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# EXISTENCE OF SOLUTIONS TO FRACTIONAL HAMILTONIAN SYSTEMS WITH LOCAL SUPERQUADRATIC CONDITIONS

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Abstract. In this article, we study the existence of solutions for the fractional Hamiltonian system

$$ {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u(t)) + L(t)u(t) = \nabla W(t, u(t)), \\ u \in H^\alpha(\mathbb{R}, \mathbb{R}^N), $$

where ${}_tD^\alpha_\infty$ and ${}_{-\infty}D^\alpha_t$ are the Liouville-Weyl fractional derivatives of order $1/2 < \alpha < 1$, $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix-valued function, which is unnecessarily required to be coercive, and $W \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$ satisfies some kind of local superquadratic conditions, which is rather weaker than the usual Ambrosetti-Rabinowitz condition.

## 1. Introduction

Fractional differential equations including both ordinary and partial ones are applied in mathematical modeling of some processes in physics, mechanics, chemistry, economics and bioengineering; see [1, 6, 7, 14, 16, 11, 24] and the references therein. Indeed, the associated fractional-order differential operators of these equations admit the characteristic of nonlocal behavior, which can provide a more realistic and practical description of these processes than the usual integer-order differential operators. Therefore, the theory of fractional differential equations is an area intensively developed during the last decades.

In recent years, fractional differential equations including both left and right fractional derivatives are also gradually investigated. Apart from their possible applications, the research of these equations is a relatively new and interesting field in the theory of fractional differential equations. Some early works on this topic can be founded in papers [2, 5, 10] and their references.

In 2012, Jiao and Zhou [8] showed the existence of solutions for the fractional boundary value problem

$$ {}_tD^\alpha_T({}_0D^\alpha_t u(t)) = \nabla W(t, u(t)), \quad \text{a.e.} t \in [0, T], \\ u(0) = u(T) = 0, $$

where ${}_tD^\alpha_T$ and ${}_0D^\alpha_t$ are the right and left Riemann-Liouville fractional derivatives of order $\alpha \in [1/2, 1]$. Inspired by this work, in [18], the author considered the

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fractional Hamiltonian system

$$\begin{gathered} {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u(t)) + L(t)u(t) = \nabla W(t,u(t)), \\ u \in H^\alpha(\mathbb{R},\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where ${}_tD^\alpha_\infty$ and ${}_{-\infty}D^\alpha_t$ are the Liouville-Weyl fractional derivatives of order $1/2 < \alpha < 1$, $L \in C(\mathbb{R},\mathbb{R}^{N\times N})$ is a symmetric matrix-valued function, $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, and $\nabla W(t,u)$ denotes the gradient of $W(t,u)$ with respect to $u$. To be more precise, he showed that the fractional Hamiltonian system (1.1) possesses at least one nontrivial solution under the following assumptions:

(A1) There exists an $l \in C(\mathbb{R},(0,\infty))$ such that $l(t) \to +\infty$ as $t \to \infty$ and

$$(L(t)u,u) \ge l(t)|u|^2, \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}^N.$$

(A2) There exists a constant $\mu > 2$ such that

$$0 < \mu W(t,u) \le (\nabla W(t,u),u), \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}^N\backslash\{0\}.$$

(A3) $|\nabla W(t,u)| = o(|u|)$ as $u \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(A4) There exists $\overline{W} \in C(\mathbb{R}^N,\mathbb{R})$ such that

$$|W(t,u)| + |\nabla W(t,u)| \le |\overline{W}(u)|, \quad \forall t \in \mathbb{R},\, u \in \mathbb{R}^N.$$

Subsequently, the existence and multiplicity of solutions for the fractional Hamiltonian system (1.1) have been extensively investigated in many papers, see [21, 13, 22, 17, 23, 25, 4, 12, 20, 3] and the references therein. However, we note that in almost all these papers but [17, 4, 25], $L$ is required to satisfy either the coercivity condition (A1) or the uniform positive-definiteness condition

(A5) there exists $b_0 > 0$ such that

$$(L(t)u,u) \ge b_0|u|^2, \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}^N.$$

Besides, some of them (see [21, 13, 23, 25]) dealt with the case where $W$ satisfies the well-known Ambrosetti-Rabinowitz condition (A2), which is more restrictive than the following weaker superquadratic condition

(A6) $\lim_{|u|\to\infty} W(t,u)/|u|^2 = +\infty$ uniformly with respect to $t \in \mathbb{R}$.

Then, more papers were devoted to the case where $W$ satisfies the weaker superquadratic growth condition (A6) and various additional technical conditions, see, [17, 3, 4, 12, 20].

