Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 32, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

OSCILLATORY BEHAVIOR OF SOLUTIONS TO THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH A SUPERLINEAR NEUTRAL TERM

ERCAN TUNÇ, SAID R. GRACE

Abstract. This article studies the oscillatory and asymptotic behavior of solutions to a class of third-order nonlinear differential equations with superlinear neutral term. The results are obtained by a comparison with first-order delay differential equations whose oscillatory behavior is known, and by using integral criteria. Two examples are provided to illustrate the results.

1. Introduction

This article concerns the oscillatory and asymptotic behavior of solutions to third-order nonlinear differential equation with superlinear neutral term

$$\Big(x(t)+p(t)x^{\alpha}(\tau(t))\Big)'''+q(t)x^{\beta}(\sigma(t))=0, \quad t\geq t_0>0\,. \tag{1.1}$$

In this paper we use the following hypotheses:

- (H1) α and β are the ratios of odd positive integers with $\alpha\geq 1$;
- (H2) $p,q:[t_0,\infty)\to\mathbb{R}$ are real-valued continuous functions with $p(t)\geq 1$, $p(t)\not\equiv 1$ for large t, $q(t)\geq 0$, and $q(t)$ is not identically zero for large t;
- (H3) $\tau,\sigma:[t_0,\infty)\to\mathbb{R}$ are real-valued continuous functions such that $\sigma(t)\leq\tau(t)\leq t$, τ is strictly increasing, and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$. We denote by τ^{-1} the inverse function of τ.

By a solution to (1.1), we mean a function $x\in C^3([t_x,\infty),\mathbb{R})$, and which satisfies (1.1) on $[t_x,\infty)$. We consider only non-trivial solutions, i.e. those that satisfy

$$\sup_{t\geq t_1}|x(t)|>0 \quad \text{for every } t_1\geq t_x\,.$$

Moreover, we tacitly assume that (1.1) possesses solutions, and the functions p,q,τ,σ are smooth enough for the solutions to be continuous. A solution $x(t)$ of (1.1) is said to be oscillatory if it has arbitrarily large zeros on its domain $[t_x,\infty)$; i.e., for any $t_1\in[t_x,\infty)$ there exists $t_2\geq t_1$ such that $x(t_2)=0$; otherwise x is called nonoscillatory, hence eventually positive or eventually negative. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

2010 *Mathematics Subject Classification.* 34C10, 34K11, 34K40.
Key words and phrases. Oscillation of solutions; asymptotic behavior; neutral differential equation.

Submitted March 3, 2020. Published April 13, 2020.

A differential equation in which the highest order derivative of the unknown function appears both with and without delays is called a neutral differential equation. Qualitative properties of solutions such equations have been studied by many authors utilizing various methods. One reason for this is that neutral delay differential equations have applications to electric networks containing lossless transmission lines such as in high speed computers. They also occur in problems dealing with vibrating masses attached to an elastic bar and as the Euler equation for variational problems involving delay equations. See [13] for additional applications.

The problem of oscillatory and asymptotic behavior of solutions for third order neutral differential and dynamic equations has been a very active area of research over the years; see for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 17, 18, 20, 21, 22, 23, 24, 25] and their references. However, the results obtained are for the cases $\alpha = 1$ and/or $0 < \alpha < 1$, i.e., for linear neutral terms; see [2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 17, 18, 20, 21, 22, 23, 24, 25]. For the sublinear neutral term see [9]. This means that the results obtained in these papers cannot be applied to the case where $\alpha > 1$.

Motivated by the above observation, we wish to establish oscillation criteria for equation (1.1) via a comparison with first-order delay differential equations whose oscillatory behavior is known, and by using integral criteria. The results in this paper can be applied when $\lim_{t\to\infty} p(t) = \infty$ for $\alpha > 1$, and when $p(t)$ is a bounded and/or $\lim_{t\to\infty} p(t) = \infty$ for $\alpha = 1$. To the best of our knowledge, there are no results for third-order differential equations with superlinear neutral terms. So this article fills partially the gap in oscillation theory for third-order neutral differential equations. We would like to point out that the results presented in this paper can easily be extended to more general third-order differential equations with superlinear neutral term (see Remark 2.12 below).

