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S-ASYMPTOTICALLY ω-PERIODIC MILD SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This article concerns the existence of mild solutions to the semilinear fractional differential equation

$$D_t^{\alpha} u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad t \geq 0$$

with nonlocal conditions $u(0) = u_0 + g(u)$ where $D_t^{\alpha}(\cdot)$ $(1 < \alpha < 2)$ is the Riemann-Liouville derivative, $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X, $f : \mathbb{R}^+ \times X \to X$ is S-asymptotically ω-periodic with respect to the first variable. We use the Krsnoselskii's theorem to prove our main theorem. The results obtained are new even in the context of asymptotically ω-periodic functions. An application to fractional relaxation-oscillation equations is given.

1. Introduction

Consider the semilinear fractional differential equation with non-local conditions,

$$D_t^{\alpha} u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad 1 < \alpha < 2,\ t \geq 0, \tag{1.1}$$

$$u(0) = u_0 + g(u)\,, \tag{1.2}$$

where $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space $X, u_0 \in X$ and $D_t^{\alpha}(\cdot)$ is the Riemann-Liouville derivative, and $g : \mathcal{C} \to \mathcal{C}$ is a continuous mapping.

In 2012, Zhao, Chang and N'Guérékata [33] showed that there exists a mild solution $u(t)$ that is asymptotically almost automorphic. We assume that the semilinear function f is asymptotically almost automorphic. We show here that there exists a mild solution $u(t)$ that is S-asymptotically ω-periodic, if the semilinear function. We assume that f is S-asymptotically ω-periodic function, a concept introduced in 2008, by Henriquez, Pierri and Tabos [19]. Both sets containing each of these type functions also contains the set of asymptotically ω-periodic functions. Cuevas and de Souza [9] proved the existence and uniqueness of an S-asymptotically ω-periodic solution of an equivalent problem with local conditions assuming a Lipschitz condition. Our results consider non-local conditions and provide assumptions where the Lipschitz condition is not necessary.

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Many real world phenomena can be described very successfully by models using mathematical tools of fractional calculus, such as dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, modeling of earthquakes, fluid dynamics, traffic models, measurements of viscoelastic material properties and viscoplasticity; see [1, 9] and references therein.

A fractional oscillator equation is a generalization of the classical harmonic oscillator equation by replacing the second-order derivative by a fractional order derivative; that is

$$D_t^\alpha u(t) + c^2 u(t) = f(t), \quad 1 < \alpha < 2,\ t \geq 0,\ c \in \mathbb{R}.$$

Damping effects can be expanded to fractional relaxation-oscillation and diffusion-wave phenomena, which include generalized equations (1.1) and (1.2); see [3, 8, 22]

The paper is organized as follows. In Section 2, we recall some properties of S-asymptotically ω-periodic functions and derive a variation of constants formula. In Section 3 we prove our main results and present an example in Section 4.

2. Preliminaries

In what follows, $(X, \|\cdot\|)$ will denote a complex Banach space, $BC(\mathbb{R}^+, X)$ will be the space of all bounded and continuous functions $f : \mathbb{R}^+ \to X$, $C_0(\mathbb{R}^+, X)$ the space of all continuous functions $f : \mathbb{R}^+ \to X$ such that $\lim_{t\to\infty} \|f(t)\| = 0$. Both spaces are Banach spaces equipped with the supremum norm.

2.1. S-asymptotically ω-periodic functions.

Definition 2.1 (Fréchet)**.** Let $g \in BC(\mathbb{R}^+, X)$ and $\omega > 0$. We say that a continuous and bounded function $f : [0, \infty) \to X$ is asymptotically ω-periodic if it admits the decomposition

$$f = g + h,$$

where $g \in P_\omega(X)$ and $h \in C_0(\mathbb{R}^+, X)$. The set of all such functions is denoted:

$$AP_\omega(X) := P_\omega(X) \oplus C_0(\mathbb{R}^+, X).$$

Definition 2.2 ([19])**.** A function $f \in BC(\mathbb{R}^+, X)$ is said to be S-asymptotically ω-periodic if there exists $\omega > 0$ such that

$$\lim_{t\to\infty} (f(t + \omega) - f(t)) = 0$$

In this case we say that ω is an asymptotic period of f. The set of all such functions is denoted by $SAP_\omega(X)$.

Additionally, if we set the shift operator $\Pi_\omega : BC(\mathbb{R}^+, X) \to BC(\mathbb{R}^+, X)$ with $\Pi_\omega f(t) = f(t + \omega)$, then

$$SAP_\omega(X) = (\Pi_\omega - I)^{-1} C_0(\mathbb{R}^+, X).$$

Remark 2.3 ([19])**.** It is easy to check that $AP_\omega(X) \subset SAP_\omega(X)$. The inclusion is strict. Indeed we have the following example.