In the recent paper [19], the author obtained the existence of nontrivial homoclinic solutions for the following second-order Hamiltonian system without the Ambrosetti-Rabinowitz condition (A2) on $W$.

$$\ddot{u} - L(t)u + \nabla W(t,u) = 0. \tag{1.2}$$

This second-order Hamiltonian system can be viewed as a special case of the fractional Hamiltonian system (1.1) with $\alpha = 1$. More precisely, in [19], $W$ is only required to satisfy the weaker superquadratic condition (A6) locally with respect to $t$ (see (W4) in [19]). Motivated by the above results, we study the existence of solutions for the fractional Hamiltonian system (1.1) when $L$ is not required to be coercive, and $W$ satisfies some kind of local superquadratic conditions similar to that in [19]. Before presenting our hypotheses, we introduce the following notation. For two $N \times N$ symmetric matrices $M_1$ and $M_2$, we say that $M_1 \ge M_2$ if

$$\min_{u\in\mathbb{R}^N,|u|=1} (M_1 - M_2)u \cdot u \ge 0$$

and that $M_1 \ngeq M_2$ if $M_1 \geq M_2$ does not hold.

Now we make the following asumptions:

(A7) There exists $l_0 > 0$ such that $(L(t)u, u) \geq l_0|u|^2$ for all $t \in \mathbb{R}$ and all $u \in \mathbb{R}^N$.

(A8) There exists a constant $r_0 > 0$ such that
$$\lim_{|s|\to\infty} \text{meas}\left(\{t \in (s - r_0, s + r_0) : L(t) \ngeq MI_N\}\right) = 0, \quad \forall M > 0,$$
where meas denotes the Lebesgue measure in $\mathbb{R}$ and $I_N$ is the identity matrix in $\mathbb{R}^N$.

(A9) $W(t, 0) \equiv 0$ and $\nabla W(t, u) = o(|u|)$ as $u \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(A10) There exists $\overline{W} \in C(\mathbb{R}^N, \mathbb{R}^+)$ such that
$$|W(t, u)| + |\nabla W(t, u)| \leq \overline{W}(u), \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}^N.$$

(A11) There exists a constant $K_0 > 0$ such that $W(t, u) \geq 0$ for all $(t, u) \in \mathbb{R}\times\mathbb{R}^N$ with $|u| \geq K_0$.

(A12) There exist $b_1, b_2 \in \mathbb{R}$ $(b_1 < b_2)$ such that $\lim_{|u|\to\infty} |W(t, u)|/|u|^2 = \infty$ uniformly with respect to $t \in (b_1, b_2)$.

(A13) $\widetilde{W}(t, u) \geq 0$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$, and there exists a nonnegative function $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{s\to+\infty} h(s) = \infty$ such that
$$\frac{|W(t, u)|}{|u|^2} \geq \frac{1}{4\beta_2^2} \quad \text{implies} \quad |W(t, u)| \leq \frac{|u|^2}{h(|u|)}\widetilde{W}(t, u),$$
where $\widetilde{W}(t, u) = (\nabla W(t, u), u) - 2W(t, u)$, and $\beta_2$ is the embedding constant given by (2.8) in the next section.

Our main result reads as follows.

**Theorem 1.1.** *Suppose that* (A7)–(A13) *are satisfied. Then the fractional Hamiltonian system* (1.1) *possesses a nontrivial solution.*

**Remark 1.2.** In Theorem 1.1, $W$ is allowed to be sign-changing and only required to be superquadratic at infinitely with respect to $u$ when $t$ belongs to some finite interval. This is in sharp contrast with the aforementioned references. To the best of our knowledge, there is no literature concerning the existence of solutions for the fractional Hamiltonian system (1.1) in this situation.

**Remark 1.3.** Evidently, condition (A12) is weaker than the usual superquadratic condition (A6). Meanwhile, it is easy to check that $W$ will satisfy our conditions (A11)–(A13) with $h(s) = (\mu - 2)s^2$ if it satisfies the well-known Ambrosetti-Rabinowitz condition (A2). Besides, our conditions (A7) and (A8) are also weaker than the coercivity conditions (A1) as well as the conditions ($L_1$) and ($L_2$) in [25]. Hence, both [18, Theorem 1.1] and the existence result [25, Theorem 1.1] are covered by Theorem 1.1. Indeed, there are many functions $W$ which satisfy the conditions of Theorem 1.1 but do not satisfy the corresponding conditions of the related results in [21, 13, 22, 17, 23, 25, 4, 12, 20, 3]. For example, let
$$W(t, u) = (|\cos t| + \cos t)|u|^2 \ln(1 + |u|^2), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N.$$
Then it is easy to see that $W$ satisfies the conditions of Theorem 1.1 by choosing $b_1 = -\pi/2$, $b_2 = \pi/2$, $\overline{W}(u) = 4|u|^3(1 + |u|^2)^{-1} + 2(|u|^2 + 2|u|)\ln(1 + |u|^2)$ and

$h(s) = s^4/[(1+s^2)\ln(1+s^2)]$. However, it does not satisfy neither (A6) nor (A2) since $W(\pi, u) \equiv 0$ for all $u \in \mathbb{R}^N$. Also let

$$L(t) = \begin{cases} [(k^2+1)^2(t-k)+m_0]I_N, & k \le t < k+\frac{1}{k^2+1}, \\ [(k^2+1)^2+m_0]I_N, & k+\frac{1}{k^2+1} \le t < k+\frac{k^2}{k^2+1}, \\ [(k^2+1)^2(k+1-t)+m_0]I_N, & k+\frac{k^2}{k^2+1} \le t < k+1, \end{cases}$$

where $k \in \mathbb{Z}$, $m_0 \in \mathbb{R}$ and $I_N$ is the $N \times N$ identity matrix. Evidently, $L$ satisfies our conditions (A7) and (A8) but does not satisfy the corresponding conditions in [18, Theorem 1.1] or [25, Theorem 1.1].