2. Main results

For proving our result we use the additional hypotheses:

(H4) For every set of positive constants c, d, θ with $0 < \theta < 1$, we have

$$\Psi(t) := \frac{1}{p(\tau^{-1}(t))}\Big[1 - (\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)})^{2/\alpha\theta}\frac{c^{\frac{1}{\alpha}-1}}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\Big] \geq 0 \tag{2.1}$$

and

$$\Omega(t) := \frac{1}{p(\tau^{-1}(t))}\Big[1 - \frac{d^{\frac{1}{\alpha}-1}}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\Big] \geq 0 \tag{2.2}$$

for all sufficiently large t.

Note that if $\alpha > 1$, these assumptions require $\lim_{t\to\infty} p(t) = \infty$. The following lemma will play an important role in establishing our main results.

Lemma 2.1 ([1, Lemma 2.2.3]). *Let $f \in C^n([t_0,\infty),(0,\infty))$, $f^{(n)}(t)f^{(n-1)}(t) \leq 0$ for $t \geq t_x \geq t_0$, and assume that $\lim_{t\to\infty} f(t) \neq 0$. Then for every $\lambda \in (0,1)$, there exists a $t_\lambda \in [t_x,\infty)$ such that, for all $t \in [t_\lambda,\infty)$,*

$$f(t) \geq \frac{\lambda}{(n-1)!}t^{n-1}\left|f^{(n-1)}(t)\right|.$$

To abbreviate notation we define

$$z(t) = x(t) + p(t)x^\alpha(\tau(t)).$$

The following lemma follows from Kiguradze [15], so we omit its proof.

Lemma 2.2. *Suppose that* (H1)-(H3) *are satisfied and x is an eventually positive solution of equation* (1.1). *Then, there exists $t_1 \in [t_0, \infty)$ such that for $t \geq t_1$, the corresponding function z satisfies one of the following two cases:*

(A) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, *and* $z'''(t) \leq 0$,
(B) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, *and* $z'''(t) \leq 0$.

Lemma 2.3. *Let $x(t)$ be a positive solution of* (1.1) *with $z(t)$ satisfying case (A) of Lemma 2.2 for $t \geq t_1 \geq t_0$. Then, for every θ with $0 < \theta < 1$, we have*

$$z(t) \geq \frac{\theta}{2} t z'(t) \tag{2.3}$$

for all large t.

Proof. Note that in case (A), $z' > 0$ and z'' is decreasing. Then by integration we have

$$z'(t) = z'(t_1) + \int_{t_1}^{t} z''(s)\, ds \geq (t - t_1) z''(t) \quad \text{for } t \geq t_1 \geq t_0\,.$$

Then for $t \geq t_2 = t_1 + 1$, we have

$$\Big(\frac{z'(t)}{t - t_1}\Big)' = \frac{(t - t_1) z''(t) - z'(t)}{(t - t_1)^2} \leq 0\,.$$

hence $z'(t)/(t - t_1)$ is non-increasing for $t \geq t_2$. Using this monotonicity and $t_2 = t_1 + 1$, we have

$$\begin{aligned}
z(t) &= z(t_2) + \int_{t_2}^{t} \frac{(s - t_1) z'(s)}{s - t_1}\, ds \geq \frac{z'(t)}{t - t_1} \int_{t_2}^{t} (s - t_1)\, ds \\
&= \frac{z'(t)}{t - t_1} \Big[\frac{(t - t_1)^2 - (t_2 - t_1)^2}{2}\Big] \\
&= \frac{z'(t)}{t - t_1} \Big[\frac{(t - t_1 + 1)(t - t_2)}{2}\Big] \\
&\geq \frac{z'(t)}{t - t_1} \Big[\frac{(t - t_1)(t - t_2)}{2}\Big] \\
&= z'(t) \frac{(t - t_2)}{2} \geq z'(t) \frac{\theta}{2} t, \quad \text{for } t \geq \theta + t_2\,.
\end{aligned}$$

Then (2.3) follows. $\square$

Lemma 2.4. *Suppose that* (H1)–(H3) *and* (2.1) *hold, and that x is an eventually positive solution of* (1.1) *with $z(t)$ satisfying case (A) of Lemma 2.2. Then*

$$z'''(t) + q(t) \Psi^{\beta/\alpha}(\sigma(t)) z^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \leq 0, \tag{2.4}$$

for large t.

