2\int_{\mathbb{R}^N} V(x)|u|^2 dx \\
&- \frac{8t^2 A}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx - 2t^2\int_{\mathbb{R}^N} (W*|u|^2)|u|^2 dx \\
&= \int_{\mathbb{R}^N} |xu_0|^2 dx + 4\int_0^t \tau\theta(\tau)d\tau,
\end{aligned}
$$

where $\theta(\tau)$ is defined by (5.2).

2. Assume that u is a blowup solution of (1.1), $u_0 \in \Lambda$ and $xu_0 \in L^2(\mathbb{R}^N)$. Using $E(u) = E(u_0)$, we have

$$\begin{aligned} B(t) &= \int_{\mathbb{R}^N} |xu|^2 dx - 4(T-t)\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx \\ &\quad + 4(T-t)^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx + 4(T-t)^2 \int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx \\ &\quad - 4(T-t)^2 \int_{\mathbb{R}^N} \Big[\frac{2A}{p+1}|u|^{p+1} + \frac{1}{2}(W*|u|^2)|u|^2 - V(x)|u|^2\Big] dx \\ &= \int_{\mathbb{R}^N} |xu|^2 dx - 4(T-t)\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx + 8(T-t)^2 E(u_0) \end{aligned}$$

and

$$\begin{aligned} B'(t) &= \frac{d}{dt}\int_{\mathbb{R}^N} |xu|^2 dx + 4\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx \\ &\quad - 4(T-t)\frac{d}{dt}\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx - 16(T-t)E(u_0) \\ &= -4(T-t)\frac{d}{dt}\Im \int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx - 16(T-t)E(u_0) \\ &= -4(T-t)\theta(t). \end{aligned}$$

Integrating from 0 to t, we obtain

$$\begin{aligned} B(t) &= B(0) - 4\int_0^t (T-\tau)\theta(\tau)d\tau \\ &= \int_{\mathbb{R}^N} |xu_0|^2 dx - 4T\frac{d}{dt}\Im \int_{\mathbb{R}^N} \bar{u}_0(x\cdot\nabla u_0)dx + 8T^2 E(u_0) \\ &\quad - 4\int_0^t (T-\tau)\theta(\tau)d\tau, \end{aligned}$$

where $\theta(\tau)$ is defined by (5.2). $\square$

5.2. **Applications of the pseudo-conformal conservation law.** As the application of Theorem 5.1, we have

Theorem 5.2. *Assume that u is the solution of* (1.1)*, $N = 4$ and the conditions in* (C1) *hold. Moreover, suppose that $p = 4\alpha - 1 + \frac{4}{N} = 4\alpha$, $V(x) \geq 0$ and $0 \leq 2V + (x\cdot\nabla V) \leq k_1 V$ for some $k_1 < 2$, $W \leq 0$, $2W + (x\cdot\nabla W) \geq 0$ and $\|u_0\|_{L^2} < 1$ such that $\frac{2A}{p+1}\|u_0\|_{L^2} C_s^{1/2} \leq 1$. Then*

$$\int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx + \int_{\mathbb{R}^N} V(x)|u|^2 - \frac{2A}{p+1}|u|^{p+1} - \frac{1}{2}(W*|u|^2)|u|^2 dx \leq \frac{C}{t^{2-k_1}},$$
$$\lim_{t\to+\infty} \int_{\mathbb{R}^N} |\nabla u|^2 dx = 2E(u_0).$$

Proof. Let u be the global solution of (1.1), $u_0 \in \Lambda$ and $xu_0 \in L^2(\mathbb{R}^N)$, $W(x) \le 0$. and $2W + (x \cdot \nabla W) \ge 0$. Then Theorem 5.1 implies

$$
\begin{aligned}
&4t^2\Big[\int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx + \int_{\mathbb{R}^N} V(x)|u|^2 - \frac{2A}{p+1}|u|^{p+1} - \frac{1}{2}(W * |u|^2)|u|^2 dx\Big] \\
&\le \int_{\mathbb{R}^N} |xu_0|^2 dx + 4\int_0^t \tau \int_{\mathbb{R}^N} \Big[(1-2\alpha)N|\nabla(|u|^{2\alpha})|^2 + \frac{[N(p-1)-4]A}{p+1}|u|^{p+1} \\
&\quad - [(W + \frac{x\cdot\nabla W}{2}) * |u|^2]|u|^2 + [2V + (x\cdot\nabla V)]|u|^2\Big]\, dx\, d\tau.
\end{aligned} \tag{5.4}
$$

Since $p = 4\alpha - 1 + \frac{4}{N} = 4\alpha$, we have

$$
\int_{\mathbb{R}^N} |u|^{p+1} dx \le \|u_0\|_{L^2} C_s^{1/2} \int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx.
$$

Using this inequality in (5.4) we obtain

$$
\begin{aligned}
&4t^2\Big[\Big(1 - \frac{2|A|}{p+1}\|u_0\|_{L^2} C_s^{1/2}\Big)\int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx \\
&+ \int_{\mathbb{R}^N} V(x)|u|^2 + \frac{1}{2}(|W| * |u|^2)|u|^2 dx\Big] \\
&\le \int_{\mathbb{R}^N} |xu_0|^2 dx + 16(2\alpha-1)(\frac{2|A|}{p+1}\|u_0\|_{L^2} C_s^{1/2} - 1)\int_0^t \tau \int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2\, dx\, d\tau \\
&\quad + 4\int_0^t \tau \int_{\mathbb{R}^N} \Big[[2V + (x\cdot\nabla V)]|u|^2\Big]\, dx\, d\tau \\
&\le \int_{\mathbb{R}^N} |xu_0|^2 dx + 4k_1 \int_0^t \tau \int_{\mathbb{R}^N} V(x)|u|^2\, dx\, d\tau.
\end{aligned}
$$