## 2. Preliminary results

In this section, we present some preliminaries of fractional calculus (cf. [16, 11]). The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as

$$ {}_{-\infty}I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-\xi)^{\alpha-1} u(\xi) d\xi, \tag{2.1}$$

$$ {}_xI_\infty^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (\xi-x)^{\alpha-1} u(\xi) d\xi. \tag{2.2}$$

The Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals

$$ {}_{-\infty}D_x^\alpha u(x) = \frac{d}{dx}({}_{-\infty}I_x^{1-\alpha} u(x)), \tag{2.3}$$

$$ {}_xD_\infty^\alpha u(x) = -\frac{d}{dx}({}_xI_\infty^{1-\alpha} u(x)). \tag{2.4}$$

Expression (2.3) and (2.4) can be written in an alternative form as

$$ {}_{-\infty}D_x^\alpha u(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{u(x)-u(x-\xi)}{\xi^{\alpha+1}} d\xi, \tag{2.5}$$

$$ {}_xD_\infty^\alpha u(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{u(x)-u(x+\xi)}{\xi^{\alpha+1}} d\xi. \tag{2.6}$$

Next we will briefly introduce some fractional spaces (see [7, 18] for more details). For $\alpha > 0$, define the semi-norm

$$|u|_{I_{-\infty}^\alpha} = \|{}_{-\infty}D_x^\alpha u\|_{L^2}$$

and the norm

$$\|u\|_{I_{-\infty}^\alpha} = \Big(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^\alpha}^2\Big)^{1/2}.$$

Let

$$I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^N)}^{\|\cdot\|_{I_{-\infty}^\alpha}},$$

where $C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ denotes the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^N$ with vanishing property at infinity.

Also we can define the fractional Sobolev space $H^\alpha(\mathbb{R}, \mathbb{R}^N)$ in terms of the Fourier transform. Recall that the Fourier transform $\widehat{u}(\tau)$ of $u(x)$ is defined by

$$\widehat{u}(\tau) = \int_{-\infty}^\infty e^{-ix\tau} u(x) dx.$$

For $0 < \alpha < 1$, define the semi-norm

$$|u|_\alpha = \||\tau|^\alpha \widehat{u}\|_{L^2}$$

and the norm

$$\|u\|_\alpha = \left(\|u\|_{L^2}^2 + |u|_\alpha^2\right)^{1/2},$$

and let

$$H^\alpha(\mathbb{R}, \mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^N)}^{\|\cdot\|_\alpha}.$$

We note that a function $u \in L^2(\mathbb{R}, \mathbb{R}^N)$ belongs to $I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^N)$ if and only if $|\tau|^\alpha \widehat{u} \in L^2(\mathbb{R}, \mathbb{R}^N)$. Indeed,

$$|u|_{I_{-\infty}^\alpha} = \||\tau|^\alpha \widehat{u}\|_{L^2}.$$

Hence, we can say that $I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^N)$ and $H^\alpha(\mathbb{R}, \mathbb{R}^N)$ are equivalent with equivalent norms.

Let $C(\mathbb{R}, \mathbb{R}^N)$ denote the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^N$, then we have the following lemma.

**Lemma 2.1** ([18, Theorem 2.1])**.** *If $\alpha > 1/2$, then $H^\alpha(\mathbb{R}, \mathbb{R}^N) \subset C(\mathbb{R}, \mathbb{R}^N)$ and there is a constant $C_\alpha$ such that*

$$\|u\|_{L^\infty} = \sup_{x \in \mathbb{R}} |u(x)| \le C_\alpha \|u\|_\alpha, \quad \forall u \in H^\alpha(\mathbb{R}, \mathbb{R}^N). \tag{2.7}$$

**Remark 2.2.** From Lemma 2.1, we know that $H^\alpha(\mathbb{R}, \mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}, \mathbb{R}^N)$ for all $q \in [2, \infty)$, since

$$\int_{\mathbb{R}} |u(t)|^q dx \le \|u\|_{L^\infty}^{q-2} \|u\|_{L^2}^2 \le C_\alpha^q \|u\|_\alpha^q, \quad \forall u \in H^\alpha(\mathbb{R}, \mathbb{R}^N),$$

where $C_\alpha$ is the embedding constant given in (2.7).