Proof. Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Then, from the definition of z, we have

$$x^{\alpha}(\tau(t)) = \frac{1}{p(t)} (z(t) - x(t)) \leq \frac{z(t)}{p(t)},$$

from which and the fact that $\tau(t) \leq t$ is strictly increasing, it is easy to see that

$$x(\tau^{-1}(t)) \leq \frac{z^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}. \tag{2.5}$$

From the definition of z and (2.5), we obtain

$$\begin{aligned} x^\alpha(t) &= \frac{1}{p(\tau^{-1}(t))}\Big[z(\tau^{-1}(t)) - x(\tau^{-1}(t))\Big] \\ &\ge \frac{1}{p(\tau^{-1}(t))}\Big[z(\tau^{-1}(t)) - \frac{z^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\Big]. \end{aligned} \tag{2.6}$$

Since $z(t)$ satisfies case (A), (2.3) holds, and so we obtain

$$\Big(\frac{z(t)}{t^{2/\theta}}\Big)' = \frac{z'(t) - \frac{2}{\theta t} z(t)}{t^{2/\theta}} \le 0.$$

Therefore $z(t)/t^{2/\theta}$ is decreasing. Since $\tau(t) \le t$ and τ is strictly increasing, it follows that τ^{-1} is increasing and $t \le \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \le \tau^{-1}(\tau^{-1}(t)). \tag{2.7}$$

Since $z(t)/t^{2/\theta}$ is decreasing, it follows that

$$\frac{(\tau^{-1}(\tau^{-1}(t)))^{2/\theta} z(\tau^{-1}(t))}{(\tau^{-1}(t))^{2/\theta}} \ge z(\tau^{-1}(\tau^{-1}(t))).$$

Using this inequality in (2.6), we obtain

$$\begin{aligned} x^\alpha(t) &\ge \frac{1}{p(\tau^{-1}(t))}\Big[z(\tau^{-1}(t)) - \frac{(\tau^{-1}(\tau^{-1}(t)))^{2/\alpha\theta}}{(\tau^{-1}(t))^{2/\alpha\theta}} \frac{z^{1/\alpha}(\tau^{-1}(t))}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\Big] \\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\Big[1 - (\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)})^{2/\alpha\theta} \frac{z^{\frac{1}{\alpha}-1}(\tau^{-1}(t))}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\Big]. \end{aligned} \tag{2.8}$$

Since $z(t)$ is positive and increasing for $t \ge t_1$, there exist a $t_2 \in [t_1, \infty)$ and a constant $c > 0$ such that

$$z(t) \ge c \quad \text{for } t \ge t_2. \tag{2.9}$$

Using this inequality in (2.8) yields

$$x^\alpha(t) \ge \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\Big[1 - (\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)})^{2/\alpha\theta} \frac{c^{\frac{1}{\alpha}-1}}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\Big] = \Psi(t) z(\tau^{-1}(t)),$$

with $\Psi(t)$ defined by (2.1). Using this inequality in (1.1) gives

$$z'''(t) \le -q(t)\Psi^{\beta/\alpha}(\sigma(t)) z^{\beta/\alpha}(\tau^{-1}(\sigma(t))), \tag{2.10}$$

and (2.4) holds. This completes the proof. $\square$

Lemma 2.5. *Suppose that* (H1)–(H3) *and* (2.2) *hold, and x is an eventually positive solution of* (1.1) *with $z(t)$ satisfying case (B) of Lemma 2.2. Then, $z(t)$ either satisfies the inequality*

$$z'''(t) + q(t)\Omega^{\beta/\alpha}(\sigma(t)) z^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \le 0, \tag{2.11}$$

for large t, or $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Proof. Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \ge t_1 \ge t_0$. Proceeding as in the proof of Lemma 2.4, we again see that (2.6) and (2.7) hold. Since $z'(t) < 0$, it follows from (2.7) that

$$z(\tau^{-1}(t)) \ge z(\tau^{-1}(\tau^{-1}(t))).$$

Substituting this inequality in (2.6) yields

$$x^{\alpha}(t) \geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\Big[1 - \frac{z^{\frac{1}{\alpha}-1}(\tau^{-1}(t))}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\Big]. \tag{2.12}$$

Since $z(t)$ satisfies case (B) of Lemma 2.2, there exists a constant κ such that

$$\lim_{t\to\infty} z(t) = \kappa < \infty.$$

Case (i): $\kappa > 0$. Then there exists $t_2 \geq t_1$ such that

$$z(t) \geq \kappa \quad \text{for } t \geq t_2. \tag{2.13}$$

Then

$$z^{\frac{1}{\alpha}-1}(t) \leq \kappa^{\frac{1}{\alpha}-1}.$$

Using this inequality in (2.12) gives

$$x^{\alpha}(t) \geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\Big[1 - \frac{\kappa^{\frac{1}{\alpha}-1}}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\Big] = \Omega(t)z(\tau^{-1}(t)),$$

with $\Omega(t)$ defined by (2.2). Using this inequality in (1.1) yields

$$z'''(t) \leq -q(t)\Omega^{\beta/\alpha}(\sigma(t))z^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \tag{2.14}$$

for $t \geq t_3 \geq t_2$, hence (2.11) holds.