Example 2.4 ([19])**.** Let $f : \mathbb{R}^+ \to c_0$ where $c_0 = \{x = (x_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} x_n = 0\}$ equipped with the norm $\|x\| = \sup_{n\in\mathbb{N}} |x(n)|$, and

$$\Big(f(t) = \frac{2tn}{n^2 + t^2}\Big)_{n\in\mathbb{N}}.$$

It is clear that $f(t)$ is uniformly continuous and $f \in SAP_\omega(X)$. But $f \notin AP_\omega(X)$, because even though each coordinate $f_n \in AP_\omega(X)$, $f_n(n) = 1 \Rightarrow \|f(n)\| = 1$

for all $n \in \mathbb{N}$ on this *infinite dimensional space*. Therefore, there does not exist $h(t)$ such that $\lim_{t\to\infty} \|h(t)\| = 0$. The function f above is a piecewise continuous function that is bounded and non-convergent. Other examples of S-asymptotically ω-periodic functions can be found in [5, 31].

It is proved in [19] that $SAP_\omega(X)$ the space of all S-asymptotically ω-periodic functions on X is a Banach space if equipped with the supremum norm.

Definition 2.5 ([19, 31])**.** A continuous function $f : [0, \infty) \times X \to X$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for every bounded set $K \subset X$, the set $\{f(t, x) : \ t \geq o, x \in K\}$ is bounded and $\lim_{t\to\infty} \|f(t + \omega, x) - f(t, x)\| = 0$ uniformly in $x \in K$.

Definition 2.6 ([19, 31])**.** A continuous function $f : [0, \infty) \times X \to X$ is said to be asymptotically uniformly continuous on bounded sets if for every $\epsilon > 0$ and every bounded set $K \subset X$, there exist $L_{\epsilon,K} > 0$, $\delta_{\epsilon,K} > 0$ such that for every $t > L_{\epsilon,K}$ $\|f(t, x) - f(t, y)\| < \epsilon$ and for every $x, y \in K$ such that $\|x - y\| < \delta_{\epsilon,K}$.

Lemma 2.7 ([5, 19])**.** *If $f : [0, \infty) \times X \to X$ is a function which is uniformly S-asymptotically ω-periodic and asymptotically uniformly continuous on bounded sets and $u(t) \in SAP_\omega(X)$, then the Nemytski operator $\mathcal{N}(\cdot) := f(\cdot, u(\cdot))$ is also in $SAP_\omega(X)$.*

Proof. Let $K = \overline{\mathcal{R}(u)}$ be the closure of the range of the function u. Since $\mathcal{R}(u)$ is a bounded set, it follows that $\sum(\cdot)$ is a bounded function. It is also obviously continuous. Let $\epsilon > 0$. From 2.5, there exists $T > 0$ such that for all $t > T$,

$$\|f(t + \omega, u(t + \omega)) - f(t, u(t + \omega))\| < \frac{\epsilon}{2}.$$

From Definition 2.6, there exists $\delta_{\epsilon,K} > 0$, $L_{\epsilon,K} > 0$ such that for all $t > L_{\epsilon,K} > 0$,

$$\|f(t, u(t + \omega)) - f(t, u(t))\| < \frac{\epsilon}{2},$$

if $\|u(t+\omega) - u(t)\| < \delta_{\epsilon,K}$. Let $t > \max\{T, L_{\epsilon,K}\}$. Then combining all of the above, gives

$$\begin{aligned}
&\|f(t + \omega, u(t + \omega)) - f(t, u(t))\| \\
&\leq \|f(t + \omega, u(t + \omega)) - f(t, u(t + \omega))\| + \|f(t, u(t + \omega)) - f(t, u(t))\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}\,.
\end{aligned}$$

The proof is complete. □

2.2. **A variation of constants formula.** Let us recall sectorial operators:

Definition 2.8 ([7, 32])**.** A closed and linear operator A is said to be sectorial if there exist $0 < \theta < \frac{\pi}{2}$, $M > 0$ and $\tau \in \mathbb{R}$ such that its resolvent exists outside the sector $\tau + S_\theta := \{\tau + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \tau\}$ and

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \tau|}, \quad \lambda \notin \tau + S_\theta,$$

where A generates a family of strongly continuous operators $E_\alpha : \mathbb{R}^+ \to \mathbf{B}(X)$ defined as

$$E_\alpha(t) := \frac{1}{2\pi i} \int_\phi e^{t\lambda} (\lambda^\alpha - A)^{-1} \lambda^{\alpha-1} d\lambda$$

are on a suitable path ϕ outside the sector $\tau + S_\theta$.