Denoting

$$
A_1(t) = 4\int_0^t \tau \int_{\mathbb{R}^N} V(x)|u|^2\, dx\, d\tau,
$$

we have

$$
A_1'(t) \le \frac{k_1}{t} A_1(t) + \frac{C_0}{t}.
$$

Using the Gronwall's inequality, we obtain

$$
A_1(t) \le t^{k_1}[A_1(1) + C_0(\frac{1}{k_1} - \frac{t^{-k_1}}{k_1})] \le C' t^{k_1}
$$

i.e.,

$$
\int_{\mathbb{R}^N} |\nabla(|u|^{2\alpha})|^2 dx + \int_{\mathbb{R}^N} V(x)|u|^2 - \frac{2A}{p+1}|u|^{p+1} - \frac{1}{2}(W * |u|^2)|u|^2 dx \le \frac{C}{t^{2-k_1}}.
$$

Therefore,

$$
\lim_{t\to+\infty} \int_{\mathbb{R}^N} [|\nabla(|u|^{2\alpha})|^2 + V(x)|u|^2 - \frac{2A}{p+1}|u|^{p+1} - \frac{1}{2}(W * |u|^2)|u|^2] dx = 0.
$$

Noticing that $E(u) = E(u_0)$, we obtain

$$
\lim_{t\to+\infty} \int_{\mathbb{R}^N} |\nabla u|^2 dx = 2E(u_0).
$$

□

6. Appendix

proof of Lemma 2.1. (i) Multiplying (2.1) by $2\bar{u}$, and taking the imaginary part, we obtain

$$\frac{d}{dt}|u|^2 = \nabla[2\Im(\bar{u}\cdot\nabla u)].$$

Integrating over $\mathbb{R}^N \times [0,t]$, we have

$$\int_{\mathbb{R}^N} |u|^2 dx = \int_{\mathbb{R}^N} |u_0|^2 dx,$$

which implies that $m(u) = m(u_0)$.

(ii) Multiplying (2.2) by $2\bar{u}_t$, taking the real part and integrating it over $\mathbb{R}^N$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla(|u|^{2\alpha})|^2 + V(x)|u|^2 \\
&- \frac{2A}{p+1}|u|^{p+1} - \frac{1}{2}(W*|u|^2)|u|^2 dx\Big] = 0,
\end{aligned}$$

which implies that $E(u) = E(u_0)$.

(iii) Multiplying $\frac{d}{dt}|u|^2$ by $|x|^2$, integrating over $\mathbb{R}^N$ by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^N} |x|^2|u|^2 dx = \int_{\mathbb{R}^N} |x|^2\nabla\cdot[2\Im(\bar{u}\nabla u)]dx = -4\Im\int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx.$$

(iv) Let $u(t,x) = a(t,x) + ib(t,x)$. We have

$$\begin{aligned}
&\frac{d}{dt}\Im\int_{\mathbb{R}^N} \bar{u}(x\cdot\nabla u)dx \\
&= N\int_{\mathbb{R}^N}(a_t b - b_t a)dx - \int_{\mathbb{R}^N}\sum_{k=1}^N[\nabla b\cdot\nabla(x_k\cdot b_{x_k}) + \nabla a\cdot\nabla(x_k\cdot a_{x_k})]dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N}\sum_{k=1}^N x_k(|u|^2)_{x_k}\cdot 2\alpha|u|^{2\alpha-2}\Delta(|u|^{2\alpha})dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N}\sum_{k=1}^N\big[-x_k(|u|^2)_{x_k}V(x) + Ax_k(|u|^2)_{x_k}|u|^{p-1} + x_k(|u|^2)_{x_k}(W*|u|^2)\big]dx \\
&= -N\int_{\mathbb{R}^N}|\nabla u|^2 dx + N\int_{\mathbb{R}^N}2\alpha|u|^2|u|^{2\alpha-2}\Delta(|u|^{2\alpha})dx - N\int_{\mathbb{R}^N}V(x)|u|^2 dx \\
&\quad + N\int_{\mathbb{R}^N}[A|u|^{p-1}|u|^2 + (W*|u|^2)|u|^2]dx + (N-2)\int_{\mathbb{R}^N}[|\nabla u|^2 + |\nabla(|u|^{2\alpha})|^2]dx \\
&\quad - N\int_{\mathbb{R}^N}[\frac{2A}{p+1}|u|^{p+1} + (W*|u|^2)|u|^2 - V(x)|u|^2]dx \\
&\quad + \int_{\mathbb{R}^N}(x\cdot\nabla V)|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N}[(x\cdot\nabla W)*|u|^2]|u|^2 dx \\
&= -2\int_{\mathbb{R}^N}|\nabla u|^2 dx - [(2\alpha-1)N+2]\int_{\mathbb{R}^N}|\nabla(|u|^{2\alpha})|^2 dx + \int_{\mathbb{R}^N}(x\cdot\nabla V)|u|^2 dx \\
&\quad + \int_{\mathbb{R}^N}\frac{NA(p-1)}{p+1}|u|^{p+1}dx - \frac{1}{2}\int_{\mathbb{R}^N}[(x\cdot\nabla W)*|u|^2]|u|^2 dx.
\end{aligned}$$

□

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