Case (ii): $\kappa = 0$. Then $\lim_{t\to\infty} z(t) = 0$. Since $0 < x(t) \leq z(t)$ on $[t_1, \infty)$, we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof. $\square$

Theorem 2.6. *Let* (H1)–(H4) *hold. If*

$$\int_{t_0}^{\infty} q(s)\Psi^{\beta/\alpha}(\sigma(s))ds = \infty \tag{2.15}$$

and

$$\int_{t_0}^{\infty} q(s)\Omega^{\beta/\alpha}(\sigma(s))ds = \infty\,, \tag{2.16}$$

then every solution $x(t)$ of (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$, and assume (2.1) and (2.2) hold for $t \geq t_1$. The proof when $x(t)$ is eventually negative is similar, so we omit it. Then, from Lemma 2.2, $z(t)$ satisfies either case (A) or case (B) for $t \geq t_1$.

First, we consider case (A). From Lemma 2.4, we see that inequalities (2.9) and (2.10) hold for $t \geq t_3 \geq t_2$. Using (2.9) in (2.10) gives

$$z'''(t) \leq -c^{\beta/\alpha}q(t)\Psi^{\beta/\alpha}(\sigma(t)) \quad \text{for } t \geq t_3. \tag{2.17}$$

Integrating from t_3 to t yields

$$z''(t) \leq z''(t_3) - c^{\beta/\alpha}\int_{t_3}^{t} q(s)\Psi^{\beta/\alpha}(\sigma(s))ds \to -\infty \quad \text{as } t \to \infty,$$

which contradicts $z''(t)$ being positive.

Now we consider case (B). From Lemma 2.5, we again have case (i) or case (ii). In case (i), we see that (2.13) and (2.14) hold for $t \geq t_3$. Using (2.13) in (2.14), we arrive at

$$z'''(t) \leq -\kappa^{\beta/\alpha}q(t)\Omega^{\beta/\alpha}(\sigma(t)) \quad \text{for } t \geq t_3. \tag{2.18}$$

Integrating from t_3 to t yields

$$z''(t) \le z''(t_3) - \kappa^{\beta/\alpha} \int_{t_3}^{t} q(s)\Omega^{\beta/\alpha}(\sigma(s))ds \to -\infty \quad \text{as } t \to \infty,$$

which contradicts $z''(t)$ being positive. In case (ii), as in Lemma 2.5, we see that $x(t) \to 0$ as $t \to \infty$. This completes the proof. $\square$

Next, we establish a new oscillation criterion for (1.1) via a comparison with first-order delay differential equations whose oscillatory behavior is known.

Theorem 2.7. *Let* (H1)–(H3), (2.1) *and* (2.2) *hold. If there exist constants* λ_1, λ_2 *in* $(0, 1)$ *such that the first-order delay differential equations*

$$w'(t) + \frac{\lambda_1^{\beta/\alpha}}{2^{\beta/\alpha}}(\tau^{-1}(\sigma(t)))^{2\beta/\alpha} q(t)\Psi^{\beta/\alpha}(\sigma(t))w^{\beta/\alpha}(\tau^{-1}(\sigma(t))) = 0, \tag{2.19}$$

for some constant $\theta \in (0, 1)$, *and*

$$y'(t) + \frac{\lambda_2^{\beta/\alpha}}{2^{\beta/\alpha}}(\tau^{-1}(\sigma(t)))^{2\beta/\alpha} q(t)\Omega^{\beta/\alpha}(\sigma(t))y^{\beta/\alpha}(\tau^{-1}(\sigma(t))) = 0 \tag{2.20}$$

are oscillatory, then a solution $x(t)$ *of* (1.1) *is either oscillatory, or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1 \ge t_0$, and assume that (2.1) and (2.2) hold for $t \ge t_1$. Then, from Lemma 2.2, $z(t)$ satisfies either case (A) or case (B) for $t \ge t_1$.