Theorem 2.9 ([2, 7, 33]). *The equation*

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad 1 < \alpha < 2,\ t \geq 0,$$
$$u(0) = u_0 + g(u),$$

where A is sectorial with $0 < \theta < \pi(1 - \frac{\alpha}{2}) < \pi/2$, has a mild solution generated by A:

$$u(t) = E_\alpha(t)[u_0 + g(u)] + \int_0^t E_\alpha(t-s) f(s, u(s))\, ds, \quad 0 \leq t \leq T,$$

Proof. By applying the definition of the Riemann-Liouville derivative,

$$D_t^\alpha(r(t)) = \frac{d^m}{dt^m} \int_0^t \frac{(t-s)^{m-\alpha-1}}{\Gamma(m-\alpha)} r(s)\, ds, \quad m-1 < \alpha < m\,,$$

to equation (1.1) after using the Riemann-Liouville derivative $D_t^{1-\alpha}(\cdot)$ on both sides of equation (1.1) with $\beta = 1 - \alpha$ and since $m = 2$ ($m = 0$ for β),

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad 1 < \alpha < 2,\ t \geq 0,$$
$$u(0) = u_0 + g(u)$$

implies

$$u'(t) = \int_0^t \frac{(t-s)^{-\beta-1}}{\Gamma(-\beta)} Au(s)\, ds + f(t, u(t)), \quad -1 < \beta < 0,\ t \geq 0,$$
$$u(0) = u_0 + g(u)\,,$$

which implies

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s) ds + f(t, u(t)), \quad 1 < \alpha < 2,\ t \geq 0 \tag{2.1}$$
$$u(0) = u_0 + g(u)\,. \tag{2.2}$$

Then integrating by t, we have

$$u(t) = u_0 + g(u) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s) ds + \int_0^t f(s, u(s)) ds \tag{2.3}$$

for $1 < \alpha < 2$ and $t \geq 0$.

Now we use Laplace transforms $\hat{r}(\lambda) = \int_0^\infty e^{i\lambda t} r(t) dt$ to find the sectorial resolvent and its mild solution. The Laplace transform of equation 2.3 is

$$\hat{u}(\lambda) = \frac{u_0 + g(u)}{\lambda} + \frac{1}{\lambda^\alpha} A\hat{u}(\lambda) + \frac{1}{\lambda} \hat{f}(\lambda, \hat{u}(\lambda)).$$

Then $\hat{u} = [(\lambda^\alpha - A)^{-1} \lambda^{\alpha-1}](u_0 + g(u) + \hat{f})$. Let $\hat{E}_\alpha(\lambda) = [(\lambda^\alpha - A)^{-1} \lambda^{\alpha-1}]$. Then there exists the mild solution

$$u(t) = E_\alpha(t)[u_0 + g(u)] + \int_0^t E_\alpha(t-s) f(s, u(s)) ds, \quad 0 \leq t \leq T,$$

where the family of sectorial operators

$$E_\alpha(t) := \frac{1}{2\pi i} \int_\phi e^{t\lambda} (\lambda^\alpha - A)^{-1} \lambda^{\alpha-1} d\lambda$$

are on a suitable path ϕ outside the sector $\tau + S_\theta$. □

The previous proof connects theorems and lemmas from references [7, 33], and shows that (2.1), (2.2) is equivalent to (1.1), (1.2).

Lemma 2.10 ([2, 7, 33]). *Let $A : D(A) \subset X \to X$ be a sectorial operator in a complex Banauch space x satisfying $\tau + S_\theta := \{\tau + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \tau\}$ and*

$$\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda - \tau|}, \quad \lambda \notin \tau + S_\theta$$

for some $M > 0$, $\tau < 0$ and $0 < \theta < \pi(1 - \frac{\alpha}{2}) < \pi/2$. Then there exists $C > 0$ such that

$$\|E_\alpha(t)\|_{B(X)} \le \frac{CM}{1 + |\tau| t^\alpha}, \quad t \ge 0\,.$$

Theorem 2.11 (Krasnosel'skii fixed point theorem). *Let M be a closed convex and non-empty subset of a Banach space X and A, B two operators such that*

(i) *$Ax + By \in M$ whenever $x, y \in M$;*
(ii) *A is compact and continuous*
(iii) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

3. Main results

Lemma 3.1. *Suppose $h(t) \in SAP_\omega(X)$. Then the function $F : [0, \infty) \to X$ defined by*

$$F(t) := \int_0^t E_\alpha(t - \xi) h(\xi) d\xi,$$

is also in $SAP_\omega(X)$, where the family of operators generated by the sectorial operator A,

$$E_\alpha(t) := \frac{1}{2\pi i} \int_\phi e^{t\lambda} (\lambda^\alpha - A)^{-1} \lambda^{\alpha - 1} d\lambda, \quad 1 < \alpha < 2,$$

are on a suitable path ϕ outside the sector $\tau + S_\theta$, (as in Definition 2.5).