Now we can introduce the following fractional space, which will serve as the variational space for the fractional Hamiltonian system (1.1). Define

$$X^\alpha = \Big\{u \in H^\alpha(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t) \cdot u(t) dt < \infty\Big\},$$

then $X^\alpha$ is a reflexive and separable Hilbert space equipped with the inner product

$$\langle u, v\rangle_{X^\alpha} = \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t),\ {}_{-\infty}D_t^\alpha v(t)) + L(t)u(t)v(t) dt, \quad \forall u, v \in X^\alpha,$$

and the corresponding norm is

$$\|u\|_{X^\alpha}^2 = \langle u, u\rangle_{X^\alpha}, \quad \forall u \in X^\alpha.$$

**Lemma 2.3** ([18, Lemma 2.3])**.** *Suppose that* (A7) *is satisfied, then $X^\alpha$ is continuously embedded into $H^\alpha(\mathbb{R}, \mathbb{R}^N)$.*

**Remark 2.4.** From Lemma 2.1, Remark 2.2 and Lemma 2.3, we know that $X^\alpha$ is continuously embedded into $L^q(\mathbb{R}, \mathbb{R}^N)$ for all $q \in [2, \infty]$. Hence, for all $q \in [2, \infty]$, there exists $\beta_q > 0$ such that

$$\|u\|_{L^q} \le \beta_q \|u\|_{X^\alpha} \quad \forall u \in X^\alpha. \tag{2.8}$$

We further have the following compact embedding result.

**Lemma 2.5.** *Suppose that* (A7) *and* (A8) *hold. Then $X^\alpha$ is compactly embedded into $L^2(\mathbb{R}, \mathbb{R}^N)$.*

*Proof.* The proof for the case $\alpha = 1$ was given in [9, Lemma 2.2]. Here we will use the similar skill to give the proof for our case $1/2 < \alpha < 1$. Let $\{u_k\} \subset X^\alpha$ be a sequence such that $u_k \rightharpoonup u$ in $X^\alpha$. We need to prove that $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^N)$. Suppose, without loss of generality, that $u_k \rightharpoonup 0$ in $X^\alpha$. The Sobolev embedding theorem implies $u_k \to 0$ in $L^2_{loc}(\mathbb{R}, \mathbb{R}^N)$. Thus it suffices to show that, for any $\epsilon > 0$, there is $r_1 > 0$ such that $\int_{\mathbb{R}\setminus(-r_1,r_1)} |u_k|^2 dt < \epsilon$. For any $s \in \mathbb{R}$, we denote by $\mathcal{B}_{r_0}(s)$ the interval in $\mathbb{R}$ centered at $s$ with radius $r_0$, i.e., $\mathcal{B}_{r_0}(s) := (s - r_0, s + r_0)$, where $r_0$ is the constant given in (A8). Let $\{s_i\} \subset \mathbb{R}$ be a sequence of points such that $\mathbb{R} = \cup_{i=1}^{\infty} \mathcal{B}_{r_0}(s_i)$ and each $t \in \mathbb{R}$ is contained in at most two such intervals. For any $r_1 > 0$ and $M > 0$, let

$$\mathcal{P}(r_1, M) = \{t \in \mathbb{R} \setminus (-r_1, r_1) : L(t) \geq MI_n\},$$
$$\mathcal{Q}(r_1, M) = \{t \in \mathbb{R} \setminus (-r_1, r_1) : L(t) \ngeq MI_n\}.$$

Then

$$\int_{\mathcal{P}(r_1,M)} |u_k|^2 dt \leq \frac{1}{M} \int_{\mathcal{P}(r_1,M)} (L(t)u_k, u_k) dt \leq \frac{1}{M} \int_{\mathbb{R}} (L(t)u_k, u_k) dt,$$

and this can be made arbitrarily small by choosing $M$ large. Besides, for fixed $M > 0$,

$$\begin{aligned}
\int_{\mathcal{Q}(r_1,M)} |u_k|^2 dt &\leq \sum_{i=1}^{\infty} \int_{\mathcal{Q}(r_1,M)\cap\mathcal{B}_{r_0}(s_i)} |u_k|^2 dt \\
&\leq \sum_{i=1}^{\infty} \Big( \int_{\mathcal{Q}(r_1,M)\cap\mathcal{B}_{r_0}(s_i))} |u_k|^4 dt \Big)^{1/2} \big( \operatorname{meas}(\mathcal{Q}(r_1, M) \cap \mathcal{B}_{r_0}(s_i)) \big)^{1/2} \\
&\leq \varepsilon_{r_1} \sum_{i=1}^{\infty} \Big( \int_{\mathcal{B}_{r_0}(s_i)} |u_k|^4 dt \Big)^{1/2} \\
&\leq \varepsilon_{r_1} \sum_{i=1}^{\infty} \int_{\mathcal{B}_{r_0}(s_i)} |u_k|^4 dt \\
&\leq 2\varepsilon_{r_1} \int_{\mathbb{R}} |u_k|^4 dt \\
&\leq 2\beta_4^4 \epsilon_{r_1} \|u_k\|_{X^\alpha}^4,
\end{aligned}$$

where $\varepsilon_{r_1} = \sup_{i\in\mathbb{N}} (\operatorname{meas}(\mathcal{Q}(r_1, M) \cap \mathcal{B}_{r_0}(s_i)))^{1/2}$ and $\beta_4$ is the embedding constant given in (2.8). By (A8), $\varepsilon_{r_1} \to 0$ as $r_1 \to \infty$. Noting that $\{u_k\}$ is bounded in $X^\alpha$, we can make this term small by choosing $r_1$ large. This completes the proof. □

With the help of Lemma 2.5, we can prove the following lemma.