First we consider case (A). Proceeding as in the proof of Lemma 2.4, we again arrive at (2.10) for $t \ge t_3 \ge t_2$. Now $z(t) > 0$ and $z'(t) > 0$ on $[t_3, \infty) \subseteq [t_2, \infty)$, so

$$\lim_{t\to\infty} z(t) \ne 0,$$

and hence by Lemma 2.1 and case (A), for every λ, $0 < \lambda < 1$, there exists $t_\lambda \ge t_3$ such that

$$z(t) \ge \frac{\lambda}{2} t^2 z''(t) \quad \text{for } t \ge t_\lambda, \tag{2.21}$$

from which we see that

$$z(\tau^{-1}(\sigma(t))) \ge \frac{\lambda}{2}(\tau^{-1}(\sigma(t)))^2 z''(\tau^{-1}(\sigma(t))) \quad \text{for } t \ge t_5, \tag{2.22}$$

where $\tau^{-1}(\sigma(t)) \ge t_\lambda$ for $t \ge t_5 \ge t_\lambda$. Using (2.22) in (2.10) gives

$$z'''(t) + \frac{\lambda^{\beta/\alpha}}{2^{\beta/\alpha}}(\tau^{-1}(\sigma(t)))^{2\beta/\alpha} q(t)\Psi^{\beta/\alpha}(\sigma(t))(z''(\tau^{-1}(\sigma(t))))^{\beta/\alpha} \le 0,$$

for every λ with $0 < \lambda < 1$. Letting $w(t) = z''(t)$ in the above inequality, we see that w is a positive solution of the first-order delay differential inequality

$$w'(t) + \frac{\lambda^{\beta/\alpha}}{2^{\beta/\alpha}}(\tau^{-1}(\sigma(t)))^{2\beta/\alpha} q(t)\Psi^{\beta/\alpha}(\sigma(t))w^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \le 0 \quad \text{for } t \ge t_5. \tag{2.23}$$

Integrating from $t \ge t_5$ to u and letting $u \to \infty$, we obtain

$$w(t) \ge \int_t^\infty \frac{\lambda^{\beta/\alpha}}{2^{\beta/\alpha}}(\tau^{-1}(\sigma(s)))^{2\beta/\alpha} q(s)\Psi^{\beta/\alpha}(\sigma(s))w^{\beta/\alpha}(\tau^{-1}(\sigma(s)))ds$$

for $t \geq t_5$. The function $w(t)$ is decreasing on $[t_5, \infty)$ for every $\lambda \in (0,1)$, and so by [19, Theorem 1], there exists a positive solution of equation (2.19). This contradicts the fact that equation (2.19) is oscillatory.

Now we consider case (B). From Lemma 2.5, we again have case (i) or case (ii). In case (i), we again have $\lim_{t\to\infty} z(t) \neq 0$ for $t \geq t_2$ and (2.14) holds for $t \geq t_3$. Since $\lim_{t\to\infty} z(t) \neq 0$ for $t \geq t_3$, by Lemma 2.1, for every λ, with $0 < \lambda < 1$, there exists $t_\lambda \geq t_3$ such that (2.21) holds for $t \geq t_\lambda$. Using (2.21) in (2.14) yields

$$z'''(t) + \frac{\lambda^{\beta/\alpha}}{2^{\beta/\alpha}}(\tau^{-1}(\sigma(t)))^{2\beta/\alpha} q(t)\Omega^{\beta/\alpha}(\sigma(t))(z''(\tau^{-1}(\sigma(t))))^{\beta/\alpha} \leq 0,$$

for every λ with $0 < \lambda < 1$ and for $t \geq t_5 \geq t_\lambda$. Letting $y(t) = z''(t)$ in the above inequality, we see that y is a positive solution of the first-order delay differential inequality

$$y'(t) + \frac{\lambda^{\beta/\alpha}}{2^{\beta/\alpha}}(\tau^{-1}(\sigma(t)))^{2\beta/\alpha} q(t)\Omega^{\beta/\alpha}(\sigma(t))y^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \leq 0. \tag{2.24}$$

for $t \geq t_5$. As in case (A), we see that there exists a positive solution of equation (2.20), which contradicts that (2.20) is oscillatory.

In case (ii), as in Lemma 2.5, we see that $x(t) \to 0$ as $t \to \infty$. This completes the proof. $\square$

It is well known from [16] (see also [1, Lemma 2.2.9] that if

$$\liminf_{t\to\infty} \int_{\zeta(t)}^{t} R(s)ds > \frac{1}{e}, \tag{2.25}$$

then the first-order delay differential equation

$$x'(t) + R(t)x(\zeta(t)) = 0 \tag{2.26}$$

is oscillatory, where $R, \zeta \in C([t_0, \infty), \mathbb{R})$ with $R(t) \geq 0$, $\zeta(t) \leq t$, and $\lim_{t\to\infty} \zeta(t) = \infty$. Thus, from Theorem 2.7, we have the following oscillation result.