Proof. Let us write

$$\begin{aligned}
F(t + \omega) - F(t) &= \int_0^{t+\omega} E_\alpha(t + \omega - \xi) h(\xi) d\xi - \int_0^t E_\alpha(t - \xi) h(\xi) d\xi \\
&= \int_{-\omega}^t E_\alpha(t - \xi) h(\xi + \omega) d\xi - \int_0^t E_\alpha(t - \xi) h(\xi) d\xi \\
&= \int_{-\omega}^t E_\alpha(t - \xi)[h(\xi + \omega) - h(\xi)] d\xi + \int_{-\omega}^0 E_\alpha(t - \xi) h(\xi) d\xi.
\end{aligned}$$

Let $\epsilon > 0$ be given. Since $h(t) \in SAP_\omega(X)$, there exists $T > 0$ such that for every $\xi > T$, we have $\|h(\xi + \omega) - h(\xi)\| < \epsilon$. This implies

$$\begin{aligned}
&\|F(t + \omega) - F(t)\| \\
&\le \int_{-\omega}^T \|E_\alpha(t - \xi)[h(\xi + \omega) - h(\xi)]\| d\xi + \int_T^t \|E_\alpha(t - \xi)[h(\xi + \omega) - h(\xi)]\| d\xi \\
&\quad + \int_{-\omega}^0 \|E_\alpha(t - \xi) h(\xi)\| d\xi \\
&\le 2\|h\|_\infty \int_{-\omega}^T \|E_\alpha(t - \xi)\| d\xi + \epsilon \int_T^t \|E_\alpha(t - \xi)\| d\xi + \|h\|_\infty \int_{-\omega}^0 \|E_\alpha(t - \xi)\| d\xi
\end{aligned}$$

$$\begin{aligned}
&\le 2\|h\|_\infty \int_{t-T}^{t+\omega} \|E_\alpha(\xi)\| d\xi + \epsilon \int_0^{t-T} \|E_\alpha(\xi)\| d\xi + \|h\|_\infty \int_t^{t+\omega} \|E_\alpha(\xi)\| d\xi \\
&\le 3\|h\|_\infty \int_{t-T}^{t+\omega} \|E_\alpha(\xi)\| d\xi + \epsilon \int_0^\infty \|E_\alpha(\xi)\| d\xi \\
&\le 3\|h\|_\infty \int_{t-T}^{t+\omega} \frac{CM}{1+|\tau|\xi^\alpha} d\xi + \epsilon \int_0^\infty \frac{CM}{1+|\tau|\xi^\alpha} d\xi \\
&\le 3\|h\|_\infty (T+\omega) \frac{CM}{1+|\tau|(t-T)^\alpha} + \epsilon\, CM|\tau|^{\frac{-1}{\alpha}} \frac{\pi/\alpha}{\sin(\pi/\alpha)},
\end{aligned}$$

where the constants $C > 0$, $M > 0$, and $\tau < 0$ are given by Lemma 2.10. Thus $\|F(t+\omega) - F(t)\| \to 0$ as $t \to \infty$. The proof is now complete. □

We use the following assumptions:

- (A1) The operator A is of sectorial of type $\tau < 0$, which generates a strongly continuous family of linear operators $E_\alpha(t)_{t\ge 0} \subset \mathbf{B}(X)$.
- (A2) $f : [0,\infty) \times X \to X$ is a function which is uniformly S-asymptotically ω-periodic and asymptotically uniformly continuous on bounded sets.
- (A3) There exists $L_f > 0$ such that $\|f(t,x) - f(t,y)\| < L_f\|x-y\|$, for all $t \ge 0, x, y \in X$.
- (A3') There exists $c_f > 0$ such that $\|f(t,x)\| < c_f(1 + \|x\|)$ for all $t \ge 0$,
- (A4) There exists $L_g > 0$ such that for all $u, v \in \mathcal{C} := BC([0,\infty), X) \to \mathcal{C}$, $\|g(u) - g(v)\| < L_g\|u-v\|_\infty$. We assume $CML_g < 1$.

Remark 3.2. It is clear that (A3) implies (A3'). Indeed by (A3), we obtain

$$\|f(x)\| \le \|f(x) - f(0)\| + \|f(0)\| \le L_f\|x\| + \|f(0)\| \le c_f(\|x\| + 1)$$

where $c_f = \max\{L_f, \|f(0)\|\}$.

Now we state and prove our first result.