**Lemma 2.6.** *Suppose that* (A7)–(A10) *are satisfied. If $u_k \rightharpoonup u$ in $X^\alpha$, then*

$$\int_{\mathbb{R}} |\nabla W(t, u_k) - \nabla W(t, u)|^2 dt \to 0 \quad \text{as } k \to \infty.$$

*Proof.* The proof is similar to that in [18, Lemma 2.7], and we give it here for the readers' convenience. Assume that $u_k \rightharpoonup u$ in $X^\alpha$. Then there exists a constant $d_1 > 0$ such that, by Banach-Steinhaus theorem and (2.7),

$$\sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty} \leq d_1, \quad \|u\|_{L^\infty} \leq d_1. \tag{2.9}$$

By (A9), there exists $0 < \delta_1 < d_1$ such that

$$|\nabla W(t,u)| \le |u|, \quad \forall t \in \mathbb{R},\ |u| < \delta_1. \tag{2.10}$$

By (A10), there is $K_1 > 0$ such that

$$|\nabla W(t,u)| \le K_1, \quad \forall t \in \mathbb{R},\ \delta_1 \le |u| \le d_1. \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$|\nabla W(t,u)| \le d_2|u|, \quad \forall t \in \mathbb{R},\ |u| \le d_1, \tag{2.12}$$

where $d_2 = \max\{K_1/\delta_1, 1\}$. Since, by Lemma 2.5, $u_k \to u$ in $L^2(\mathbb{R},\mathbb{R}^N)$, passing to a subsequence if necessary, we may assume that

$$u_k \to u \text{ a.e. in } \mathbb{R} \quad \text{and} \quad \sum_{k=1}^{\infty} \|u_k - u\|_{L^2} < \infty,$$

then

$$v := \sum_{k=1}^{\infty} |u_k - u| \in L^2(\mathbb{R},\mathbb{R}^N).$$

It follows from (2.9) and (2.12) that

$$\begin{aligned}
\int_{\mathbb{R}} |\nabla W(t,u_k) - \nabla W(t,u)|^2 dt &\le d_2^2 \int_{\mathbb{R}} (|u_k| + |u|)^2 dt \\
&\le d_2^2 \int_{\mathbb{R}} (|u_k - u| + 2|u|)^2 dt \\
&\le 2d_2^2 \int_{\mathbb{R}} (|u_k - u|^2 + 4|u|^2) dt \\
&\le 2d_2^2 \int_{\mathbb{R}} (v^2 + 4|u|^2) dt.
\end{aligned}$$

Thus, the proof will be complete by using the Lebesgue's convergence theorem. □

We will use the following well-known mountain pass theorem to prove Theorem 1.1 in the next section.

**Theorem 2.7** ([15, Theorem 2.2]). *Let $E$ be a real Banach space and functional $I \in C^1(E,\mathbb{R})$ satisfying the Palais-Smale condition. Suppose that $I(0) = 0$ and*

(1) *There exist constants $\rho, \eta > 0$ such that $\inf I\big|_{\partial B_\rho} \ge \eta$,*
(2) *There exists an $e \in E \backslash \overline{B}_\rho$ such that $I(e) \le 0$.*

*Then, $I$ possesses a critical value $c \ge \eta$. Moreover $c$ can be characterized as*

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u), \tag{2.13}$$

*where $\Gamma = \{g \in C([0,1],E) : g(0) = 0, g(1) = e\}$.*

**Remark 2.8.** We recall that any sequence $\{u_k\} \subset E$ satisfying

$$\sup_{k \in \mathbb{N}} |I(u_k)| < +\infty, \quad \|I'(u_k)\|_{E^*}(1 + \|u_k\|_E) \to 0 \quad \text{as } k \to +\infty$$

is called a Cerami sequence. If any Cerami sequence $\{u_k\}$ has a convergent subsequence, we say that $I$ satisfies the Cerami condition. It is worth to point out that Theorem 2.7 still holds true under the Cerami condition, since a similar deformation lemma can be proved with the usual Palais-Smale condition replaced by the Cerami condition. In the next section, we will use the version of Theorem 2.7 under the Cerami condition to give the proof of our main result.