Corollary 2.8. *Let* (H1)–(H4) *be satisfied and* $\alpha = \beta$. *If*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} (\tau^{-1}(\sigma(s)))^2 q(s)\Psi(\sigma(s))ds > \frac{2}{e} \tag{2.27}$$

and

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} (\tau^{-1}(\sigma(s)))^2 q(s)\Omega(\sigma(s))ds > \frac{2}{e}\,, \tag{2.28}$$

then a solution $x(t)$ *of* (1.1) *either oscillates, or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. From (2.27), one can choose a positive constant λ_1 with $0 < \lambda_1 < 1$ such that

$$\liminf_{t\to\infty} \lambda_1 \int_{\tau^{-1}(\sigma(t))}^{t} (\tau^{-1}(\sigma(s)))^2 q(s)\Psi(\sigma(s))ds > \frac{2}{e}\,. \tag{2.29}$$

Now, in view of (2.25)–(2.26), inequality (2.29) ensures that (2.19) is oscillatory in the case when $\alpha = \beta$. Again, in view of (2.25)–(2.26), inequalities (2.28) ensures that (2.20) is oscillatory in the case when $\alpha = \beta$. So, by Theorem 2.7, the conclusion holds. $\square$

From Theorem 2.7, we have the following result.

Corollary 2.9. *Let* (H1)–(H4) *hold and* $\beta < \alpha$. *If*

$$\int_{t_0}^{\infty} (\tau^{-1}(\sigma(s)))^{2\beta/\alpha} q(s) \Psi^{\beta/\alpha}(\sigma(s)) ds = \infty \tag{2.30}$$

and

$$\int_{t_0}^{\infty} (\tau^{-1}(\sigma(s)))^{2\beta/\alpha} q(s) \Omega^{\beta/\alpha}(\sigma(s)) ds = \infty, \tag{2.31}$$

then a solution $x(t)$ *of equation* (1.1) *either oscillates, or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$, and that assume (2.1) and (2.2) hold for $t \geq t_1$. Proceeding as in the proof of Theorem 2.7, we again arrive at (2.23) and (2.24) for $t \geq t_5$. Using that $w(t) := z''(t)$ is positive and decreasing, and noting that $\tau^{-1}(\sigma(t)) \leq t$, we have

$$w(\tau^{-1}(\sigma(t))) \geq w(t)$$

and so, (2.23) can be written as

$$w'(t) + \frac{\lambda^{\beta/\alpha}}{2^{\beta/\alpha}} (\tau^{-1}(\sigma(t)))^{2\beta/\alpha} q(t) \Psi^{\beta/\alpha}(\sigma(t)) w^{\beta/\alpha}(t) \leq 0,$$

or

$$\frac{w'(t)}{w^{\beta/\alpha}(t)} + \frac{\lambda^{\beta/\alpha}}{2^{\beta/\alpha}} (\tau^{-1}(\sigma(t)))^{2\beta/\alpha} q(t) \Psi^{\beta/\alpha}(\sigma(t)) \leq 0 \quad \text{for } t \geq t_5. \tag{2.32}$$

Integration from t_5 to ∞ gives

$$\int_{t_5}^{\infty} (\tau^{-1}(\sigma(s)))^{2\beta/\alpha} q(s) \Psi^{\beta/\alpha}(\sigma(s)) ds \leq (\frac{2}{\lambda})^{\beta/\alpha} \frac{w^{1-\frac{\beta}{\alpha}}(t_5)}{1-\frac{\beta}{\alpha}} < \infty,$$

which contradicts (2.30). Using the similar arguments, the remainder of proof follows from inequality (2.24) and case (ii) in Theorem 2.7; we omit the details. $\square$

We conclude this paper with the following examples and remarks to illustrate the above results. Our first example is concerned with the equation with superlinear neutral term in the case where $p(t) \to \infty$ as $t \to \infty$, and the second example deals with the equation with linear neutral term in the case where p is a constant function.