Theorem 3.3. *Under assumptions* (A1)–(A4), (1.1)-(1.2) *possesses a unique solution in* $SAP_\omega(X)$ *provided* $CM\Big(L_g + L_f|\tau|^{\frac{-1}{\alpha}} \frac{\pi/\alpha}{\sin(\pi/\alpha)}\Big) < 1$.

Proof. Consider the operator $\Omega : SAP_\omega(X) \to SAP_\omega(X)$ defined by

$$\Omega u(t) := E_\alpha(t)[u_0 + g(u)] + \int_0^t E_\alpha(t-\xi) f(\xi, u(\xi)) d\xi.$$

In view of Lemmas 2.7 and 3.1, Ω is well-defined.

Now if $u, v \in SAP_\omega(X)$, we obtain

$$\begin{aligned}
&\|(\Omega u)(t) - (\Omega v)(t)\| \\
&\le \|E_\alpha(t)\|\|g(u) - g(v)\| + \int_0^t \|E_\alpha(t-\xi)\|\|f(\xi, u(\xi)) - f(\xi, v(\xi))\| d\xi \\
&\le \Big(\frac{CM}{1+|\tau|t^\alpha} L_g + L_f \int_0^t \frac{CM}{1+|\tau|\xi^\alpha} d\xi\Big) \|u-v\|_\infty \\
&\le CM\Big(L_g + L_f|\tau|^{\frac{-1}{\alpha}} \frac{\pi/\alpha}{\sin(\pi/\alpha)}\Big) \|u-v\|_\infty\,.
\end{aligned}$$

Therefore $\|\Omega u - \Omega v\|_\infty \le \gamma_{f,g,\alpha}\|u-v\|_\infty$, where

$$\gamma_{f,g,\alpha} = CM\Big(L_g + L_f|\tau|^{\frac{-1}{\alpha}} \frac{\pi/\alpha}{\sin(\pi/\alpha)}\Big) < 1\,.$$

We conclude the existence of a unique solution using the Banach's fixed point theorem. □

Remark 3.4. When equation (1.2) is the local condition $g(u) = 0$, we recover the results by Cuevas and de Souza [9].

Theorem 3.5. *Assume* (A1), (A2), (A3'), (A4). *Then problem* (1.1)-(1.2) *has at least one mild solution* $u(t) \in SAP_\omega(X)$ *if we assume that* $E_\alpha(t)$ *is compact for any* $t > 0$.

Proof. Note that (A4) implies the existence a constant $c_g > 0$ such that $\|g(u)\| \leq c_g(1 + \|u\|)$ for any $u \in BC([0,\infty), X)$, as in Remark 3.2.

We consider the same operator Ω as in the previous theorem and use several steps to achieve our conclusion.

Step 1. Let $B_\rho := \{u \in SAP_\tau(X) : \|u\|_\infty \leq \rho\}$, where

$$\rho > \max\Big\{\frac{CM\big(\alpha\sin(\pi/\alpha)c_g + c_f|\tau|^{-1/\alpha}\pi\big)}{\alpha\sin(\pi/\alpha) - CM\big(\alpha\sin(\pi/\alpha) + \alpha\sin(\pi/\alpha)c_g + c_f|\tau|^{-1/\alpha}\pi\big)}\,, 0\Big\}$$

Define the operators $P, Q : SAP_\tau(X) \to SAP_\tau(X)$ by

$$(Pv)(t) : E_\alpha(t)[v_0 + g(v)],$$
$$(Qu)(t) := \int_0^t E_\alpha(t-\xi) f(\xi, u(\xi)) d\xi\,.$$

Using (A3') we obtain

$$\begin{aligned}
&\|(Pv)(t) + (Qu)(t)\| \\
&\leq \|E_\alpha(t)\|\|u_0 + g(v)]\| + \int_0^t \|E_\alpha(t-\xi) f(\xi, u(\xi))\| d\xi \\
&\leq CM\Big(\frac{1}{1+|\tau|t^\alpha}(\|v_0\| + \|g(v)\|) + \int_0^t \frac{1}{1+|\tau|(t-\xi)^\alpha}\ \|f(\xi, u(\xi))\| d\xi\Big) \\
&\leq CM\Big(\|v_0\| + \|g(v)\| + c_f(1+\|u\|)\int_0^t \frac{1}{1+|\tau|\xi^\alpha}\ d\xi\Big) \\
&\leq CM\Big(\|v_0\| + c_g(1+\|v\|) + c_f(1+\|u\|)|\tau|^{\frac{-1}{\alpha}}\frac{\pi/\alpha}{\sin(\pi/\alpha)}\Big) \\
&\leq CM\Big[\rho + \Big(c_g + \frac{c_f|\tau|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\Big)(1+\rho)\Big] \leq \rho.
\end{aligned}$$

We conclude that For all $u, v \in B_\rho$, $Pv + Qv \in B_\rho$.