## 3. Proof of the main result

To prove Theorem 1.1 via variational methods, we first define the variational functional $I$ on $X^\alpha$ associated with the fractional Hamiltonian system (1.1) by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \Big[\frac{1}{2}|{}_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t),u(t)) - W(t,u(t))\Big]dt \\ &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t,u(t))dt. \end{aligned} \tag{3.1}$$

Under the conditions of Theorem 1.1, it is standard to show that $I \in C^1(X^\alpha, \mathbb{R})$ with the Frechét derivative $I'$ given by

$$\begin{aligned} &I'(u)v \\ &= \int_{\mathbb{R}} \big[({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t),v(t)) - (\nabla W(t,u(t)),v(t))\big]dt \end{aligned} \tag{3.2}$$

for all $u, v \in X^\alpha$. Particularly,

$$I'(u)u = \|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} (\nabla W(t,u(t)),u(t))dt, \quad \forall u \in X^\alpha. \tag{3.3}$$

Here, we say that $u \in X^\alpha$ is a solution of the fractional Hamiltonian system (1.1) if

$$\int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t),v(t)) - (\nabla W(t,u(t)),v(t))]\,dt = 0$$

for every $v \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$. Evidently, any critical point of $I$ is a solution of the fractional Hamiltonian system (1.1).

Before applying Theorem 2.7 to prove our main result, we need to establish the following lemmas.

**Lemma 3.1.** *Suppose that* (A7)–(A9) *are satisfied. Then there exist positive constants* $\rho, \eta$ *such that* $\inf_{\|u\|_{X^\alpha}=\rho} I(u) \geq \eta$.

*Proof.* Note that (A9) implies that there exists $\delta_2 > 0$ such that

$$|W(t,u)| \leq \frac{1}{4\beta_2^2}|u|^2, \quad \forall t \in \mathbb{R},\ |u| \leq \delta_2, \tag{3.4}$$

where $\beta_2$ is the embedding constant given in (2.8). Taking $\rho = \delta_2/\beta_\infty$ with $\beta_\infty$ being given in (2.8). For any $u \in X^\alpha$ with $\|u\|_{X^\alpha} = \rho$, by (2.8), we have

$$\|u\|_{L^\infty} \leq \beta_\infty \|u\|_{X^\alpha} = \delta_2. \tag{3.5}$$

Then it follows from (2.8), (3.1), (3.4) and (3.5) that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t,u)dt \\ &\geq \frac{1}{2}\|u\|_{X^\alpha}^2 - \frac{1}{4\beta_2^2}\|u\|_{L^2}^2 \\ &\geq \frac{1}{4}\|u\|_{X^\alpha}^2 = \frac{\rho^2}{4} \end{aligned}$$

for all $u \in X^\alpha$ with $\|u\|_{X^\alpha} = \rho$. We complete the proof by taking $\eta = \rho^2/4$. □

**Lemma 3.2.** *Suppose that* (A7)–(A12) *are satisfied. Then there exists* $e \in X^\alpha$ *such that* $\|e\|_{X^\alpha} > \rho$ *and* $I(e) < 0$*, where* $\rho$ *is given in Lemma 3.1.*

*Proof.* Choose $\varphi_0 \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ such that $|\varphi_0(t)| \le 1$ for all $t \in \mathbb{R}$, and

$$|\varphi_0(t)| = \begin{cases} 0, & t \in (-\infty, b_1] \cup [b_2, \infty), \\ 1, & t \in [(3b_1 + b_2)/4, (b_1 + 3b_2)/4]. \end{cases}$$

For any $\lambda > 0$, let

$$\mathcal{I}_\lambda = \{t : |\lambda\varphi_0(t)| \le K_0\} \cap [b_1, b_2], \quad \mathcal{J}_\lambda = \{t : |\lambda\varphi_0(t)| > K_0\} \cap [b_1, b_2],$$

where $K_0$ is the constant of (A11). Noting that $|\lambda\varphi_0(t)| > K_0$ for all $t \in [(3b_1 + b_2)/4, (b_1 + 3b_2)/4]$ whenever $\lambda > K_0$, we have $\operatorname{meas}(\mathcal{I}_\lambda) \le b_2 - b_1$ and $[(3b_1 + b_2)/4, (b_1 + 3b_2)/4] \subset \mathcal{J}_\lambda$ for $\lambda > K_0$. Hence, combining this with (3.1), (A9), (A11) and (A12), we have

$$\begin{aligned} I(\lambda\varphi_0) &= \frac{\lambda^2}{2}\|\varphi_0\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, \lambda\varphi_0)dt \\ &= \frac{\lambda^2}{2}\|\varphi_0\|_{X^\alpha}^2 - \int_{\mathcal{I}_\lambda} W(t, \lambda\varphi_0)dt - \int_{\mathcal{J}_\lambda} W(t, \lambda\varphi_0)dt \\ &\le \frac{\lambda^2}{2}\|\varphi_0\|_{X^\alpha}^2 + K_2(b_2 - b_1) - \int_{(3b_1+b_2)/4}^{(b_1+3b_2)/4} W(t, \lambda\varphi_0)dt \\ &= \lambda^2\Big[\frac{1}{2}\|\varphi_0\|_{X^\alpha}^2 + \frac{K_2(b_2 - b_1)}{\lambda^2} - \int_{(3b_1+b_2)/4}^{(b_1+3b_2)/4} \frac{W(t, \lambda\varphi_0)}{\lambda^2|\varphi_0|^2}dt\Big] \\ &= -\infty, \quad \text{as } \lambda \to +\infty, \end{aligned}$$

where $K_2 = \max\{|W(t, u)| : t \in [b_1, b_2], |u| \le K_0\}$. Thus, we can finish the proof by taking $e = \lambda_0\varphi_0$ with $\lambda_0 > \max\{K_0, \rho/\|\varphi_0\|_{X^\alpha}\}$ large enough. ☐