Example 2.10. Consider the third-order differential equation with superlinear neutral term

$$z'''(t) + \frac{t}{2} x^3(\frac{t}{4}) = 0, \quad t \geq 1, \tag{2.33}$$

with

$$z(t) = x(t) + t x^3(\frac{t}{2}).$$

Here $p(t) = t$, $q(t) = t/2$, $\tau(t) = t/2$, $\sigma(t) = t/4$, $\alpha = 3$, and $\beta = 3$. Then, it is easy to see that conditions (H1)–(H3) hold, and

$$\tau^{-1}(t) = 2t, \quad \tau^{-1}(\tau^{-1}(t)) = 4t, \quad \tau^{-1}(\sigma(t)) = t/2.$$

It follows from (2.15) and (2.16) that

$$\int_{t_0}^{\infty} q(s) \Psi^{\beta/\alpha}(\sigma(s)) ds = \int_{1}^{\infty} (1 - \frac{2^{2/3\theta}}{c^{2/3} s^{1/3}}) ds = \infty,$$

and

$$\int_{t_0}^{\infty} q(s)\Omega^{\beta/\alpha}(\sigma(s))ds = \int_{1}^{\infty}(1-\frac{1}{d^{2/3}s^{1/3}})ds = \infty;$$

thus (2.15) and (2.16) hold. Then by Theorem 2.6, a solution $x(t)$ of equation (2.33) is either oscillatory, or satisfies $\lim_{t\to\infty} x(t) = 0$.

Example 2.11. Consider the third-order differential equation with linear neutral term

$$z'''(t) + (1+t^{\mu})x^{1/5}(\frac{t}{3}) = 0, \quad t \ge 1, \tag{2.34}$$

with

$$z(t) = x(t) + 20x(\frac{t}{2}).$$

Here $p(t) = 20$, $q(t) = 1 + t^{\mu}$ with $\mu \ge 0$, $\tau(t) = t/2$, $\sigma(t) = t/3$, $\alpha = 1$, and $\beta = 1/5$. Then, it is easy to see that (H1)–(H3) hold, and

$$\tau^{-1}(t) = 2t,\ \tau^{-1}(\tau^{-1}(t)) = 4t, \ \text{ and } \ \tau^{-1}(\sigma(t)) = 2t/3.$$

Choosing $\theta = 1/2$, it follows from (2.30) and (2.31) that

$$\int_{t_0}^{\infty}\big(\tau^{-1}(\sigma(s))\big)^{2\beta/\alpha} q(s)\Psi^{\beta/\alpha}(\sigma(s))ds = \big(\frac{2}{3}\big)^{2/5}\big(\frac{1}{100}\big)^{1/5}\int_{1}^{\infty} s^{2/5}(1+s^{\mu})ds = \infty$$

and

$$\int_{t_0}^{\infty}\big(\tau^{-1}(\sigma(s))\big)^{2\beta/\alpha} q(s)\Omega^{\beta/\alpha}(\sigma(s))ds = \big(\frac{2}{3}\big)^{2/5}\big(\frac{19}{400}\big)^{1/5}\int_{1}^{\infty} s^{2/5}(1+s^{\mu})ds = \infty;$$

thus (2.30) and (2.31) hold. Then by Corollary 2.9, a solution $x(t)$ of equation (2.34) either oscillates, or satisfies $\lim_{t\to\infty} x(t) = 0$.

Remark 2.12. The results of this paper can be easily extended to the third-order differential equation with superlinear neutral term

$$(r(t)(z''(t))^{\gamma})' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \ge t_0 > 0, \tag{2.35}$$

under the two conditions

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt = \infty, \quad \int_{t_0}^{\infty} r^{-1/\gamma}(t)dt < \infty,$$

where $r \in C([t_0,\infty),(0,\infty))$, γ is the ratio of odd positive integers, and the other functions and constant β in the equation are defined as in this paper.

Remark 2.13. It would be of interest to study the oscillatory behavior of all solutions of (1.1) for $p(t) \le -1$ with $p(t) \not\equiv -1$ for large t.

Acknowledgments. The authors would like to express their gratitude to Prof. Julio G. Dix and the anonymous referees for the careful reading of the original manuscript. Their comments helped us improve the presentation of the results, and accentuate important details.

References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan; *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 2010.

[2] B. Baculíková, J. Džurina; *Oscillation of third-order neutral differential equations*, Math. Comput. Model., **52** (2010), 215–226.

[3] G. E. Chatzarakis, J. Džurina, I. Jadlovská; *Oscillatory properties of third-order neutral delay differential equations with noncanonical operators*, Mathematics, **7** (2019), No. 12, 12 pp.

[4] Da-X. Chen, Jie-C. Liu; *Asymptotic behavior and oscilation of solutions of third-order nonlinear neutral delay dynamic equations on time scales*, Canad. Appl. Math. Quart., **16** (2008), 19–43.