Step 2. The operator P is contractive. Indeed, for $u, v \in SAP_\tau(X)$ we have

$$\|(Pu)(t) + (Pv)(t)\| \leq \|E_\alpha(t)\|\|g(u) - g(v)\| \leq CM\frac{1}{1+|\tau|t^\alpha}L_g\|u - v\|_\infty\,.$$

Therefore

$$\|Pu - Pv\|_\infty \leq CML_g\|u - v\|_\infty\,.$$

We conclude by using the assumption $CML_g < 1$.

Step 3. The operator Q is continuous on B_ρ. Let $(u_n) \subset B_\rho$ such that $u_n \to u$ in B_ρ. Then in view of Definition 2.6, $f(\xi, u_n(\xi)) \to f(\xi, u(\xi))$ as $n \to \infty$ for all

$\xi \in [0,\infty)$. Now we have

$$\begin{aligned}
\|(Qu_n)(t)-(Qu)(t)\| &= \|\int_0^t E_\alpha(t-\xi)[f(\xi,u_n(\xi))-f(\xi,u(\xi))]d\xi\| \\
&\le CM\int_0^t \frac{1}{1+|\tau|(t-\xi)^\alpha}[\|f(\xi,u_n(\xi))\|+\|f(\xi,u(\xi))\|]d\xi \\
&\le CMc_f\int_0^t \frac{1}{1+|\tau|\xi^\alpha}[2+\|u_n(\xi)\|+\|u(\xi)\|]d\xi \\
&\le 2CMc_f(1+\rho)|\tau|^{-1/\alpha}\frac{\pi}{\alpha\sin(\pi/\alpha)} \\
&\le \frac{2CMc_f(1+\rho)|\tau|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)} < \infty\,.
\end{aligned}$$

Therefore, $Qu_n \to Qu$ as $n\to\infty$ by the Lebesgues's Dominated Convergence Theorem.

Step 4. The set (Qu_n) where $(u_n)\subset B_\rho$ is uniformly bounded. Indeed for all n, we have

$$\begin{aligned}
\|(Qu_n)(t)\| &= \|\int_0^t E_\alpha(t-\xi)f(\xi,u_n(\xi)d\xi\| \\
&\le CM\int_0^t \frac{1}{1+|\tau|(t-\xi)^\alpha}\|f(\xi,u_n(\xi))\|d\xi \\
&\le CMc_f\int_0^t \frac{1}{1+|\tau|\xi^\alpha}[1+\|u_n(\xi)\|]d\xi \\
&\le CMc_f(1+\rho)|\tau|^{-1/\alpha}\frac{\pi}{\alpha\sin(\pi/\alpha)} \\
&\le \frac{CMc_f(1+\rho)|\tau|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\,.
\end{aligned}$$

This shows that (Qu_n) is uniformly bounded.

Step 5. (Qu_n) with $(u_n)\subset B_\rho$ is equicontinuous. Indeed taking t_1,t_2 such that $0\le t_1<t_2$, we have

$$\begin{aligned}
&\|(Qu_n)(t_1)-(Qu_n)(t_2)\| \\
&= \|\int_0^{t_1} E_\alpha(t_1-\xi)f(\xi,u_n(\xi))d\xi - \int_0^{t_2} E_\alpha(t_2-\xi)f(\xi,u_n(\xi))d\xi\| \\
&= \|\int_0^{t_1} [E_\alpha(t_2-\xi)-E_\alpha(t_1-\xi)]f(\xi,u_n(\xi))d\xi - \int_{t_1}^{t_2} E_\alpha(t_2-\xi)f(\xi,u_n(\xi))d\xi\| \\
&\le \|\int_0^{t_1} [E_\alpha(t_2-\xi)-E_\alpha(t_1-\xi)]f(\xi,u_n(\xi))d\xi\| + \|\int_{t_1}^{t_2} E_\alpha(t_2-\xi)f(\xi,u_n(\xi))d\xi\| \\
&\le CMc_f\Big(\int_0^{t_1}(\frac{1}{1+|\tau|(t_2-\xi)^\alpha}-\frac{1}{1+|\tau|(t_1-\xi)^\alpha})[1+\|u_n(\xi)\|]d\xi \\
&\quad + \int_{t_1}^{t_2}\frac{1}{1+|\tau|(t_2-\xi)^\alpha}[1+\|u_n(\xi)\|]d\xi\Big) \\
&\le CMc_f(1+\rho)\Big(\int_0^{t_1}\frac{1}{1+|\tau|(t_2-\xi)^\alpha}d\xi - \int_0^{t_1}\frac{1}{1+|\tau|(t_1-\xi)^\alpha}d\xi
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\frac{1}{1+|\tau|(t_2-\xi)^\alpha}d\xi\Big)\\
&\le CMc_f(1+\rho)\Big(\int_0^{t_2}\frac{1}{1+|\tau|(\xi)^\alpha}d\xi-\int_0^{t_1}\frac{1}{1+|\tau|(\xi)^\alpha}d\xi\Big)\\
&\le CMc_f(1+\rho)\int_{t_2}^{t_1}\frac{1}{1+|\tau|(\xi)^\alpha}d\xi\\
&<\frac{2CMc_f(1+\rho)}{\alpha}|\tau|^{-1/\alpha}\frac{\pi}{\alpha\sin(\pi/\alpha)}<\infty\,.
\end{aligned}
$$