**Lemma 3.3.** *Suppose that* (A7)–(A13) *are satisfied. Then* $I$ *satisfies the Cerami condition.*

*Proof.* We follow partially the idea of the proof in [19, Lemma 2.4]. Let $\{u_k\} \subset X^\alpha$ be a Cerami sequence, i.e., $I(u_k)$ is bounded and $\|I'(u_k)\|_{(X^\alpha)^*}(1 + \|u_k\|_{X^\alpha}) \to 0$ as $k \to +\infty$. Then there exists a constant $D_0 > 0$ such that

$$|I(u_k)| \le D_0, \text{ for every } k \in \mathbb{N}, \quad (I'(u_k), u_k) \to 0. \tag{3.6}$$

We firstly prove that $\{u_k\}$ is bounded in $X^\alpha$. Argue indirectly. Suppose that there exists $\{u_k\} \subset X^\alpha$ satisfying (3.6) but $\|u_k\|_{X^\alpha} \to \infty$ as $k \to \infty$. For each $k \in \mathbb{N}$, let $w_k = u_k/\|u_k\|_{X^\alpha}$, then

$$\|w_k\|_{X^\alpha} = 1, \quad \forall k \in \mathbb{N}. \tag{3.7}$$

Combining (3.1), (3.3) and (3.6), one has

$$3D_0 \ge 2I(u_k) - (I'(u_k), u_k) = \int_{\mathbb{R}} \widetilde{W}(t, u_k)dt \tag{3.8}$$

for $k$ large enough and it follows from (3.1) and (3.6) that

$$\Big|\frac{1}{2} - \int_{\mathbb{R}} \frac{W(t, u_k)}{\|u_k\|_{X^\alpha}^2}dt\Big| \le \frac{D_0}{\|u_k\|_{X^\alpha}^2} \to 0 \quad \text{as } k \to \infty,$$

which implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}} \frac{W(t, u_k)}{\|u_k\|_{X^\alpha}^2}dt = \frac{1}{2}. \tag{3.9}$$

By (A9), there exists $\delta_2 > 0$ such that

$$|W(t,u)| \le \frac{1}{32\beta_2^2}|u|^2, \quad \forall t \in \mathbb{R},\ |u| < \delta_2, \tag{3.10}$$

where $\beta_2$ is the embedding constant given in (2.8). Since $\lim_{s\to+\infty} h(s) = \infty$, for the embedding constant $\beta_\infty$ given in (2.8), there exists a constant $R > 0$ such that

$$h(s) \ge 32D_0\beta_\infty^2, \quad \forall s \ge R. \tag{3.11}$$

Let

$$\begin{gathered}
\Lambda_k = \{t \in \mathbb{R} : |u_k| \ge R\}, \quad \Omega_k = \{t \in \mathbb{R} : |u_k| < \delta_2\}, \\
\Theta_k = \{t \in \mathbb{R} : \delta_2 \le |u_k| < R\}, \quad \Upsilon_k = \big\{t \in \mathbb{R} : \frac{|W(t,u_k)|}{|u_k|^2} \le \frac{1}{4\beta_2^2}\big\}, \\
\Upsilon_k^c = \big\{t \in \mathbb{R} : |u_k| = 0 \text{ or } \frac{|W(t,u_k)|}{|u_k|^2} > \frac{1}{4\beta_2^2}\big\}.
\end{gathered} \tag{3.12}$$

Then we infer from (A9), (2.8), (3.7), (3.10) and (3.12) that

$$\begin{aligned}
\int_{\Upsilon_k} \frac{|W(t,u_k)|}{\|u_k\|_{X^\alpha}^2} dt &= \int_{\Upsilon_k} \frac{|W(t,u_k)|}{|u_k|^2}|w_k|^2 dt \\
&\le \frac{1}{4\beta_2^2}\int_{\Upsilon_k} |w_k|^2 dt \\
&\le \frac{1}{4\beta_2^2}\int_{\mathbb{R}} |w_k|^2 dt \\
&\le \frac{\beta_2^2\|w_k\|_{X^\alpha}^2}{4\beta_2^2} = \frac{1}{4}
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
\int_{\Upsilon_k^c\cap\Omega_k} \frac{|W(t,u_k)|}{\|u_k\|_{X^\alpha}^2} dt &\le \frac{1}{32\beta_2^2}\int_{\Upsilon_k^c\cap\Omega_k} \frac{|u_k|^2}{\|u_k\|_{X^\alpha}^2} dt \\
&\le \frac{1}{32\beta_2^2\|u_k\|_{X^\alpha}^2}\int_{\mathbb{R}} |u_k|^2 dt \\
&\le \frac{\beta_2^2\|u_k\|_{X^\alpha}^2}{32\beta_2^2\|u_k\|_{X^\alpha}^2} \le \frac{1}{32}.
\end{aligned} \tag{3.14}$$