[5] P. Das; *Oscillation criteria for odd order neutral equations*, J. Math. Anal. Appl., **188** (1994), 245–257.

[6] J. Džurina, S. R. Grace, I. Jadlovská; *On nonexistence of Kneser solutions of third-order neutral delay differential equations*, Appl. Math. Lett., **88** (2019), 193–200.

[7] Z.Došlá, P. Liška; *Comparison theorems for third-order neutral differential equations*, Electron. J. Differ. Equ., **2016** (2016), No. 38, 1–13.

[8] S. R. Grace, J. R. Graef, M. A. El-Beltagy; *On the oscillation of third order neutral delay dynamic equations on time scales*, Comput. Math. Appl., **63** (2012), 775–782.

[9] S. R. Grace, J. R. Graef, E. Tunç; *Oscillatory behaviour of third order nonlinear differential equations with a nonlinear nonpositive neutral term*, Journal of Taibah University for Science, **13** (2019), 704–710.

[10] J. R. Graef, R. Savithri, E. Thandapani; *Oscillatory properties of third order neutral delay differential equations*, Proceedings of the fourth international conference on dynamical systems and differential equations, May 24–27, 2002, Wilmington, NC, USA, pp. 342–350.

[11] J. R. Graef, E. Tunç, S. R. Grace; *Oscillatory and asymptotic behavior of a third-order nonlinear neutral differential equation*, Opuscula Math., **37** (2017), 839–852.

[12] J. R. Graef, P. W. Spikes, M. K. Grammatikopoulos; *Asymptotic behavior of nonoscillatory solutions of neutral delay differential equations of arbitrary order*, Nonlinear Anal., **21** (1993), 23–42.

[13] J. K. Hale, S. M. Verduyn Lunel; *Introduction to Functional Differential Equations*, Springer, New York, 1993.

[14] Y. Jiang, C. Jiang, T. Li; *Oscillatory behavior of third-order nonlinear neutral delay differential equations*, Adv. Differ. Equ., **2016** (2016), Article ID 171, 12 pages.

[15] I. T. Kiguradze; *On the oscillatory character of solutions of the equation* $d^m u/dt^m + a(t)|u|^n \operatorname{sign} u = 0$, Mat. Sb. (N.S.), **65** (1964), 172–187.

[16] R. G. Koplatadze, T. A. Chanturiya; *Oscillating and monotone solutions of first-order differential equations with deviating argument* (in Russian), Differ. Uravn., **18** (1982), 1463–1465.

[17] T. Li, Yu. V. Rogovchenko; *Asymptotic behavior of higher-order quasilinear neutral differential equations*, Abstr. Appl. Anal., **2014** (2014), Article ID 395368, 11 pages.

[18] B. Mihalíková, E. Kostiková; *Boundedness and oscillation of third order neutral differential equations*, Tatra Mt. Math. Publ., **43** (2009), 137–144.

[19] Ch. G. Philos; *On the existence of nonoscillatory solutions tending to zero at* ∞ *for differential equations with positive delays*, Arch. Math. (Basel), **36** (1981), 168–178.

[20] S. H. Saker, J. R. Graef; *Oscillation of third-order nonlinear neutral functional dynamic equations on time scales*, Dynam. Syst. Appl., **21** (2012), 583–606.

[21] Y. Sun, T. S. Hassan; *Comparison criteria for odd order forced nonlinear functional neutral dynamic equations*, Appl. Math. Comput., **251** (2015), 387–395.

[22] Y. Sun, Y. Zhao; *Oscillatory behavior of third-order neutral delay differential equations with distributed deviating arguments*, J. Inequal. Appl., **2019** (2019), Article ID 207, 16 pages.

[23] E. Thandapani, T. Li; *On the oscillation of third-order quasi-linear neutral functional differential equations*, Arch. Math. (Brno), **47** (2011), 181–199.

[24] E. Thandapani, S. Padmavathy, S. Pinelas; *Oscillation criteria for odd-order nonlinear differential equations with advanced and delayed arguments*, Electron. J. Differ. Equ., **2014** (2014), No. 174, 1–13.

[25] E. Tunç; *Oscillatory and asymptotic behavior of third-order neutral diferential equations with distributed deviating arguments*, Electron. J. Differ. Equ., **2017** (2017), No. 16, 1–12.

Ercan Tunç
Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, 60240, Tokat, Turkey
Email address: ercantunc72@yahoo.com

Said R. Grace
Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12221, Egypt
Email address: saidgrace@yahoo.com