Since

$$
\lim_{t_1\to t_2}[\frac{CMc_f(1+\rho)}{\alpha}\int_{t_2}^{t_1}\frac{1}{1+|\tau|(\xi)^\alpha}d\xi]=0,
$$

we conclude the equicontinuity of (Qu_n).

Step 6. Q is compact. First, we show that the set $\{(Qu)(t) : u(t) \in B_\rho\}$ is relatively compact in X for each $t > 0$. To this end, fix $t > 0$ and ϵ_0 such that $0 < \epsilon_0 < t$. We have

$$
\big\{(Q_{\epsilon_0}u)(t):=\int_0^{t-\epsilon_0}E_\alpha(t-\epsilon_0-\xi)f(\xi,u(\xi))d\xi\big\}
$$

is uniformly bounded for $u \in B_\rho$. This with the assumption that $E_\alpha(\epsilon_0)$ is compact yield the set $\{E_\alpha(\epsilon_0)(Q_{\epsilon_0}u)(t) : u \in B_\rho\}$ is relatively compact.

Since from Definition 2.5, $E_\alpha(0) = I$ and $E_\alpha(t)x$ is continuous for every $x \in X$, we obtain

$$
R(\epsilon_0)(Q_{\epsilon_0}u)(t)=E_\alpha(\epsilon_0)\int_0^{t-\epsilon_0}E_\alpha(t-\epsilon_0-\xi)f(\xi,u(\xi))d\xi\},
$$

which shows that

$$
\lim_{\epsilon_0\to 0}E_\alpha(\epsilon_0)(Q_{\epsilon_0}u)(t)=(Qu)(t)\,.
$$

We conclude that $\{(Qu)(t) : u(t) \in B_\rho\}$ is relatively compact in X. Finally, Q is compact as claimed. From all of the above, we conclude that problem (1.1)-(1.2) has at least one mild solution $u(t) \in SAP_\omega(X)$, using the Krasnosel'ski's fixed point theorem. □

These results are new even in the context of asymptotically ω-periodic functions.

4. An Example

As an application, we investigate the following fractional relaxation-oscillation equations, that are similar to those introduced in [2, 9, 33].

Example 4.1.

$$
\begin{gathered}
D_t^\alpha u(t,x)=\frac{\partial^2}{\partial x^2}u(t,x)-\mu v(t,x)+D_t^{\alpha-1}\Big(\beta u(t,x)(\cos t+\cos(3t))\\
+\beta(-1)^n[\ln(1+t)-(2n+1)]\sin(u(t,x))\Big),\\
\text{for } e^{2n}-1\le t\le e^{2n+2}-1,\ n\in\mathbb{N}\,,\\
u(t,0)=u(t,\pi)=0,\quad 1<\alpha<2,\ t\ge 0,\ x\in[0,\pi],\\
u(0,\eta)=u_0(\eta)+g(u),\quad \eta\in[0,\pi],
\end{gathered}
$$

where $u_0 \in L^2[0,\pi]$. Let $X = (L^2[0,\pi]; \|\cdot\|_2)$, define the linear operator A defined on X by $Au = u'' - \mu u, (\mu > 0)$ with domain

$$D(A) := \{u \in X : u'' \in X, u(0) = u(\pi) = 0\}\,.$$

Also, let $g(u)$ be a function that satisfies (A4). It is well-known that $\Delta u = u''$ is an infinitesimal generator of a analytic semigroup on $L^2[0,\pi]$; then A is a sectorial of type $\tau = -\mu$. The equations above can be formulated into (1.1)-(1.2) where $u(t) = u(t,\cdot)$. Let us consider the nonlinearity, for all $u \in X, t \geq 0, s \in [0,\pi]$ and $\beta \in \mathbb{R}$ with $u \in SAP_{2\pi}$. Therefore two cases follow.