Combining (A9), (A13), (2.8), (3.7), (3.8), (3.11) and (3.12), we have

$$\begin{aligned}
&\int_{\Upsilon_k^c\cap(\Lambda_k\cup\Theta_k)} \frac{|W(t,u_k)|}{\|u_k\|_{X^\alpha}^2} dt \\
&= \int_{\Upsilon_k^c\cap\Lambda_k} \frac{|W(t,u_k)|}{|u_k|^2}|w_k|^2 dt + \int_{\Upsilon_k^c\cap\Theta_k} \frac{|W(t,u_k)|}{\|u_k\|_{X^\alpha}^2} dt \\
&\le \|w_k\|_{L^\infty}^2 \int_{\Upsilon_k^c\cap\Lambda_k} \frac{\widetilde{W}(t,u_k)}{h(|u_k|)} dt + \int_{\Upsilon_k^c\cap\Theta_k} \frac{|u_k|^2\widetilde{W}(t,u_k)}{h(|u_k|)\|u_k\|_{X^\alpha}^2} dt \\
&\le \frac{\beta_\infty^2\|w_k\|_{X^\alpha}^2}{32D_0\beta_\infty^2}\int_{\mathbb{R}} \widetilde{W}(t,u_k) dt + \frac{R_k}{\|u_k\|_{X^\alpha}^2}\int_{\mathbb{R}} \widetilde{W}(t,u_k) dt \\
&\le \frac{3D_0}{32D_0} + \frac{3D_0R_k}{\|u_k\|_{X^\alpha}^2} \\
&\le \frac{3}{32} + \frac{1}{32} = \frac{1}{8}
\end{aligned} \tag{3.15}$$

for $k$ lager enough, where $R_k := \max_{t\in\Upsilon_k^c\cap\Theta_k} |u_k|^2/h(|u_k|)$, and we use the fact that $\lim_{k\to\infty} 3D_0R_k/\|u_k\|^2_{X^\alpha} = 0$ since $R_k \le \max_{\delta_2\le s\le R} s^2/h(s) < +\infty$. Combining (3.13)–(3.15), we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{|W(t,u_k)|}{\|u_k\|^2_{X^\alpha}}dt\\
&= \int_{\Upsilon_k} \frac{|W(t,u_k)|}{\|u_k\|^2_{X^\alpha}}dt + \int_{\Upsilon_k^c\cap\Omega_k} \frac{|W(t,u_k)|}{\|u_k\|^2_{X^\alpha}}dt + \int_{\Upsilon_k^c\cap(\Lambda_k\cup\Theta_k)} \frac{|W(t,u_k)|}{\|u_k\|^2_{X^\alpha}}dt\\
&\le \frac{1}{4} + \frac{1}{32} + \frac{1}{8} = \frac{13}{32}
\end{aligned}
$$

for $k$ lager enough, which is in contradiction to (3.9). Thus $\{u_k\}$ is bounded in $X^\alpha$. Since $X^\alpha$ is a Hilbert space, passing to a subsequence if necessary, we may assume that there is $u \in X^\alpha$ such that

$$u_k \rightharpoonup u \quad \text{in } X^\alpha, \tag{3.16}$$

which together with (3.6) yields

$$(I'(u_k) - I'(u))(u_k - u) \to 0. \tag{3.17}$$

Moreover, by Lemma 2.6 and the Hölder inequality,

$$\int_{\mathbb{R}} (\nabla W(t,u_k) - \nabla W(t,u), u_k - u)\, dt \to 0. \tag{3.18}$$

By (3.2),

$$\|u_k - u\|^2_{X^\alpha} = (I'(u_k) - I'(u))(u_k - u) + \int_{\mathbb{R}} (\nabla W(t,u_k) - \nabla W(t,u), u_k - u)\, dt.$$

Combining this with (3.17) and (3.18), we have $\|u_k - u\|^2_{X^\alpha} \to 0$ as $k \to +\infty$. The proof is complete. □

Now we are in a position to give the proof of our main result.

*Proof of Theorem 1.1.* Let $E = X^\alpha$ and $I$ be the functional defined on $E$ by (3.1), then Lemmas 3.1 and 3.2 show that $I$ satisfies the conditions (1) and (2) in Theorem 2.7. Besides, it follows from Lemma 3.3 that $I$ satisfies the Cerami condition. Therefore, by Theorem 2.7 and Remark 2.8, $I$ possesses a critical value $c \ge \eta > 0$. Thus, there exists a critical point $u \in X^\alpha\backslash\{0\}$ of $I$ with $I(u) = c$, which is a nontrivial solution of the fractional Hamiltonian system (1.1). The proof of Theorem 1.1 is complete. □

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