Case 1.

$$\begin{aligned}\|f(t,u(s)) - f(t,v(s))\| &= \|\beta(u(s)-v(s))(\cos t + \cos(3t)) \\ &\quad + \beta(-1)^n[\ln(1+t) - (2n+1)](\sin(u(s)) - \sin(v(s)))\| \\ &\leq |\beta|\ (2\|u(s)-v(s)\|_\infty + \|\sin(u(s)) - \sin(v(s))\|_\infty).\end{aligned}$$

Therefore,

$$\|f(t,u(s)) - f(t,v(s))\| \leq 3\ |\beta|\ \|u(s)-v(s)\|_\infty,$$

or

$$\|f(t,u(s)) - f(t,v(s))\| \leq 3\ |\beta|\ \|\sin(u(s)) - \sin(v(s))\|_\infty\,.$$

In either inequality, we assume

$$|\beta|\ < |\mu|^{\frac{1}{\alpha}} \frac{\sin(\pi/\alpha)}{\pi/\alpha} \frac{1 - CML_g}{3CM};$$

when by Theorem 3.3, problem (1.1)-(1.2) has a unique S-asymptotically 2π-periodic solution.

Case 2. Since

$$\begin{aligned}\|f(t,u(s))\| &= \|\beta(u(s))(\cos t + \cos(3t)) + \beta(-1)^n[\ln(1+t) - (2n+1)](\sin(u(s))\| \\ &\leq |\beta|\ (2\|u(s)\|_\infty + \|\sin(u(s))\|_\infty) \\ &\leq 3|\beta|(1 + \|u(s)\|_\infty) \quad \Rightarrow \quad \exists c_f = 3|\beta|,\end{aligned}$$

by Theorem 3.5, problem (1.1)-(1.2) has at least one S-asymptotically 2π-periodic solution.

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5. Addendum posted on April 18, 2020

In response to a reader's comments, we want to make the following corrections:
(1) Change the title of subsection 2.2 to "Application of the Laplace transform and subsequent sectorial solutions"
(2) Page 3 line -3: change "defined as" to "defined for $\{\lambda^\alpha : \operatorname{Re}\lambda > \mu\} \subset \rho(A)$ as"
(3) Page 3 line -1: change "are on a suitable" to "which are on a suitable"
(4) Page 4: line -2: delete "The previous proof ... equivalent to (1.1), (1.2)"
(5) Add the condition $g(u) = -u_0$ to the assumptions of Theorem 2.9, and replace its proof by the following.

Proof of Theorem 2.9. By applying the Riemann-Liouville derivative, $D_t^{1-\alpha}(\cdot)$, to both sides of (1.1) with $\beta = 1-\alpha$, and since $m = 2$ ($m = 0$ for β), from

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad 1 < \alpha < 2,\ t \geq 0,$$
$$u(0) = u_0 + g(u) = 0$$

we obtain

$$u'(t) = \int_0^t \frac{(t-s)^{-\beta-1}}{\Gamma(-\beta)} Au(s)\,ds + f(t, u(t)), \quad -1 < \beta < 0,\ t \geq 0,$$
$$u(0) = u_0 + g(u) = 0\,.$$

Recall that the Riemann-Loiuville derivative is

$$D_t^\beta(r(t)) = \frac{d^m}{dt^m} \int_0^t \frac{(t-s)^{m-\beta-1}}{\Gamma(m-\beta)} r(s)\,ds, \quad m-1 < \beta < m\,.$$

Therefore,

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds + f(t, u(t)), \quad 1 < \alpha < 2,\ t \geq 0, \tag{5.1}$$
$$u(0) = u_0 + g(u) = 0\,. \tag{5.2}$$

Now we use Laplace transforms to find the sectorial resolvent and its mild solution. Since $\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds$ is a convolution, the Laplace transform of (5.1)–(5.2) is

$$\lambda\hat{u}(\lambda) - \big(u_0 + g(u)\big) = \frac{A\hat{u}(\lambda)}{\lambda^{\alpha-1}} + \hat{f}(\lambda, \hat{u}(\lambda))$$

which implies $\hat{u} = [(\lambda^\alpha - A)^{-1}\lambda^{\alpha-1}](u_0 + g(u) + \hat{f})$.

Let $\hat{E}_\alpha(\lambda) = [(\lambda^\alpha - A)^{-1}\lambda^{\alpha-1}]$. Then we obtain the mild solution

$$u(t) = E_\alpha(t)[u_0 + g(u)] + \int_0^t E_\alpha(t-s) f(s, u(s))ds, \quad t \geq 0,$$

where the family of sectorial operators

$$E_\alpha(t) := \frac{1}{2\pi i} \int_\phi e^{t\lambda} (\lambda^\alpha - A)^{-1} \lambda^{\alpha-1} d\lambda$$

are defined on a suitable path ϕ outside the sector $\tau + S_\theta$. □

End of addendum

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